$\aleph_0$-CATEGORICAL STRUCTURES SMOOTHLY APPROXIMATED BY FINITE SUBSTRUCTURES

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ABSTRACT

A classification is given of primitive $\aleph_0$-categorical structures which are smoothly approximated by a chain of finite homogeneous substructures. The proof uses the classification of finite simple groups and some representation theory. The main theorems give information about a class of structures more general than the $\aleph_0$-categorical, ω-stable structures examined by Cherlin, Harrington, and Lachlan.

1. *Introduction*

This paper consists of a classification of a class of $\aleph_0$-categorical structures. In [**7**, **8**, **16**, **17**] various families of $\aleph_0$-categorical structures which are nice model-theoretically were examined. Here, we use group-theoretic methods to describe a more general class of structures. In particular, this sheds light on the automorphism groups of $\aleph_0$-categorical, ω-stable structures.

All structures in this paper will have countable (which for us includes finite) domains, and will have finitary relations and possibly functions and constants from some first order language. A structure is said to be *relational* if its language has only relations. The automorphism group of a structure is the group of all permutations of its domain which preserve its relations and functions and fix the constants. A first order theory is said to be $\aleph_0$-*categorical* if it has a countable model and all its countable models are isomorphic. By a theorem due to Ryll-Nardzewski [**20**] and independently to Engeler and Svenonius, the following are equivalent for a structure $\mathcal{M}$ with domain M and automorphism group $\operatorname{Aut}\mathcal{M}$:

(i) $\mathcal{M}$ is $\aleph_0$-categorical;

(ii) $\operatorname{Aut}\mathcal{M}$ has finitely many orbits on M^k for all $k \in \mathbb{N}\backslash\{0\}$.

An $\aleph_0$-categorical relational structure $\mathcal{M}$ is said to be *homogeneous* if, whenever A, B are finite subsets of $\mathcal{M}$ and ϕ: $A \to B$ is an isomorphism, there is $\hat{\phi} \in \operatorname{Aut}\mathcal{M}$ extending ϕ. It is well-known that an $\aleph_0$-categorical relational structure $\mathcal{M}$ is homogeneous if and only if, modulo $\operatorname{Th}(\mathcal{M})$, every formula with parameters in $\mathcal{M}$ is equivalent to a quantifier-free formula. We say that $\mathcal{M}$ is a *transitive* structure if $\operatorname{Aut}\mathcal{M}$ is transitive on M, and that $\mathcal{M}$ is a *primitive* structure if $\operatorname{Aut}\mathcal{M}$ is a primitive permutation group. If $\mathcal{M}$ is $\aleph_0$-categorical, then it is primitive if and only if there is no non-trivial equivalence relation on $\mathcal{M}$ definable by a parameter-free formula. For other model-theoretic notions, such as stability and ω-stability, we refer to [**6**].

In [**7**, **8**, **16**, **17**] the following classes are examined: in [**8**] and [**16**], the class $\mathcal{H}$ of those $\aleph_0$-categorical structures which are stable and homogeneous over a finite relational language; in [**17**], the class $\mathcal{D}$ of those $\aleph_0$-categorical, ω-stable structures in which the coordinatising strictly minimal sets are indiscernible; and in [**7**], the class $\mathcal{C}$ of all $\aleph_0$-categorical, ω-stable structures. The containments

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$\mathscr{H} \subset \mathscr{D} \subset \mathscr{C}$ hold (with $\subset$ being used in its strict sense). Here we consider the class $\mathscr{S}$ of $\aleph_0$-categorical, smoothly approximable structures (the term 'smoothly approximable' will be defined later). It follows from Corollary 7.4 of [7] that $\mathscr{C} \subseteq \mathscr{S}$ (see also Corollary 5.5 below). In this paper we classify the primitive structures in $\mathscr{S}$, some of which do not lie in $\mathscr{C}$ (see Proposition 5.8 below). We also obtain a coordinatisation theorem for transitive structures in $\mathscr{S}$, analogous to Theorem 4.1 of [7]. Our main results are stated in Theorems 1.1 and 1.2 below.

Let L be a fixed relational language, and let $\mathscr{M}$, $\mathscr{N}$ be L-structures having, as always in this paper, domains M and N respectively. Following [7], we say that $\mathscr{N}$ *is a finite homogeneous substructure of* $\mathscr{M}$ (written $\mathscr{N} \subseteq_{\mathrm{hom}} \mathscr{M}$) if the following condition holds:

(I) $\mathscr{M}$ is $\aleph_0$-categorical, $\mathscr{N}$ is a finite substructure of $\mathscr{M}$, and if $\bar{a}$, $\bar{b} \in N^r$ for some $r > 0$ then $\bar{a}$, $\bar{b}$ lie in the same $\mathrm{Aut}\,\mathscr{M}$-orbit if and only if they lie in the same $(\mathrm{Aut}\,\mathscr{M})_N$-orbit (where $(\mathrm{Aut}\,\mathscr{M})_N$ is the setwise stabilizer of N in $\mathrm{Aut}\,\mathscr{M}$).

This is equivalent to the following definition, which is that given in [7].

(II) Let $\mathscr{M}$ be $\aleph_0$-categorical and $\mathscr{N}$ a finite substructure of $\mathscr{M}$. Let $\mathscr{M}^*$ be the expansion of $\mathscr{M}$ to the canonical language L^*, and let $\mathscr{N}^*$ be the L^*-substructure of $\mathscr{M}^*$ with domain N. Then $\mathscr{N}$ is a homogeneous substructure of $\mathscr{M}$ if, whenever $\bar{a}$, $\bar{b} \in N^r$ for some $r > 0$, then $\bar{a}$, $\bar{b}$ lie in the same $\mathrm{Aut}\,\mathscr{M}^*$-orbit if and only if they lie in the same $\mathrm{Aut}\,\mathscr{N}^*$-orbit.

To see the equivalence of (I) and (II), note that the group induced on $\mathscr{N}^*$ by $(\mathrm{Aut}\,\mathscr{M})_N$ is equal to $\mathrm{Aut}\,\mathscr{N}^*$.

Note that if $\mathscr{M}_0 \subseteq_{\mathrm{hom}} \mathscr{M}_1 \subseteq_{\mathrm{hom}} \mathscr{M}_2$, then $\mathscr{M}_0 \subseteq_{\mathrm{hom}} \mathscr{M}_2$.

Next, we shall say that an L-structure $\mathscr{M}$ is *smoothly approximated* if the following hold:

(i) $\mathscr{M}$ is $\aleph_0$-categorical with countably infinite domain;

(ii) there is a chain $\mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \dots$ of finite substructures of $\mathscr{M}$ with $\mathscr{M} = \bigcup \{\mathscr{M}_i \colon i < \omega\}$ and $\mathscr{M}_i \subseteq_{\mathrm{hom}} \mathscr{M}$ for all $i < \omega$.

Note that if $\mathscr{M}$ is smoothly approximated by a chain $\{\mathscr{M}_i \colon i < \omega\}$ then $\mathscr{M}_i \subseteq_{\mathrm{hom}} \mathscr{M}_j$ for all $i < j$. An essential point in our work is that, as shown in § 4 (in Claims 1 and 2 of the proof of Theorem 1.2), the $\mathscr{M}_i$ will have large automorphism groups, so can be examined group-theoretically.

We prove two theorems describing infinite, primitive, smoothly approximated structures. The first gives an implicit description, and is analogous to the Coordinatisation Theorem of [7]. The second gives an explicit group-theoretic description.

An $\aleph_0$-categorical structure $\mathscr{M}$ with full automorphism group G will be called *classical* if one of (a), (b), (c) or (d) below holds.

(a) The group G is the full symmetric group in its natural action of countably infinite degree.

(b) We have one of:

(i) $\mathrm{PGL}(\aleph_0, q) \leqslant G \leqslant \mathrm{P\Gamma L}(\aleph_0, q)$,

(ii) $\mathrm{PGU}(\aleph_0, q) \leqslant G \leqslant \mathrm{P\Gamma U}(\aleph_0, q)$,

(iii) $\mathrm{PSp}(\aleph_0, q) \leqslant G \leqslant \mathrm{P\Gamma Sp}(\aleph_0, q)$,

(iv) $\mathrm{PO}(\aleph_0, q) \leqslant G \leqslant \mathrm{P\Gamma O}(\aleph_0, q)$.

Here q is a prime power, and G has its natural action on the set of 1-spaces of its natural associated vector space (in the presence of a form, the action is on an orbit of 1-spaces). In the unitary case, the associated vector space is defined over $\mathrm{GF}(q^2)$. In the orthogonal case (as in (d) (iv) below), only one kind of orthogonal geometry arises (see Point 1 at the end of § 2). Note that $\mathrm{P\Gamma O}(\aleph_0, q)$ is defined to be the *full* automorphism group of $\mathrm{PO}(\aleph_0, q)$. It includes a diagonal automorphism of order $(2, q-1)$ which, when q is odd, is induced by a matrix multiplying the form by a non-square.

(c) Let V be an infinite-dimensional vector space over a field $\mathrm{GF}(q)$ of characteristic 2, having a non-singular orthogonal geometry given by a quadratic form $Q(\ )$ with associated bilinear form $(\ ,\ )$. Let v be a non-singular vector of V, and let $\mathrm{PO}(V)_{\langle v\rangle} \leqslant G \leqslant \mathrm{P\Gamma O}(V)_{\langle v\rangle}$. Thus $\mathrm{PSp}(\aleph_0, q) \leqslant G \leqslant \mathrm{P\Gamma Sp}(\aleph_0, q)$. Then G, in its action on either of two orbits of non-singular 2-spaces containing v, gives a classical structure. (This action, which we regard as classical for convenience, arises from the actions of finite odd-dimensional orthogonal groups in even characteristic on either of the two orbits on non-singular hyperplanes in the associated odd-dimensional vector space. The action can also be viewed as the symplectic group of dimension $\aleph_0$ over $\mathrm{GF}(q)$ acting on an orbit of non-singular quadratic forms.)

(d) $G = V \rtimes H$, where V is a vector space of countable dimension over a finite field $\mathrm{GF}(q)$ (over $\mathrm{GF}(q^2)$ in the unitary case), and H satisfies one of:

(i) $\mathrm{GL}(\aleph_0, q) \leqslant H \leqslant \Gamma\mathrm{L}(\aleph_0, q)$,

(ii) $\mathrm{GU}(\aleph_0, q) \leqslant H \leqslant \Gamma\mathrm{U}(\aleph_0, q)$,

(iii) $\mathrm{Sp}(\aleph_0, q) \leqslant H \leqslant \Gamma\mathrm{Sp}(\aleph_0, q)$,

(iv) $\mathrm{O}(\aleph_0, q) \leqslant H \leqslant \Gamma\mathrm{O}(\aleph_0, q)$.

The action of G is the natural affine one (so we may identify M with V, and if $g = (v, h) \in V \rtimes H$, then $(x)g = (x)h + v$ for all $x \in V$).

We shall refer to classical structures of Type (a), (b)(ii), etc.

Next, $\mathcal{M}$ is *almost classical* if it is transitive and there is an $(\mathrm{Aut}\,\mathcal{M})$-invariant partition of its domain into finitely many blocks, each of which carries a classical structure (this means that the group induced on a block has the same orbits on finite ordered sets as one of those in (a)–(d) above). Suppose now that $\mathcal{N}$ is $\aleph_0$-categorical, and $A \subseteq N$ is finite and *algebraically closed* (that is, the pointwise stabiliser in $\mathrm{Aut}\,\mathcal{N}$ of A has no finite orbits on $N \backslash A$). Then the *Grassmannian* $\mathrm{Gr}(\mathcal{N}; A)$ is the structure $\mathcal{M}$ whose domain M is the orbit of A under $\mathrm{Aut}\,\mathcal{N}$, with relations corresponding to orbits of $\mathrm{Aut}\,\mathcal{N}$ on M^r $(r > 0)$. Two $\aleph_0$-categorical structures will be said to be *equivalent* if their automorphism groups are isomorphic as permutation groups. Our main theorem is the following. It is a refinement of an earlier conjecture of Lachlan.

THEOREM 1.1. *Let $\mathcal{M}$ be an infinite, primitive, smoothly approximated structure. Then there are an almost classical structure $\mathcal{N}$ and a finite algebraically closed set $A \subseteq N$ such that $\mathcal{M}$ is equivalent to* $\mathrm{Gr}(\mathcal{N}; A)$.

An example at the end of § 4 shows that Theorem 1.1 has no natural converse.

In § 2 we give a list of certain classes Σ_i $(1 \leqslant i \leqslant 6)$ of finite primitive permutation groups. Each class Σ_i consists of infinitely many infinite families, each parametrised by several fixed natural numbers, and one natural number n (playing the role of dimension) which ranges freely. For each $i = 1, \ldots, 6$ there is a corresponding class Σ_i^∞ of structures, obtained from Σ_i by replacing n by $\aleph_0$ in the obvious places (though care is needed with certain actions on hyperplanes, as explained in § 2). Our group-theoretic analogue of Theorem 1.1 is the following, and in fact Theorem 1.1 is obtained as a corollary of Theorem 1.2. Note that we sometimes refer to the Σ_i (or Σ_i^∞) as classes of structures, and sometimes as classes of permutation groups. This looseness is justified by the notion of *permutation structure,* defined towards the end of this introduction.

THEOREM 1.2. *The infinite primitive smoothly approximated structures are precisely the members of* $\Sigma_1^\infty, \ldots, \Sigma_6^\infty$.

The proof of Theorem 1.2 depends on the following proposition. If G is a permutation group on X, let $s_k(G)$ denote the number of G-orbits on X^k.

PROPOSITION 1.3. *There is a function* $f\colon \mathbb{N} \to \mathbb{N}$ *such that if* G *is a primitive permutation group on a finite set* X *with* $|X| > f(s_5(G))$, *then* $(X\,;G)$ *is one of the permutation groups in* $\Sigma_1, \ldots, \Sigma_6$.

A proof of Proposition 1.3 is given in § 3. It uses the O'Nan–Scott Theorem, the classification of finite simple groups, and recent work of Aschbacher and Liebeck on maximal subgroups of finite classical groups. In § 4 we give proofs of Theorems 1.1 and 1.2, which are easy corollaries of Proposition 1.3. In § 5 we obtain a description of transitive, smoothly approximated structures analogous to Theorem 4.1 of [7]. We also make in § 5 some other model-theoretic observations about smoothly approximated structures.

REMARKS. 1. Since by [7], any $\aleph_0$-categorical, ω-stable structure is smoothly approximated, Theorem 1.2 gives an explicit group-theoretic description of the primitive $\aleph_0$-categorical, ω-stable structures. (It is straightforward to determine precisely which structures in Theorem 1.2 are ω-stable.)

2. Recall that the *rank* of a transitive permutation group G on X is the number of G-orbits on X^2. We hope to write up a proof of the following result: for any $r \in \mathbb{N}$, all but finitely many faithful, primitive permutation groups of finite degree and rank at most r are known. This result is stronger than Proposition 1.3, and the list of permutation groups arising is much longer than that of Proposition 1.3.

Our group-theoretic and model-theoretic notation is fairly standard. Relational structures are usually denoted by $\mathcal{M}$ or $\mathcal{N}$, and have domains M or N respectively. If $\bar{x} = (x_1, \ldots, x_n)$ is a tuple, then $l(\bar{x}) = n$. If $\mathcal{M}$ is an $\aleph_0$-categorical structure and A is a finite subset of M, then $\operatorname{acl}(A)$ (the *algebraic closure* of A) is defined to be the union of the finite orbits of the pointwise stabiliser of A in $\operatorname{Aut} \mathcal{M}$. A relation is said to be *0-definable* in $\mathcal{M}$ if it is definable by a formula without parameters.

If G is a permutation group on a set X, we say that G is *closed* if, when X is given a structure over a language with a relation corresponding naturally to each G-orbit on X^k for all $k \in \mathbb{N}\setminus\{0\}$, G is the full automorphism group of the resulting relational structure. (Equivalently, if $\mathrm{Sym}(X)$ is given the topology of pointwise convergence, G is closed if it is a closed subset of $\mathrm{Sym}(X)$.) We also talk of the *closure* $\bar{G}$ of a permutation group G. When it is convenient to regard a structure $\mathcal{M}$ independently of its language, we often use the notion of a *permutation structure* $\mathcal{M} = (M\,; G)$, defined in [8]. Here, $G \leqslant \mathrm{Sym}(M)$, $s_k(G)$ is finite for all k, and G is closed. Note that for $\aleph_0$-categorical structures, questions about stability, ω-stability and smooth approximation are really questions about permutation structures, as they are independent of the language. If $\mathcal{M} = (M\,; G)$ is a permutation structure, then the *canonical language* for $\mathcal{M}$ is the language L with a relation symbol for each G-orbit on finite ordered sets. Note that over the canonical language L, $\mathcal{M}$ will be a homogeneous structure. Two permutation structures are isomorphic if, regarded as canonical structures over some relational language, they are isomorphic relational structures (or equivalently if their automorphism groups are isomorphic as permutation groups: here, we say that two permutation groups G on Ω, H on Σ are isomorphic if there are a bijection ϕ: $\Omega \to \Sigma$ and an isomorphism ψ: $G \to H$ such that $(\omega g)\phi = (\omega\phi)(g\psi)$ for all $\omega \in \Omega$, $g \in G$).

If G and H are groups, then $G \rtimes H$ denotes a semidirect product of G by H, and $G * H$ denotes a central product of the groups. If G is a permutation group on X and $x \in X$ then G_x is the stabiliser of x in G. If $A \subseteq X$, then G_A, $G_{(A)}$ denote the setwise and pointwise stabilisers respectively of A in G. We let $V(n, q)$ denote an n-dimensional vector space over $\mathrm{GF}(q)$, and $\mathrm{PG}(V)$ denote the corresponding projective space. If G is a group, then the *socle* of G, written $\mathrm{Soc}(G)$, is the direct product of the minimal normal subgroups of G. Finally, we define the *product* action of the permutation group $G \operatorname{wr} H$ on X^Y, where G and H are permutation groups on X and Y respectively. Here, the base group G^Y acts coordinatewise; that is, if $\phi \in X^Y$, $f \in G^Y$, and $d \in Y$, then $(\phi f)(d) = \phi(d)f(d)$. Also, H acts by permuting the coordinates (so if $\phi \in X^Y$, $d \in Y$, and $h \in H$, then $(\phi h)(d) = \phi(dh^{-1})$).

We have not attempted to give detailed background for the group theory used. Some general information on the classical groups and their associated natural representations can be found in § 3 of [8], or in [5]. In [5, Chapter 12] there is useful information on the automorphism groups of the finite classical groups. The survey paper of Cameron [4] can be used as a reference for permutation group notation.

For model theorists not familiar with the classical geometries, we describe briefly the symplectic geometry, which yields the simplest examples of smoothly approximated unstable structures. Let $V(\aleph_0, q)$ be equipped with a bilinear form $(\ ,\ )$ which is alternating (so $(v, v) = 0$ for all $v \in V$) and non-singular. There is a standard basis $\{e_i, f_i\colon i < \omega\}$ of V, with $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_j) = (f_i, f_j) = 0$ for all $i, j < \omega$. The group of invertible linear transformations of V which preserve $(\ ,\ )$ is denoted by $\mathrm{Sp}(\aleph_0, q)$. The classical structures of Type (b)(iii) arise by taking the projective group $\mathrm{PSp}(\aleph_0, q) := \mathrm{Sp}(\aleph_0, q)/Z(\mathrm{Sp}(\aleph_0, q))$ acting on the corresponding projective space, and the classical structures of Type (d)(iii) arise by taking $V(\aleph_0, q) \rtimes \mathrm{Sp}(\aleph_0, q)$ acting on the vector space as a subgroup of the affine group. In each case we may extend the symplectic group by outer automorphisms to obtain other classical structures of the same types.

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2. *A list of primitive permutation groups*

We give below six classes of primitive permutation groups of finite degree. Each class Σ_i contains infinitely many infinite families, each parametrised by several fixed natural numbers and one natural number n (playing the role of dimension) which ranges freely. We also sometimes regard the Σ_i as classes of permutation structures $(X\,;G)$, or as homogeneous structures over the canonical language.

The groups in families Σ_1, Σ_2 and Σ_3 have non-abelian socles, while those in Σ_4, Σ_5 and Σ_6 have elementary abelian socles. The family Σ_3 is built from Σ_1 and Σ_2 via a standard wreath product construction. Family Σ_5 is built from Σ_4, and Σ_6 from both Σ_4 and Σ_5.

Σ_1: Groups A_n and S_n, in the action on the set of t-subsets of $\{1, \ldots, n\}$, for fixed t.

Σ_2: The classical simple groups (linear, symplectic, orthogonal or unitary groups) possibly extended by outer automorphisms, defined over a fixed finite field F and with dimension n, acting on an orbit of t-subspaces (fixed t) of the associated projective space $\mathrm{PG}(n-1, F)$. In the presence of a form, the subspaces will be totally isotropic or non-degenerate, with the exception that for orthogonal groups of even dimension in characteristic 2, we also allow non-singular 1-spaces. We also include in Σ_2 the following two cases.

(i) Any group G satisfying $\mathrm{PSL}(n, F).2 \leqslant G \leqslant \mathrm{P\Gamma L}(n, F).2$ in its natural action on an orbit of pairs $\{U, W\}$ of subspaces, where $\mathrm{Dim}\, U + \mathrm{Dim}\, W = n$ and either $U \leqslant W$ or $U \cap W = \{0\}$. Here .2 refers to the graph automorphism.

(ii) An orthogonal group of odd dimension and even characteristic, acting on either of two orbits of non-singular hyperplanes of the associated vector space.

Σ_3: All groups $G \leqslant G_1 \operatorname{wr} S_m$ in the (primitive) product action, where m is fixed, G_1 is one of the permutation groups in Σ_1 or Σ_2, $\mathrm{Soc}(G)$ is a direct product of m copies of $\mathrm{Soc}(G_1)$, and G induces G_1 on each coordinate and a transitive subgroup of S_m on the set of coordinates.

Σ_4: Affine groups $G = V(n, F) \rtimes G(n, F)$ with $V(n, F)$ identified with the set X. Here F is a fixed finite field, $G(n, F)$ is a classical group of dimension n over F, and $G(n, F)$ has the natural irreducible module $V(n, F)$. Note that if F is of even characteristic, then we take $\mathrm{O}(2m+1, F)$ to have natural module $V(2m, F)$.

Σ_5: Let F be a fixed finite field, and let G_1, G_2 have faithful irreducible representations on the F-vector spaces V_1, V_2 respectively, where $V_1 \rtimes G_1$ is in Σ_4, and both F and $\mathrm{Dim}_F\, V_2$ are fixed. Then $G_1 * G_2$ has a natural representation on $V_1 \otimes V_2$. The set Σ_5 consists of all those subgroups $(V_1 \otimes V_2) \rtimes H$ of

$(V_1 \otimes V_2) \rtimes (G_1 * G_2)$, with $V_1 \otimes V_2$ identified with X, such that

(i) H induces G_i on V_i for $i = 1, 2$,

(ii) the action of H on $V_1 \otimes V_2$ is irreducible.

Σ_6: Let t be a fixed natural number and F a fixed finite field. Let $V_1 \rtimes H$ be a group in Σ_4 or Σ_5 (so V_1 is an FH-module). Then $H \operatorname{wr} S_t$ has a natural imprimitive action on the direct sum of t copies of V_1. Let V_i be the ith copy, and H_i the ith copy of H in the base group H^t, acting on V_i. Then Σ_6 consists of those groups $V \rtimes G_0$, where $G_0 \leq H \operatorname{wr} S_t$, and

(i) G_0 acts irreducibly on V;

(ii) G_0 induces the action of H_i on V_i;

(iii) G_0 induces a transitive subgroup of S_t on $\{V_1, \ldots, V_t\}$.

Note that (i), (ii) and (iii) imply quite severe restrictions—see Point 6 below.

To form the classes $\Sigma_1^\infty, \ldots, \Sigma_6^\infty$ of families of $\aleph_0$-categorical structures, we take the dimension n to be $\aleph_0$. Seven points, however, need further discussion, and, in particular, the class Σ_6^∞ needs further restrictions.

Point 1. If the characteristic is odd, only one kind of orthogonal geometry arises in the classes Σ_i^∞. Essentially this is because an $O^+(2n+2, q)$ geometry contains a $2n$-space with $O^-(2n, q)$ induced on it, and an $O^-(2n+2, q)$ geometry contains a $2n$-space with $O^+(2n, q)$ induced on it. Thus, in the affine case Σ_4^∞, the union of a chain of finite O^+-geometries, each a homogeneous substructure of the next, is the same structure as the union of a chain of finite O^--geometries (each homogeneous in the next), or as the union of a chain of odd-dimensional orthogonal geometries.

Point 2. In characteristic 2, the union of a chain of odd-dimensional orthogonal geometries gives a different kind of countable structure, since it has a radical. In the affine cases $(\Sigma_4^\infty, \ldots, \Sigma_6^\infty)$ this does not affect us, since the corresponding orthogonal group is reducible on the natural module. In the remaining cases Σ_2^∞ and Σ_3^∞ (apart from those in Case (ii) of Σ_2^∞—see Point 4 below) we can replace the orthogonal group by the symplectic group acting naturally on the quotient space by the radical.

Point 3. In Σ_2^∞, a structure whose automorphism group is a projective linear group acting on the set of k-spaces will have automorphism group $\mathrm{PGL}(\aleph_0, F)$. Note however that $\mathrm{PGL}(\aleph_0, F)$ and $\mathrm{P\Gamma L}(\aleph_0, F)$, in their actions on $\mathrm{PG}(\aleph_0, F)$, give different permutation structures, as they have different orbits on finite ordered sets. Similar remarks apply to other classes, and to other classical groups.

Point 4. The infinite-dimensional structures corresponding to Σ_2(ii) (that is, the members of Σ_2^∞(ii)) are precisely the classical structures of Type (c) defined in the Introduction. Note that if H is a permutation group in Σ_2^∞(ii), then Σ_3^∞ contains, for example, the permutation group $H \operatorname{wr} S_m$ in the natural primitive product action.

Point 5. There are also infinite-dimensional structures corresponding to Σ_2(i). Let V be a vector space over a finite field $\mathrm{GF}(q)$, and let $\{e_i\colon i < \omega\}$ be a basis of V. For each $i < \omega$, call a subspace of V of codimension r *good* if it contains all but finitely many of the e_i, and let $\hat{V}$ be the set of good hyperplanes in V. Let V^* be the $\aleph_0$-dimensional vector space consisting of all linear functionals $V \to \mathrm{GF}(q)$ whose kernel lies in $\hat{V}$. There is a map which takes e_i to the functional $\hat{e}_i$ for all $i<\omega$, where $(e_j)\hat{e}_i = \delta_{ij}$ for all $i, j<\omega$; this map induces an involutory incidence-preserving map α: $\mathrm{PG}(V) \cup \hat{V} \to \mathrm{PG}(V) \cup \hat{V}$ with $(\mathrm{PG}(V))\alpha = \hat{V}$, $\hat{V}\alpha = \mathrm{PG}(V)$. Let K satisfy $\mathrm{PGL}(V) \leqslant K \leqslant \mathrm{P\Gamma L}(V)$, and let $H = \{g \in K\colon \langle e_i \rangle g = \langle e_i \rangle$ for all but finitely many $i<\omega\}$. Then the map $h \mapsto \alpha h \alpha$ induces an automorphism of H, which we also denote by α. Fix $r \in \mathbb{N}\backslash\{0\}$, and let M be one of the following two sets:

(a) all pairs $\{U, W\}$ where U is an r-dimensional subspace of V, W is a subspace of V of codimension r, and $U \cap W = 0$;

(b) as for (a), but with $U \leqslant W$.

Let $G = \overline{H\langle \alpha \rangle}$ in its natural action on M. Then G acts primitively on M, and $\mathcal{M} = (M\,; G)$ is a structure in $\Sigma_2^\infty(\mathrm{i})$.

Point 6. We specify further the structures in Σ_6^∞. Graph automorphisms of general linear groups arise here too.

First, fix a finite field F, and let V_1 and V_2 be vector spaces over F with bases $\{e_i\colon i<\omega)$ and $\{f_i\colon i<\omega\}$ respectively. Let L be a group with $\mathrm{GL}(V_1, F) \leqslant L \leqslant \Gamma\mathrm{L}(V_1, F)$, and let H consist of all elements of L which fix all but finitely many of the e_i. Identify V_2 with the dual space V_1^*, so that $f_i(e_j) = \delta_{ij}$ for $i, j<\omega$. Now H admits an outer automorphism α of order 2 corresponding to the vector space isomorphism from V_1 to V_2 which takes each e_i to f_i. The group $H\langle \alpha \rangle$ acts naturally as an irreducible imprimitive linear group on $V_1 \oplus V_2$, with H inducing its natural action on V_1, its dual action on V_2, and α interchanging e_i and f_i for each $i<\omega$. Let G_0 be the closure of H in the action on $V_1 \oplus V_2$. We call permutation structures of the form $(V_1 \oplus V_2\,; (V_1 \oplus V_2) \rtimes G_0)$ *special.* We also include as special the permutation structures $(V_1 \oplus V_2\,; (V_1 \oplus V_2) \rtimes L)$, where L induces a tensor product action on each $V_i = U_i \otimes W_i$, with an infinite-dimensional linear group (isomorphic to the subgroup of index 2 in the group G_0 of the last sentence) acting diagonally, inducing its natural action on U_1 and its dual action on U_2, interchanged by the graph automorphism.

Now Σ_6^∞ consists of all closed affine permutation groups

$$G = (V_1 \oplus \ldots \oplus V_t) \rtimes G_0,$$

where $G_0 \leqslant A \operatorname{wr} S_t$ in the imprimitive linear action, and all the following hold:

(i) G_0 is irreducible on $V_1 \oplus \ldots \oplus V_t$;

(ii) the action of A on each V_i is of type Σ_4^∞ or Σ_5^∞ or is special, and is induced by G_0;

(iii) if B is the subgroup of G_0 which fixes each of $V_1, \ldots, V_t$ setwise, and if $i, j \in \{1, \ldots, t\}$ with $i \neq j$, then $|B^{V_i} : B^{V_i}_{(V_j)}|$ is finite.

We remark that there are corresponding restrictions on the structures in Σ_6. Here 'finite index' in (iii) is replaced by 'bounded index' (so it does not increase with $\operatorname{Dim} V_i$). These restrictions arise essentially because if V_1 and V_2 are n-dimensional vector spaces over F, and ϕ is an isomorphism between V_1 and V_2, then $\mathrm{GL}(n, F)$, in its natural diagonal action on $V_1 \oplus V_2$ which commutes with ϕ, is reducible with invariant subspace $\langle v + v\phi\colon v \in V \rangle$. Thus, diagonal actions can only arise from outer automorphisms, and outer automorphisms of classical groups are well understood (see Carter [5, Chapter 12]).

Point 7. Because of the notion of isomorphism between permutation structures which we are using (see § 1), the permutation structure arising from the action of $\mathrm{PSL}(n, F)$ on 1-spaces of $V(n, F)$ is isomorphic to the permutation structure arising from the action on hyperplanes, even though the corresponding permutation groups are not equivalent. A similar remark applies in Σ_4 to the dual action of $\mathrm{SL}(n, F)$ on $V(n, F)$, and also to structures built from these actions.

3. *Proof of Proposition* 1.3

In this section we prove

PROPOSITION 1.3. *There is a function f: $\mathbb{N} \to \mathbb{N}$ such that if G is a primitive permutation group on a finite set X with $|X| > f(s_5(G))$, then $(X ; G)$ is one of the permutation groups in $\Sigma_1, \ldots, \Sigma_6$.*

In the proof of Proposition 1.3 we consider permutation groups $(X ; G)$ with a fixed value of $s_5(G)$, and show that if $|X|$ is sufficiently large then $(X ; G)$ lies in Σ_i for some i. Various parameters associated with the group G arise in the course of the proof. We say that such a parameter $N = N(G)$ is *small* if, for any fixed $r \in \mathbb{N}$, there exists a constant K_1 such that for any primitive $(X ; G)$ with $s_5(G) \leqslant r$, we have $N(G) < K_1$ (so roughly speaking, $s_5(G) \to \infty$ as $N(G) \to \infty$). And we say that $N(G)$ is *unbounded* if, once we fix any $r \in \mathbb{N}$, then for any K_2 there exists $h \in \mathbb{N}$ such that whenever $(X ; G)$ is primitive with $|X| \geqslant h$ and $s_5(G) \leqslant r$, we have $N(G) > K_2$ (roughly speaking, $N \to \infty$ as $|X| \to \infty$). We say N is *bounded* if it is not unbounded.

As described in § 2, each of the families Σ_i is parametrised by several natural numbers. Precisely one of these parameters is unbounded, and the rest are small.

Recall that the *socle* of a group is the product of its minimal normal subgroups. It is well known (see, for example, § 4 of **[4]**) that if G is a group with a faithful primitive permutation representation, the socle of G is a direct product of isomorphic simple groups. Let $\mathrm{Soc}(G)$ denote the socle of G. Using the method of **[8]** (which uses **[3, 14, 21]**), we first reduce to the case when the socle is abelian.

LEMMA 3.1. *There is a function F: $\mathbb{N} \to \mathbb{N}$ such that if G is a primitive permutation group with non-abelian socle acting on a finite set X with $|X| > F(s_5(G))$, then G is one of the permutation groups in Σ_1, Σ_2 or Σ_3.*

Proof. This is an application of the O'Nan–Scott Theorem **[19]**, and is implicit in **[8]**. By the remarks after Lemma 8 of **[8]**, we may suppose that there is an almost simple primitive group G_1 of degree n_1 such that $G \leqslant G_1 \operatorname{wr} S_m$ in the product action of degree n_1^m, with $\mathrm{Soc}(G) = \mathrm{Soc}(G_1)^m$, G inducing the primitive group G_1 on each coordinate, m small, and G acting transitively on the set of coordinates. Since m is small, the degree n_1 of G_1 is unbounded. Also, G_1 can be taken to satisfy the hypotheses of the lemma (that is, n_1 is large relative to $s_5(G_1)$ in the action on a coordinate), since G does. By Lemma 11 of **[8]**, if $\mathrm{Soc}(G)$ is a simple group of Lie type, then its defining field is small. The proof is completed in Lemmas 13 and 14 of **[8]**.

For the rest of this section we will suppose that G is a primitive permutation group with abelian socle, and that G acts on a set X of unbounded size, with a fixed upper bound for $s_5(G)$. Then $\mathrm{Soc}(G)$ may be identified with a vector space V of dimension n over a prime field $F = \mathrm{GF}(p)$. Since $\mathrm{Soc}(G)$ acts regularly on X, we may also identify X with V. If $H := G_0$ (the stabiliser of the zero vector), then $G = V \rtimes H$, and H acts irreducibly on V with a bounded number of orbits on vectors. Since $s_3(G)$ increases with p, p is small. Let $S = \mathrm{Soc}(H/Z(H))$. Then $S = N_1 \times \ldots \times N_t \times T$, with $N_1, \ldots, N_t$ non-abelian simple groups, and T abelian (possibly trivial). For $i = 1, \ldots, t$, let R_i be the full preimage of N_i under the natural map $H \to H/Z(H)$, and let W be the full preimage of T. Clearly if $j \neq i$ then $\langle R_j, W\rangle \leq C_H(R_i')$. Hence, as $R_i = \langle R_i', Z(H)\rangle$, we have $[R_i, R_j] = [R_i, W] = 1$. Thus $R_1, \ldots, R_t$, W generate a central product. Put

$$R := R_1 * \ldots * R_t * W$$

(so $R \leq H$).

Lemma 3.2. *Suppose that V is not a homogeneous FR-module. Then there is an H-invariant direct decomposition $V = V_1 \oplus \ldots \oplus V_r$ for $r > 1$, with H isomorphic to a subgroup of $\mathrm{GL}(V_1)$ wr S_r in the imprimitive linear action. We may suppose that if H induces H_i on V_i, then*

(i) *V_i is a homogeneous irreducible FH_i-module,*

(ii) *r is small,*

(iii) *H induces a transitive group on $\{V_1, \ldots, V_r\}$,*

(iv) *for each i, $s_4(H_i)$ (in the action on V_i) is small.*

Proof. If V is not a homogeneous FR-module then there is an H-invariant decomposition as above, and if r is chosen to be maximal then (i) will be satisfied. For (ii), note that if $v = v_1 + \ldots + v_r$ with $v_i \in V_i$, then the H-orbit of v varies with the number of v_i which are non-zero. If (iii) failed, then H would be reducible on V. Part (iv) is obvious.

Lemma 3.3. *Suppose that V is a homogeneous but not absolutely irreducible FR-module. Then there is an extension field K of F with $[K:F]$ small, and K-vector spaces V_1 and A such that $V \cong V_1 \otimes_K A$ with $H \leq H_1 * H_2$ preserving this K-tensor decomposition in its action on V. Also*

(i) *$\mathrm{Dim}_K V_1$ is unbounded and $\mathrm{Dim}_K A$ is small,*

(ii) *H_1 is absolutely irreducible on V_1, and H_2 is irreducible on A (over K),*

(iii) *R acts faithfully and absolutely irreducibly on V_1, and trivially on A,*

(iv) *$Z(H) = Z(H_1)$, and $S = \mathrm{Soc}(H_1/Z(H_1))$, with R equal to the preimage of S under the map $H_1 \to H_1/Z(H)$.*

Proof. If V is an irreducible FR-module, then simply put $V_1 = V$, $K = \mathrm{Hom}_{FR}(V, V)$, and let A be a 1-dimensional K-vector space. Thus, we may suppose that V is a reducible FR-module. By Clifford's Theorem $V = \bigoplus_{i=1}^r V_i$, where the V_i are isomorphic irreducible FR-modules. Let $K = \mathrm{Hom}_{FR}(V_1, V_1)$. By Schur's Lemma, K is a field containing F and V has a K-space structure with $H \leq \Gamma\mathrm{L}(V, K)$. Here $|K|$ is small, since $s_3(G)$ increases with $|K|$. Put $A := \mathrm{Hom}_{FR}(V_1, V)$, $M := C_{\mathrm{GL}(V,F)}(R)$ and $L := C_{\Gamma\mathrm{L}(V,F)}(M)$. The following observations are part of Lemma 3.13 of [1].

(a) $L \cong \mathrm{GL}(V_1, K)$.

(b) The representation of $L * M$ on V is K-equivalent to the tensor product representation on $V_1 \otimes_K A$.

(c) $\Gamma(\{V_1, A\})\pi = N_{\Gamma\mathrm{L}(V,F)}(L * M) \cap N_{\Gamma\mathrm{L}(V,F)}(L)$. The notation $\Gamma(\{V_1, A\})\pi$ is explained in [1]. Essentially it is the largest group preserving the tensor product decomposition, possibly extended by an involution interchanging V_1 and A (this involution is excluded in our case, since it does not normalise L).

Now H normalises R, so normalises M and L, and hence lies in $\Gamma(\{V_1, A\})\pi$. Thus, there are irreducible $H_1 \leqslant \Gamma\mathrm{L}(V_1, K)$, $H_2 \leqslant \Gamma\mathrm{L}(A, K)$, with $H \leqslant H_1 * H_2$ acting naturally on $V_1 \otimes_K A$, with H inducing H_1 on V_1 and H_2 on A. By choice of A, R centralises A, so it acts faithfully on V_1. Since $\mathrm{Hom}_{FR}(V_1, V_1) = K$, H_1 is absolutely irreducible on V_1 (formally, V_1 is an absolutely irreducible $K(H_1 \cap \mathrm{GL}(V_1, K))$-module). Thus (ii) and (iii) hold. If $\mathrm{Dim}_K(V_1)$ were bounded, then $|R|$ would be bounded, so $C_H(R)$ would be unbounded, contradicting the fact that $C_H(R) \leqslant K^*$. Thus $\mathrm{Dim}_K(V_1)$ is unbounded. We can identify V with the set of $\mathrm{Dim}_K(V_1) \times \mathrm{Dim}_K(A)$ matrices over K, with $H_1 \cap \mathrm{GL}(V_1, K)$ acting by premultiplication, and $H_2 \cap \mathrm{GL}(A, K)$ acting by postmultiplication. Matrices of different ranks lie in different orbits of $H \cap \mathrm{GL}(V, K)$, so $\mathrm{Dim}_K(A)$ is small, proving (i). To see that $Z(H) = Z(H_1)$, note that $Z(H)$ induces scalars on V, so $Z(H) \leqslant H_1$, and that H_1 centralises H_2. To see that $S = \mathrm{Soc}(H_1/Z(H))$, note that $S \leqslant \mathrm{Soc}(H_1/Z(H))$, and that any normal subgroup of $H_1/Z(H)$ is normal in $H/Z(H)$.

In view of Lemmas 3.2 and 3.3, and of the classes Σ_5 and Σ_6, we now suppose that V is an absolutely irreducible KR-module. Set $S = Q_1 \times \ldots \times Q_m$, where each Q_i is either a non-abelian minimal normal subgroup of $H/Z(H)$ or an abelian characteristically simple group, and if $i \neq j$ and Q_i, Q_j are abelian, then $(|Q_i|, |Q_j|) = 1$. Let P_i be the full preimage of Q_i under the natural map $H \to H/Z(H)$. Then, since W (the preimage of the abelian part of S) is nilpotent, it can be seen that $R = P_1 * \ldots * P_m$. Note that $|K|$ is small, since $s_3(G)$ is small.

LEMMA 3.4. *For each $i = 1, \ldots, m$ there is an absolutely irreducible $K(P_1 \cap \mathrm{GL}(V, K))$-module V_i such that $P_i \leqslant \Gamma\mathrm{L}(V_i, K)$, $V \cong V_1 \otimes \ldots \otimes V_m$, and $H \leqslant H_1 * \ldots * H_m$ acting naturally on the tensor decomposition with the following properties (each case holds for any $i \in \{1, \ldots, m\}$):*

(i) *$H_i \leqslant \Gamma\mathrm{L}(V_i, K)$, and H induces H_i on V_i;*

(ii) *$Z(H) = Z(H_i)$, $Q_i = \mathrm{Soc}(H_i/Z(H_i))$, and P_i is the full preimage of Q_i under the map $H_i \to H_i/Z(H_i)$;*

(iii) *m is small;*

(iv) *all but one of* $\mathrm{Dim}\, V_1, \ldots, \mathrm{Dim}\, V_m$ *is small;*

(v) *H_i has a small number of orbits on vectors of V_i.*

Proof. We again use Lemma 3.13 of [1] (as quoted above). There is a K-tensor decomposition $V = V_1 \otimes_K A$ with P_1 acting absolutely irreducibly on V_1, and $P_2 * \ldots * P_m$ acting irreducibly on A. An induction on m completes the proof of (i), and (ii) follows as in Lemma 3.3. Note that since each P_i is normal in H, there are no group elements interchanging the V_i. To prove (iii), note that if

$$\mathrm{GL}(n_1, q) * \ldots * \mathrm{GL}(n_m, q)$$

acts naturally on $V(n_1, q) \otimes \ldots \otimes V(n_m, q)$, then the number of orbits on vectors is at least $q^{n_1 \cdots n_m}/q^{n_1^2 + \cdots + n_m^2}$, which grows with m. Part (iv) follows by the matrix argument given in the proof of Lemma 3.3(i), and Part (v) is immediate.

In view of Lemma 3.4, and of class Σ_5, we now suppose that the group S is characteristically simple.

LEMMA 3.5. *The group S is not elementary abelian.*

Proof. Suppose for a contradiction that S is an elementary abelian r-group for some prime r. Clearly R is nilpotent, so $R = R_1 \times R_2$, where R_1 is an r-group and R_2 is an r'-group; furthermore $R_2 \leqslant Z(H) \leqslant K^*$, so $|R_2|$ is small and V is an absolutely irreducible KR_1-module.

Since $Z(R_1) \leqslant K^*$, it is cyclic, so $Z(R_1) \cong Z_{r^c}$, where r^c is small. Also $R_1/Z(R_1) \cong (Z_r)^m$ for some m. By Lemma 2.27(f) and Theorem 2.31 of **[13]**, $\dim_K V = r^{m/2}$. Put $n = r^{m/2}$. Since $Z(R_1)$ contains the Frattini subgroup of R_1, Theorem 12.2.2 of **[11]** gives

$$\begin{aligned} |H| &\leqslant |K|.|\mathrm{Aut}\, R_1| \\ &\leqslant |K|.r^{(m+c)c}.|\mathrm{GL}(m, r)| \\ &< |K|.r^{cm+c^2+m^2} \\ &< r^{2m^2} \quad \text{(since } r^c \text{ is small and } m \text{ is unbounded)} \\ &= n^{4m} \\ &= n^{8 \log_r n}. \end{aligned}$$

However $|V| \geqslant 2^n$, so $|V|/|H| \to \infty$, which contradicts the fact that $s_2(G)$ is small.

In view of Lemmas 3.4 and 3.5, we suppose that S is a direct product of isomorphic non-abelian simple groups $N_1, \ldots, N_t$, with full preimages $R_1, \ldots, R_t$ in H, and that $N_1 \times \ldots \times N_t$ is a minimal normal subgroup of $H/Z(H)$.

LEMMA 3.6. *Under these conditions, $t = 1$.*

Proof. Suppose that $t > 1$. Since R_i (for $1 \leqslant i \leqslant t$) has non-abelian centraliser in R, it is reducible. Thus by Clifford's Theorem, V is a direct sum of at least two irreducible KR_i-submodules; for each i let V_i be one of these. Since $R/Z(H)$ is a minimal normal subgroup of $H/Z(H)$, the V_i may be chosen to have equal dimension. Thus the conditions of 3.17 of **[1]** apply, and there is a K-tensor product decomposition $V = V_1 \otimes \ldots \otimes V_t$ with $H \leqslant H_1 \operatorname{wr} S_t$ acting naturally, with $\mathrm{Soc}(H_1/Z(H_1)) = N_1$. Now t is small, as in Lemma 3.4(iii). Hence $\mathrm{Dim}_K(V_1)$ is unbounded. If

$$M = N_H(V_1) \cap \ldots \cap N_H(V_t) \cap \mathrm{GL}(V, K)$$

then $|H : M| \leqslant t! \log |K|$, so M has a small number of orbits on vectors. A matrix argument as in the proof of Lemma 3.3(i) now forces a contradiction.

LEMMA 3.7. *Suppose that S is a non-abelian simple group,* $\mathrm{Soc}(H/Z(H)) = S$, *R is the preimage of S under the map $H \to H/Z(H)$, and V is an absolutely irreducible KR-module with* $\mathrm{Dim}_K V = n$. *Then*

(i) $S \leqslant H/Z(H) \leqslant \mathrm{Aut}\, S$;

(ii) *there are a subfield L of K, and an n-dimensional classical group R_1 over L such that*

(a) *$R \cong R_1 * K_1$, where K_1 is a subgroup of the multiplicative group of K, and*

(b) *R has the natural action on $V = V(n, L) \otimes_L K$.*

Proof. (i) This is immediate, since $C_H(R)$ acts as scalars on V.

(ii) First observe that since $s_5(G)$ is small, we have $|H| > |K|^{3n}$. We now imitate the proof of [**18**, Theorem 4.1]. Suppose first that S is an alternating group A_m. Then as in (a) of [**18**, 4.1], we see that n is $m+1$ or $m+2$ and that V is an irreducible constituent of the natural permutation module of S_m. It follows that H has an unbounded number of orbits on vectors, which is a contradiction.

Next, if S is a sporadic group or a group of Lie type in characteristic other than p, then as in [**18**, 4.1] we have $|H| < |K|^{3n}$. Similarly, if S is an exceptional group of Lie type in characteristic p, the same conclusion holds.

Thus S is a classical group in characteristic p. Let d be the dimension of the natural vector space associated with S. If $d < n$, we see as in [**18**, 4.1] that $|H| < |K|^{3n}$, which is a contradiction. Hence $d = n$ and S is a classical group of dimension n over a subfield L of K. The result now follows, taking R_1 to be a quasisimple subgroup of H for which $R_1/Z(R_1) = S$.

Proof of Proposition 1.3. By Lemma 3.1 we may suppose that G has abelian socle, so $G = V \rtimes H$ where $V = V(n, F)$ and $H \leqslant \mathrm{GL}(V)$. If H is imprimitive as a linear group on V, then by Lemma 3.2 (together with the later reductions), $(X ; G) \in \Sigma_6$. If H is primitive but R is not absolutely irreducible on V, then $(X ; G)$ is in Σ_5 by Lemma 3.3 and later reductions. If V is an absolutely irreducible KR-module but S is not a non-abelian simple group, then by Lemmas 3.4, 3.5 and 3.6, $S = N_1 \times N_2$ where $|S : N_1|$ is small and N_1 is non-abelian simple, and $(X ; G) \in \Sigma_5$. Thus, we may suppose that S is non-abelian simple, and V is an absolutely irreducible $K\bar{S}$-module (where $\bar{S}$ is the full covering group of S). By Lemma 3.7, $(X ; G)$ now lies in Σ_4.

4. *Proofs of the theorems*

We first prove the easier half of Theorem 1.2.

LEMMA 4.1. *Let $\mathcal{M} = (M ; G)$ be a permutation structure in Σ_i^∞ for $i \in \{1, \ldots, 6\}$. Then $\mathcal{M}$ is primitive and smoothly approximated by a chain of finite homogeneous substructures.*

Proof. We must show that if $(M ; G) \in \Sigma_i^\infty$ then the corresponding natural chain of finite structures $(M_n ; G_n)$ $(n < \omega)$ in Σ_i smoothly approximates $(M ; G)$. The primitivity of $(M ; G)$ will then follow from the primitivity of the finite structures.

It is necessary to check that

(a) $(M_n\,;G_n)\subseteq_{\text{hom}}(M_{n+1}\,;G_{n+1})$ for each $n<\omega$,

(b) $\bigcup(\mathcal{M}_n\colon n<\omega)=\mathcal{M}$,

(c) $\bigcup(\mathcal{M}_n\colon n<\omega)$ is $\aleph_0$-categorical.

Parts (b) and (c) are in each case straightforward, so we consider only (a). We use definition (I) of homogeneous substructure, given in § 1.

Case (i). $(M\,;G)\in\Sigma_1^\infty$. This is straightforward.

Case (ii). $(M\,;G)\in\Sigma_2^\infty$. In the cases other than Σ_2^∞(i) and Σ_2^∞(ii), condition (I) is an immediate consequence of Witt's Theorem (see, for example, [2, Chapter 7]).

Suppose first that $(M\,;G)\in\Sigma_2^\infty$(i). We adopt the notation of Point 5 Case (b) of § 2 (we omit Case (a), which is similar). Thus, V is a vector space over GF(q) with basis $(e_i\colon i<\omega)$, $G\leqslant \mathrm{P\Gamma L}(\aleph_0,q)\langle\alpha\rangle$, and M consists of pairs $\{U,W\}$ where U is an r-subspace of V, W is a good subspace of V of codimension r, and $U\leqslant W$. For convenience we shall assume that $G\leqslant \mathrm{PGL}(\aleph_0,q)$. (Note here that field automorphisms cause no more than notational problems; they give easily analysed split extensions of the corresponding subgroups.)

For each $n<\omega$, let $V_n=\langle e_1,\dots,e_n\rangle$, and let K_n be the group induced by G on V_n. Then $K_n\leqslant \mathrm{PGL}(n,q)$, and K_n admits a graph automorphism α_n induced in the natural way by the dual map on the basis $(e_1,\dots,e_n)$. Let

$$M_n=\{\{U,W\}\colon\ U<W<V_n,\ \mathrm{Dim}\,U=r,\ \mathrm{Dim}\,W=n-r\},$$

and let $\mathcal{M}_n=(M_n\,;K_n\langle\alpha_n\rangle)$. We can regard $\mathcal{M}_n$ as a substructure of $\mathcal{M}_{n+1}$ by identifying each subspace W of codimension r in V_n with $W\oplus\langle e_{n+1}\rangle$, and embedding $K_n\langle\alpha_n\rangle$ into $K_{n+1}\langle\alpha_{n+1}\rangle$ in the obvious way (so the image of K_n preserves the decomposition $V_{n+1}=V_n\oplus\langle e_{n+1}\rangle$, and the image of α_n is α_{n+1}). To check that $\mathcal{M}_n\subseteq_{\text{hom}}\mathcal{M}_{n+1}$, we shall verify condition (I) of the definition of homogeneous substructures.

Let $\{U_1,W_1\},\dots,\{U_t,W_t\}\in M_n$, and let $g\in\mathrm{Aut}\,\mathcal{M}_{n+1}$ with $\{U_i,W_i\}g\in M_n$ for $i=1,\dots,t$. We must show that there is $h\in\mathrm{Aut}\,\mathcal{M}_n$ such that $\{U_i,W_i\}g=\{U_i,W_i\}h$ for $1\leqslant i\leqslant t$. By replacing g by $g\alpha_{n+1}$ if necessary, we may suppose that $g\in K_{n+1}$. Let ϕ be the natural map $\mathrm{GL}(n+1,q)\to\mathrm{PGL}(n+1,q)$, and let $\hat g\in\mathrm{GL}(n+1,q)$ satisfy $\hat g\phi=g$. Let A be a matrix representing $\hat g$ with respect to the basis $(e_1,\dots,e_{n+1})$, and let B be the matrix obtained from A by putting the $(n+1,n+1)$-entry equal to 1, and all other entries in the last row or column equal to 0. Then the image under ϕ of the element represented by B fixes $\mathcal{M}_n$ and agrees with g on $U_1,\dots,U_t,W_1,\dots,W_t$, as required.

Suppose next that $(M\,;G)\in\Sigma_2^\infty$(ii). Let q be a power of 2, and let $V(2n+1,q)$ be an orthogonal space invariant under $\mathrm{O}(2n+1,q)$. Then $(M_n\,;G_n)$ is the permutation structure arising from the action of $G_n:=\mathrm{PO}(2n+1,q)$ on either orbit of hyperplanes (again, we assume for convenience that G_n does not contain field automorphisms).

We may describe $\mathcal{M}_n$, $\mathcal{M}_{n+1}$ and the embedding $\mathcal{M}_n\subseteq\mathcal{M}_{n+1}$ as follows for either of the two orbits on hyperplanes. Let $d,d',e_1,\dots,e_{n+1},f_1,\dots,f_{n+1}$ be a basis of a vector space $V_{2n+4}=V(2n+4,q)$ having an O^- geometry with quadratic form Q and bilinear form (,), satisfying the conditions $(e_i,f_j)=\delta_{ij}$,

$$(e_i,e_j)=(f_i,f_j)=(d,e_i)=(d,f_i)=(d',e_i)=(d',f_i)=Q(e_i)=Q(f_i)=0$$

for $1 \leqslant i, j \leqslant n+1$, $Q(d') = 1 = (d, d')$, $Q(d) \neq 0$ and $\langle d, d' \rangle$ anisotropic. Let V_{2n+2} be the subspace with basis $d, d', e_1, \ldots, e_n, f_1, \ldots, f_n$. The stabiliser of $\langle d \rangle$ in $\mathrm{PO}^-(2n+2, q)$ is just $\mathrm{PO}(2n+1, q)$, so our action on an orbit of hyperplanes of $V(2n+1, q)$ can be identified with the action of $\mathrm{PO}^-(2n+2, q)_{\langle d \rangle}$ on an orbit of non-singular 2-spaces. Let $\mathcal{M}_n$ be the resulting structure, and let $\mathcal{M}_{n+1}$ be the corresponding structure consisting of 2-spaces in V_{2n+4} in the orbit of $\langle d, d' \rangle$ under $\mathrm{PO}^-(2n+4, q)_{\langle d \rangle}$. We must show that $\mathcal{M}_n \subseteq_{\mathrm{hom}} \mathcal{M}_{n+1}$. This, however, is now a trivial consequence of Witt's Theorem.

Case (iii). $(M\,; G) \in \Sigma_3^\infty$. If $(M_n\,; G_n) \in \Sigma_3$, then the permutation group induced on each coordinate lies in Σ_2, and $\mathrm{Soc}(G_n)$ is the direct product of the socles of the groups induced on each coordinate. Thus, the result in this case follows easily from that of Case (ii).

Case (iv). $(M\,; G) \in \Sigma_4^\infty$. This is handled by Witt's Theorem.

Case (v). $(M\,; G) \in \Sigma_5^\infty$. Consider first the case where $G = (V_1 \otimes V_2) \rtimes (\mathrm{GL}(\aleph_0, q) * H)$ acting naturally on $V_1 \otimes V_2$, where $V_1 = V(\aleph_0, q)$, $V_2 = V(t, q)$, and H is an irreducible subgroup of $\mathrm{GL}(V_2)$. Let $\mathcal{M}_n = (V(n, q) \otimes V_2\,; (V(n, q) \otimes V_2) \rtimes (\mathrm{GL}(n, q) * H))$ for each $n < \omega$. We may identify $\mathcal{M}_n$ with the set of all $n \times t$ matrices over $\mathrm{GF}(q)$, with $\mathrm{GL}(n, q)$ acting by premultiplication and H by postmultiplication, and embed $\mathcal{M}_n$ into $\mathcal{M}_{n+1}$ by adjoining an extra row and column of zeros to each matrix in M_n. To show that $\mathcal{M}_n \subseteq_{\mathrm{hom}} \mathcal{M}_{n+1}$ we must check condition (I) in the definition of homogeneous substructure. This is done as for Σ_2^∞(i), by replacing a matrix in $\mathrm{GL}(n+1, q)$ by the corresponding matrix with $(n+1, n+1)$-entry 1, all other entries in the last row or column 0, and the remaining entries unchanged. Other classical groups acting on V_1 are handled similarly: if, for example, the group induced on V_1 is symplectic, let $M_n = V(2n, q) \otimes V(t, q)$, $M_{n+1} = V(2n+2, q) \otimes V(t, q)$, and in the last argument replace a matrix in $\mathrm{Sp}(2n+2, q)$ by the matrix obtained by putting all the entries in the last two rows or columns equal to 0, except for a 2×2 submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at the bottom right. Note that field automorphisms again cause no difficulties; neither does the possibility that the group induced on the tensor product is a subgroup of a corresponding central product (to see this last point, note that closed subgroups of small index in infinite-dimensional classical groups are very restricted).

Case (vi). $(M\,; G) \in \Sigma_6^\infty$. First, suppose that $\mathcal{M}$ is a special structure of the form $(V_1 \oplus V_2\,; (V_1 \oplus V_2) \rtimes H\langle \alpha \rangle)$, where V_1, V_2 are infinite-dimensional over $\mathrm{GF}(q)$, the closure of H on V_1 is $\mathrm{GL}(V_1)$, and α is the graph automorphism (as described in Point 5 of § 2; note that again we assume for simplicity that field automorphisms do not arise). Then $\mathcal{M}_n = (W_1(n, q) \oplus W_2(n, q)\,; (W_1 \oplus W_2) \rtimes H_n\langle \alpha \rangle)$, where $W_1(n, q)$ has basis $(e_1, \ldots, e_n)$, W_2 has basis $(f_1, \ldots, f_n)$ identified with the dual basis of $(e_1, \ldots, e_n)$, $H_n = \mathrm{GL}(W_1)$ with its dual action on W_2, and α swaps each pair (e_i, f_i). An element of H_n may be regarded as a pair $(A, (A')^{-1})$, where A acts by premultiplication on W_1 and $(A')^{-1}$ by postmultiplication on W_2. To show that $\mathcal{M}_n \subseteq_{\mathrm{hom}} \mathcal{M}_{n+1}$, we check condition (I). As before, replace a pair $(A, (A')^{-1})$ in H_{n+1} by $(B, (B')^{-1})$, where B is obtained from A by putting the $(n+1, n+1)$-entry equal to 1 and all other entries in the last row or column equal to 0. The other special structures (those built up using tensor products) are handled similarly. More general structures in Σ_6^∞ are treated almost exactly as in Σ_3^∞.

LEMMA 4.2. *Let $\mathcal{M} = (M; G)$ be smoothly approximated by a chain $(\mathcal{M}_i = (M_i; G_i)$: $i < \omega)$ of finite homogeneous substructures. Let L be the canonical language for $\mathcal{M}$, and for each i let $\mathcal{M}_i^*$ be the L-substructure of $\mathcal{M}$ with domain M_i. Then*

(i) $\mathrm{Aut}(\mathcal{M}_i^*) = G_i$ *for all* $i < \omega$,

(ii) $\mathcal{M}_i^*$ *is a homogeneous L-structure for all* $i < \omega$.

Proof. (i) If $g \in G_i$, then since g extends to an element of G, g induces an L-automorphism of M_i, so $g \in \mathrm{Aut}\, \mathcal{M}_i^*$. A similar argument shows that $\mathrm{Aut}\, \mathcal{M}_i^* \subseteq G_i$.

(ii) By (i), together with the definition of homogeneous substructure in terms of automorphism groups, $\mathcal{M}_i^* \subseteq_{\mathrm{hom}} \mathcal{M}$. Since $\mathcal{M}$ is a homogeneous L-structure, and any homogeneous substructure of a homogeneous L-structure is homogeneous, the result follows.

COROLLARY 4.3. *Let $\mathcal{M} = (M; G)$ and $\mathcal{N} = (N; H)$ be infinite permutation structures smoothly approximated by chains $(\mathcal{M}_i$: $i < \omega)$ and $(\mathcal{N}_i$: $i < \omega)$ respectively of finite homogeneous substructures, with $\mathcal{M}_i = (M_i; G_i)$ and $\mathcal{N}_i = (N_i; H_i)$ for all i. Suppose also that for each i, $(M_i; G_i) \cong (N_i; H_i)$. Then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic permutation structures.*

Proof. Let L be the canonical language for $\mathcal{M}$. By Lemma 4.2, we may regard $\mathcal{M}_i$ as a homogeneous L-structure. We construct an isomorphism f: $\mathcal{M} \to \mathcal{N}$ as a union of a chain of isomorphisms f_i: $\mathcal{M}_i \to \mathcal{N}_i$. Suppose that we have constructed $f_1, \ldots, f_k$, each extending the previous isomorphism. Since $\mathcal{M}_{k+1} \cong \mathcal{N}_{k+1}$, we may regard $\mathcal{N}_{k+1}$ as a homogeneous L-structure isomorphic to $\mathcal{M}_{k+1}$. It follows that the isomorphism f_k extends to an L-isomorphism f_{k+1}: $\mathcal{M}_{k+1} \to \mathcal{N}_{k+1}$, as required.

Proof of Theorem 1.2. By Lemma 4.1 it suffices to prove the following. Let $\mathcal{M}$ be an infinite primitive structure smoothly approximated by a chain $\{\mathcal{M}_i$: $i < \omega\}$ of finite homogeneous substructures; then $\mathcal{M} \in \Sigma_i^\infty$ for some $i \in \{1, \ldots, 6\}$.

Claim 1. *For all i, $k < \omega$, $s_k(\mathrm{Aut}\, \mathcal{M}_i) \leqslant s_k(\mathrm{Aut}\, \mathcal{M}) < \omega$.*

Proof of claim. The second inequality follows from the fact that $\mathcal{M}$ is $\aleph_0$-categorical. The first inequality comes from condition (I) of the definition of finite homogeneous substructure (see § 1).

Claim 2. *There is $N \in \mathbb{N}$ such that for all $i > N$, $\mathcal{M}_i$ is primitive.*

Proof of claim. We use a criterion for primitivity given by Higman in [**12**]: that is, $\mathrm{Aut}\, \mathcal{M}_i$ is primitive if and only if for any two $\mathrm{Aut}\, \mathcal{M}_i$-orbits Ω_1, Ω_2 on the set of unordered 2-subsets of M_i, and any $\{x, y\} \in \Omega_1$, there exist $r \in \mathbb{N}$ and a sequence $x = u_0, u_1, \ldots, u_r = y \in M_i$ such that $\{u_i, u_{i+1}\} \in \Omega_2$ for each $i = 0, \ldots, r-1$. Since $\mathcal{M}$ is primitive, we may choose N such that for any two orbits Ω_1, Ω_2 of $\mathrm{Aut}\, \mathcal{M}$ on unordered 2-subsets of M, there is a path $u_0, \ldots, u_r$ as above and lying in M_N. The claim now follows from the definition of homogeneous substructure.

Claim 3. *There are some infinite subsequence $\{\mathcal{M}_{p_j}$: $j < \omega\}$ of $\{\mathcal{M}_i$: $i < \omega\}$, and L-structures $\mathcal{N}_j$ with $\mathcal{N}_j \cong \mathcal{M}_{p_j}$ and $\mathcal{N}_j \subseteq_{\mathrm{hom}} \mathcal{N}_{j+1}$ (for all $j < \omega$) such that some structure in $\bigcup (\Sigma_i^\infty$: $1 \leqslant i \leqslant 6)$ is smoothly approximated by the $\mathcal{N}_j$.*

Proof of claim. First, by Proposition 1.3 and Claims 1 and 2 above, we may suppose that all the $\mathcal{M}_i$ lie in $\bigcup_{k=1}^{6} \Sigma_k$. By the pigeon-hole principle, there is an infinite subsequence of the $\mathcal{M}_i$ consisting of structures all lying in the same Σ_k. The claim now follows by inspecting the families in each Σ_k and applying Lemma 4.1. For example, if infinitely many $\mathcal{M}_i$ lie in Σ_1, then the same value of t (in the definition of Σ_1) occurs for infinitely many of the $\mathcal{M}_i$; this is because the permutation rank increases with t (for the orbit of a pair of t-subsets varies with the size of their intersection). If we choose the $\mathcal{N}_i$ to be increasingly large members of Σ_1 realising this value of t, and each isomorphic to a member of $\{\mathcal{M}_i\colon i<\omega\}$ then this case of the claim will follow from Case (i) of Lemma 4.1. If, say, infinitely many of the $\mathcal{M}_i$ lie in Σ_2(i), then we may suppose that:

(i) the field GF(q) is fixed (as the number of orbits on M_i^4 increases with q);

(ii) if U is the smaller subspace in a pair $\{U, W\}$ which corresponds to a point of M_i, then Dim U is fixed;

(iii) always $U \leqslant W$, or always $U \cap W = 0$.

These observations prove Claim 3 in this case. A similar but easier argument deals with Σ_2(ii). In each case Lemma 4.1 is used. Suppose that all the $\mathcal{M}_i$ lie in Σ_3. Then we may suppose that one value of m always occurs, and that the same subgroup of S_m operates on the set of coordinates; also, that the action on a coordinate is the same for all the $\mathcal{M}_i$ (except for the variation of the free parameter n in the descriptions of Σ_1 and Σ_2). Furthermore, $\mathrm{Soc}(G) = \mathrm{Soc}(G_1)^m$. There is a bound (independent of n) on the number of subgroups of the wreath product G_1 wr S_m which satisfy these conditions, so we may suppose that there is a subsequence $(\mathcal{M}_{p_j}\colon j<\omega)$ consisting of structures in the same family. Similar arguments verify the claim for Σ_4, Σ_5 and Σ_6. Recall here Remark 5 at the end of § 2, which puts tight restrictions on the structures in Σ_6 and Σ_6^∞.

Given that there is an infinite subsequence $\{\mathcal{M}_{p_j}\colon j<\omega\}$ of structures as in Claim 3, the result follows from Corollary 4.3. For by Claim 3 there is a structure $\mathcal{N} \in \Sigma_i^\infty$ smoothly approximated by a chain $(\mathcal{N}_i\colon i<\omega)$ with $\mathcal{N}_i \cong \mathcal{M}_{p_i}$ for all i, and by Corollary 4.3 we have $\mathcal{N} \cong \mathcal{M}$.

Proof of Theorem 1.1. It suffices to run through the list of smoothly approximated structures $\mathcal{M}$ which arise in Theorem 1.2.

(i) $\mathcal{M} \in \Sigma_1^\infty$. Then there are $t \in \mathbb{N}$ and a countably infinite set X such that Aut $\mathcal{M}$ is isomorphic as a permutation group to the action of Sym(X) on the set of t-subsets of X. Let $\mathcal{N}$ be the trivial structure of domain X, and let A be a t-subset of X. Then $\mathcal{M}$ and $\mathrm{Gr}(\mathcal{N}; A)$ are isomorphic permutation structures.

(ii) $\mathcal{M} \in \Sigma_2^\infty$. If $\mathcal{M}$ is not of type (i) or (ii) in Σ_2^∞, then there are a classical structure $\mathcal{N}$ of dimension $\aleph_0$, and a finite-dimensional totally singular or non-singular subspace A, such that $\mathcal{M} \cong \mathrm{Gr}(\mathcal{N}; A)$. Structures in Σ_2^∞(ii) are by definition classical of Type (c), so it remains to consider structures in Σ_2^∞(i).

Let $\mathcal{M} = (M; G) \in \Sigma_2^\infty$(i). We shall use the notation of Remark 4 of § 2 (so there is a vector space V over GF(q) with basis $\{e_i\colon i<\omega)$, and a group $H \leqslant \mathrm{P\Gamma L}(\aleph_0, q)$ with an outer automorphism α so that $H\langle\alpha\rangle$ acts on $V \cup \hat{V}$).

Now let W be a vector space over GF(q) disjoint from V, with a basis $\{f_i\colon i<\omega\}$. The structure $\mathcal{N}$ will have domain $\mathrm{PG}(V) \cup \mathrm{PG}(W)$ (the union of the projective spaces). Let $\phi\colon V \to W$ be the isomorphism with $e_i\phi = f_i$ for all $i<\omega$, and let $\hat{\phi}$ be the corresponding map $\mathrm{PG}(V) \to \mathrm{PG}(W)$. The group $H\langle\alpha\rangle$ acts on N as follows. Elements of H act on $\mathrm{PG}(V)$ in the natural way. If $u, v \in \mathrm{PG}(W)$ and $h \in H$, then $uh = v$ just in case $((u\hat{\phi}^{-1})\alpha)h = (v\hat{\phi}^{-1})\alpha$. Also, if $u \in \mathrm{PG}(V)$ then $u\alpha = u\hat{\phi}$, and if $u \in \mathrm{PG}(W)$ then $u\alpha = u\hat{\phi}^{-1}$. It can be checked that this makes $(\mathrm{PG}(V) \cup \mathrm{PG}(W)\,;\overline{H\langle\alpha\rangle})$ into a permutation structure.

If $\mathcal{M}$ is as in Case (a) in § 2, Point 5, then let

$$A = \langle e_0, \ldots, e_{r-1}\rangle \cup \langle f_0, \ldots, f_{r-1}\rangle.$$

By considering the stabiliser of A in $\overline{H\langle\alpha\rangle}$, we see that $\mathcal{M}$ and $\mathrm{Gr}(\mathcal{N}\,;A)$ are isomorphic as permutation structures. A similar argument applies if $\mathcal{M}$ satisfies Case (b).

(iii) $\mathcal{M} \in \Sigma_3^\infty$. Suppose that $\mathrm{Aut}\,\mathcal{M} \leqslant G_0 \,\mathrm{wr}\, S_r$, where G_0 is a permutation group as in (i) above. Let $\mathcal{N}$ be a structure consisting of an equivalence relation with r countable classes, and let A be a set containing t points from each class. Then $\mathcal{M} \cong \mathrm{Gr}(\mathcal{N}\,;A)$. The cases when G_0 is classical are handled similarly. For example, suppose that $G_0 = \mathrm{PSp}(\aleph_0, q)$ with its action on each coordinate being the action on the set of totally isotropic t-dimensional subspaces of the natural module $V(\aleph_0, q)$. Then identify $\mathcal{N}$ with the disjoint union of r infinite-dimensional projective spaces over $\mathrm{GF}(q)$, each with a non-singular symplectic form, and let A be the disjoint union of r isotropic t-spaces, one from each vector space. It will not matter how these r-spaces are chosen, essentially because in Σ_3 we have $\mathrm{Soc}(G) = \mathrm{Soc}(G_0)^r$.

(iv) $\mathcal{M} \in \Sigma_4^\infty$. Let $\mathcal{N}$ be an infinite-dimensional affine space over the appropriate field (possibly with a form on it) and let 0 be the zero vector. Then $\mathcal{M} \cong \mathrm{Gr}(\mathcal{N}\,;\{0\})$.

(v) $\mathcal{M} \in \Sigma_5^\infty$. Now as a permutation structure, $\mathcal{M}$ is isomorphic to

$$(V_1 \otimes V_2\,;(V_1 \otimes V_2) \rtimes H),$$

where V_1 is an $\aleph_0$-dimensional vector space over $\mathrm{GF}(q)$, V_2 is an r-dimensional vector space over $\mathrm{GF}(q)$, and $H \leqslant H_1 * H_2$ inducing H_1 on V_1 and H_2 on V_2. Choose an H_2-orbit $\Omega = \{C_1, \ldots, C_s\}$ of ordered bases of V_2, and write $C_i \sim C_j$ if there is $\alpha \in Z(H_2)$ such that $C_j = (u_1\alpha, \ldots, u_r\alpha)$ where $C_i = (u_1, \ldots, u_r)$. Let $\{\mathcal{B}_1, \ldots, \mathcal{B}_t\}$ be a complete set of representatives of the $\sim$-classes of Ω. Also let $\mathcal{B}_i = (v_{i1}, \ldots, v_{ir})$ for each $i \in \{1, \ldots, t\}$. Choose disjoint copies W_{ij} of $V(\aleph_0, q)$ for all $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r$. Let $N = \bigcup (W_{ij}\colon\ 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant r)$.

We must describe the action of $(V_1 \otimes V_2) \rtimes H$ on N. For $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r$, let $\phi_{ij}\colon V_1 \to W_{ij}$ be an isomorphism. Let $u \in V_1$ and let $z \in V_1 \otimes V_2$. For $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant r$, we must define $(\phi_{ij}(u))z$. We can write z uniquely as

$$z = u_1 \otimes v_{i1} + \ldots + u_r \otimes v_{ir}.$$

Then $(\phi_{ij}(u))z = \phi_{ij}(u + u_j)$. To define $(\phi_{ij}(u))h$ (for $h \in H$) note that there is a unique way of writing $h = h_1h_2$ $(h_1 \in H_1,\ h_2 \in H_2)$ such that $\mathcal{B}_ih_2 \in \{\mathcal{B}_1, \ldots, \mathcal{B}_t\}$. Suppose $\mathcal{B}_ih_2 = \mathcal{B}_k$. Then define $(\phi_{ij}(u))h_1h_2 = \phi_{kj}(uh_1)$. This makes $\mathcal{N} = (N\,;(V_1 \otimes V_2) \rtimes H)$ into a permutation structure. Let

$$A := \{\phi_{ij}(0)\colon\ 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant r\}.$$

Then A is an algebraically closed subset of $\mathcal{N}$, and $\mathcal{M}$ is isomorphic to $\mathrm{Gr}(\mathcal{N}\,;A)$ as a permutation structure (to see this, note that the stabiliser of A in $\mathrm{Gr}(\mathcal{N}\,;A)$ is $H_1 * H_2$). It is possible that $\mathrm{Aut}\,\mathcal{N}$ is not transitive on the set of blocks of $\mathcal{N}$. In this case, choose any H_2-orbit on the blocks, let N^* be the union of these blocks, let $\mathcal{N}^*$ be the permutation structure induced on N^*, and put $A^* = A \cap N^*$. It is easily checked that $\mathcal{N}^*$ is almost classical and that $\mathcal{M}$ and $\mathrm{Gr}(\mathcal{N}^* ; A^*)$ are isomorphic permutation structures.

(vi) $\mathcal{M} \in \Sigma_6^\infty$. Now $\mathcal{M}$ has automorphism group $G = (V_1 \oplus \ldots \oplus V_r) \rtimes H$, where $H \leqslant H_1 \operatorname{wr} S_r$ and H induces H_1 on V_1. Suppose (using (iv) and (v) above) that each permutation structure $(V_i ; V_i \rtimes H_i)$ is of the form $\mathrm{Gr}(\mathcal{N}_i ; A_i)$, where $N_1, \ldots, N_r$ are pairwise disjoint. Let $N = \bigcup_{i=1}^r N_i$, with $\operatorname{Aut} \mathcal{M}$ acting naturally on N, and let $A = \bigcup_{i=1}^r A_i$. Then $\mathcal{N}$ is almost classical and the permutation structures $\mathcal{M}$ and $\mathrm{Gr}(\mathcal{N} ; A)$ are isomorphic, as required.

We conclude this section with a counter-example to a converse of Theorem 1.1. The example gives an $\aleph_0$-categorical permutation structure which is primitive and is a Grassmannian of an almost classical structure, but is not smoothly approximated. We assume familiarity with the theorem of Fraissé ([**9**], also [**10**, Chapter 11, § 1]) which says that, over any fixed relational language, countably infinite homogeneous structures correspond exactly to amalgamation classes of finite structures.

Example. Let L be a language with two binary relations Exy and Rxy, and let $\mathcal{F}$ be the class of all isomorphism types of finite L-structures $\mathcal{N}$ such that

(i) E defines an equivalence relation on N with at most two classes,

(ii) R is symmetric and irreflexive, and N carries a bipartite graph with edges given by R, each edge containing a vertex from each E-class.

Then $\mathcal{F}$ is hereditary and has the amalgamation property, so by Fraissé's Theorem there is a unique countable homogeneous L-structure Γ whose finite substructures are the members of $\mathcal{F}$ (Γ is sometimes called 'the random bipartite graph'). Now $\operatorname{Aut} \Gamma$ is transitive but imprimitive with the two E-classes as a block system, and the group induced on each E-class is k-transitive for all $k < \omega$ (but is not closed). If Γ were smoothly approximated, there would be a finite homogeneous substructure Δ containing elements x, y_1, y_2, z_1, z_2 with y_1, y_2, z_1, z_2 all in the E-class not containing x, and x adjacent to y_1 and y_2 but not to z_1 and z_2. Because Δ is a homogeneous substructure, the E-classes give an $(\operatorname{Aut} \Gamma)_\Delta$-congruence on Δ; the two classes in Δ are interchanged by an element of $(\operatorname{Aut} \Gamma)_\Delta$ and the full symmetric group is induced on each class. The set of neighbours of x must have the same number of $\operatorname{Aut} \Delta$-translates as x. Since $\operatorname{Aut} \Delta$ induces the full symmetric group on each E-class, it follows that the two E-classes of Δ have different sizes, which is a contradiction.

5. *Concluding results*

We wish to derive an analogue of Theorem 1.1 for structures which are transitive but not primitive. It is convenient to work with a generalisation of the notion of smooth approximation. We say that a structure $\mathcal{M}$ is a *k-homogeneous substructure* of a structure $\mathcal{N}$ if

(i) $\mathcal{M}$ is finite, $\mathcal{N}$ is $\aleph_0$-categorical, and $\mathcal{M} \leqslant \mathcal{N}$;

(ii) if L^* is the canonical language for $\mathcal{N}$, and $\mathcal{M}^*$, $\mathcal{N}^*$ are the L^*-structures corresponding to $\mathcal{M}$, $\mathcal{N}$ respectively, then

(a) every automorphism of $\mathcal{M}^*$ extends to an automorphism of $\mathcal{N}^*$, and

(b) if U, V are subsets of M with $|U| \leqslant k$, and there is $\alpha \in \operatorname{Aut} \mathcal{N}^*$ with $U\alpha = V$, then there is $\beta \in \operatorname{Aut} \mathcal{M}^*$ with $\alpha|_U = \beta|_U$.

Next, we say that $\mathcal{M}$ is *weakly approximated* if it is infinite, $\aleph_0$-categorical, and there is a chain $\{\mathcal{M}_i\colon i<\omega\}$ of finite substructures of $\mathcal{M}$ with union M such that, for all $i \geqslant 1$, $\mathcal{M}_i$ is an $|M_{i-1}|$-homogeneous substructure of $\mathcal{M}$. Note that every smoothly approximated structure is weakly approximated.

PROPOSITION 5.1. *If $\mathcal{M}$ is primitive, then $\mathcal{M}$ is smoothly approximated if and only if it is weakly approximated.*

Proof. It suffices to show that each weakly approximated structure lies in Σ_i^∞ for some $i = 1, \ldots, 6$. This is proved in the manner of Theorem 1.2. The application of Proposition 1.3 runs as before. We require an analogue of Corollary 4.3, which is easy to prove; it will have essentially the same statement as before, except that $\mathcal{M}$ and $\mathcal{N}$ will be weakly approximated by the chains $(\mathcal{M}_i\colon i<\omega)$ and $(\mathcal{N}_i\colon i<\omega)$ respectively.

QUESTION. Is every weakly approximated structure smoothly approximated?

If $(M\,;G)$ is a permutation structure, and E is a 0-definable equivalence relation on $\mathcal{M}$, then $(M/E\,;G)$ is the structure whose domain is the set of E-classes, with relations corresponding to the action of G on this set. The E-class containing $x \in M$ will be denoted by x/E.

LEMMA 5.2. *Let $\mathcal{M} = (M\,;G)$ be a weakly approximated permutation structure:*

(i) *if $A \subseteq M$ is finite, then $(M\,;G_{(A)})$ is weakly approximated;*

(ii) *if S is an infinite class of a 0-definable equivalence relation on $\mathcal{M}$, then $(S\,;G_S)$ is weakly approximated;*

(iii) *if A is a finite algebraically closed subset of M, then the Grassmannian $(\mathrm{Gr}(\mathcal{M}\,;A)\,;G)$ is weakly approximated;*

(iv) *if E is a 0-definable equivalence relation on $\mathcal{M}$ with infinitely many classes, then the quotient structure $(M/E\,;G)$ is weakly approximated.*

Proof. Parts (i), (ii) and (iii) are straightforward, so we consider only (iv). Let $\mathcal{M}$ have a weak approximation $(\mathcal{M}_i\colon i<\omega)$. Suppose that for some k we have a chain $\mathcal{N}_0 \leqslant \ldots \leqslant \mathcal{N}_{k-1}$ of substructures of $\mathcal{M}/E$, where $\mathcal{N}_i$ is an $|N_{i-1}|$-homogeneous substructure of $\mathcal{M}/E$ for $1 \leqslant i \leqslant k-1$. Put $c = |N_{k-1}|$. Then there is $l \in \mathbb{N}$ such that:

(a) $N_{k-1} \subseteq M_l/E$;

(b) whenever there are $x_1, \ldots, x_c$, $y_1, \ldots, y_c \in M$ and $\alpha \in \operatorname{Aut} \mathcal{M}$ with x_i/E, $y_i/E \in M_l/E$ and $x_i\alpha = y_i$ for $1 \leqslant i \leqslant c$, there are also $u_1, \ldots, u_c$, $v_1, \ldots, v_c \in M_l$ with $u_i/E = x_i/E$, $v_i/E = y_i/E$ and $u_i\alpha = v_i$ for $1 \leqslant i \leqslant c$.

Note that the notation $\mathcal{M}_l/E$ makes sense, since E defines a partition of M_l invariant under $\operatorname{Aut} \mathcal{M}_l$.

Now put $\mathcal{N}_k = \mathcal{M}_l/E$. Then $\mathcal{N}_{k-1} \subseteq \mathcal{N}_k$, and $\mathcal{N}_k$ is an $|N_{k-1}|$-homogeneous substructure of $\mathcal{M}/E$, as required.

REMARK. The notion of weak approximation is introduced because Part (iv) of Lemma 5.2 is used in the proof of the next theorem. Parts (i)–(iii) of the lemma would be true with 'smooth approximation' replacing 'weak approximation' throughout, but we do not know this for (iv).

Our next result is suggested by the Coordinatisation Theorem for $\aleph_0$-categorical, ω-stable structures (Theorem 4.1 of [7]). The difference is that their 'rank one' is replaced by our 'almost classical'. Recall the notion of an *extension by definitions* of a structure [7]. If $\mathcal{M} = (M\,; G)$ is a permutation structure, then an extension by definitions is a permutation structure obtained from $\mathcal{M}$ by repeating the following process a finite number of times: pick $n \in \mathbb{N}\backslash\{0\}$, pick a 0-definable (possibly trivial) equivalence relation E on M^n, and adjoin to M a point for each E-class, with G acting naturally on the resulting structure with domain $M \dot\cup M^n/E$. If $\mathcal{M}^*$ is an extension by definitions of $\mathcal{M}$, and S is a 0-definable subset of $\mathcal{M}^*$, we say that S *coordinatises* $\mathcal{M}$ if, for all $x \in M$, $\mathrm{acl}(x) \cap S \neq \varnothing$.

THEOREM 5.3. *Let $\mathcal{M}$ be a transitive weakly approximated structure. Then there are an extension by definitions $\mathcal{M}^*$ of $\mathcal{M}$, and an almost classical structure $\mathcal{N}$ which is 0-definable in $\mathcal{M}^*$, such that $\mathcal{N}$ coordinatises $\mathcal{M}$.*

Proof. We first suppose that $\mathcal{M}$ is primitive. By Proposition 5.1 and Theorem 1.1, there is an almost classical structure $\mathcal{N}$ with a finite algebraically closed set A such that $\mathcal{M} \cong \mathrm{Gr}(\mathcal{N}\,; A)$. Suppose that for some (and hence for all) $x \in N$, $|\mathrm{acl}\{x\}| > 1$. Then there is a 0-definable equivalence relation $\approx$ on N with $x \approx y$ if and only if $y \in \mathrm{acl}\{x\}$. The $\approx$-classes are finite, and it is easily checked that $\mathcal{N}/\approx$ is almost classical, and that $\mathcal{M}$ is a Grassmannian of $\mathcal{N}/\approx$. Hence, we may suppose that $\mathrm{acl}\{x\} = x$ for all $x \in N$. It follows that every finite subset of $N\backslash\{x\}$ has a disjoint translate under $(\mathrm{Aut}\,\mathcal{N})_x$. Thus, for all $x \in N$ there are $(\mathrm{Aut}\,\mathcal{N})$-translates A_1, A_2 of A with $A_1 \cap A_2 = \{x\}$.

Let ϕ: $\mathcal{M} \to \mathrm{Gr}(\mathcal{N}\,; A)$ be an isomorphism, and for $m \in M$ identify $\phi(m)$ with the corresponding subset of N. By the last paragraph, there are m_1, $m_2 \in M$ with $|\phi(m_1) \cap \phi(m_2)| = 1$. Let Ω be the $(\mathrm{Aut}\,\mathcal{M})$-orbit of (m_1, m_2), and write $(m_1, m_2) \sim (n_1, n_2)$ if and only if $\phi(m_1) \cap \phi(m_2) = \phi(n_1) \cap \phi(n_2)$. Clearly $\sim$ is an equivalence relation on Ω, and furthermore it is $(\mathrm{Aut}\,\mathcal{M})$-invariant. Hence $\sim$ is 0-definable, and we may identify N with $\Omega/\sim$, as required.

Suppose now that $\mathcal{M}$ is transitive but imprimitive. Choose a 0-definable equivalence relation E on M maximal subject to the condition that M/E is infinite, and let F be a minimal 0-definable equivalence relation with $E < F$ ($E \neq F$). Then M/F is finite (possibly of size 1), and if $S \in M/F$ then S/E is infinite and carries a primitive permutation structure. Furthermore, by Proposition 5.1 and Lemma 5.2(ii) and (iv), the permutation structure on S/E lies in Σ_i^∞ for some $i = 1, \ldots, 6$. Clearly any coordinatisation of $\mathcal{M}/E$ gives a coordinatisation of $\mathcal{M}$.

Let $S_1, \ldots, S_t$ be the F-classes of $\mathcal{M}/E$. Since the structures S_i/E are isomorphic, by Theorem 1.1 there are a Grassmannian $\mathrm{Gr}(\mathcal{N}\,; A)$ of an almost classical structure $\mathcal{N}$, and isomorphisms ϕ_i: $S_i/E \to \mathrm{Gr}(\mathcal{N}\,; A)$ for $1 \leqslant i \leqslant t$. Let $x \in N$, and let $\bar{m} = (m_{11}, m_{12}, \ldots, m_{t1}, m_{t2})$ where, for $1 \leqslant i \leqslant t$,

$$m_{i1}, m_{i2} \in S_i/E \quad \text{and} \quad \phi_i(m_{i1}) \cap \phi_i(m_{i2}) = \{x\}.$$

Let Ω be the $(\mathrm{Aut}\,\mathcal{M})$-orbit of $\bar{m}$. Then there is a 0-definable equivalence relation on Ω such that N may be identified with the set of classes. Thus, $\mathcal{N}$ coordinatises M/E, and is identified with a 0-definable subset of an extension by definitions of $\mathcal{M}$. Since $\mathcal{N}$ coordinatises $\mathcal{M}/E$, it coordinatises $\mathcal{M}$, as required.

Our next few results confirm that the notion of smooth approximation coincides with a notion discussed in [**7**].

Proposition 5.4. *Let $\mathcal{M}$ be an $\aleph_0$-categorical structure over a language L. Then the following are equivalent:*

(i) *$\mathcal{M}$ is smoothly approximated;*

(ii) *for every $\sigma \in \mathrm{Th}(\mathcal{M})$, there is a finite $\mathcal{N} \subseteq_{\mathrm{hom}} \mathcal{M}$ with $\mathcal{N} \models \sigma$.*

Proof. First, notice that we may assume that L is the canonical language for $\mathcal{M}$. Essentially, this is because any sentence in another language can be translated into a sentence in the canonical language, and vice versa.

(i)$\Rightarrow$(ii). Let $(\mathcal{M}_i\colon\ i<\omega)$ be a smooth approximation for $\mathcal{M}$ by finite homogeneous substructures, and let $\sigma \in \mathrm{Th}(\mathcal{M})$. Since $\mathcal{M}$ is homogeneous, $\mathrm{Th}(\mathcal{M})$ is $\forall\exists$-axiomatised, so we may suppose that

$$\sigma = \bigwedge_{i=1}^{t} (\forall \bar{x}_i \exists \bar{y}_i \phi_i(\bar{x}_i, \bar{y}_i)).$$

Here $l(\bar{x}_i) = p_i$, $l(\bar{y}_i) = q_i$, and we may suppose that $p_1, \dots, p_s > 0$ and $p_{s+1}, \dots, p_t = 0$. For $i = 1, \dots, s$, let $\bar{a}_{i1}, \dots, \bar{a}_{ir_i}$ be a complete set of representatives of the $(\mathrm{Aut}\,\mathcal{M})$-orbits on M^{p_i}, and let $\bar{b}_{i1}, \dots, \bar{b}_{ir_i} \in M^{q_i}$ be chosen so that $M \models \phi_i(\bar{a}_{ij}, \bar{b}_{ij})$ for all $1 \leq j \leq r_i$. Also for $s+1 \leq k \leq t$ choose $\bar{b}_k$ with $\mathcal{M} \models \phi_k(\bar{b}_k)$. There is a finite $\mathcal{M}_l \subseteq_{\mathrm{hom}} \mathcal{M}$ with

$$\{\bar{a}_{ij}, \bar{b}_{ij}\colon\ 1 \leq i \leq s,\ 1 \leq j \leq r_i\} \cup \{\bar{b}_k\colon\ s+1 \leq k \leq t\} \subseteq M_l.$$

Then $\mathcal{M}_l \models \sigma$.

(ii)$\Rightarrow$(i). Let $(\sigma_i\colon\ i<\omega)$ enumerate $\mathrm{Th}(\mathcal{M})$, and let $\tau_i = \bigwedge_{j=0}^{i} \sigma_j$. We construct a sequence $(\mathcal{M}_i\colon\ i<\omega)$ inductively. Suppose we have found finite

$$\mathcal{M}_0 \subseteq_{\mathrm{hom}} \mathcal{M}_1 \subseteq_{\mathrm{hom}} \cdots \subseteq_{\mathrm{hom}} \mathcal{M}_{i-1}$$

with $\mathcal{M}_j \models \tau_j$ and $\mathcal{M}_j \subseteq_{\mathrm{hom}} \mathcal{M}$ for all $j<i$. Then by (ii) there is some $\mathcal{N}_i \subseteq_{\mathrm{hom}} \mathcal{M}$ such that $\mathcal{N}_i \models \tau_i$ and $\mathcal{M}_{i-1}$ is isomorphic to a substructure $\mathcal{N}_{i-1}$ of $\mathcal{N}_i$. Since $\mathcal{M}$ is homogeneous, $\mathcal{N}_i$ is homogeneous; hence, as $\mathcal{N}_{i-1} \subseteq_{\mathrm{hom}} \mathcal{M}$, it follows that $\mathcal{N}_{i-1} \subseteq_{\mathrm{hom}} \mathcal{N}_i$. Thus, there is $\mathcal{M}_i \cong \mathcal{N}_i$ such that $\mathcal{M}_{i-1} \subseteq_{\mathrm{hom}} \mathcal{M}_i \subseteq_{\mathrm{hom}} \mathcal{M}$. Now let $\mathcal{M}^* = \bigcup(\mathcal{M}_i\colon\ i<\omega)$. Since $\mathrm{Th}(\mathcal{M})$ is $\forall\exists$-axiomatised, $\mathcal{M}^* \models \mathrm{Th}(\mathcal{M})$. Hence by $\aleph_0$-categoricity $\mathcal{M}^* \cong \mathcal{M}$, so $\mathcal{M}$ also has a smooth approximation.

Corollary 5.5. *Every $\aleph_0$-categorical, ω-stable structure over a language with just finitely many function symbols is smoothly approximated.*

Proof. Apply Proposition 5.4 above and Corollary 7.4 of [**7**].

Corollary 5.6. *If $\mathcal{M}$ is smoothly approximated, then $\mathrm{Th}(\mathcal{M})$ is not finitely axiomatisable.*

Proof. By Proposition 5.4, any finite subset of Th($\mathcal{M}$) would have a finite model.

Finally, we make some stability theoretic observations about smoothly approximated structures. The *strict order property* and the *independence property* are defined by Shelah in [**22**, Chapter II, § 4]. From his definitions it follows easily that if $\mathcal{M}$ is $\aleph_0$-categorical, then Th($\mathcal{M}$) has the strict order property if and only if the following holds: there are $r \in \mathbb{N}\setminus\{0\}$, a finite set $A \subseteq M$, and a formula $\phi(\bar{x}, \bar{y})$ (with $l(\bar{x}) = l(\bar{y}) = r$) in the language of $\mathcal{M}$ with parameters in A, such that, if we write $\bar{x} < \bar{y}$ if and only if $\mathcal{M} \models \phi(\bar{x}, \bar{y})$, then $<$ defines a partial ordering on M^r which contains an infinite chain. Also, Th($\mathcal{M}$) has the independence property if and only if the following holds: there are $r \in \mathbb{N}\setminus\{0\}$, a finite set A, an infinite set $\{\bar{a}_i\colon i < \omega\} \subseteq M^r$, and a formula $\psi(x, \bar{y})$ with parameters in A, such that for all finite $S \subseteq \omega$ there is $b_S \in M$ with $\mathcal{M} \models \psi(b_S, \bar{a}_i)$ if and only if $i \in S$. In [**22**, Chapter II, Theorem 4.7], Shelah shows that every unstable theory has the strict order property or the independence property (possibly both), and that a stable theory has neither. The following proof was suggested by A. Pillay.

PROPOSITION 5.7. *A weakly approximated structure cannot have the strict order property.*

Proof. Let $\mathcal{M} = (M\,; G)$ be weakly approximated by a chain $(\mathcal{M}_i\colon i < \omega)$, and suppose that Th($\mathcal{M}$) has the strict order property witnessed by $\phi(\bar{x}, \bar{y})$ and A as above. Let $r = l(\bar{x})$, and let $c = s_{r+|A|}(G)$. There are $\bar{b}_0, \ldots, \bar{b}_c \in M^r$ with $\mathcal{M} \models \phi(\bar{b}_i, \bar{b}_j)$ if and only if $i < j$. For some k, there is an $(|A| + 2r)$-homogeneous substructure $\mathcal{M}_k$ of $\mathcal{M}$ containing $A, \bar{b}_0, \ldots, \bar{b}_c$. However, since in a finite poset, elements of different height lie in different orbits, $s_{|A|+r}(\operatorname{Aut} \mathcal{M}_k) \geqslant c + 1$. This is a contradiction (by the analogue for weak approximation of Claim 1 of the proof of Theorem 1.2).

Next, we confirm the claim, made in the Introduction, that there are smoothly approximated structures which are not ω-stable.

PROPOSITION 5.8. *If $\mathcal{M}$ is a classical structure which is not of Type* (a), (b)(i) *or* (d)(i), *then* Th($\mathcal{M}$) *has the independence property.*

Proof. We shall suppose that $\mathcal{M} = (M\,; G)$ is of Type (d)(iii), with $M = V(\aleph_0, q)$ and $G = V(\aleph_0, q) \rtimes \operatorname{Sp}(\aleph_0, q)$ acting naturally. The other classical structures are handled similarly.

There is a basis $\{e_i, f_i\colon i < \omega\}$ of V such that, with respect to the bilinear form $(\ ,\ )$ invariant under $\operatorname{Sp}(\aleph_0, q)$, $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_j) = (f_i, f_j) = 0$ for all $i, j < \omega$. Let S be a finite subset of ω, and put $f_S = \sum_{i \in S} f_i$. Then for all $i < \omega$, $(e_i, f_S) = 1$ if and only if $i \in S$. Since $(\ ,\ )$ is invariant under the stabiliser of the zero vector, it follows by $\aleph_0$-categoricity that, whatever the language for $\mathcal{M}$, there is a formula $\phi(x, y)$ witnessing the independence property for Th($\mathcal{M}$).

REMARKS. 1. Lachlan's Conjecture [**15**] holds for smoothly approximated structures: that is, every smoothly approximated stable structure is ω-stable. This is verified by inspection for primitive structures, and proved in general by induction on the height of the lattice of 0-definable equivalence relations.

2. We do not know any neat group-theoretic way of distinguishing those smoothly approximated structures which are ω-stable. For example, the following primitive structure $\mathcal{M}$ (from $\Sigma_2^\infty(\mathrm{i})$) is unstable, but can be represented as a Grassmannian of an almost classical structure whose blocks of imprimitivity carry an ω-stable classical structure. Let $V = V(\aleph_0, q)$, let $\{e_i : i < \omega\}$ be a basis of V, and let M consist of pairs $\{U, W\}$ where U is a 1-dimensional subspace of V, W is a subspace of codimension 1 containing all but finitely many of the e_i, and $U \leqslant W$. We take G to be the usual group of permutations of M, and put $\mathcal{M} = (M\,; G)$.

PROPOSITION 5.9. *The above structure $\mathcal{M}$ is unstable* (*and so has the independence property*).

Proof. For each $i = 2, 3, 4, \dots,$ define

$$U_i = \langle e_i \rangle \quad \text{and} \quad W_i = \langle e_0 + e_1, \dots, e_0 + e_{i-1}, e_i, e_{i+1}, \dots \rangle.$$

Then $\{U_i, W_i\} \in M$, and $U_i \leqslant W_j$ if and only if $i \geqslant j$. It follows that if $i < j$ then the pairs $(\{U_i, W_i\}, \{U_j, W_j\})$ and $(\{U_j, W_j\}, \{U_i, W_i\})$ are in different orbits of $\operatorname{Aut}\mathcal{M}$. By Ramsey's Theorem, we may suppose (by taking a subsequence of $(\{U_i, W_i\} : i < \omega)$ and relabelling) that all pairs $(\{U_i, W_i\}, \{U_j, W_j\})$ (for $i < j$) lie in the same $(\operatorname{Aut}\mathcal{M})$-orbit. It follows by $\aleph_0$-categoricity that, whatever the language of $\mathcal{M}$, there is a formula $\phi(x, y)$ with $\mathcal{M} \models \phi(\{U_i, W_i\}, \{U_j, W_j\})$ if and only if $i < j$. This proves that $\mathrm{Th}(\mathcal{M})$ is unstable.

Added in proof (*June 1989*). It is worth mentioning that any group interpretable in a smoothly approximated structure is nilpotent-by-finite. For, by Proposition 5.7, its theory cannot have the strict order property, so we can apply Theorem 1.2 of a recent paper by the third author (H. D. Macpherson, 'Absolutely ubiquitous structures and $\aleph_0$-categorical groups', *Quart. J. Math. Oxford* (2) 39 (1988) 483–500). Extraspecial groups provide examples of smoothly approximated groups which are not ω-stable.

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