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0-EFFICIENT TRIANGULATIONS OF 3-MANIFOLDS

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Abstract

0-efficient triangulations of 3-manifolds are defined and studied. It is shown that any triangulation of a closed, orientable, irreducible 3-manifold M can be modified to a 0-efficient triangulation or M can be shown to be one of the manifolds $S^3, \mathbb{R}\mathrm{P}^3$ or $L(3,1)$. Similarly, any triangulation of a compact, orientable, irreducible, ∂-irreducible 3-manifold can be modified to a 0-efficient triangulation. The notion of a 0-efficient ideal triangulation is defined. It is shown if M is a compact, orientable, irreducible, ∂-irreducible 3-manifold having no essential annuli and distinct from the 3-cell, then $\mathring{M}$ admits an ideal triangulation; furthermore, it is shown that any ideal triangulation of such a 3-manifold can be modified to a 0-efficient ideal triangulation. A 0-efficient triangulation of a closed manifold has only one vertex or the manifold is S^3 and the triangulation has precisely two vertices. 0-efficient triangulations of 3-manifolds with boundary, and distinct from the 3-cell, have all their vertices in the boundary and then just one vertex in each boundary component. As tools, we introduce the concepts of barrier surface and shrinking, as well as the notion of crushing a triangulation along a normal surface. A number of applications are given, including an algorithm to construct an irreducible decomposition of a closed, orientable 3-manifold, an algorithm to construct a maximal collection of pairwise disjoint, normal 2-spheres in a closed 3-manifold, an alternate algorithm for the 3-sphere recognition problem, results on edges of low valence in minimal triangulations of 3-manifolds, and a construction of irreducible knots in closed 3-manifolds.

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1. Introduction

This is the first in a series of papers concerning triangulations of 3-manifolds. Our aim is to give a theory of *efficient* triangulations of 3-manifolds, which has applications to finiteness theorems, knot theory, Dehn fillings, decision problems, algorithms, computational complexity and Heegaard splittings. A motivation for these investigations is to achieve properties of triangulations which have features similar to those of geometric structures on manifolds. For example, with triangulations, the analogue of a stable minimal surface is a normal surface and the analogue of an unstable surface of index one is an almost normal surface. Our work very strongly uses normal and almost normal surfaces and is guided by the similarities to minimal surfaces. An appropriately efficient triangulation has features similar to those of a complete hyperbolic structure of finite volume on a 3-manifold. In the latter, any essential surface can be homotoped to a (stable) minimal surface; hence, by the Gauss-Bonnet formula, there can not be any essential surfaces of nonnegative Euler characteristic. In a manifold with a 1-efficient triangulation, we obtain similar results, limiting embedded normal surfaces with nonnegative Euler characteristic.

In this paper, we define and study 0-efficient triangulations of 3-manifolds. A triangulation of a closed, orientable 3-manifold is 0-*efficient* if and only if the only embedded, normal 2-spheres are vertex-linking. For a compact, orientable 3-manifold with nonempty boundary, a triangulation is 0-*efficient* if and only if the only properly embedded, normal disks are vertex-linking. We show that a closed 3-manifold with a 0-efficient triangulation is irreducible, not $\mathbb{R}P^3$ and the triangulation has precisely one-vertex or the manifold is S^3 and the triangulation has precisely two vertices. In the case of a compact 3-manifold with boundary, no component of which is a 2-sphere, a 0-efficient triangulation implies that the manifold is irreducible and ∂-irreducible and the triangulation has all vertices in the boundary and has precisely one vertex in each boundary component.

One of the main theorems in Section 5, is that any triangulation of a closed, orientable, irreducible 3-manifold can be modified to a 0-efficient triangulation or it can be shown the 3-manifold is one of S^3, $\mathbb{R}P^3$, or $L(3,1)$. Similarly, it is shown that any triangulation of a compact, orientable irreducible, ∂-irreducible 3-manifold with nonempty boundary can be modified to a 0-efficient triangulation.

We show that a minimal triangulation of a closed, orientable, irreducible 3-manifold is 0-efficient, except for $\mathbb{R}\mathrm{P}^3$ and $L(3,1)$, and therefore has just one vertex or is S^3. The 3-sphere has two distinct, one-tetrahedron triangulations, both are 0-efficient but one has two vertices. For the other exceptions, there are two distinct, two-tetrahedra (minimal) triangulations of $\mathbb{R}\mathrm{P}^3$, neither are 0-efficient, and four distinct, two-tetrahedra (minimal) triangulations of $L(3,1)$, only one of which is 0-efficient. Unlike the case of 2-manifolds, where one immediately has from Euler characteristic that a minimal triangulation of a closed 2-manifold with nonpositive Euler characteristic has just one vertex, there does not seem to be an immediate way to see that a minimal triangulation of an irreducible 3-manifold, other than one of $S^3, \mathbb{R}\mathrm{P}^3$ and $L(3,1)$, has just one vertex. One-vertex triangulations are very interesting and have a number of applications. In particular, one-vertex triangulations of 3-manifolds appear to be very well-suited for normal surface theory. See [5, 6, 7, 9, 12, 13] for numerous applications of one-vertex and efficient triangulations.

We also study 0-efficient (no normal 2-spheres), ideal triangulations and show that the interior of any compact, orientable, irreducible, ∂-irreducible 3-manifold that has no essential annuli admits an ideal triangulation; furthermore, in such a 3-manifold, any ideal triangulation can be modified to a 0-efficient, ideal triangulation. It follows that a minimal, ideal triangulation of such a bounded 3-manifold is 0-efficient.

Our main technique is to identify a normal 2-sphere, which is not vertex-linking and which bounds a 3-cell, and crush such a 2-sphere and the 3-cell it bounds to a point. This gives us back our manifold but wrecks havoc with the triangulation. Most work then is to show that we can recover a "nicer" triangulation; we must do this with great care not to add tetrahedra (or new normal 2-spheres). To achieve this we introduce in Section 4 the notion of crushing a triangulation along a normal surface. This has turned out to be a useful tool throughout our work.

In the sequel to this paper, we show that further modifications of the triangulation can be performed for closed, orientable 3-manifolds which are not only irreducible but also assumed atoroidal. In such cases, we show that a triangulation can be obtained so that any embedded, normal torus is of a very special form or it can be shown the 3-manifold is S^3, a lens space or a small Seifert fiber space. This property along with 0-efficiency is called 1-efficiency. The reader can immediately see our motivation if they recall that in Haken's theory of normal surfaces, there is a finite constructible collection of fundamental normal surfaces and every normal surface is a sum of these fundamental surfaces. Hence, if we have control of all surfaces with nonnegative Euler characteristic, then we can expect a bounded number of such sums giving all normal surfaces of bounded genus, up to isotopy. A similar result works for almost normal surfaces and this completes the solution of the Waldhausen Conjecture that a closed, orientable 3-manifold has only a finite number (up to homeomorphism and isotopy) of Heegaard splittings of bounded genus [10]. This was shown in the case of Haken manifolds by K. Johannson [16].

One of the main applications in this first paper is an algorithm to determine the prime decomposition of a 3-manifold. Given a closed, orientable 3-manifold, M, we give an algorithm to write M as a connected sum decomposition of 3-manifolds where it is known that each factor is either, S^3, $S^2 \times S^1$, $\mathbb{R}P^3$, $L(3,1)$, or has a 0-efficient triangulation. Now, a 0-efficient triangulation is precisely the environment for implementation of the 3-sphere recognition algorithm, [18]. Hence, we arrive at the prime decomposition of the given 3-manifold M. Ben Burton and David Letscher have written a program to implement this algorithm. We observe certain methods which have implications toward the complexity of these algorithms. In particular, we obtain results similar to those announced by A. Casson who observed that the algorithm for achiving a prime decomposition can be implemented in time essentially of order $p(t)\mathcal{O}(3^t)$, where t is the number of tetrahedra in a given triangulation for M and $p(t)$ is a low degree polynomial in t.

We provide the background from normal and almost normal surface theory needed in this paper in Section 2, along with our generalized idea of triangulations. However, we assume the reader has some familiarity with normal and almost normal surface theory. In Section 3, we define and provide a study of barrier surfaces and shrinking within the context of normal and almost normal surface theory. These ideas come from geometric surface theory and are consistent with the usage there. More complete developments of barriers and shrinking can be found in [11, 5].

We would like to thank Ben Burton, Dave Letscher and Eric Sedgwick for many helpful conversations throughout the development of this theory. Also, we want to acknowledge Andrew Casson for helpful comments and ideas on this project. In particular, Andrew Casson announced a beautiful program to prove geometrization for irreducible, atoroidal 3-manifolds with tori boundaries at a meeting in Montreal in 1995. His method involved a version of efficient *ideal* triangulations and became our model for efficient triangulations of closed and bounded 3-

manifolds. Finally, we wish to thank Thomas Faulkenberry, who has been so helpful in drafting many of the figures which appear in this paper.

2. Triangulations, cell-decompositions and normal surfaces

In this section we give some basic definitions and results needed for the work presented in this paper. Our approach to triangulations is more general than the notion one typically finds. We develope a broad study of triangulations from this point of view in [5]. At various places in our proofs and constructions, we use cell-decompositions having more general cells than triangulations; however, a major feature of our methods is that these various cell-decompositions are, themselves, quite nice. We shall assume the reader has a basic understanding of normal surface theory. The references [8, 9, 14] are sources to review normal surface theory from our point of view. While we use very little normal surface theory here, it is an indispensable part of our program and serves to organize what might otherwise be quite messy.

2.1 Triangulations and cell-decompositions

We assume most readers are familiar with cell-decompositions; however, since we are using them in a slightly broader sense than usual, we include a brief discussion with some definitions.

Let $\mathbf{\Delta} = \{\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_t\}$ be a pairwise-disjoint collection of oriented, compact, convex, linear cells. Suppose Φ is a family of affine isomorphisms pairing faces of the cells in $\mathbf{\Delta}$ so that if $\phi \in \Phi$, then ϕ is an orientation-reversing affine isomorphism from a face $\sigma_i \in \widetilde{\Delta}_i$ to a face $\sigma_j \in \widetilde{\Delta}_j$, possibly $i = j$. We use $\mathbf{\Delta}/\Phi$ to denote the space obtained from the disjoint union of the $\widetilde{\Delta}_i$ by setting $x \in \widetilde{\sigma}_i$ equal to $\phi(x) \in \widetilde{\sigma}_j$, with the identification topology. Then $\mathbf{\Delta}/\Phi$ is a 3-manifold, except possibly at the images of the vertices of the $\widetilde{\Delta}_i$. (In a completely general setting, the identification space $\mathbf{\Delta}/\Phi$ may not be a 3-manifold at the image of the centers of some edges, as well as the images of the vertices; however, we have avoided this problem by orienting the $\widetilde{\Delta}_i$ and choosing the affine isomorphisms $\phi \in \Phi$ orientation-reversing.) We collect all this information into a single symbol $\mathcal{T}$ and call $\mathcal{T}$ a *cell-decomposition* of $\mathbf{\Delta}/\Phi$; in this case, we also use just $|\mathcal{T}|$ to denote the space $\mathbf{\Delta}/\Phi$. A *cell* (*tetrahedron*)*, face, edge*, or *vertex* in this cell decomposition is, respectively, the image of a cell (tetrahedron), face, edge, or vertex from the collection $\mathbf{\Delta} = \{\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_t\}$. We will denote the image of the faces by $\mathcal{T}^{(2)}$, the image of the edges by $\mathcal{T}^{(1)}$ and the image of the vertices by $\mathcal{T}^{(0)}$. We call $\mathcal{T}^{(i)}$ the *i-skeleton of* $\mathcal{T}$; but, generally, we just refer to these as the faces, edges or vertices of $\mathcal{T}$. We will denote the image of $\widetilde{\Delta}_i$ by Δ_i and call $\widetilde{\Delta}_i$ the *lift of* Δ_i. A cell is the quotient of a unique cell and a face is the quotient of one or two faces; edges and vertices may be the quotient of a number of edges or vertices, respectively. While the cells are not necessarily embedded, the interior of each cell is embedded. We define the *order* or *valence of an edge* e of $\mathcal{T}$ to be the number of edges in the collection $\mathbf{\Delta} = \{\widetilde{\Delta}_1, \ldots \ \ldots, \widetilde{\Delta}_t\}$, which are identified to e. If the link (here we mean the boundary of a small regular neighborhood and not the combinatorial link) of each vertex is either a 2-sphere or a 2-cell, then the underlying point set is an oriented 3-manifold, possibly with boundary, and we say $\mathcal{T}$ is a *cell-decomposition* of the 3-manifold $M = |\mathcal{T}|$. If each cell in $\mathbf{\Delta}$ is a tetrahedron, then we say $\mathcal{T}$ is a *triangulation* of the 3-manifold $M = |\mathcal{T}|$. In the literature, one is likely to find this notion of a triangulation referred to as a *pseudo*-triangulation and the term triangulation reserved to mean that tetrahedra are embedded and if two simplices meet at all, then they meet in a face of each. Our cells (tetrahedra) are not necessarily embedded; however, they are embedded on the interiors of each cell (tetrahedron), face and edge and the intersection of two cells is a union of sub-cells of each. If the link of some vertex is a closed surface, distinct from the 2-sphere, we say $\mathcal{T}$ is an *ideal cell-decomposition* (or *ideal triangulation*) of the 3-manifold $M = |\mathcal{T}| \setminus |\mathcal{T}^{(0)}|$. In this case the vertices of $\mathcal{T}$ are called *ideal vertices* and the *index of an ideal vertex* is the genus of its linking surface. In some cases where we have a mix of genus zero and higher genus ideal vertices, we include the genus zero vertices into our manifold. Generally, however, we don't have genus zero vertices in an ideal triangulation.

Similarly, we use cell-decompositions of surfaces. If $\boldsymbol{\sigma} = \{\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n\}$ is a pairwise disjoint collection of compact, convex, planar polygons and Ψ is a family of linear isomorphisms pairing edges of the polygons in $\boldsymbol{\sigma}$ so that $\psi \in \Psi$, then ψ is a linear isomorphism of an edge e_i of $\widetilde{\sigma}_i$ to an edge e_j of $\widetilde{\sigma}_j$, possibly $i = j$. We use $\boldsymbol{\sigma}/\Psi$ to denote the space obtained from the disjoint union of the $\widetilde{\sigma}_i$ by setting $x \in \widetilde{e}_i$ equal to $\psi(x) \in \widetilde{e}_j$, with the identification topology. We have that $\boldsymbol{\sigma}/\Psi$ is always a 2-manifold, possibly nonorientable. In this situation, we say we have a cell-decomposition of the 2-manifold $\boldsymbol{\sigma}/\Psi$. If each $\widetilde{\sigma}_i$ is a triangle, we say we have a *triangulation* of the 2-manifold $\boldsymbol{\sigma}/\Psi$. Similar to the case for 3-manifolds, in a cell-decomposition of a 2-manifold our cells are not embedded; however, the open cells are embedded.

In our cell-decompositions, an edge can be a simple closed curve, an edge in a cell with end points (vertices) identified. In fact, we will be working toward having triangulations with just one vertex. In this case, every edge is a simple closed curve. Faces can take on some interesting configurations. In Figure 1 we give the possibilities. In Figure 1, parts (4) and (5), we have two edges identified to give faces which are *cones* (the latter is a *pinched cone*); in (6), we have a face which is a Möbius band; and in (7) and (8), we have all three edges identified, giving in (7) the classical *dunce hat* (see for example Figure 2 (5)) and in (8) a spine for $L(3,1)$ (see for example the classical presentation for $L(3,1)$ in Figure 17). In the case of a triangulation of a 2-manifold, we only have those possibilities (1), (2), (3), (4) and (6) in Figure 1 for the triangles. As for tetrahedra, we have in Figure 2 the seven distinct identifications of a single tetrahedron to give an orientable 3-manifold. In Figure 2, the edge e in part (4) has order 1 and in part (5) e has order 5.

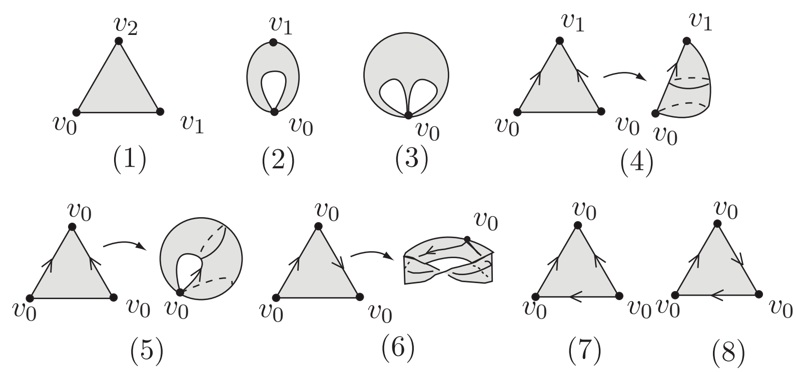


Figure 1: Possible configurations for faces in a triangulation.

2.2 Normal surfaces

A cell-decomposition (or ideal cell-decomposition) of a 3-manifold distinguishes certain surfaces, the normal surfaces relative to that cell-decomposition. They are analogous to stable (minimal) surfaces in geometric analysis and are determined by how they meet the cells of the cell-decomposition. However, since our cells are not embedded, we provide the definition of a normal surface in this setting. Furthermore, in this paper, we, generally, only use normal surface theory in triangula-

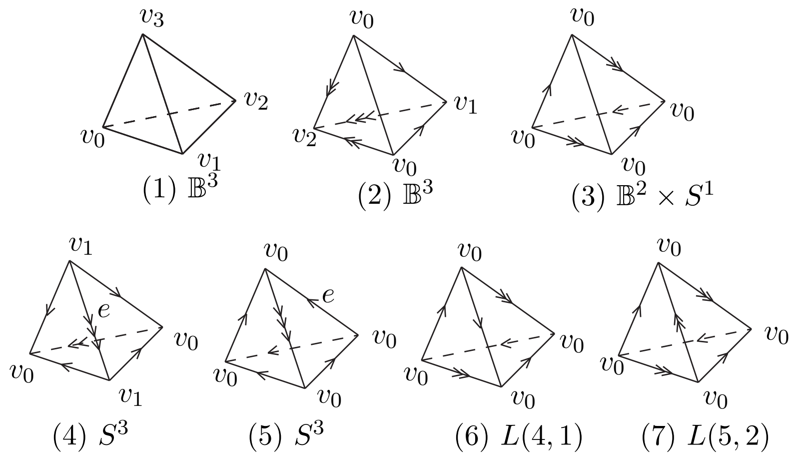


Figure 2: One-tetrahedron triangulations of orientable 3-manifolds.

tions. Even when we use more general cell decompositions (truncated tetrahedra, truncated prisms, regions between parallel quads and triangles), we restrict our considerations to very special types of normal surfaces.

If $\widetilde{\Delta}$ is a compact, convex, linear cell and $\widetilde{\sigma}$ is a face of $\widetilde{\Delta}$, we say a spanning arc in $\widetilde{\sigma}$ is a *normal arc* if its end points are in distinct edges of $\widetilde{\sigma}$. A *normal curve* in the boundary of a compact, convex, linear cell is a curve which meets each face in a collection of normal arcs. The elementary components of normal surface theory are the normal disks in the cells of the cell-decomposition. We call a properly embedded disk in a compact, convex, linear cell a *normal disk* if its boundary is a normal curve and it meets no edge more than once. If $\mathcal{T}$ is a cell-decomposition of the manifold M, then an isotopy of M is called a *normal isotopy* if it is invariant on the cells, faces, edges and vertices of $\mathcal{T}$. Up to normal isotopy there are only finitely many equivalence classes of normal disks in a compact, convex, linear cell; these are called *normal disk types.*

Suppose $\mathcal{T}$ is a cell-decomposition or ideal cell-decomposition of the 3-manifold M and S is a properly embedded surface transverse to the 2-skeleton of $\mathcal{T}$. Suppose c is a component of S in the cell Δ_i. Then c is the image of a properly embedded surface, $\widetilde{c}$ in $\widetilde{\Delta}_i$. We will call $\widetilde{c}$ the *lift of c.*

Now, if $\mathcal{T}$ is a cell-decomposition of the 3-manifold M, we say a surface F is a *normal surface* in M (with respect to $\mathcal{T}$) if F meets each cell of $\mathcal{T}$ in the images of a collection of normal disks in the cells of $\mathbf{\Delta} = \{\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_n\}$. That is, the surface F is a normal surface if and only if the lift of every component of F in a cell of $\mathbf{\Delta}$ is a normal disk. The elementary components of normal surface theory for triangulations are the *normal triangles* and *normal quadrilaterals* (*normal quads*) in a tetrahedron. The normal triangles and normal quads are shown in Figure 3. There are four types of normal triangles and three types of normal quads in each tetrahedron (no identification). Also, in Figure 3, we give some examples of normal disks in cells which are truncated tetrahedra and truncated-prisms. In a truncated tetrahedra, besides normal triangles and normal quads, one may have other normal disks. For example, we show a normal hexagon and a normal octagon in a truncated tetrahedra. We also show some normal quads in a truncated prism and normal triangles and normal quads in a truncated tetrahedron, which are also normal in the tetrahedron before truncation. In Figure 4, we show a 2-sphere made up of two normal triangles (Figure 4(1)), a normal torus made up of a normal quad (Figure 4(2)), and a Möbius band made up of a single normal quad (Figure 4(3)). In a tetrahedron in M, after identifications, a normal triangle can take on any of the possible orientable identifications of a triangle shown in Figure 1, Parts (1)-(6). (A normal triangle can identify to a Möbius band but then the resulting identification space is not a manifold at the associated vertex. It can be made into an ideal triangulation of a manifold; but the manifold is nonorientable.) Similarly, a normal quadrilateral, after identification, can take on numerous forms, including some which are nonorientable. The normal disk types give a normal surface F a cell-decomposition made up of normal quads and normal triangles; we call this the *cell-decomposition induced on* F(or the induced cell-decomposition). Finally, we note that an embedded normal surface must be properly embedded.

If $\mathcal{T}$ is a cell-decomposition, as we noticed above, there are only finitely many normal isotopy classes of normal disks. If n is the number of normal isotopy classes of normal disks in $\mathcal{T}$ and we select an ordering of these normal disk types, say $d_1, \ldots, d_n$, then a normal isotopy class of a normal surface has a parameterization as an n-tuple of nonnegative integers $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$, where x_i is the number of elementary disks of type $d_i, 1 \leq i \leq n$. For a triangulation there are four normal triangle types and three normal quad types in each tetrahedra; so, $n = 7t$ for a triangulation $\mathcal{T}$ with t tetrahedra. Associated with the cell-decomposition $\mathcal{T}$ is a system of linear equations. Non-negative integer solutions to this system give a parameterization of the normal

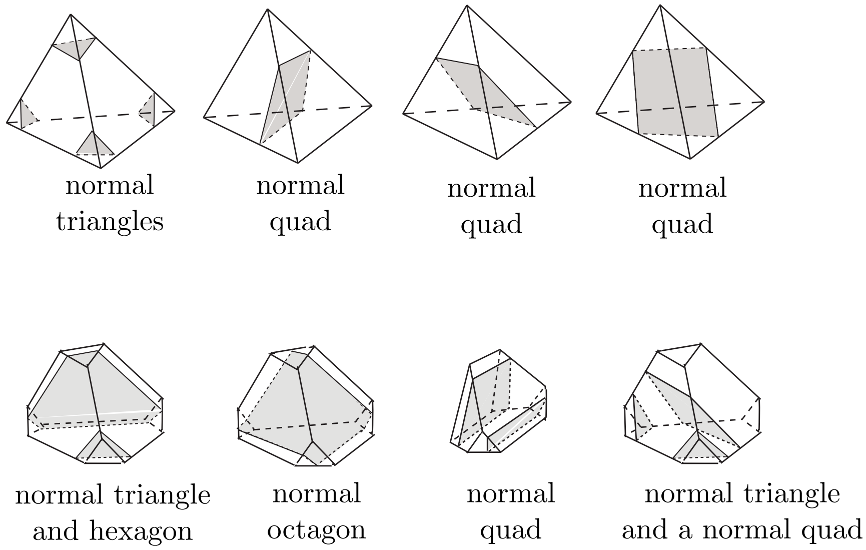


Figure 3: Examples of normal disk types (normal triangles and normal quads for a triangulation).

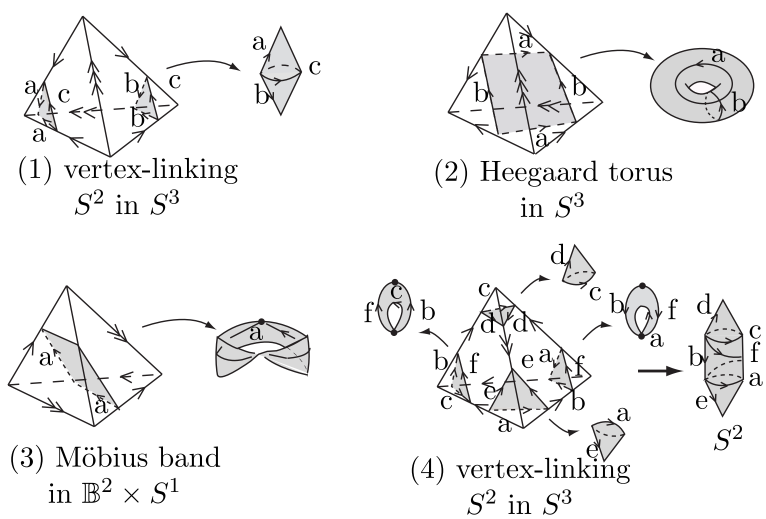


Figure 4: Examples of normal surfaces in triangulated 3-manifolds.

isotopy classes of normal surfaces. We add the condition $x_i \geq 0, \forall i$, and obtain a cone in the positive-orthant of $\mathbb{R}^n$. We will refer to this cone as the *solution cone* and write $\mathcal{S}(M, \mathcal{T})$. There are a finite number of nonnegative integer lattice points (solutions) in the solution cone so that any integer lattice point in the cone can be written as a finite linear combination of these using only nonnegative integer coefficients. These are a Hilbert basis for the nonnegative integer lattice points in the solution cone and are called *fundamental solutions* after Haken [4]. If, in addition to the above equations, we add the equation $\sum_1^n x_i = 1$, we determine a compact, convex linear cell. The rational points in this cell correspond to projective classes of normal isotopy classes of normal surfaces in $\mathcal{S}(M, \mathcal{T})$. We denote this compact, convex linear cell by $\mathcal{P}(M, \mathcal{T})$ and call it the *projective solution space* (of $\mathcal{S}(M, \mathcal{T})$). If F is a normal surface in M, we do not distinguish and let F denote not only the surface F but its normal isotopy class, and its representation as an n-tuple in $\mathbb{R}^n$. We denote the projective class of F by $\overline{F} \in \mathcal{P}(M, \mathcal{T})$. The *carrier* of a normal surface F is the unique minimal face of $\mathcal{P}(M, \mathcal{T})$ which contains $\overline{F}$ and is denoted $\mathcal{C}(F)$. If the projective representation of the normal isotopy class of the normal surface F is at a vertex of the projective solution space, we say F is a *vertex solution*. A vertex solution is a fundamental solution but the converse is not necessarily true. We have algebraic characterizations of fundamental and vertex solutions. A solution F is a fundamental solution if and only if it can *not* be written as a nontrivial sum $F = X + Y$; a solution F is a vertex solution if and only kF can *not* be written as a nontrivial sum $kF = nX + mY$ for some positive integers k, n and m.

In a triangulation $\mathcal{T}$, a normal surface is embedded if and only if it does not meet a tetrahedron in more than one quad type. This is sometimes call the *quadrilateral condition* or the *compatibility condition*. Any normal surface determines a unique nonnegative, integer n-tuple in $\mathbb{R}^n$; but, this correspondence is not one-one. However, if an integer lattice point represents an embedded normal surface, then there is a unique such embedded, normal surface associated with that lattice point. One can realize a face (or a cone) of embedded normal surfaces by adding the conditions that two of the quadrilateral types in each tetrahedron are zero. There are a maximum 3^t such faces. These contain the representations of all embedded normal surfaces.

In a cell-decomposition or an ideal cell-decomposition of the 3-manifold M, the boundary of a small regular neighborhood of a vertex is normally isotopic to a normal surface, each component of which is called a *vertex-linking surface.* If $\mathcal{T}$ is a cell-decomposition of the 3-manifold M, a vertex-linking surface is either a disk (the vertex is in ∂M) or a 2-sphere (the vertex is in $\overset{\circ}{M}$, the interior of M). If $\mathcal{T}$ is an ideal cell-decomposition and v is an ideal vertex, a vertex-linking surface about v is a closed, orientable 2-manifold possibly having genus $g \geq 1$, where g is the index of the vertex v. In this case we sometimes refer to the vertex-linking surface as a *surface-at-infinity.* If $\mathcal{T}$ is a triangulation or an ideal triangulation, the entire collection of elementary triangles form an embedded, normal surface, each component of which is a vertex-linking surface. The elementary disk types in a vertex-linking surface of a more general cell-decomposition do not allow such a simple combinatorial description. The vertex-linking surfaces give examples of normal surfaces.

If S is a properly embedded surface in a 3-manifold M and $N(S)$ is a small regular neighborhood of S, the manifold $M' = M \setminus \overset{\circ}{N}(S)$ is said to be obtained from M by *splitting along* S. If S is one-sided in M, then S is nonorientable and there is a copy, say S', of the orientable double cover of S in $\partial M'$. If S is two-sided, then there are two homeomorphic copies, S' and S'', of S in $\partial M'$. They are in the same component of M' if and only if S does not separate M. If S is a normal surface, then we choose $N(S)$ in such a way that the components of its frontier, S' and S'', are normally isotopic to S (or if S is one-sided, then $S' = 2S$).

If S is a normal surface and M' is obtained by splitting M along S, there is a natural cell-decomposition on M'. A cell in this decomposition is just a component of a $\widetilde{\Delta}_i$ split along the normal disks in the lifts of S and the face identifications are just the face identifications for $\mathcal{T}$ restricted to the faces of our new cells. We call this the *induced cell-decomposition* on M', the manifold obtained by splitting M along S. Let $\mathcal{C}$ denote the induced cell-decomposition on M'. A normal disk in a cell in $\mathcal{C}$, which misses $S' \cup S''$, also is a normal disk in $\mathcal{T}$; however, there are, possibly, numerous distinct disk types of this sort in $\mathcal{C}$, which all represent the same disk type in $\mathcal{T}$. Let $d_{1,1}, \ldots, d_{1,n_1}, d_{2,1}, \ldots, d_{n,1}, \ldots, d_{n,n_n}$ denote the normal disk types in $\mathcal{C}$, which miss $S' \cup S''$, where $d_{i,1}, \ldots, d_{i,n_i}$ are all normally isotopic to the normal disk type d_i in $\mathcal{T}$. If F is a normal surface in X, which misses $S' \cup S''$, and $(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{i,1}, \ldots, x_{i,n_i}, \ldots, x_{n,n_n})$ is a parametrization of F as a normal surface in $\mathcal{C}$, then $(x_1, \ldots, x_n)$, where $\sum_{j=1}^{j=n_i} x_{i,j} = x_i$, is the parametrization of F as a normal surface in $\mathcal{T}$. We can think of the lattice point $(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{i,1}, \ldots, x_{i,n_i}, \ldots, x_{n,n_n})$ as a point in the sub-cone of the solution cone $\mathcal{S}(X, \mathcal{C})$, where we set all normal disk types in $\mathcal{C}$ that meet $S' \cup S''$ equal to zero. We call the parametrization $(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{i,1}, \ldots, x_{i,n_i}, \ldots, x_{n,n_n})$ of F, a *re-writing* of the parametrization $(x_1, \ldots, x_n)$ of F.

Finally, if S is a two-sided normal surface, M' is the manifold obtained by splitting M along S and S' and S'' the copies of S in $\partial M'$, then there is a natural identification of S' and S'' to recover the 3-manifold M with S' and S'' being identified to S in M. We will refer to this as *re-attaching along S' and S''*. In addition, if S^* is a normal surface in the cell-decomposition induced on M' (M split along S) and possibly now S^* meets $S' \cup S''$, then when we re-attach along S' and S'', we get a subcomplex, denoted $S \cup S^*$, which is the image of $S' \cup S'' \cup S^*$ in M. We will call this subcomplex the *piecewise linear normal surface* obtained from S and S^*.

We list below some existence results for normal surfaces. We say a 2-sphere S embedded in a 3-manifold M is *inessential in M* if S bounds a 3-cell in M; otherwise S is *essential*. A properly embedded disk D in a 3-manifold M is *inessential in M* if ∂D bounds a disk D' in ∂M; otherwise, D is *essential in M*.

Theorem 2.1 ([4])**.** *Let M be a 3-manifold. If there is an essential, properly embedded disk in M, then for any cell-decomposition $\mathcal{T}$ of M there is an essential, normal disk embedded in M.*

Theorem 2.2 ([17, 19])**.** *Let M be a 3-manifold. If there is an essential, embedded 2-sphere in M, then for any cell-decomposition $\mathcal{T}$ of M there is an essential, embedded, normal 2-sphere in M.*

Theorem 2.3 (Kneser's Finiteness Theorem [17])**.** *Suppose $\mathcal{T}$ is a triangulation of the compact 3-manifold M. There is a nonnegative integer N_0 so that whenever $F_1, \ldots, F_n$ is a pairwise disjoint collection of normal surfaces in M and $n \geq N_0$, then for some $i \neq j, F_i = F_j$.*

The equality, $F_i = F_j$, can be interpreted as F_i is normally isotopic to F_j or, equivalently, they have the same parameterization. A similar result is true for cell-decompositions; however, Theorem 2.3 is quite familiar and is the form in which we use this result. If M is closed, then one has, for example, $N_0 \leq 5t$, where t is the number of tetrahedra in $\mathcal{T}$.

There are numerous other existence results for normal surfaces, not needed for this work. See, for example, [4, 8, 9, 14, 15].

In this work we also use almost normal surfaces in triangulations.

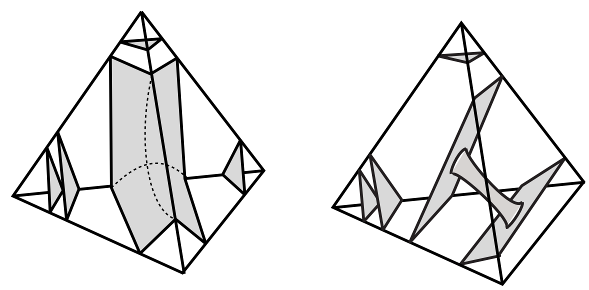
The elementary components of almost normal surface theory include the normal triangles and normal quads of normal surface theory but allow more general elementary components, normal octagons and normal tubes. A *normal octagon* is a properly embedded disk in a tetrahedron having boundary consisting of eight normal arcs in the boundary of the tetrahedron; whereas, a *normal tube* is a properly embedded annulus in a tetrahedron formed from two disjoint normal triangles, two disjoint normal quads or a normal triangle and a disjoint normal quad by joining them via a tube parallel to an edge of the tetrahedron. A normal octagon and a normal tube are shown in Figure 5. There are three types of normal octagons and twenty-five types of normal tubes in each tetrahedron.

If $\mathcal{T}$ is a triangulation or ideal triangulation of the 3-manifold M, we say a surface F is *almost normal* (with respect to $\mathcal{T}$) if and only if the lift of every component of F in a tetrahedron is either a normal triangle or a normal quadrilateral, except for at most one tetrahedron, where we allow the lift of F in the exceptional tetrahedron to be precisely one of:

- a collection of normal triangles and one normal octagon, or
- a collection of normal triangles and normal quads with one normal tube.

In the case of a normal tube, we do not allow that the normal tube is along an edge between two copies of the same normal surface. So, an almost normal surface never contains both a normal octagon and a normal tube but may contain one of them. An almost normal surface with a normal octagon is called an *octagonal almost normal surface* and one with a normal tube is called a *tubed almost normal surface*. A compression of the tube gives a normal surface and does not give two copies of the same normal surface.

Finally, we have from [8, 14] that if M has an essential 2 sphere, then for any triangulation of M there is a vertex solution which is an essential, embedded, normal 2-sphere. Similarly, if M has an essential, properly embedded disk, then for any triangulation of M there is a vertex solution which is an essential, normal disk. Under the appropriate hypothesis (which is one of the motivations of this work), one can also conclude that if there are almost normal 2-spheres, then there is one that is a vertex-solution. We will need additional results analogous to these for this work. See Propositions 5.7, 5.12 and 5.19.



(A) normal octagon (B) normal tube

Figure 5: A normal octagon and a normal tube between two normal quads.

3. Barriers and shrinking

In this section we collect a number of facts, which we need later and which are, in their own right, very useful to many applications. The basic notions are that of a "barrier surface" and the operation of "shrinking a surface" to a (possibly disconnected) normal surface. Similar results may be found in [11] and a complete study of barrier surfaces and shrinking is in [5]. The ideas comes from geometric analysis and minimal surface theory and were introduced into normal surface theory in [18]. Given a surface which is not normal, one wants to try to normalize the surface preserving some control. The control comes from the barrier surface and the shrinking is defined in terms of a sequence of permissable moves and a complexity. Traditionally, this complexity has been the weight (area) of the surface; the generality in which we use this notion requires a new complexity. Each permissable move reduces the complexity; if there are no permissable moves, then the surface is *stable*. It will follow that a stable surface has components that are either normal surfaces or are 0-weight 2-spheres or 0-weight disks, each properly embedded in a cell in our cell-decomposition.

Suppose $\mathcal{T}$ is a cell-decomposition (or ideal cell-decomposition) of the 3-manifold M and S is a properly embedded surface in M, which is transverse to the 2-skeleton of $\mathcal{T}$, $\mathcal{T}^{(2)}$. The *weight* of S is the cardinality of $S \cap \mathcal{T}^{(1)}$, $\mathrm{wt}(S) = |S \cap \mathcal{T}^{(1)}|$. Now, recall that the surface S is a normal surface if and only if the lift of every component of the intersection of S with a cell is a normal disk (and in the case of a triangulation, a normal triangle or a normal quad). So, for normal surface theory, we would like the lifts of the components of S in the various cells to be disks, in general, and normal disks, in particular. If the lifts of all the components of S in the cells are not disks, we need a measure of this variance. We define the *local Euler number of* S, written $\lambda_\chi(S)$, to be the sum $\lambda_\chi(S) = \Sigma_{c \neq S^2}(1 - \chi(\widetilde{c}))$, where c runs over all non-2-sphere components of S in the cells of $\mathcal{T}$. Notice that $\lambda_\chi(S) = 0$ if and only if each non spherical component of S in a cell of the decomposition $\mathcal{T}$ lifts to a disk. Finally, it is convenient to clean up some of the 0-weight intersections with the faces of $\mathcal{T}$. A 0-weight curve of intersection of S with $\mathcal{T}^{(2)}$ is a simple closed curve lying entirely in the interior of a face of $\mathcal{T}$. Let $\sigma(S)$ denote the number of 0-weight curves of the intersection of S with faces of $\mathcal{T}$, which are also in $\overset{\circ}{M}$, the interior of the 3-manifold M. We define the *complexity of* S to be $C(S) = (\mathrm{wt}(S), \sigma(S), \lambda_\chi(S))$, where we consider the set of triples under lexicographical order from the left.

3.1 Shrinking surfaces: normalization

In this work we use four basic moves in shrinking (normalizing) a properly embedded surface S: a compression, an isotopy, a ∂-compression and finally a "cleaning up" move, which is not really necessary but brings a nice order to the notion of stable surface. Of course, a ∂-compression is relevant only when S (and hence, the manifold M) has nonempty boundary. We show these moves in Figures 6-8.

We begin with a properly embedded surface S meeting the 2-skeleton of the cell-decomposition transversely. To keep notation simple, we refer to the surface at each step of the shrinking as S, understanding that it may have changed considerably from the original surface S. The target of shrinking is to arrive at a surface (a stable surface) having components which are normal surfaces or are properly embedded, 0-weight 2-spheres and 0-weight disks, each of which is contained entirely in some cell of our cell-decomposition. Hence, the lifts of the components of S in a cell will be normal disks and properly embedded, 0-weight 2-spheres and 0-weight disks in the cell. Recall that normal disks are characterized by their boundary curves in the cells, which are made up of a finite number of normal spanning arcs in the faces of the cells and which do not meet an edge in the cell more than once. Finally, we use the terms "compression" and "∂-compression" in a more general context than usual; we do not require the boundary of a compressing disk to be an essential curve in our surface nor do we require the arc in which a ∂-compressing disk meets our surface to be an essential arc in the surface. Hence, a compression or a ∂-compression may split off only a trivial bit of the surface.

The *normal moves* are:

1. *A compression in the interior of a cell. This move reduces the local Euler number and does not change weight or the number of* 0-*weight curves of intersection of* S *with the faces of* $\mathcal{T}$. (See Figure 6.)

   A compression reducing the local Euler number can be made in the interior of some cell whenever the local Euler number is not zero, $\lambda_\chi(S) \neq 0$. In this case, there is a component c of the intersection of S with some cell, say Δ_i and for $\widetilde{c}$ the lift of c, we have $1 - \chi(\widetilde{c}) > 0$; hence, there is a compression of $\widetilde{c}$ along an essential curve in $\widetilde{c}$ in $\widetilde{\Delta}_i$. It follows, there is a disk $\widetilde{D}$ embedded in $\widetilde{\Delta}_i$ so that $\widetilde{D} \cap \widetilde{c} = \partial\widetilde{D}$ and $\partial\widetilde{D}$ is not trivial in $\widetilde{c}$. Of course, it is possible that $\widetilde{D}$ meets other lifts of the components of S in Δ_i. However, if this is the case, then we may assume the intersection of the lifts of the components of S in Δ_i meet $\widetilde{D}$ in simple closed curves in the interior of $\widetilde{D}$. Either we can change our choice of $\widetilde{D}$ to eliminate such intersections or there is a lift $\widetilde{c}'$ of a component c' of S meeting Δ_i and a disk $\widetilde{D}' \subset \widetilde{D}$ so that $\partial\widetilde{D}'$ is an essential curve in $\widetilde{c}'$ (in particular, $1-\chi(\widetilde{c}') > 0$) and $\widetilde{D}'$ does not meet any other lifts of the components of S in Δ_i. Let's assume that D is such an innermost disk so we don't have to drag the prime notation along. We let D denote the image of $\widetilde{D}$ in Δ_i and compress c along D (which induces a compression of $\widetilde{c}$ along $\widetilde{D}$). Notice such a compression does not affect the weight but since $\partial\widetilde{D}$ is essential in $\widetilde{c}$, the compression decreases the local Euler number; furthermore, this move does not affect the intersection of S with the interior of the faces or the edges of $\mathcal{T}$. Hence, it reduces the complexity of the surface S.

   Note that this move may be an essential compression of the surface S and thereby a change of its topological type. If S were incompressible, we could argue that we still have a component, after this move, homeomorphic with S; and if M were also irreducible, we could argue that we have a surface isotopic with S. In our case we do not care. We wait until we arrive at a stable surface and then make an analysis.

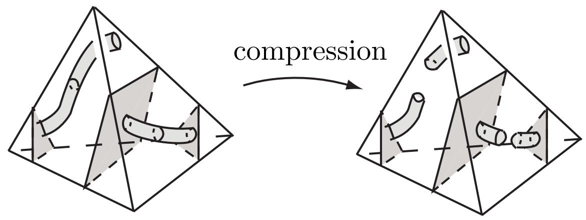


Figure 6: A compression in the interior of a cell. This move reduces the local Euler number and does not change weight or the 0-weight curves in the interior of faces of $\mathcal{T}$.

2. *An isotopy reducing the number of times the boundary of a lift of a component meets an edge of a cell, where the edge is in the interior of M. This move reduces* $\mathrm{wt}(S)$. (See Figure 7.)

   At this stage we may assume all the lifts of the components of S in a cell of $\mathcal{T}$ are either properly embedded disks or 2-spheres. Now, suppose $\widetilde{c}$ is a lift of a component of S in a cell Δ_i; $\widetilde{c}$ is a properly embedded disk; and $\widetilde{c}$ meets an edge e of $\widetilde{\Delta}_i$, the lift of Δ_i, more than once. Then if we consider the curve $\partial\widetilde{c}$, it divides the edge e into a number of subarcs and there is at least one, say $\widetilde{\beta}$ which has both its end points in $\partial\widetilde{c}$. Hence, there is a disk $\widetilde{D}$ embedded in $\widetilde{\Delta}_i$ so that $\widetilde{D} \cap \widetilde{c}$ is an arc $\widetilde{\alpha} \subset \partial\widetilde{D}$, $\widetilde{\alpha} \cup \widetilde{\beta} = \partial\widetilde{D}$ and $\widetilde{\alpha} \cap \widetilde{\beta} = \partial\widetilde{\alpha} = \partial\widetilde{\beta}$. See Figure 7. However, it is possible that $\widetilde{D}$ meets other lifts in $\widetilde{\Delta}_i$ of the components of S in Δ_i. If this is the case, then $\widetilde{D}$ meets lifts other than $\widetilde{c}$ in simple closed curves in the interior of $\widetilde{D}$ and spanning arcs in $\widetilde{D}$ having both their end points in $\widetilde{\beta}$. Standard techniques allow us to choose $\widetilde{D}$ so that there are no such simple closed curve components. Hence, if there are spanning arcs remaining, we can choose one $\widetilde{\alpha}'$, which is "outermost" in the sense that there is a disk $\widetilde{D}' \subset \widetilde{D}$ and a lift $\widetilde{c}'$ of a component c' of S in Δ_i so that $\widetilde{D}' \cap \widetilde{c}' = \widetilde{\alpha}'$, $\partial\widetilde{D}' = \widetilde{\alpha}' \cup \widetilde{\beta}'$, where $\widetilde{\beta}' \subset \widetilde{\beta}$ and $\widetilde{\alpha}' \cap \widetilde{\beta}' = \partial\widetilde{\alpha}' = \partial\widetilde{\beta}'$. Furthermore, $\widetilde{D}'$ does not meet any other lifts of components of S in Δ_i. As above, having demonstrated that we can find such an outermost disk, we assume the original disk $\widetilde{D}$ has this property so we do not need to drag along the prime notation.

   We consider the image D of $\widetilde{D}$ in Δ_i. Then D is an embedded disk in Δ_i; D only meets S in c; $D \cap c = \alpha \subset \partial D$ is a spanning arc of c; D meets the boundary of Δ_i in the arc β in the edge e of Δ_i (e is also used for the image of the edge e in $\widetilde{\Delta}_i$); and $\beta \subset \partial D$, where $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$. See Figure 7. Since we have assumed the edge e containing β is in the interior of the 3-manifold, then there is an isotopy of S, splitting c into two disks and reducing $\mathrm{wt}(S)$. Of course, this move may increase the value $\sigma(S)$ and the local Euler number; however, it reduces the complexity of S.

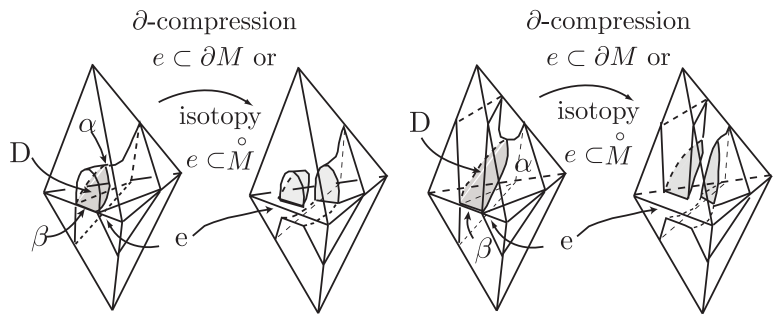


Figure 7: An isotopy or ∂-compression reducing the number of times a lift of a component meets an edge of a cell. This move reduces $\mathrm{wt}(S)$.

3. *A ∂-compression reducing the number of times a lift of a component meets an edge of a cell, where the edge is in the boundary of M. This moves reduces* $\mathrm{wt}(S)$. (See Figure 7.)

   Instead of, as above, where the edge e is in the interior of M, we now have the edge e in ∂M. In this case, the move must be accomplished by a ∂-compression rather than an isotopy. This move reduces $\mathrm{wt}(S)$; however, as above in the isotopy move, it may increase the value $\sigma(S)$ and the local Euler number. In any case, it reduces the complexity of S. Again, see Figure 7.

   Note that as above when we had a compression, a ∂-compression can change the topological type of S. However, if S were ∂-incompressible, then we would still have a component topological equivalent to the one before the ∂-compression and if M were ∂-irreducible and irreducible, we would have a component isotopic to the one before the ∂-compression. But just as above, in our case, this does not matter and we wait until we have a stable situation to make an analysis.

4. *A compression eliminating* 0*-weight simple closed curve components from the intersection of S with the faces of $\mathcal{T}$ in the interior of M.* (See Figure 8.)

   We may assume there are no essential compressions in the interior of any cell; actually, at this stage, we may assume the lift of any component of S in a cell is either a normal disk or a properly embedded, 0-weight 2-sphere or 0-weight disk in the cell. If there is a 0-weight simple closed curve common to S and a face σ of a cell, say Δ_i, then there is an innermost one. Such a 0-weight curve bounds a component c of S in Δ_i, which is a 0-weight disk, properly embedded in Δ_i, and having its boundary entirely in the interior of the face σ. Furthermore, its boundary bounds a disk D in the interior of σ. If σ is in the interior of the manifold, then there is a similarly embedded disk c' on the other side of this face, $\partial D = \partial c'$ and $c \cup c'$ is a small 2-sphere, which can be isotoped entirely into the interior of one of the cells or we can perform a compression along the disk D in σ and create two 0-weight 2-spheres, one in each cell sharing the face, σ. We choose to do the latter and therefore get two 2-spheres, each embedded entirely in the interior of a cell; this is one of our stable situations. Thus we eliminate all 0-weight simple closed curves in the interior of faces of $\mathcal{T}$, which are also in the interior of M. These moves reduce the value $\sigma(S)$ and do not affect the weight or the local Euler number.

   Note that we could just throw away all of the 0-weight pieces, which in practice is essentially what we do; but what we have done here reduces our work later when we need to analyze what we have after we shrink a surface.

As mentioned above, compressions and ∂-compressions can alter the properties of the surface S. We can place conditions on the surface and the 3-manifold so we can make these modifications and still maintain certain properties of the surface; for example, as indicated above, the surface is incompressible and ∂-incompressible and the manifold is irreducible and ∂-irreducible. However, these normal moves can be made on any surface. The moves never increase weight. We call a sequence

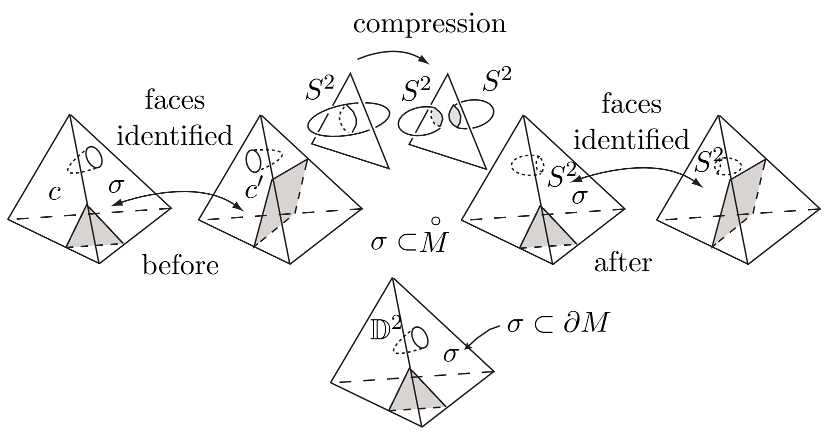


Figure 8: A compression removing a simple closed curve component from the intersection of S with a face in the interior of the manifold.

of these normal moves a *shrinking* of the surface S and we allow the surface S to be compressed and ∂-compressed, possibly resulting in a number of distinct components. After a finite number of steps, the components will either be normal or will be properly embedded, 0-weight 2-spheres and 0-weight disks contained entirely in various cells of our cell-decomposition. We say S has been *shrunk* to the resulting surfaces or sometimes we throw away the 0-weight components and say S has been shrunk to the resulting normal surface(s). Of course, it may be that S has shrunk until it disappears (has only 0-weight components).

3.2 Barrier surfaces

Suppose B be a properly embedded surface in a 3-manifold M and let N be a component of the complement of B, $M \setminus B$. We say B is a *barrier surface for* N, or simply a *barrier*, if any properly embedded, compact surface F in N can be shrunk in N.

We give criteria for a surface B, properly embedded in M, to be a barrier surface for a component N of its complement in M. Suppose Δ is a cell of $\mathcal{T}$ and C is the closure of a component of $\Delta \cap N$, $\widetilde{\Delta}$ the lift of Δ and $\widetilde{C}$ the lift of C. Let $b = C \cap B$ and $\widetilde{b}$ denote the lift of b. A collection of pairwise disjoint disks in $\widetilde{C}$ is said to be a *complete system of compressing disks for B in C* if:

- a disk, $\widetilde{D}$, in the collection meets $\widetilde{b}$ only in its boundary, which is an essential curve in $\widetilde{b}$; i.e., $\widetilde{D}$ is an essential compressing disk for $\widetilde{b}$ in $\widetilde{C}$ (see Figure 9(A)),
- a disk, $\widetilde{D}$, in the collection meets $\widetilde{b}$ in a properly embedded arc $\widetilde{\alpha}$ and meets boundary of $\widetilde{\Delta}$ in an arc $\widetilde{\beta}$, which is entirely in the interior of an edge of $\widetilde{\Delta}$, $\widetilde{\alpha} \cup \widetilde{\beta} = \partial\widetilde{D}$ and $\widetilde{\alpha} \cap \widetilde{\beta} = \partial\widetilde{\alpha} = \partial\widetilde{\beta}$; i.e., $\widetilde{D}$ is a ∂-compression (not necessarily essential) of $\widetilde{b}$ in $\widetilde{C}$ (see Figure 9(B)), and
- each component remaining after $\widetilde{b}$ has been compressed and ∂-compressed along the collection of disks is either a normal disk for $\mathcal{T}$ or a properly embedded 0-weight disk $\widetilde{E}$ having its boundary entirely in the interior of a face σ of $\widetilde{\Delta}$, $\partial\widetilde{E}$ bounds a disk $\widetilde{E}' \subset \sigma$ and the 2-sphere $\widetilde{E} \cup \widetilde{E}'$ bounds a 3-cell in $\widetilde{C}$ (see Figure 9(C)).

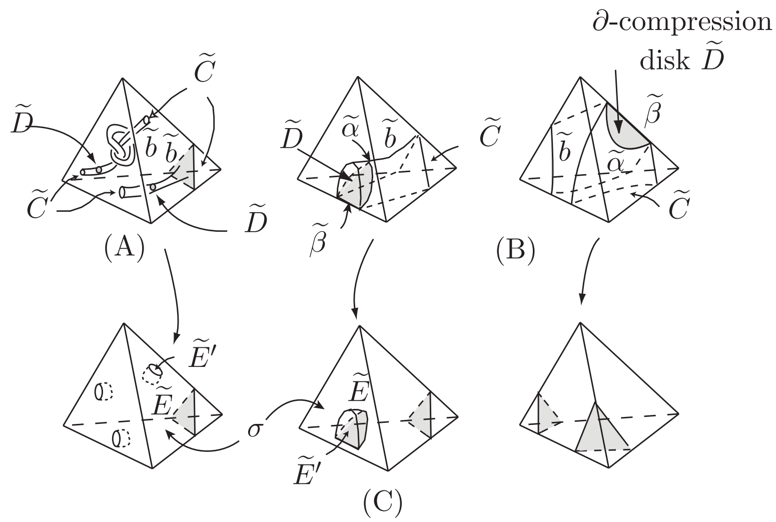


Figure 9: Complete system of compressing disks for a barrier surface.

Lemma 3.1. *Suppose $\mathcal{T}$ is a cell-decomposition (or ideal cell-decomposition) of the 3-manifold M and B is a properly embedded surface in M. The surface B is a barrier surface for the component N of $M \setminus B$ if for each cell Δ of $\mathcal{T}$ and the closure of each component C of $\Delta \cap N$, there is a complete system of compressing disks for B in C.*

Proof. Suppose F is a properly embedded surface in N. The surface F misses B (N is a component of $M \setminus B$) and we may assume that F is transverse to $\mathcal{T}^{(2)}$, the 2-skeleton of $\mathcal{T}$. We choose a properly embedded surface S that has minimal complexity among all surfaces that can be obtained by shrinking F missing B; i.e., sequences of compressions and ∂-compressions, missing B, and isotopies, which are identity on B. We claim each component of S is either normal or is a 0-weight 2-sphere or disk properly embedded in a cell of $\mathcal{T}$.

To see this, suppose Δ is a cell of $\mathcal{T}$. Let C be a component of $N \cap \Delta$. Let $\widetilde{C}$ be the lift of C; let $b = B \cap C$ and let $\widetilde{b}$ be the lift of b. By hypothesis, there is a complete system of disks for B in C. Suppose $c = S \cap C$ and $\widetilde{c}$ is the lift of c. If $\widetilde{c}$ meets the complete system of compressing disks for B in C, it does so in a very nice way. Specifically, if $\widetilde{D}$ is a compressing disk in $\widetilde{C}$ for $\widetilde{b}$, any component of $\widetilde{c} \cap \widetilde{D}$ is a simple closed curve in the interior of $\widetilde{D}$. If $\widetilde{D}$ is a ∂-compressing disk in $\widetilde{C}$ for $\widetilde{b}$, any component of $\widetilde{c} \cap \widetilde{D}$ is either a simple closed curve in the interior of $\widetilde{D}$ or a spanning arc; furthermore, the spanning arcs have both end points in the same edge of $\widetilde{\Delta}$. It follows that none of these intersections of $\widetilde{c}$ with the complete system of compressing disks are essential in $\widetilde{c}$ or we could make a sequence of moves, as defined above, to reduce the complexity of S. Hence, we can assume that $\widetilde{c}$ misses a complete system of compressing disks for B in C. This is true in every cell of $\mathcal{T}$. Notice *a priori* it may seem we have to go back to a cell in which we have made these moves before. But this could only happen if we reduce the weight of S, which would contradict our choice of S.

Now, for any cell Δ and any component C of $N \cap \Delta$, if c is a component of S meeting C and $\widetilde{c}$ is the lift of c, we can make all compressions and ∂-compressions along a complete system of disks for B in C, missing $\widetilde{c}$. Hence, $\widetilde{c}$ lies in a component of $\widetilde{\Delta}$ determined by normal disks and properly embedded 0-weight disks parallel into the interior of a face of $\widetilde{\Delta}$ through a 3-cell in $\widetilde{C}$. It follows, there is no obstruction to making further moves on $\widetilde{c}$ to reduce complexity unless each component of $\widetilde{c}$ is a normal disk or a properly embedded, 0-weight 2-sphere or 0-weight disk lying entirely in a cell of $\mathcal{T}$. q.e.d.

We now list several examples of barrier surfaces. Additional examples are given in [11] and [5]. As above, if S is a two-sided, normal surface in M and M' is the manifold obtained from M by splitting along S, then we let S' and S'' denote the copies of S in $\partial M'$. Furthermore, if S^* is a normal surface in the induced cell structure on M', we let $S \cup S^*$ denote the piecewise normal surface obtained from S and S^*. Similarly, if $\mathcal{K}$ is a subcomplex of the cell-decomposition induced on M'

by $\mathcal{T}$, then when we re-attach along S' and S'', we get a subpolyhedron, denoted $S \cup |\mathcal{K}|$, which is the image of $S' \cup S'' \cup |\mathcal{K}|$ in M.

Theorem 3.2. *Suppose $\mathcal{T}$ is a cell-decomposition (or ideal cell-decomposition) of the 3-manifold M.*

(1) *If S is a normal surface or an almost normal surface in M and B is the boundary of a small regular neighborhood of S in M, then B is a barrier surface for each component of its complement not meeting S. Often in this case, we just say the normal surface or almost normal surface S is a barrier surface for each component of its complement.*

(2) *If S is a two-sided, normal surface in M and S^* is a normal surface in the induced cell structure on M split along S and B is the boundary of a small regular neighborhood (in M) of the piecewise normal surface $S \cup S^*$, then B is a barrier surface for each component of its complement not meeting $S \cup S^*$.*

(3) *If X is a finite union of normal surfaces in M, which meet transversely, and B is the boundary of a small regular neighborhood of X, then B is a barrier surface for each component of its complement not meeting X.*

(4) *If $\mathcal{K}$ is a subcomplex of $\mathcal{T}$ and B is the boundary of a small regular neighborhood of the underlying point set of $\mathcal{K}$, $|\mathcal{K}|$, then B is a barrier surface for any component of its complement not meeting $|\mathcal{K}|$.*

(5) *If S is a normal surface in M and $\mathcal{K}$ is a subcomplex of the cell structure induced by $\mathcal{T}$ on M split along S and B is the boundary of a small regular neighborhood of $S \cup |\mathcal{K}|$, then B is a barrier surface for any component of its complement not meeting $S \cup |\mathcal{K}|$.*

(6) *If S is a two-sided, normal surface in M and $\mathcal{K}$ is a subcomplex of the cell structure induced by $\mathcal{T}$ on M', the manifold obtained by spliting M along S, and F is the frontier of a small regular neighborhood of $|\mathcal{K}|$, then F is a barrier surface in M' for any component of its complement not meeting $|\mathcal{K}|$.*

Proof. The proof is straight forward. In situations 3, 4, 5 and 6 there are numerous cases to consider; but listing the cases is the only task. q.e.d.

Barrier surfaces are used as a tool throughout this work and appear in the middle of certain arguments; however, we give here some general notions, which exhibit the use of barrier surfaces.

Suppose $B_1, \ldots, B_n$ is a pairwise disjoint collection of 3-cells in S^3. We call the 3-manifold $M = S^3 \setminus \bigcup_{i=1}^n \overset{\circ}{B}_i$ a *punctured* 3*-sphere*. In particular, the collection may be empty; so, we allow that the 3-sphere, itself, is a punctured 3-sphere, of course, without any punctures. In the case we know the boundary is not empty, we may also say we have a *punctured* 3*-cell*.

In our definition of a barrier surface B, we have that whenever B is a barrier surface for a component of its complement, then in all cases B is two-sided. So, if B is a barrier surface for the component N of its complement, then by taking a small regular neighborhood of B, we have a copy of B in N. We can then shrink this copy. We use the phrase "shrink B in N" in this situation. Also, if the barrier surface B is a normal surface or has components which are normal surfaces, then we may say shrink B, understanding that each component of B which is normal is stable and there are no normalization moves on such components. Such components survive (are never touched) in the shrinking.

Our first result uses a barrier surface to engulf the vertices of a triangulation.

Proposition 3.3. *Suppose $\mathcal{T}$ is a triangulation of the closed, orientable, irreducible* 3*-manifold M and suppose S is a normal* 2*-sphere in M which bounds a* 3*-cell E in M. Then there is a normal* 2*-sphere S' bounding a* 3*-cell E' in M, which contains E and all the vertices of $\mathcal{T}$ or $M = S^3$.*

Proof. If all the vertices of $\mathcal{T}$ are in E, then there is nothing to prove. Otherwise, split M along S and let M' denote the component not meeting E. We will continue to use S to denote the copy of S in the boundary of M'. There is a subcomplex Λ of the 1-skeleton of the induced cell structure on M' so that each component of Λ is a tree and meets S in precisely one point and Λ contains all vertices of $\mathcal{T}$ not in E. By the above theorem, the frontier B of a small regular neighborhood $N = N(S \cup \Lambda)$ of $S \cup \Lambda$ is a barrier surface in the component of its complement in M' not meeting $S \cup \Lambda$. Note that N is a punctured 3-cell in M' (actually, B is isotopic to S and in this case $N = S^2 \times I$), which contains all the vertices of $\mathcal{T}$ not in E.

If B is normal, then it is itself stable and does not shrink. In this case, we let $E' = E \cup N$ and let $S' = B$. So, we may assume B is not normal. We can shrink B in M'; furthermore, the point of B being a barrier is that this shrinking will not meet $S \cup \Lambda$. A shrinking (a finite sequence of normal moves) involves either a compression or an isotopy move (M is closed). So, assume we are at a stage in our shrinking where we have a finite number of pairwise disjoint 2-spheres $S_1, \ldots, S_n$, and a punctured 3-cell P_k with $S \cup \Lambda \subset P_k$ and ∂P_k includes S and the spheres $S_1, \ldots, S_n$. If the collection $S_1, \ldots, S_n$ is not stable, then there is either an isotopy normal move or a compression normal move on one of these 2-spheres.

An isotopy move is across an edge in N missing $S \cup \Lambda$; hence, we have an isotopy move of some 2-sphere, say S_i, in ∂P_k. We get a new collection of 2-spheres $S_1, \ldots, \mathcal{S}'_i, \ldots, S_n$ where S'_i replaces S_i and $\mathrm{wt}(S'_i) < \mathrm{wt}(S_i)$. We let P_{k+1} denote the image of P_k under this isotopy. Then P_{k+1} is a punctured 3-cell containing $S \cup \Lambda$.

If there is a compression on one of the 2-spheres, say S_i, then let D denote the compressing disk. Not only does D not meet any 2-sphere in the collection except for S_i, which it meets in its boundary, D does not meet $S \cup \Lambda$. If $D \subset P_k$, then D splits P_k into two punctured 3-cells, one, say P_{k+1} containing $S \cup \Lambda$, and ∂D splits S_i into two 2-spheres, S'_i and S''_i, with, say $S'_i \subset P_{k+1}$. We have a new collection of 2-spheres, $S_{j_1} \ldots, S'_i, \ldots, S_{j_m}$, which along with S make up the boundary of our new punctured 3-cell P_{k+1}. If D is not in P_k, then again ∂D splits S_i into two 2-spheres, S'_i and S''_i; however, in this case, a compression is adding a 2-handle to S_i and we get a new punctured 3-cell, P_{k+1}, containing P_k and having both S'_i and S''_i in its boundary. Also, $S \cup \Lambda \subset P_{k+1}$.

It follows that in shrinking B and in the stable situation we have a punctured 3-cell P, $S \cup \Lambda \subset P$ and the boundary of P consists of S along with possibly some other normal 2-spheres and possibly some 0-weight 2-spheres entirely in the interior of cells in the induced cell structure on M'. Each 0-weight 2-sphere bounds a 3-cell whose interior misses P. We fill in these 2-spheres with these 3-cells and continue to call our punctured 3-cell P. Now, since M is irreducible, each normal 2-sphere in the boundary of P bounds a 3-cell in M. If such a boundary component, other than S, bounds a 3-cell whose interior misses P we add that 3-cell to P. We will continue to call the punctured 3-cell P. So, we now have that $S \cup \Lambda \subset P$ and any 2-sphere in boundary of P other than S does not bound a 3-cell whose interior misses P. If S is the only boundary component of P, then M is S^3. If S' is a component of the boundary of P distinct from S, then by M irreducible, S' bounds a 3-cell, say E', in M. But then we have $E \cup P \subset E'$. So, such an S'

and E' satisfy the conclusions of our proposition. q.e.d.

There is a useful variation to the previous proposition when we do not assume the 3-manifold M is irreducible; namely, we have either there is a collection $S_1 \dots, S_n$ of normal 2-spheres bounding a punctured 3-cell P in M, where P contains E and all the vertices of $\mathcal{T}$, or M is S^3. We also can use Proposition 3.3 by, say, choosing S to be a vertex-linking normal 2-sphere and E the 3-cell it bounds to conclude that for any triangulation $\mathcal{T}$ of a closed, orientable, irreducible 3-manifold M, there is a normal 2-sphere bounding a 3-cell containing all the vertices of $\mathcal{T}$ or it follows that M is S^3. There are triangulations of S^3 for which there is no normal 2-sphere bounding a 3-cell containing all the vertices of the triangulation. For example, the triangulation given in Figure 2(4) is such a triangulation. There also are more interesting ones.

Suppose F is a closed, orientable surface. Let $F \times [0,1]$ be the product of F with the unit interval and let $\gamma_1, \dots, \gamma_n$ be a finite, pairwise disjoint collection of simple closed curves in $F \times 0$; it is not necessary that the γ_i be essential. Choose small regular neighborhoods $N(\gamma_1), \dots, N(\gamma_n)$ of the $\gamma_i, 1 \leq i \leq n$, in $F \times 0$ so that $N(\gamma_i) \cap N(\gamma_j) = \emptyset, i \neq j$. Let $D_1 \times [0,1], \dots, D_n \times [0,1]$ be a collection of 2-handles, where $D_i, 1 \leq i \leq n$, is a 2-cell. A 3-manifold is obtained by attaching the 2-handles, $D_i \times [0,1]$ along the γ_i; i.e., identifying the annulus $\partial D_i \times [0,1]$ with the annulus $N(\gamma_i)$ for $1 \leq i \leq n$. $F \times 1$ is a component of the boundary of this 3-manifold. There may be some number of 2-sphere components in the boundary as well. We may or may not fill in some of the 2-sphere boundary components with 3-cells (3-handles). We call the resulting 3-manifold, say H, a *compression body* and denote the boundary component $F \times 1$ by $\partial_+ H$ and denote the remaining boundary, which may not be connected, by $\partial_- H$. A component of $\partial_- H$, which is not a 2-sphere, is incompressible in H. If $\partial_- H = \emptyset$, then H is a handlebody and if each component of $\partial_- H$ is a 2-sphere, then H is a punctured handlebody. Finally, $F \times [0,1]$ is itself a compression body, as is a punctured $F \times [0,1]$.

We have the following result which is analogous to Proposition 3.3.

Proposition 3.4. *Suppose $\mathcal{T}$ is a triangulation of the closed, orientable 3-manifold M and suppose F is a normal, two-sided surface, embedded in M. Then there are compression bodies H' and H'' embedded in M so that $H' \cap H'' = F = \partial_+ H' = \partial_+ H''$, each component of $\partial_- H'$ and $\partial_- H''$ is normal and $H' \cup H''$ contains all vertices of $\mathcal{T}$.*

Proof. The proof is very similar to the proof of Proposition 3.3. Let

M' denote the manifold we get by splitting M along F; let F' and F'' denote the copies of F in $\partial M'$. There are disjoint subcomplexes Λ' and Λ'' of the 1-skeleton of the induced cell structure on M' so that each component of Λ' and Λ'' is a tree and meets F' and F'', respectively, in precisely one point, $\Lambda' \cap F'' = \emptyset = \Lambda'' \cap F'$ and $\Lambda' \cup \Lambda''$ contains all vertices of $\mathcal{T}$. Let B' and B'' be the boundaries of small regular neighborhoods of $F' \cup \Lambda'$ and $F'' \cup \Lambda''$, respectively. Then $B' \cup B''$ is a barrier surface for the components of their complements not meeting $F' \cup \Lambda' \cup F'' \cup \Lambda''$; furthermore, B' and F' are the boundaries of a compression body as well as B'' and F''; actually, in these cases the compression bodies are products. We shrink $B' \cup B''$. In shrinking $B' \cup B''$ we obtain two compression bodies G' and G'' so that $\partial_+ G' = F'$, $F' \cup \Lambda' \subset G'$, $\partial_+ G'' = F''$, $F'' \cup \Lambda'' \subset G''$ and each component of $\partial_- G'$ and of $\partial_- G''$ is either a normal surface or a 0-weight 2-sphere contained entirely in the interior of a cell in the induced cell structure on M'. Any such 0-weight 2-sphere bounds a 3-cell missing $F' \cup \Lambda'$ and $F'' \cup \Lambda''$. We fill in these 0-weight 2-spheres with such 3-cells.

Now, when we reattach F' and F'' to get M and set H' equal to the image of G' and set H'' equal to the image of G'', we get the desired compression bodies. q.e.d.

The following is essentially a direct generalization of Proposition 3.3 for engulfing the vertices of a triangulation by a handlebody; it uses the previous proposition and has numerous useful variants.

Proposition 3.5. *Suppose $\mathcal{T}$ is a triangulation of the closed orientable, irreducible 3-manifold M and suppose F is an embedded normal surface, which bounds a handlebody H in M. If F is incompressible in $M \setminus \overset{\circ}{H}$ and is not contained in a 3-cell in M, then there is a normal surface F' embedded in M, F' is parallel to F and bounds a handlebody H' in M so that $H \subset H'$ and H' contains all the vertices of $\mathcal{T}$.*

Proof. We can repeat the argument used in the proof of Proposition 3.4. In this case we split M along F and consider only the component that does not meet the handlebody H, call it M'. We denote the copy of F in $\partial M'$ by F'. We have the subcomplex Λ' as above and we let B' denote the boundary of a small regular neighborhood of $F' \cup \Lambda'$. B' is a barrier surface in the component of its complement not meeting $F' \cup \Lambda'$. Furthermore, F' and B' bound a compression body which is homeomorphic to $F' \times I$. We shrink B'. However, since F' is incompressible in M' (hence, B' is incompressible in M'), each normal move which is a compression is an inessential compression. It follows that in the stable situation we have a surface which is a copy of F' and every other component is either a normal 2-sphere or a 0-weight 2-sphere contained entirely in the interior of a cell in the induced cell decomposition of M'. Since M is irreducible and F is not contained in a 3-cell, we can fill in each 2-sphere boundary component with a 3-cell whose interior does not meet the compression body. It follows that we have a compression body G' with $\partial_+ G' = F'$, $\partial_- G'$ a normal surface isotopic to F' and $F' \cup \Lambda' \subset G'$. When we reattach M' to H to get M, the image of G' along with H gives us the desired handlebody H'.

q.e.d.

The next application is referred to as a "double barrier" argument.

Proposition 3.6. *Suppose $\mathcal{T}$ is a triangulation of the compact, orientable 3-manifold M and suppose K and L are disjoint subcomplexes in $\mathcal{T}$. Then there is a normal surface F in M separating K and L.*

Proof. Let B_K and B_L denote the frontiers of small regular neighborhoods of K and L, respectively, chosen so that $B_K \cap B_L = \emptyset$. Then B_K and B_L are barrier surfaces in the component of the complement of $B_K \cup B_L$ not meeting $K \cup L$. Furthermore, B_K separates K and L. We shrink B_K. In shrinking B_K, we have compressions, ∂-compressions and isotopy moves. An isotopy move or ∂-compression occurs through the interior of an edge that meets B_K and so is away from K or L. A compression is entirely in the interior of a cell or the face of a cell and so does not run through K or L. So, in our stable situation we have components which are either normal surfaces or 0-weight 2-spheres and disks which are properly embedded in the cells of our induced cell decomposition; furthermore, the union of these components separate K from L. We wish to eliminate the 0-weight 2-spheres and disks. None of the 0-weight 2-spheres and disks separate any components of K from L; so, we can discard these components. Since we must have K separated from L, we have the desired normal surface. q.e.d.

Notice that in shrinking a surface, we do not increase the genus of the surface, even in the bounded case. Hence, in the previous proposition, the separating normal surface may be found so that its genus is no more than the minimal genus of the surfaces B_K and B_L. In particular, if one of K or L is simply connected, then we can separate K and L by normal 2-spheres.

Finally, we have:

Proposition 3.7. *Suppose $\mathcal{T}$ is a triangulation of the compact, orientable 3-manifold M and S is a closed, two-sided normal surface in M. Let M' be the manifold obtained by splitting M along S. Suppose $D_1, \ldots, D_n$ is a collection of pairwise disjoint, properly embedded disks in M', which are normal in the induced cell-decomposition on M'. Furthermore, suppose the D_i are all on the same side of S (only meet S', say, and so not meet S'' in M'). Then there is a compression body H embedded in M, $\partial_+ H = S$, each component of $\partial_- H$ is a normal surface in M and $D_1 \cup \cdots \cup D_n \subset H$.*

Proof. The proof of this proposition follows along the very same lines as 3.4, except we replace the graph Λ in that argument with the subcomplex $D_1 \cup \cdots \cup D_n$. q.e.d.

More discussions of barriers and shrinking appear in [5].

4. Crushing triangulations

In the Introduction we pointed out that our techniques evolve from the idea of finding a non-vertex-linking, normal 2-sphere, which bounds a 3-cell in our manifold, and then "crushing" the 2-sphere (and 3-cell) to a point. Generally speaking, this is what we do; however, while crushing a cell to a point gives us back our manifold, in general, it wrecks havoc with the triangulation. Of course, crushing a 3-cell, which is bounded by a normal 2-sphere, to a point induces a cell-decomposition on the resulting manifold and there are many straight forward techniques to construct a triangulation from such a cell-decomposition. We emphasize that this is not what we do. Such methods generally result in having a large number of tetrahedra and our notion of a simpler triangulation is one with fewer tetrahedra. Our main task then, in this situation, is to reconstruct a triangulation of our manifold, which is simpler than the original triangulation. In this section we organize such a construction, which we think of as crushing a triangulation along a normal surface. This has turned out to be a very useful concept.

Suppose $\mathcal{T}$ is a triangulation of a closed, orientable 3-manifold or an ideal triangulation of the interior of a compact, orientable 3-manifold M. Suppose S is a normal surface embedded in M and X is the closure of a component of the complement of S; furthermore, suppose X does not contain any vertices of $\mathcal{T}$. In this situation, we give sufficient conditions for constructing a particularly nice *ideal* triangulation of $\overset{\circ}{X}$. We will organize our construction into a theorem at the end of this section.

Notice that the manifold X has a nicely described cell-decomposition, say $\mathcal{C}$, induced from the triangulation $\mathcal{T}$ and the way in which the normal surface S sits in $\mathcal{T}$ (no vertices are in X). The induced cells are of four types: truncated tetrahedra, truncated prisms, triangular parallel regions, and quadrilateral parallel regions. We say the truncated tetrahedra in $\mathcal{C}$ are *cells of Type* I, the truncated prisms are *cells of Type* II, the parallel triangular regions are *cells of Type* III and the parallel quadrilateral regions are *cells of Type* IV. See Figure 10.

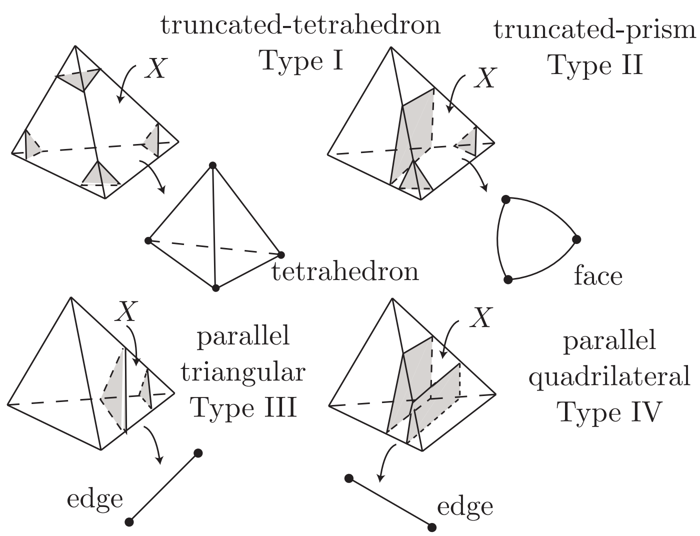


Figure 10: Cells in induced cell-decomposition of X.

We show that under the right conditions, one can get an ideal triangulation of $\overset{\circ}{X}$ by essentially replacing the truncated tetrahedra in $\mathcal{C}$ by tetrahedra; replacing the truncated prisms in $\mathcal{C}$ by faces and replacing the parallel product cells by edges (again, see Figure 10). However, as one might expect, things are in general a bit more complicated than this; hence, to see that something like this works, we need to analyze the total structure of the collection of parallel triangular and quadrilateral regions and truncated prisms in X.

First, we define what we will call a product region for X, determined from the cell-decomposition $\mathcal{C}$. Notice that in cells of Type III and Type IV in $\mathcal{C}$ there are quadrilateral faces in the 2-skeleton of $\mathcal{C}$ (and in faces of the tetrahedra of $\mathcal{T}$), which are complementary to the faces in S. See Figure 11. Also, in the cells of Type II, the truncated prisms, there are quadrilateral faces in the 2-skeleton of $\mathcal{C}$ (and, again, in the faces of tetrahedra of $\mathcal{T}$), which are complementary to the faces in S. We will call these quadrilaterals *trapezoids* to distinguish them from the normal quadrilaterals in S, which also are in the faces of cells of Types II and IV in $\mathcal{C}$. We let $\mathbb{P}(\mathcal{C})$ denote the following union. $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in $S\} \cup \{$cells of Type III and Type IV in $\mathcal{C}\} \cup \{$all trapezoidal faces of $\mathcal{C}\}$. Each component of $\mathbb{P}(\mathcal{C})$ is an I-bundle. In earlier drafts, we only considered cells of Type III and Type IV in $\mathcal{C}$ in defining $\mathbb{P}(\mathcal{C})$; the method we are using here was suggested by Nathan Dunfield, Marc Culler and Peter Shalen and supported by a number of other colleagues.

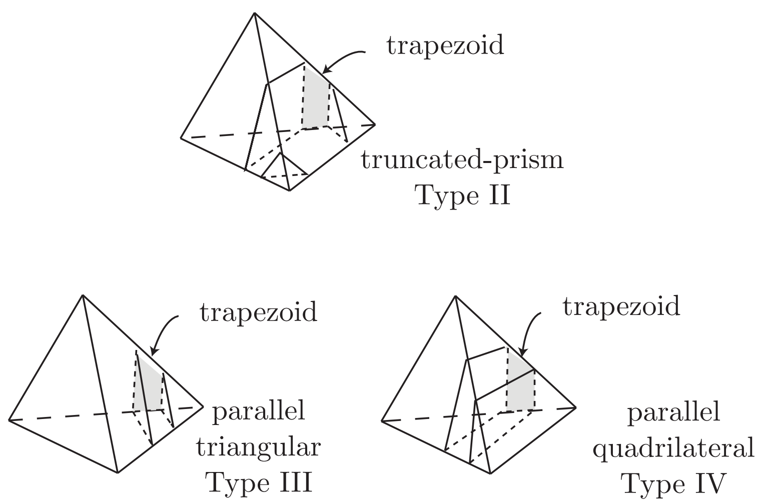


Figure 11: Trapezoidal faces in the 2-skeleton of the cell-decomposition of X.

Suppose $\mathbb{P}(\mathcal{C}) \neq X$ and all of the components of $\mathbb{P}(\mathcal{C})$ are product I-bundles. In applications, we will need to establish that these conditions are satisfied. Under these assumptions, each component of $\mathbb{P}(\mathcal{C})$, say $\mathbb{P}_i$, for some $i = 1, \ldots, k$, where k is the number of components of $\mathbb{P}(\mathcal{C})$, has the structure $\mathbb{P}_i = K_i \times [0,1]$, where K_i is isomorphic to a subcomplex of the induced normal cell structure on S. Let $K_i^\varepsilon = K_i \times \varepsilon, \varepsilon = 0, 1$. Then K_i^0 and K_i^1 are disjoint, isomorphic subcomplexes of the induced normal cell structure on S. Some of these product components may be just edges in the 1-skeleton of $\mathcal{C}$ or made up entirely of trapezoids, in which case $K_i^\varepsilon, \varepsilon = 0, 1$ are points or graphs, respectively. The ideal situation would be that each K_i^ε is simply connected and therefore a cellular planar complex. (We have assumed $\mathbb{P}(\mathcal{C}) \neq X$.) While, in general, we can not have each K_i^ε simply connected, in practice there is a construction which alters the products $K_i \times [0,1]$ to a new collection of products $D_j \times [0,1]$, where $D_j \times \varepsilon, \varepsilon = 0, 1$ is a subcomplex of S and the inclusion homomorphism $\pi_1(D_j \times \varepsilon)$ into $\pi_1(S)$ is injective for $\varepsilon = 0, 1$.

For now, we simply make this an assumption, which in practice will need to be established. We assume there is a pairwise disjoint collection of spaces $D_j, 1 \leq j \leq k'$, along with pairwise disjoint embeddings $D_j \times [0,1]$ into X so that $D_j \times [0,1]$ is a subcomplex of X and for $D_j^\varepsilon = D_j \times \varepsilon, \varepsilon = 0, 1$, we have D_j^ε is embedded as a subcomplex of the induced cell structure on S; the inclusion of $\pi_1(D_j^\varepsilon)$ into $\pi_1(S)$ is injective, $\varepsilon = 0, 1$; and, finally, $\bigcup(K_i \times [0,1]) \subset \bigcup(D_j \times [0,1])$ and the frontier of $\bigcup(D_j \times [0,1])$ is contained in the frontier of $\bigcup(K_i \times [0,1])$. While K_i^0 is isomorphic to K_i^1 for every i, it will not necessarily be the case that D_j^0 is isomorphic to D_j^1 for every j. See Figure 12.

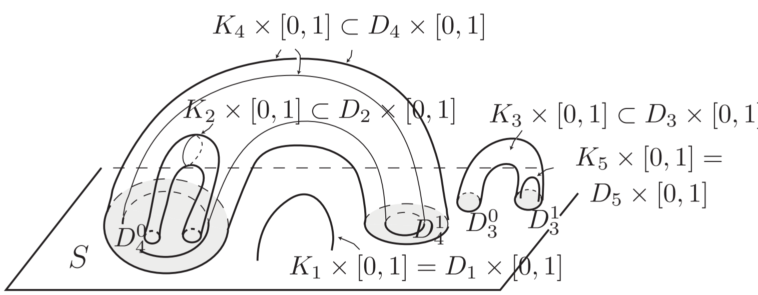


Figure 12: Product region $\mathbb{P}(X)$ in X.

Under our assumptions, we let $\mathbb{P}(X)$ be the collection of components of $\bigcup(D_j \times [0,1])$ and say that $\mathbb{P}(X)$ is an *induced product region for* X. We note, in general, there is not a unique induced product region. Each component of $\mathbb{P}(\mathcal{C})$ is contained in a component of $\mathbb{P}(X)$. It is possible that some $D_{j'} \times [0,1]$ is contained in a $D_j \times [0,1], j' \neq j$; so, $D_{j'} \times [0,1]$ is consumed and does not appear as a component of $\mathbb{P}(X)$. Also, it is possible that a number of truncated prisms and truncated tetrahedra in the cell-decomposition $\mathcal{C}$ on X are also consumed into the components of $\mathbb{P}(X)$. The product structure on the components of $\mathbb{P}(X)$, however, respects the combinatorial structure of the cells of $\mathcal{C}$ in the sense that each component of the frontier of $D_j \times [0,1]$ is either an edge in $\mathcal{C}$ (when $D_j \times [0,1]$ is itself an edge) or a collection of trapezoids. We assumed $\mathbb{P}(\mathcal{C}) \neq X$; in practice, this will give that $\mathbb{P}(X) \neq X$. Of course, this is something else we will have to establish.

If each component of $\mathbb{P}(X)$ is a product $D_i \times [0,1]$ where D_i is a simply connected planar complex (is cellular), then we say the product region $\mathbb{P}(X)$ for X is a *trivial product region for* X.

Now, consider the cells of X of Type II, truncated prisms. Each cell of Type II has two hexagonal faces. In the cell decomposition of X, these hexagonal faces can be identified to either a hexagonal face of a cell of Type I (truncated tetrahedron) or a hexagonal face of another cell of Type II. If we follow a sequence of such identifications through cells of Type II, we trace out a well-defined arc (see Figure 13 below) which either terminates at the identification of a hexagonal face of a cell of Type II with one of Type I or it forms a simple closed curve through cells of Type II, ending at the cell in which it started. We call the collection of truncated prisms identified in this way a *chain*. If a chain ends in a truncated tetrahedra, we say the chain *terminates*; otherwise, we call the chain a *cycle*. Notice, in particular, if for some truncated prism there is an identification of one of its hexagonal faces with the other, then we have a cycle of length one.

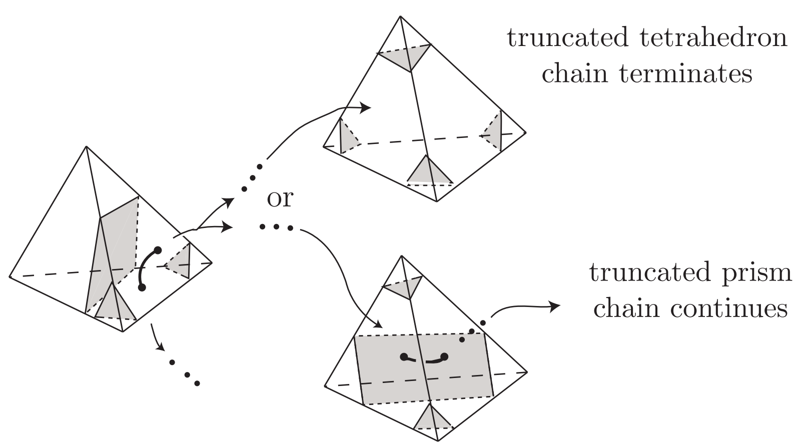


Figure 13: Chain of truncated prisms.

We will assume there are no cycles of truncated prisms, which are not in the induced product region of X, $\mathbb{P}(X)$. Later, of course, we will have the burden of proof to establish this in applications of crushing.

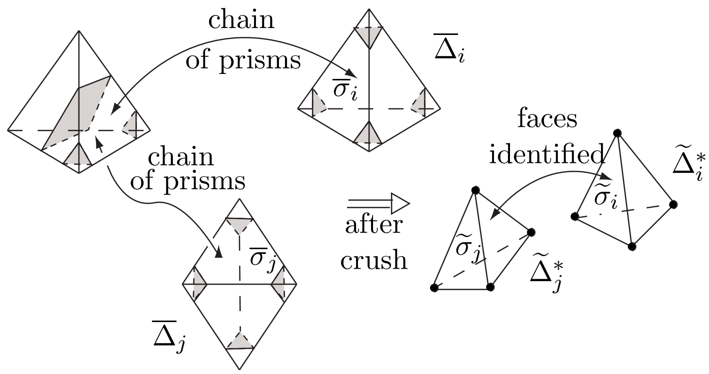


Figure 14: Face identifications induced through a chain of truncated prisms.

By our assumptions, there are truncated tetrahedra in X, which are not in $\mathbb{P}(X)$ ($X \neq \mathbb{P}(X)$), the frontier of $\mathbb{P}(X)$ consists of trapezoids, and there is no cycle of truncated prisms, which is not in $\mathbb{P}(X)$). Let $\{\overline{\Delta}_1, \ldots, \overline{\Delta}_n\}$ be the collection of truncated tetrahedra in X, which are not in $\mathbb{P}(X)$. Now, a face of a truncated tetrahedron in $\mathcal{C}$ is either shared by two truncated tetrahedra in $\mathcal{C}$ or is also a hexagonal face of a truncated prism in $\mathcal{C}$. In the last case, suppose we have a hexagonal face, say $\overline{\sigma}_i$ of the truncated tetrahedron $\overline{\Delta}_i$, which also is a hexagonal face in the first truncated prism in a chain of truncated prisms. If we follow the chain of truncated prisms, there is a last hexagonal face, say $\overline{\sigma}_j$, which also is in a truncated tetrahedron, $\overline{\Delta}_j$, possibly, $i = j$. Hence, there is an induced identification of the face $\overline{\sigma}_i$ of $\overline{\Delta}_i$ with the face $\overline{\sigma}_j$ of $\overline{\Delta}_j$, through the chain of truncated prisms. So, the faces of the truncated tetrahedra in $\{\overline{\Delta}_1, \ldots, \overline{\Delta}_n\}$ have an induced pairing. See Figure 14. Now, notice that each truncated tetrahedron in X has its triangular faces in S. We can identify each such triangular face to a point (distinct points for each triangular face) and we get tetrahedra. We will now use the notation $\widetilde{\Delta}_i^*$ for the tetrahedron coming from the truncated tetrahedron $\overline{\Delta}_i$ by identifying each of the triangular faces of $\overline{\Delta}_i$ to a point (distinct points for each triangular face) and use $\widetilde{\sigma}_i$ for the triangular face coming from the hexagonal face $\overline{\sigma}_i$. Then $\mathbf{\Delta}^*(X) = \{\widetilde{\Delta}_1^*, \ldots, \widetilde{\Delta}_n^*\}$ is a collection of tetrahedra with orientation induced by that on $\mathcal{T}$ and the induced pairings between the triangular faces $\widetilde{\sigma}_i$ and $\widetilde{\sigma}_j$ described above is a family $\mathbf{\Phi}^*$ of orientation reversing affine isomorphisms. Hence, we get a 3-complex $\mathbf{\Delta}^*(X)/\mathbf{\Phi}^*$, which is a 3-manifold except, possibly, at its vertices. We will denote the associated ideal triangulation by $\mathcal{T}^*$ and the underlying point set by $|\mathcal{T}^*|$. We call $\mathcal{T}^*$ the ideal triangulation obtained by *crushing the triangulation* $\mathcal{T}$ *along* S. We denote the image of a tetrahedron $\widetilde{\Delta}_i^*$ by Δ_i^* and, as above, call $\widetilde{\Delta}_i^*$ the lift of Δ_i^*.

We have the following theorem:

Theorem 4.1. *Suppose $\mathcal{T}$ is a triangulation of a closed, orientable 3-manifold or an ideal triangulation of the interior of a compact, orientable 3-manifold M. Suppose S is a normal surface embedded in M, X is the closure of a component of the complement of S and X does not contain any vertices of $\mathcal{T}$. Suppose there is an induced product region, $\mathbb{P}(X)$, for X. If:*

i) $X \neq \mathbb{P}(X)$,

ii) $\mathbb{P}(X)$ *is a trivial product region for X, and*

iii) *there are no cycles of truncated prisms in X, which are not in $\mathbb{P}(X)$,*

then the triangulation $\mathcal{T}$ can be crushed along S and $\mathcal{T}^*$ is an ideal triangulation of $\overset{\circ}{X}$.

Proof. The underlying point set $|\mathcal{T}^*|$ is obtained from X by identifying each component of S to a point (distinct points for distinct components), identifying each component $D_i \times [0,1]$ of $\mathbb{P}(X)$, the product region for X, to an edge $e_i = [0,1]_i$ (distinct edges for distinct components; see Figure 15), and identifying each chain of truncated prisms to a face (see Figure 14). If we look at this identification map we have the inverse image of a point in the interior of a tetrahedron Δ_i^* is just a point in the interior of the truncated tetrahedron $\overline{\Delta}_i$; the inverse image of a point in the interior of a face is either a point or an arc, the latter in the case a chain of truncated prisms is identified to a face; and the inverse image of a point in the interior of an edge is a copy $D_j \times x$ for some j and $x \in [0,1]$. Notice that in the identification of a chain of truncated prisms to a face; the associate identification of the edges is through a band of trapezoids and so there are no new identifications not already made in $D_j \times [0,1]$ for some j. Thus the identification map on $\mathring{X}$ is a cell-like map. It follows by [1, 20], that $\mathcal{T}^*$ is an ideal triangulation of $\mathring{X}$. q.e.d.

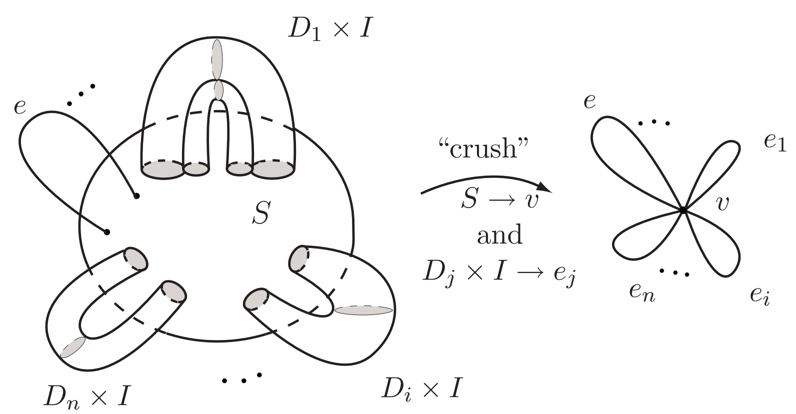


Figure 15: Components of trivial product region "crushed" to edges.

We end this section with a couple of observations. First, the tetrahedra of the ideal triangulation $\mathcal{T}^*$ are in one-one correspondence with the truncated tetrahedra of the cell-decomposition $\mathcal{C}$, which are not in the induced product region $\mathbb{P}(X)$. Hence, if t is the number of tetrahedra of $\mathcal{T}$ and t^* is the number of tetrahedra of $\mathcal{T}^*$, then $t^* \leq t$. The inequality is strict unless S is vertex-linking; in fact, $\mathcal{T}^* = \mathcal{T}$ if and only if S is vertex-linking. If each component of S is a 2-sphere and $\widehat{X}$ is the 3-manifold obtained by capping off each 2-sphere in ∂X, then $\mathcal{T}^*$ is a triangulation of $\widehat{X}$. More in the spirit of crushing, if each component of S is a 2-sphere, we can think of $|\mathcal{T}^*|$ as the the manifold obtained from X by identifying each 2-sphere in S to a point (distinct points for distinct 2-spheres). This is the same manifold as $\widehat{X}$.

5. 0-efficient triangulations

In this section we develop the concepts of 0-efficient triangulations of closed, orientable 3-manifolds and compact, orientable 3-manifolds with boundary. In a later section we define and study 0-efficient ideal triangulations. A 0-efficient triangulation severally limits the nature of embedded normal 2-spheres (closed manifolds and ideal triangulations) and normal disks (bounded manifolds). Also, in this section, and in a later section on minimal triangulations, we show that with few exceptions, 0-efficient triangulations do not have edges of order less than four.

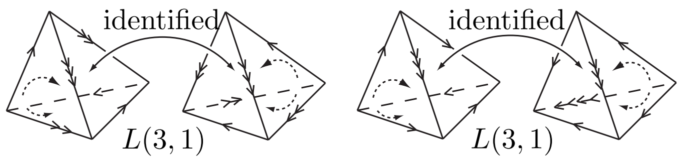
5.1 0-efficient triangulations for closed 3-manifolds

A triangulation of a closed 3-manifold is said to be 0-*efficient* if and only if the only embedded, normal 2-spheres are vertex-linking. For example, both of the one-tetrahedron triangulations of the 3-sphere are 0-efficient (see Figure 2 (4) and (5)), while only one of the two-tetrahedra, one-vertex triangulations of $L(3,1)$ is 0-efficient, see Figure 16. Neither of the two-tetrahedra, two-vertex triangulations of $L(3,1)$ are 0-efficient, which includes the two tetrahedron standard lens space presentations of $L(3,1)$ (see Figure 17). No triangulation of $\mathbb{R}\mathrm{P}^3$ can be 0-efficient; there are two two-tetrahedra triangulations of $\mathbb{R}\mathrm{P}^3$, one has one vertex and the other is the standard lens space presentation of $\mathbb{R}\mathrm{P}^3$ which has two vertices (again, see Figure 17). In [5], it is shown that S^3 and all lens spaces, except $\mathbb{R}\mathrm{P}^3$, admit infinitely many one-vertex 0-efficient triangulations. In fact, it can be shown that if there is a 0-efficient triangulation of the 3-manifold M which contains a layered solid torus, then there are infinitely many 0-efficient triangulations of M. However, there are manifolds for which the only 0-efficient triangulation we know does not have a layered solid torus as a subcomplex. We suspect that any manifold which admits a 0-efficient triangulation admits infinitely many such triangulations but we have not been able to confirm this in general.

The following proposition is quite useful and provides some insight to 0-efficient triangulations. We wish to thank Eric Sedgwick for helpful comments regarding the proof of this proposition.

Proposition 5.1. *Suppose M is a closed, orientable 3-manifold. If M has a 0-efficient triangulation, then M is irreducible and $M \neq \mathbb{R}P^3$. Furthermore, either the triangulation has one vertex or M is S^3 and the triangulation has precisely two vertices.*

Proof. Suppose $\mathcal{T}$ is a 0-efficient triangulation of M. If M were not irreducible, then M would contain an essential 2-sphere, a 2-sphere that does not bound a 3-cell. If this is the case, then as in Theorem 2.2, also [17, 4, 9], for any triangulation, in particular $\mathcal{T}$, M has an embedded,



(A) Two solid tori; 0-efficient

(B) Solid torus and 3-cell; not 0-efficient

Figure 16: Two tetrahedron, one-vertex triangulations of $L(3,1)$. (A) is 0-efficient. (B) is not 0-efficient.

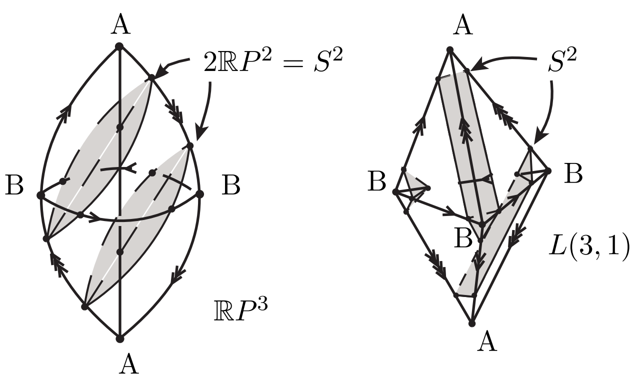


Figure 17: Standard lens space representations giving two-tetrahedron, two-vertex triangulations of $\mathbb{R}\mathrm{P}^3$ and $L(3,1)$, neither of which is 0-efficient.

essential, normal 2-sphere. However, an embedded, essential, normal 2-sphere can not be vertex-linking. So, M must be irreducible.

Now, if M contains an embedded $\mathbb{RP}^2$, then, similarly, we have for any triangulation of M there is an embedded, normal $\mathbb{RP}^2$. Since an $\mathbb{RP}^2$ can not be embedded in a 3-cell, such an $\mathbb{RP}^2$ must contain a normal quadrilateral. Hence, its double, which is an embedded normal 2-sphere, can not be vertex-linking. See Figure 17 where there is an embedded normal $\mathbb{RP}^2$ with two normal quads and its double, which is an embedded, normal 2-sphere, has four normal quads. This completes the proof of the first part of the proposition.

So, suppose $\mathcal{T}$ is 0-efficient and has more than one vertex. In this case, we first show M is the 3-sphere.

Since we are assuming $\mathcal{T}$ has more than one vertex, there is an edge e in $\mathcal{T}$ which has distinct vertices. Let $N(e)$ denote a small neighborhood of e and let S denote $\partial N(e)$; then $N(e)$ is a 3-cell and S is a 2-sphere. If S is normal, then S is vertex-linking and so, must bound a 3-cell complementary to the 3-cell $N(e)$, which contains the edge e. It follows in this case that M is the 3-sphere. So, suppose S is not normal. Then S forms a barrier surface in the component of its complement not containing e; so, we can shrink S. In shrinking S we end up with a punctured 3-cell containing the edge e and its boundary is made up of normal 2-spheres and 0-weight 2-spheres contained entirely in the interiors of tetrahedra of $\mathcal{T}$. Let P denote this punctured 3-cell containing e. Any 2-sphere in ∂P, which is embedded in a tetrahedron, bounds a 3-cell in that tetrahedra which does not meet e, which can be added to P, still giving a punctured 3-cell; and any normal 2-sphere in ∂P, which is vertex-linking, bounds a 3-cell not meeting e, which can be added to P, giving a punctured 3-cell. Thus each boundary component of P bounds a 3-cell in the complement of P. It follows that M is S^3.

We only have left to prove that a 0-efficient triangulation of S^3, not having just one vertex, has precisely two. We establish this through a series of claims.

Suppose $\mathcal{T}$ is a 0-efficient triangulation of S^3 with more than one vertex.

Claim 1. *The* 1*-skeleton of* $\mathcal{T}$ *can not have disjoint edges, one of which is an arc. See Figure* 18.

Proof. Suppose e and e' are disjoint edges of $\mathcal{T}$. By Proposition 3.6, e and e' can be separated by a normal surface; furthermore, since e, say, is an arc, then such a normal surface can be taken to be a normal

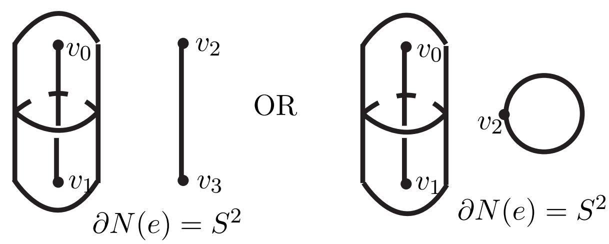


Figure 18: No disjoint edges in $\mathcal{T}$ with one an arc.

2-sphere. However such a normal 2-sphere can not be vertex-linking. Thus we arrive at a contradiction.

Claim 2. *The* 1*-skeleton of* $\mathcal{T}$ *can not have a cycle of three embedded edges between three distinct vertices; i.e., there are no triads in the* 1*-skeleton of* $\mathcal{T}$*. See Figure* 19*.*

Proof. Suppose v_0, v_1, v_2 are three distinct vertices determining the cycle of distinct edges $\overline{v_0v_1}$, $\overline{v_1v_2}$, $\overline{v_2v_0}$. If there were more than three vertices, then there is an edge e' from a vertex v, distinct from each of the vertices v_0, v_1, v_2, to one of these vertices, say v_0. Then e' and $e = \overline{v_1v_2}$ are disjoint edges, both of which are arcs. This contradicts our previous observation (see Figure 19). So, we may suppose that if there is an embedded triad in the 1-skeleton of $\mathcal{T}$, then $\mathcal{T}$ has at most three vertices. If there were only three vertices, then since a tetrahedron has four vertices, there must be an edge e' of $\mathcal{T}$ having both its vertices at the same vertex of the triad with vertices v_0, v_1, v_2, say at v_0. Again, we have an edge e' disjoint from the edge $e = \overline{v_1v_2}$ which is an arc and, hence, gives a contradiction to Claim 1 (again, see Figure 19). This shows that there are no embedded triads in the 1-skeleton of $\mathcal{T}$.

Claim 3. *If* $\mathcal{T}$ *has more than two vertices, then there is a vertex* v_0 *of* $\mathcal{T}$ *so that every edge has* v_0 *as a vertex. See Figure* 20*.*

Proof. Since we are assuming $\mathcal{T}$ has at least three vertices and we have shown that no two edges, one of which is an arc, are disjoint, we can choose notation so there are distinct vertices v_0, v_1, v_2 and edges $\overline{v_0v_1}$ and $\overline{v_0v_2}$, meeting only in v_0. Suppose some edge e' does not meet v_0. Then e' can not have both v_1 and v_2 as vertices, since this would form an embedded triad in the 1-skeleton of $\mathcal{T}$. Hence, either e' is disjoint from the two edges $\overline{v_0v_1}$ and $\overline{v_0v_2}$ or e' meets just one of the vertices v_1

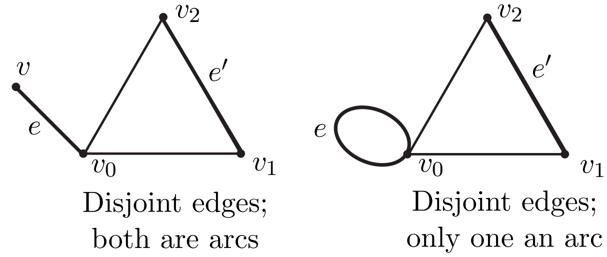


Figure 19: No embedded triads in 1-skeleton of $\mathcal{T}$.

or v_2; e' may be a loop. In either case, we have two disjoint edges, e' and one of $\overline{v_0v_1}$ or $\overline{v_0v_2}$, each of which is an arc. This contradicts an earlier claim.

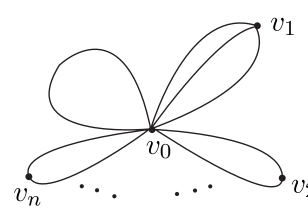


Figure 20: All edges of $\mathcal{T}$ meet at v_0.

Claim 4. *If every edge in the* 1*-skeleton of* $\mathcal{T}$ *meets a single vertex and there is more than one vertex, then* $\mathcal{T}$ *is not* 0*-efficient. See Figure* 21*.*

Proof. Suppose every edge meets the vertex v_0 and there are at least two vertices, say v_1 is a second vertex. Since each edge has a vertex at v_0, there is an edge e having v_0 and v_1 as vertices. Furthermore, every tetrahedron having v_1 as a vertex has all its other vertices at v_0.

There are two obstructions to building an edge-linking normal 2-sphere about an edge in a triangulation of a 3-manifold. One is that the ends of the edge are the same vertex; so, we choose an edge from v_1 to v_0, say $e = \overline{v_1v_0}$. The other obstruction is that, say for the edge e, there is a tetrahedron τ in $\mathcal{T}$ having e as an edge and τ has an adjacent edge e', also from v_1 to v_0, identified with e; this prevents a small neighborhood about the edge e from being normal. However, if this is the case, we will show how to use the induced triangulation of the vertex-linking normal 2-sphere, S_1, about v_1 to find another edge, say e^*, from v_1 to v_0 that does not have this obstruction. See Figure 21. In the identification of the tetrahedron τ we assume that the edge $e' = e$. Let σ denote the normal triangle at the vertex v_1 in S_1. The edge a from e to e' in this triangle is a loop in the induced triangulation on S_1. But since we are on a 2-sphere, there must always be an "innermost" loop, which is also formed by two adjacent edges in a tetrahedron from v_1 to v_0 being identified. We may just assume this is the situation for e and e' in τ; i.e., a is an "innermost" loop on S_1. We will also continue to use σ for the triangle in the vertex-linking 2-sphere with vertices on e and e'. In this situation, an edge e^* going through any vertex in the interior of the loop determined by a on the vertex-linking 2-sphere does not have an edge in S_1 a loop and so does not have an obstruction to building an embedded normal sphere about that edge. So, there is an embedded, non-vertex-linking, normal 2-sphere. This gives the desired contradiction and completes the proof of the Claim.

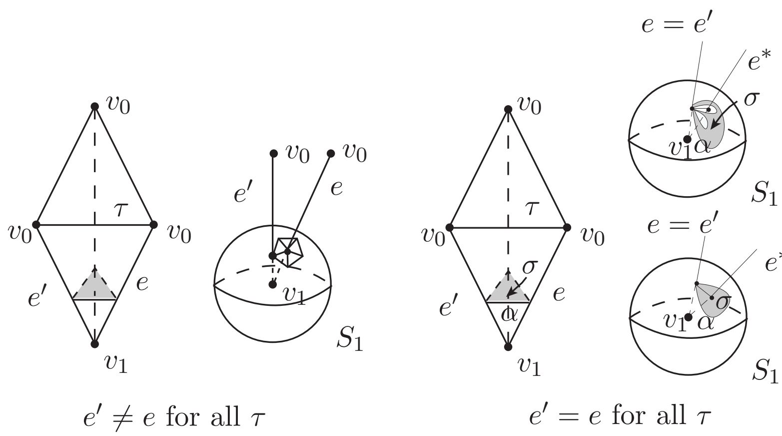


Figure 21: If all edges meet at v_0 and there is more than one vertex, then $\mathcal{T}$ is not 0-efficient.

We have shown that if $\mathcal{T}$ has more than two vertices, then the triangulation must have a special vertex which meets every edge; however, if every edge meets a vertex and there is more than one vertex, then it leads to a contradiction of 0-efficiency.

This completes the proof of Proposition 5.1. q.e.d.

Ben Burton has constructed an infinite family of two vertex, 0-

efficient triangulations of S^3 (see [5]). The one-tetrahedron, two-vertex triangulation of S^3 is dual to the natural cell complex coming from the genus one Heegaard splitting of S^3 with the two meridian disks attached to the Heegaard surface. Triangulations dual to cell subdivisions induced by a Heegaard surface with a system of meridional disks attached have precisely two vertices. By Proposition 5.1, except possibly for S^3, these are never 0-efficient.

There is almost a converse for two-vertex, 0-efficient triangulations of S^3. Suppose $\mathcal{T}$ is a two-vertex triangulation of the 3-manifold M with vertices v_1 and v_2. For each tetrahedron Δ of $\mathcal{T}$ which has both v_1 and v_2 as vertices there is either a uniquely determined normal triangle or normal quadrilateral in Δ, separating the vertices identified with v_1 from those identified with v_2. The collection of all such normal triangles and quadrilaterals forms a normal surface S. S separates the vertex v_1 from the vertex v_2 and if S meets an edge of $\mathcal{T}$ at all, then it meets it in at most one point. This construction works no matter how many vertices, as long as we divide the vertices into two disjoint sets. However, it is probably most interesting in the case of two vertices. In fact, if K and L are disjoint subcomplexes, then as observed in Proposition 3.6, there is a normal surface in M separating K from L. There is not, in general, a unique such surface as one gets in the above situation for a triangulation with two vertices. We call S a *vertex-splitting surface*.

Lemma 5.2. *If $\mathcal{T}$ is a two-vertex, 0-efficient triangulation of S^3, then the vertex-splitting surface S determines a Heegaard splitting of S^3.*

Proof. Choose an edge e of $\mathcal{T}$ joining v_1 and v_2. As observed above, e meets S precisely once. A small regular neighborhood of $S \cup e$ has two boundary components, each parallel to S and each barrier surfaces in the components of their complements not meeting $S \cup e$. So, each can be shrunk in these components of their complements, resulting in two compression bodies, H and H', where $\partial_+ H = \partial_+ H' = S$, $\partial_- H$ and $\partial_- H'$ are normal surfaces along with, possibly, some 0-weight 2-spheres contained entirely in the interior of tetrahedra of $\mathcal{T}$; furthermore, e is contained in $H \cup H'$.

We claim that there are no components in either $\partial_- H$ or in $\partial_- H'$ which are normal. For suppose there were normal components. Let $N(e)$ be a small 3-cell neighborhood of e (disjoint from $\partial_- H \cup \partial_- H'$) with boundary the 2-sphere Σ. Now, assuming there are normal surfaces in $\partial_- H$ or in $\partial_- H'$, Σ separates these normal surfaces from e.

Furthermore, Σ along with $\partial_- H \cup \partial_- H'$ are barrier surfaces. Thus we can shrink Σ in the component of the complement of $\Sigma \cup \partial_- H \cup \partial_- H'$ whose closure contains Σ but not e. But we then must have a punctured 3-cell containing e and having boundary a collection of normal 2-spheres and 0-weight 2-spheres entirely contained in the interior of tetrahedra of $\mathcal{T}$, which separate e from the normal components of $\partial_- H \cup \partial_- H'$. It follows that if $\partial_- H \cup \partial_- H'$ had any normal surfaces, then there must be a normal 2-sphere in the boundary of this punctured 3-cell. But this is impossible for $\mathcal{T}$ a 0-efficient triangulation.

It follows that both $\partial_- H \cup \partial_- H'$ is either empty or consists entirely of 2-spheres contained entirely in the interior of tetrahedra of $\mathcal{T}$. Any such 2-sphere bounds a 3-cell complementary to H and H'. Hence, we may assume that we have filled any such boundary components and therefore H and H' are handlebodies. Hence, S is a Heegaard surface, as claimed. q.e.d.

We point out that in the case, cited above, of the Burton examples of two-vertex, 0-efficient triangulations, the vertex-splitting surface determines a genus one Heegaard splitting in every case. We have not tried to get two-vertex, 0-efficient triangulations of S^3 with higher genus splitting surfaces nor have we tried to understand those conditions on a Heegaard splitting of S^3 which guarantee that the two-vertex triangulation dual to the cell decomposition coming from the Heegaard surface along with a complete system of meridional disks is 0-efficient.

We have the following observations concerning edges in 0-efficient triangulations.

Proposition 5.3. *Suppose the closed, orientable 3-manifold M has a 0-efficient triangulation $\mathcal{T}$. If $\mathcal{T}$ has an edge bounding an embedded disk in M, then $M = S^3$.*

Proof. Suppose the edge e bounds an embedded disk D (in particular, the edge has just one vertex and is an embedded simple closed curve). Let S be the boundary of a small regular neighborhood of D. Then S is a 2-sphere and bounds a 3-cell containing the edge e. Let T be the boundary of a small regular neighborhood of e which is contained entirely in the 3-cell bounded by S. Then T is a barrier surface in the component of its complement not containing e; hence, in the component containing S. So, we can shrink S obtaining a punctured 3-cell, which contains the edge e and has boundary a number of 2-spheres, each of which is either normal or contained in the interior of a tetrahedron. We may fill in those in tetrahedra with 3-cells missing e. On the other hand, if there is a normal 2-sphere then it must be vertex linking. It can not be linking the vertex of e; so, we conclude that it bounds a 3-cell not meeting e. Hence M is S^3. Of course, if there were a normal 2-sphere after shrinking, then there would necessarily be more than one vertex and we have from Proposition 5.1 that M is S^3. q.e.d.

Corollary 5.4. *Suppose the closed, orientable 3-manifold M has a 0-efficient triangulation $\mathcal{T}$.*

(1) *If $\mathcal{T}$ has an edge of order one, then $M = S^3$.*

(2) *If $\mathcal{T}$ has a face which is a cone, then $M = S^3$.*

Proof. If there is an edge, say e, of order one, then there is a single tetrahedron $\widetilde{\Delta}$ in our triangulation and a single edge $\widetilde{e}$ in $\widetilde{\Delta}$, which is the full collection of edges identified to e. Let $\widetilde{e}'$ be the edge in $\widetilde{\Delta}$ dual to $\widetilde{e}$; i.e., $\widetilde{\Delta} = \widetilde{e} * \widetilde{e}'$. Then if e' is the image of $\widetilde{e}'$, e' bounds an embedded disk in the image of $\widetilde{\Delta}$. Hence, by the previous corollary, M is S^3. This proves Part (1).

If there is a face which is a cone, then there are two possibilities (see Figure 1 (4) and (5)). There is a tetrahedron $\widetilde{\Delta}$ in the triangulation $\mathcal{T}$ and a face, $\widetilde{\sigma}$, with vertices a, b, c where the edge $\overline{ac}$ is identified with the edge $\overline{bc}, a \leftrightarrow b, c \leftrightarrow c$. In one case, Figure 1 (4), σ, the image of $\widetilde{\sigma}$, is an embedded disk (cone) having boundary the edge $\overline{ab}$. The other is similar, except the vertex c is also identified with $a = b$. The cone is not embedded but we can replace a small open disk in a neighborhood of c in the cone by a disk meeting the cone only in its boundary, still giving a disk bounded by the edge $\overline{ab}$. Hence, an edge bounds an embedded disk; by the previous proposition, $M = S^3$. This completes the proof of Part (2). q.e.d.

The two vertex, 0-efficient triangulations of S^3 constructed by Ben Burton feature these anomalies. Later, we show if the closed, orientable 3-manifold M has a 0-efficient triangulation $\mathcal{T}$, which does not allow certain local reductions in the number of tetrahedra, then we can make similar conclusions regarding edges of order two and order three. See Section 6 on minimal triangulations.

We now show that all closed, orientable, irreducible 3-manifolds, except $\mathbb{RP}^3$, admit a 0-efficient triangulation. This is an existence theorem. However, the techniques provide an algorithm to modify a given triangulation of a closed, orientable, irreducible 3-manifold to a 0-efficient triangulation of the manifold or along the way conclude that the given

3-manifold is $S^3, \mathbb{RP}^3$ or $L(3,1)$. The proof of this uses the 3-sphere recognition algorithm. In practice, this is not very reasonable; furthermore, one of our main objectives, leading to 0-efficient triangulations, is to get a practical implementation of the 3-sphere recognition algorithm. So, we show that given any triangulation of a closed, orientable 3-manifold, there is an algorithm to construct a connected sum decomposition of the 3-manifold in which each factor has a 0-efficient triangulation or is known to be $S^3, S^2 \times S^1, \mathbb{RP}^3$, or the lens space L(3,1). This decomposition also has the feature that if the manifold is given via the triangulation $\mathcal{T}$ and $\mathcal{T}$ is not 0-efficient, then the number of tetrahedra needed in the triangulations of all of the factors in this connected sum decomposition having a 0-efficient triangulation is strictly less than the number of tetrahedra in $\mathcal{T}$. This method is implemented in the computer program "REGINA" by Ben Burton and David Letscher. It can be used to decide if a given 3-manifold is the 3-sphere, the 3-sphere recognition algorithm [18, 22]. We point out here that the 3-sphere recognition algorithm really wants a 0-efficient triangulation. First, if there are non vertex-linking, normal 2-spheres, then standard methods do not provide an algorithm to find an almost normal 2-sphere. (In the original proof of the 3-sphere recognition algorithm, this was circumvented by showing that one can construct a maximal collection of pairwise disjoint, normal 2-spheres. Then the closure of a component of the complement of such a collection is either a small regular neighborhood of a vertex in the triangulation, or has multiple (more than one) boundary components and is a punctured 3-cell, or has just one boundary component, is not of the first kind, and every normal 2-sphere is parallel to the boundary component. In the last situation, one has a cell decomposition which has some of the characteristics of a 0-efficient triangulation.) Also, in the presences of non-vertex-linking normal 2-spheres, the existence of an almost normal 2-sphere does not tell one much about the topology of the given 3-manifold. Other aspects and implementation of these algorithms are discussed in [5, 6].

Theorem 5.5. *A closed, orientable, irreducible* 3*-manifold distinct from* $\mathbb{RP}^3$ *has a* 0*-efficient triangulation.*

Proof. Suppose M is a closed, orientable irreducible 3-manifold. Let $\mathcal{T}$ be any triangulation of M. If $\mathcal{T}$ is 0-efficient, there is nothing to prove; so, we assume $\mathcal{T}$ is not 0-efficient. Hence, there is a non-vertex-linking, normal 2-sphere, say S, in M. Since M is irreducible, S separates and bounds a 3-cell in M.

Let S be a maximal, non-vertex-linking, normal 2-sphere in M. Here we are using maximal in the sense that if S' is a non vertex-linking, normal 2-sphere and S' bounds a 3-cell containing S, then $S' = S$. In this situation, such a maximal non vertex-linking normal 2-sphere exists by Kneser's Theorem, Theorem 2.3. Furthermore, if S is maximal and we denote the 3-cell that S bounds by E, then by Proposition 3.3, E contains all the vertices of $\mathcal{T}$ or M

is the 3-sphere S^3. So, either we have a maximal, non vertex-linking, normal 2-sphere S bounding a 3-cell E in M, which contains all the vertices of $\mathcal{T}$, or M is S^3.

Let X be the closure of the complement of E in M. Then X is homeomorphic to M with an open 3-cell removed. We will use Theorem 4.1 to crush the triangulation $\mathcal{T}$ along S. By the comments following Theorem 4.1, if we do this, then we will have a triangulation $\mathcal{T}^*$, which is an ideal triangulation of $\mathring{X}$ and since the index of the ideal vertex is zero, $|\mathcal{T}^*|$ is homeomorphic to $\hat{X}$, which is homeomorphic to M and, therefore, $\mathcal{T}^*$ is a triangulation of M. Let $\mathcal{C}$ be the induced cell-decomposition of X. Then since all the vertices of $\mathcal{T}$ are in E, the cells of $\mathcal{C}$ are of Type I, II, III or IV (see Figure 10). To apply Theorem 4.1, we have to establish the existence of an induced product region on X and then verify the three conditions in the hypothesis of Theorem 4.1. We will do this through a sequence of claims.

Claim 1. *$\mathbb{P}(\mathcal{C})$ is a product I-bundle and $\mathbb{P}(\mathcal{C}) \neq X$.*

Proof. Recall that $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in $S\} \cup \{$cells of Type III and Type IV in $\mathcal{C}\} \cup \{$all trapezoidal faces of $\mathcal{C}\}$ and is an I-bundle. If $\mathbb{P}(\mathcal{C}) = X$, then X is an I-bundle with 2-sphere boundary; hence, X is a twisted I-bundle over $\mathbb{R}\mathrm{P}^2$ and $M = \mathbb{R}\mathrm{P}^3$. This contradicts our hypothesis. Similarly, if a component of $\mathbb{P}(\mathcal{C})$ were not a product I-bundle, then there would be a Möbius band properly embedded in X (its boundary in the 2-sphere S). However, if this were the case, then there would be an $\mathbb{R}\mathrm{P}^2$ embedded in M and since M is irreducible, then $M = \mathbb{R}\mathrm{P}^3$. However, again, this contradicts our hypothesis.

This completes the proof of Claim 1.

Claim 2. *There is a trivial induced product region $\mathbb{P}(X)$ for X.*

Before proving this claim, we will establish some notation and terminology which we will use through the remainder of this work.

If $K_i \times [0,1]$ is a component of $\mathbb{P}(\mathcal{C})$, then each component of the complement of $K_i^\varepsilon, \varepsilon = 0$ or 1, in the 2-sphere S is simply connected. For each $K_i \times [0,1]$, let D_i^0 denote the union of K_i^0 along with all components of the complement of K_i^0 in S, which do not meet K_i^1; similarly, let D_i^1 denote the union of K_i^1 along with all components of the complement of K_i^1 in S, which do not meet K_i^0. For a fixed $i, 1 \leq i \leq k$, we have that D_i^0 and D_i^1 are disjoint and simply connected. Furthermore, because of the product $K_i \times [0,1]$, we have that D_i^0 and D_i^1 are homeomorphic subcomplexes in the induced cell structure on S. Furthermore, $K_i^\varepsilon \subseteq D_i^\varepsilon, \varepsilon = 0, 1$ and we have equality if and only if K_i is, itself, simply connected. Note, while D_i^0 is homeomorphic with D_i^1, it may be the case that D_i^0 has a cell structure induced by the cell structure on S, which is quite different from the induced cell structure on D_i^1, even though K_i^0 and K_i^1 have isomorphic induced cell decompositions. Let $N_i = N(K_i \times [0,1] \cup D_i^0 \cup D_i^1)$ be a small regular neighborhood of the subcomplex $K_i \times [0,1] \cup D_i^0 \cup D_i^1$ in X. The frontier of N_i consists of one properly embedded annulus component along with a number of 2-sphere components (these 2-sphere components exists if and only if K_i is not simply connected). In our current situation (M is irreducible), each of these 2-spheres separates, bounds a 3-cell and is a barrier surface in the closure of its complement in X not meeting $K_i \times [0,1] \cup D_i^0 \cup D_i^1$. While we know each 2-sphere bounds a 3-cell, we want to show that such a 2-sphere bounds a 3-cell not meeting $K_i \times [0,1] \cup S$. We will denote the closure of a component of the complement of N_i in X, which has a 2-sphere boundary component, by $N_{i,j}$ and its boundary 2-sphere (also in the boundary of N_i) by $S_{i,j}$. We call $N_{i,j}$ a *plug for* $K_i \times [0,1]$. See Figure 22.

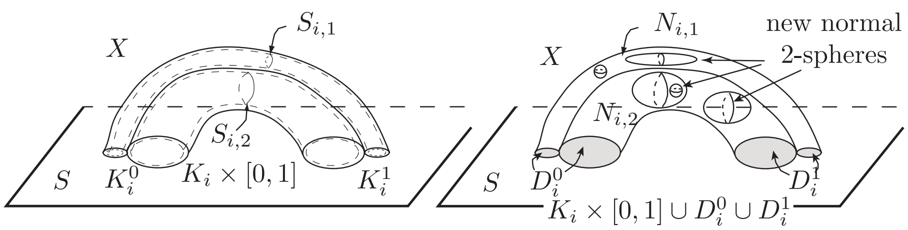


Figure 22: Plugs for product regions in X and new normal 2-spheres, which appear in shrinking $S_{i,j}$.

Using this terminology, we will establish in this claim that a plug for $K_i \times [0,1]$ is a 3-cell. We remark that there are a number of alternate ways to approach our problem at this juncture. We have chosen the following as the most straight forward and natural for this proof and for similar situations we encounter later in this work.

Proof of Claim 2. Chose some order for the components of $\mathbb{P}(\mathcal{C})$, say $K_1 \times [0,1], \ldots, K_n \times [0,1]$, where n is the number of components of $\mathbb{P}(\mathcal{C})$. If K_1 is simply connected, then let $D_1 = K_1$. If K_1 is not simply connected, let D_1^0 and D_1^1 be as defined above. Then, using the above notation, we let N_1 be a small regular neighborhood of $K_1 \times [0,1] \cup D_1^0 \cup D_1^1$. Associated with N_1, we have some set of plugs $N_{1,j}$ and boundary 2-spheres $S_{1,j}$, each of the latter being a barrier surface in the component of its complement in $N_{1,j}$ not meeting $K_1 \times [0,1] \cup D_1^0 \cup D_1^1$. Shrink each $S_{1,j}$ in $N_{1,j}$. Each $S_{1,j}$ shrinks to a collection of normal 2-spheres and possibly some 0-weight 2-spheres contained entirely in the interior of tetrahedra. Again, see Figure 22. We can fill each of the 0-weight 2-spheres in with a 3-cell which does not meet $K_1 \times [0,1] \cup S$. Now, suppose in shrinking some $S_{1,j}$ we have a normal 2-sphere. Since M is irreducible, any such 2-sphere bounds a 3-cell in M. However, by the maximality of S, such a 3-cell does not meet $K_1 \times [0,1] \cup S$. It follows that if such a normal 2-sphere occurs, then it bounds a 3-cell in X missing $K_1 \times [0,1] \cup S$. Thus each plug for N_1 is a 3-cell.

Since the frontier of each of the regions complementary to $K_1 \times [0,1]$ is a union of trapezoids, there is a simply connected planar complex D_1 and an embedding of $D_1 \times [0,1]$ into X with $D_1 \times \varepsilon = D_1^\varepsilon, \varepsilon = 0,1$, $K_1 \times [0,1] \subseteq D_1 \times [0,1]$ and the frontier of $D_1 \times [0,1]$ is contained in the frontier of $K_1 \times [0,1]$. So, we have replaced the product $K_1 \times [0,1]$ with a product $D_1 \times [0,1]$ where we have that D_1 is a simply connected planar complex.

Now, suppose for the components $K_1 \times [0,1], \ldots, K_k \times [0,1]$ of $\mathbb{P}(\mathcal{C})$, we have simply connected planar complexes $D_1, \ldots, D_{k'}, k' \leq k$ and embeddings $D_j \times [0,1], 1 \leq j \leq k'$ into X so that $D_j \times \varepsilon = D_j^\varepsilon, \varepsilon = 0,1$. Furthermore, suppose for $j \neq j'$, either $(D_j \times [0,1]) \cap (D_{j'} \times [0,1]) = \emptyset$ or $D_j \times [0,1] \subset D_{j'} \times [0,1]$ or vice versa; $\bigcup_1^k (K_i \times [0,1]) \subset \bigcup_1^{k'} (D_j \times [0,1])$ and the frontier of $\bigcup_1^{k'} (D_j \times [0,1])$ is contained in the frontier of $\bigcup_1^k (K_i \times [0,1])$. So, we have replaced a number of the $K_i \times [0,1]$ with trivial products. See Figure 23.

If $k < n$, consider the component $K_{k+1} \times [0,1]$ of $\mathbb{P}(\mathcal{C})$. If $K_{k+1} \times [0,1] \subset \bigcup_1^{k'} (D_j \times [0,1])$, then $\bigcup_1^{k+1} (K_i \times [0,1]) \subset \bigcup_1^{k'} (D_j \times [0,1])$ and the frontier of $\bigcup_1^{k'} (D_j \times [0,1])$ is contained in the frontier of $\bigcup_1^{k+1} (K_i \times [0,1])$ and there is nothing to do. So, suppose $K_{k+1} \times [0,1] \not\subset \bigcup_1^{k'} (D_j \times [0,1])$. If K_{k+1} is simply connected, we set $D_{k'+1} = K_{k+1}$; $D_{k'+1} \times [0,1]$ is disjoint from each $D_j \times [0,1], 1 \leq j \leq k'$. So, suppose K_{k+1} is not simply connected. We let $D_{k+1}^\varepsilon, \varepsilon = 0,1$ be defined as above and let $N_{k+1} = N(K_{k+1} \times [0,1] \cup D_{k+1}^0 \cup D_{k+1}^1)$ denote a small regular neighborhood of $K_{k+1} \times [0,1] \cup D_{k+1}^0 \cup D_{k+1}^1$. Then just as in the above for the case of $K_1 \times [0,1]$ and N_1, we can show that each plug for N_{k+1} is a 3-cell.

Thus there is a simply connected planar complex $D_{k'+1}$ and an embedding of $D_{k'+1} \times [0,1]$ into X so that $D_{k'+1} \times \varepsilon = D_{k+1}^\varepsilon, \varepsilon = 0,1$. Furthermore, for $j \leq k'$, either $(D_j \times [0,1]) \cap (D_{k'+1} \times [0,1]) = \emptyset$ or $D_j \times [0,1] \subset D_{k'+1} \times [0,1]$; and finally, $\bigcup_1^{k+1} (K_i \times [0,1]) \subset \bigcup_1^{k'+1} (D_j \times [0,1])$ and the frontier of $\bigcup_1^{k'+1} (D_j \times [0,1])$ is contained in the frontier of $\bigcup_1^{k+1} (K_i \times [0,1])$. Hence, we can enlarge our collection $D_1, \ldots, D_{k'}, k' \leq k$ to a collection including $D_{k'+1}$, which incorporates the product K_{k+1}. See Figure 23.

Let $\mathbb{P}(X)$ be the union of the components of $\bigcup_i (D_i \times [0,1])$. As we observed above, it is possible that some $D'_j \times [0,1]$ may actually be embedded in a $D_j \times [0,1]$, $j' \neq j$. For example, in Figure 23, we have $D_1 \times [0,1] \subset D_2 \times [0,1]$ and $D_3 \times [0,1] \subset D_4 \times [0,1]$. We also see that $K_4 \times [0,1] \subset D_2 \times [0,1]$ and therefore, is dropped. In this example, $\mathbb{P}(X)$ has three components, $D_2 \times [0,1], D_4 \times [0,1]$ and $D_5 \times [0,1]$. Also, this example shows that there is not a unique product region; had we ordered the components $K_1 \times [0,1], \ldots, K_n \times [0,1]$ of $\mathbb{P}(\mathcal{C})$ differently, say with $K_4 \times [0,1]$ first, then all the other components of $\mathbb{P}(\mathcal{C})$ would have been dropped and we would have only one component for $\mathbb{P}(X)$. In those cases where M is irreducible, if something like this happened, then it would follow that M is S^3.

This completes the proof of Claim 2.

Claim 3. *Either there is no cycle of truncated prisms in X, which is not in $\mathbb{P}(X)$, or M is the manifold S^3 or the manifold $L(3,1)$.*

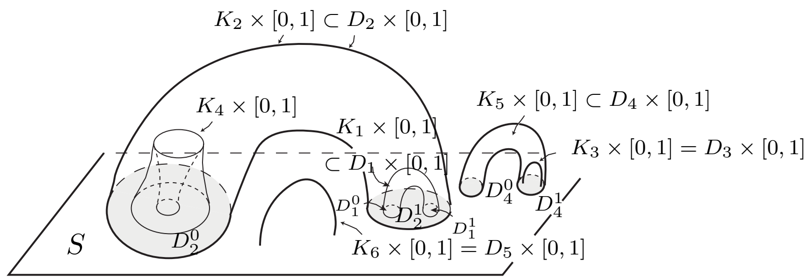
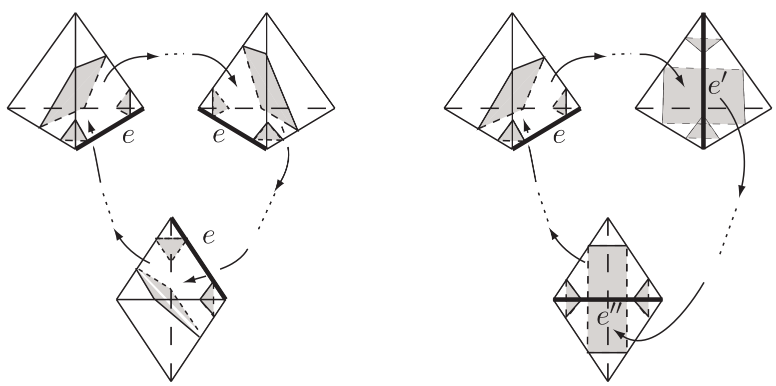


Figure 23: Building a trivial product region, $\mathbb{P}(X)$, from the combinatorial products $K_i \times [0,1]$.

Proof. There are two types of cycles of truncated prisms: one is a cycle about an edge e of $\mathcal{T}$ (see Figure 24(A)) and the other cycles about more than one edge of $\mathcal{T}$ (see Figure 24(B)).



(A) Cycle about a single edge

(B) Cycle about more than one edge

Figure 24: Cycles of truncated prisms.

If there is a complete cycle about an edge e as in Figure 24(A) (i.e., the 2-sphere S contains a thin tube of elementary quads about the edge e), then there is a properly embedded disk D in X meeting e in precisely one point and meeting S in ∂D. A surgery on S at D gives two *normal* 2-spheres S_0 and S_1, neither of which is vertex-linking and together with S bound a punctured 3-cell. See Figure 25. Since M is irreducible, each of these 2-spheres bounds a 3-cell. If either bounds a 3-cell containing S (hence, the 3-cell E and all the vertices of $\mathcal{T}$), then we have a contradiction to the maximality of the 2-sphere S. So, the only possibility is that they bound 3-cells not containing E. Hence,

M must be S^3. It follows that having chosen S a maximal 2-sphere, if there is a complete cycle of truncated prisms in the induced cell structure on X about a single edge, then we have that the manifold M is homeomorphic with S^3.

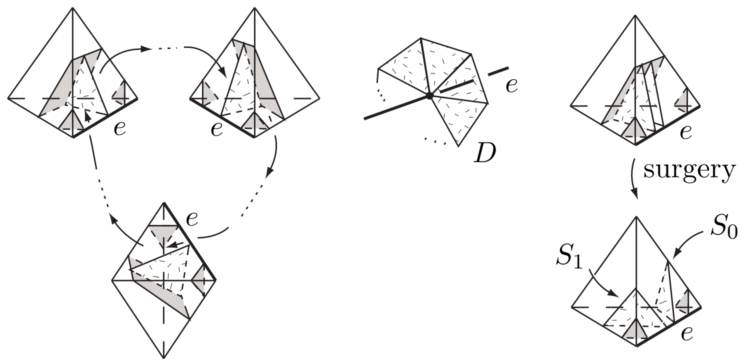


Figure 25: Cycle about a single edge gives a new normal 2-sphere.

If there is a complete cycle about more than one edge (see Figure 24(B)), then the collection of cells of Type II (truncated prisms), form a solid torus with, possibly, some self identifications in its boundary (possibly, some of the trapezoidal faces in the boundary are identified); furthermore, each hexagonal face of a truncated prism in the cycle of truncated prisms is a meridional disk for the solid torus. Because of the possible singularities, we distinguish between the cycle of truncated prisms, which we denote $\hat{\tau}$, and the cycle of truncated prisms minus the bands of trapezoids, which we denote by τ. We have that $\tau \cap S$ is either three open annuli, each meeting a meridional disk of τ precisely once, or a single open annulus, meeting a meridional disk of τ three times. See Figure 26 where we also show the trapezoidal annuli slightly shrunken into the torus τ.

First, we consider the case we have three annuli in $\tau \cap S$, say A_1, A_2 and A_3, each meeting a hexagonal face of a truncated prism in $\hat{\tau}$ precisely once. The frontier of each $A_i, i = 1, 2, 3$ in S is in the collection of trapezoids in the faces of the cycle of truncated prisms $\hat{\tau}$; thus in the induced I-bundle region $\mathbb{P}(\mathcal{C})$. Furthermore, there are two components of the frontier of each A_i and each component of the frontier of A_i separates S. Consider A_1 and denote the two components of the frontier of A_1 by a_1 and a_1'. Then both a_1 and a_1' are in trapezoids in $\hat{\tau}$ and so in $\mathbb{P}(\mathcal{C})$. By our above construction of the induced product region for X, $\mathbb{P}(X)$, it follows that a_1 and a_1' are each in a simply connected region of S common to $\mathbb{P}(X)$. Hence, we either have A_1, and therefore $\hat{\tau}$ in $\mathbb{P}(X)$ or $\mathbb{P}(X)$ meets S in the complement of A_1. But the latter is impossible, since A_2 and A_3 are in regions of S complementary to A_1. So, the only possibility is that A_1 is in $\mathbb{P}(X)$; that is, such a cycle of truncated prisms, $\hat{\tau}$, is in the induced product region for X, $\mathbb{P}(X)$. See Figure 27.

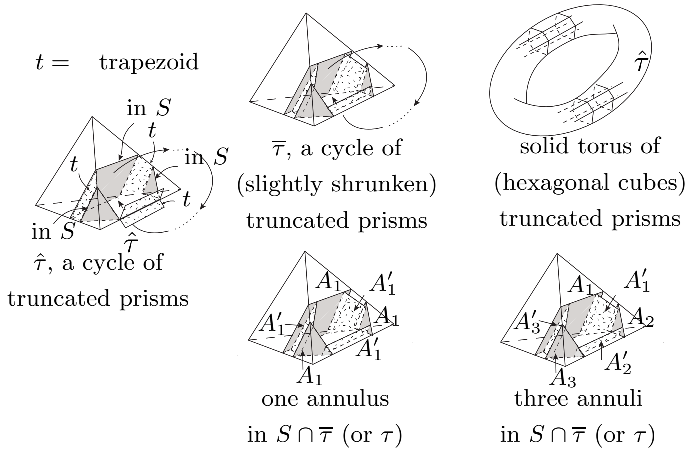


Figure 26: Cycle of truncated prisms determine a solid torus.

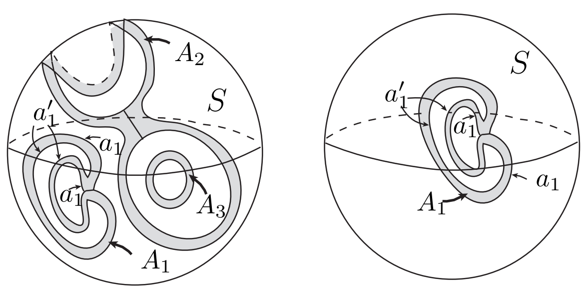


Figure 27: The (possibly singular) torus $\hat{\tau}$ meets S in (possibly singular) annuli.

While the previous observations take care of this case, it can be shown that if S is maximal, M is irreducible and there is a cycle of truncated prisms about more than one edge and there are three annuli in $\tau \cap S$, then M is S^3.

If we have just one annulus, say A_1, in $\tau \cap S$, then A_1 meets each hexagonal face in the chain of truncated prisms, $\hat{\tau}$, three times. We can not make a conclusion similar to that above when there were three annuli, since it is possible that the components of S complementary to A_1 are in the induced product region on X, $\mathbb{P}(X)$; in fact, it is necessary. However, the core of the annulus A bounds a disk in S; so, it follows that M is a connected sum with the lens space $L(3,1)$. But M is irreducible so is, itself, $L(3,1)$. This completes the proof of Claim 3.

Thus we have shown that the conditions in the hypothesis of Theorem 4.1 are satisfied; furthermore, we may assume there are no cycles of truncated prisms in X, which are not in $\mathbb{P}(X)$, since these lead to the conclusion that M is either S^3 or $L(3,1)$. So, from the truncated tetrahedra in $\mathcal{C}$, which are not in $\mathbb{P}(X)$ (and there must be some as $\mathbb{P}(X) \neq X$ and there are no complete cycles of truncated prisms, which are not in $\mathbb{P}(X)$), we get an ideal triangulation $\mathcal{T}^*$ of $\overset{\circ}{X}$. However, since S is a 2-sphere the ideal vertex has index zero and so, $|\mathcal{T}^*|$ is homeomorphic to M and $\mathcal{T}^*$ is a triangulation of M. Since S was chosen to be non-vertex-linking, there is at least one truncated prism in X and so there are strictly fewer truncated tetrahedra in $\mathcal{C}$ than there are in $\mathcal{T}$. It follows that the triangulation $\mathcal{T}^*$ of M has fewer tetrahedra than the triangulation $\mathcal{T}$.

Hence, by iterating the process if necessary, we must terminate in a desired 0-efficient triangulation of M or we have that M is homeomorphic with S^3 or the lens space $L(3,1)$. Each of these latter two manifolds admits a 0-efficient triangulation. See Figure 2(4) and (5) and Figure 16(A).

This completes the proof of Theorem 5.5. q.e.d.

Now, we have a series of results which follow from variations in the proof of Theorem 5.5. Other variations and generalizations appear in [5].

Theorem 5.6. *Suppose M is a closed, orientable, irreducible 3-manifold. Then M admits a one-vertex triangulation.*

The standard presentation of closed 2-manifolds as an identification space of a planar n-gon, leads to easy one-vertex triangulations of closed 2-manifolds distinct from S^2 and $\mathbb{RP}^2$. See Figure 28 for three distinct one-vertex triangulations of a genus two surface.

When we first started this study of triangulations of 3-manifolds, it was not exactly clear how to get a one-vertex triangulation of an arbitrary 3-manifold (we now know several ways, see [5]). Theorem 5.5, Proposition 5.1 and one-vertex triangulations of $S^3, \mathbb{RP}^3$ and $L(3,1)$ provide an argument showing that any closed, orientable, irreducible 3-manifold admits a one-vertex triangulation. However, the methods in the proof of Theorem 5.5 provide a direct proof of existence of one-vertex triangulations of a closed, orientable, irreducible 3-manifold. Furthermore, these methods enable us to do what seems obvious;

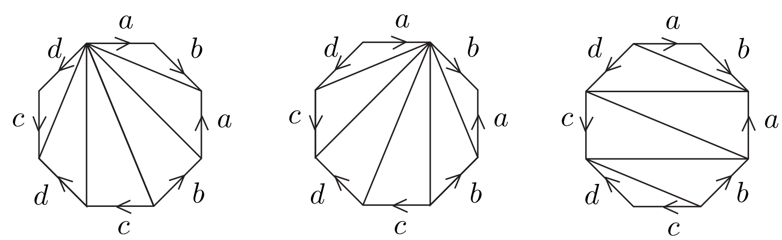


Figure 28: Distinct one-vertex triangulations of a genus 2 surface.

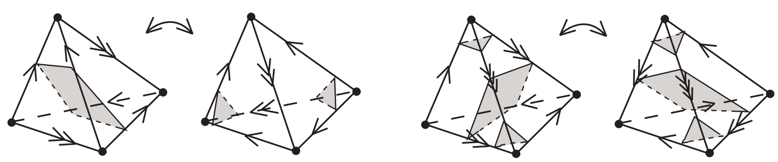
namely, crush a maximal tree in the 1-skeleton to a point. They do not require the iteration necessary in the proof of Theorem 5.5. Note, these methods do not seem to adapt to show that any closed 3-manifold admits a one-vertex triangulation; however, it can be shown, using other methods, that this is true (see [5]).

Proof. As we have said, the intuitive thing to do is to "crush" a maximal tree in the 1-skeleton of a triangulation to a point. While this seems obvious; it is not so obvious how to organize a proof without the methods used in the proof of Theorem 5.5. To this end, suppose we have a triangulation $\mathcal{T}$ of M. If $\mathcal{T}$ has just one vertex, then there is nothing to prove. So assume $\mathcal{T}$ has more than one vertex. Let S' be a vertex-linking normal 2-sphere. Then by Proposition 3.3, there is a normal 2-sphere S bounding a ball E, which contains all the vertices of $\mathcal{T}$. As in the proof of Theorem 5.5, either there is a maximal normal 2-sphere bounding a 3-cell and containing all the vertices of $\mathcal{T}$ or we have that M is S^3. We now use S to denote such a maximal 2-sphere. We wish to crush the triangulation $\mathcal{T}$ along S.

As in the proof of Theorem 5.5, let X denote the closure of the complement of E and let $\mathcal{C}$ be the cell-structure induced on X by $\mathcal{T}$. Now, if $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in $S\} \cup$ $\{$cells of Type III and Type IV in $\mathcal{C}\} \cup$ $\{$all trapezoidal faces of $\mathcal{C}\}$, then each component of $\mathbb{P}(\mathcal{C})$ is an I-bundle. If $\mathbb{P}(\mathcal{C})$ is not a product I-bundle, then there is a Möbius band in $\mathbb{P}(\mathcal{C})$ with its boundary in S. It follows, in this case, there is an $\mathbb{R}\mathrm{P}^2$ embedded in M. Similarly, if $\mathbb{P}(\mathcal{C}) = X$, then since S is a 2-sphere, we have the I-bundle $\mathbb{P}(\mathcal{C})$ is a twisted I-bundle with boundary a 2-sphere; so, is a twisted I-bundle over $\mathbb{R}\mathrm{P}^2$. It follows in both situations that $M = \mathbb{R}\mathrm{P}^3$, which admits a one-vertex triangulation. The proof now follows the proof of Theorem 5.5 and either we can crush the triangulation $\mathcal{T}$ along S or we have that M is one of S^3 or $L(3,1)$. Both of which admit one-vertex triangulations.

Let $\mathcal{T}^*$ denote the ideal triangulation of $\mathring{X}$ we achieve by crushing $\mathcal{T}$ along S. Then, just as above, $|\mathcal{T}^*|$ is homeomorphic to M. $\mathcal{T}^*$ is a one-vertex triangulation of M. In this case, we are done; we do not need to iterate, as we may have needed to do in the proof of Theorem 5.5. q.e.d.

One of the motivating features of 0-efficient triangulations is a more effective implementation of the 3-sphere recognition algorithm. Our next result provides the environment to apply the 3-sphere recognition algorithm. In particular, we show that given a triangulation of an arbitrary closed, orientable 3-manifold M, we can produce a connected sum decomposition of M into factors, each of which either has a 0-efficient triangulation or is known to be one of the manifolds $S^3, S^2 \times S^1, \mathbb{RP}^3$ or the lens space $L(3,1)$. We single out $\mathbb{RP}^3$ and $S^2 \times S^1$ because they get split off in our construction and do not admit 0-efficient triangulations. We include S^3 and $L(3,1)$, even though both admit 0-efficient triangulations, because from our construction and like $S^2 \times S^1$ and $\mathbb{RP}^3$, we know precisely the homeomorphism type of some of the factors without resorting to a recognition algorithm. Finally, the total number of tetrahedra needed in our triangulations of the components in this connected sum decomposition, which have 0-efficient triangulations, is strictly smaller than the number of tetrahedra in $\mathcal{T}$. It may be that the total number of tetrahedra needed to give our triangulations of all the factors, including those which are $S^3, S^2 \times S^1, \mathbb{RP}^3$ and $L(3,1)$, is, in general, a smaller number than the number of tetrahedra for $\mathcal{T}$. This seems likely if we use a one-tetrahedron triangulation of S^3 (or toss S^3 out of the connected sum) or the two tetrahedra, one-vertex triangulations for $\mathbb{RP}^3, S^2 \times S^1$ and $L(3,1)$. See Figure 29, which gives the unique minimal triangulation of $S^2 \times S^1$ and the two tetrahedra, one-vertex triangulation for $\mathbb{RP}^3$. See Figure 16(A), which gives a two-tetrahedron, one-vertex, 0-efficient triangulation of $L(3,1)$ and Figure 2(4) and (5) for one tetrahedra, 0-efficient triangulations of S^3. However, this is not clear and seems more a curiosity than useful.



(A) $\mathbb{RP}^2 \subset \mathbb{RP}^3$ (B) $S^2 \subset S^2 \times S^1$

Figure 29: Minimal (one-vertex) triangulations of $\mathbb{RP}^3$ and $S^2 \times S^1$.

We do not construct normal 2-spheres in M giving our decomposition; however, it is possible to do this from our methods. It also is possible to construct an irreducible decomposition. However, we must wait and apply the 3-sphere recognition algorithm to get a prime decomposition, because there may be factors which are S^3 but it is unknown at this stage. There may also be factors which have a 0-efficient triangulation and are $L(3,1)$, whose homeomorphism type is also not known at this time. We give an alternate method to construct an irreducible decomposition of a given 3-manifold. This is by construction of a maximal pairwise disjoint collection of distinct normal 2-spheres. The methods we use provide algorithms and the different techniques may lead to one being preferred to the other when we consider issues of complexity. Our proof of the decomposition theorem, Theorem 5.9 below, provides the steps for an algorithm, which has been coded by David Letscher and Ben Burton, to construct 0-efficient triangulations and implement the 3-sphere recognition algorithm. It seems to be the preferred algorithm for arriving at an irreducible decomposition of a given 3-manifold. The philosophy behind this preference is that in practice the crushing of a triangulations along normal (non vertex-linking) 2-sphere greatly reduces the number of tetrahedra; so, one should crush whenever one can. A construction of a maximal collection of normal 2-spheres is a result obtained by the authors in 1988 and is used by the second author in the proof of the 3-sphere recognition algorithm. Quite different algorithms, which produce a family of normal 2-spheres giving an irreducible or a prime decomposition of a given 3-manifold, appear in [14] and [8].

First, we give some results necessary for our proof. the next Proposition also appears in [8].

Proposition 5.7. *Given a triangulation $\mathcal{T}$ of a closed, orientable 3-manifold M, there is an algorithm to decide if $\mathcal{T}$ is 0-efficient; furthermore, the algorithm will construct a non-vertex-linking 2-sphere, if one exists.*

Proof. If there is a non-vertex-linking, normal 2-sphere, we show there must be one whose projective class is a vertex of the projective solution space; i.e., a vertex solution. Since a normal 2-sphere is not vertex-linking if and only if it meets some tetrahedron in a quadrilateral, it can be decided if a normal 2-sphere is non vertex-linking. To this end suppose Σ is a non-vertex-linking, normal 2-sphere, $\mathcal{C}(\Sigma)$ is the carrier of Σ and Σ has been chosen so that among all non-vertex-linking, normal 2-spheres in M, the dimension of $\mathcal{C}(\Sigma)$ is a minimum. If $\mathcal{C}(\Sigma)$ is not a vertex of the projective solution space, then there are normal surfaces X and Y carried by proper faces of $\mathcal{C}(\Sigma)$ and nonnegative integers k, n, and m with $k\Sigma = nX + mY$. Since, $\chi(\Sigma) > 0$, we must have either $\chi(X) > 0$ or $\chi(Y) > 0$ and so a component of one with positive Euler characteristic, say a component X' of X. The carrier of X' is also a proper face of $\mathcal{C}(\Sigma)$. X' can not be a projective plane; for then, its double would be a non vertex-linking, normal 2-sphere carried by a proper face of $\mathcal{C}(\Sigma)$ and so, as a proper face of $\mathcal{C}(\Sigma)$, has dimension less than that of $\mathcal{C}(\Sigma)$. The only other possibility is that X' is a vertex-linking 2-sphere; however, then a component of $k\Sigma$ would be vertex-linking and so Σ would be vertex-linking. This is a contradiction to our choice of Σ. Thus Σ has to be carried by a vertex of the projective solution space. The vertices of the projective solution space can be constructed and as we remarked, we can decide if a normal surface is a vertex-linking 2-sphere. q.e.d.

Lemma 5.8. *Suppose M is a closed, orientable 3-manifold, $\mathcal{T}$ is a triangulation of M and $\mathcal{S}$ is a collection of pairwise disjoint, normal 2-spheres embedded in M. If X is the closure of a component of the complement of the 2-*

spheres in $\mathcal{S}$ and ∂X is not connected, then there is a punctured 3-sphere, P_X, embedded in X, each component of ∂P_X is a normal 2-sphere, $\partial X \subset \partial P_X$ and no component of the frontier of P_X in X is normally isotopic to a component of ∂X.

Proof. We have that X is the closure of a component of the complement of the collection of normal 2-spheres $\mathcal{S}$ and X has more than one boundary component. Hence, we can find a pairwise disjoint collection, Λ_X, of arcs in the 1-skeleton of the induced cell-decomposition on X, each such arc has its end points in distinct components of ∂X and the complex $\Lambda_X \cup \partial X$ is connected and simply connected. A small regular neighborhood of the complex $\Lambda_X \cup \partial X$ is a punctured 3-sphere and its frontier, which is a single 2-sphere, is a barrier surface in the component of its complement not containing the complex $\Lambda_X \cup \partial X$. Thus we can shrink this frontier 2-sphere. We obtain a punctured 3-sphere, containing $\Lambda_X \cup \partial X$ and whose boundary consists of all the 2-spheres in ∂X, along with (possibly) some normal 2-spheres in the interior of X and some 0-weight 2-spheres contained entirely in the interior of cells in X. The 2-spheres in the interior of cells in X can be filled in with 3-cells in X; we do this and forget them. None of the normal 2-spheres in the interior of X are normally isotopic to a component of ∂X, since there is at least one edge of the collection of edges Λ meeting each boundary component and the complex $\Lambda_X \cup \partial X$ is a barrier surface. It is possible, of course, that we have no frontier of X and so X is itself a punctured 3-sphere. This completes the proof. q.e.d.

Note Lemma 5.8 can be modified to be applicable to the situation where we have a manifold with boundary and some of the 2-spheres in $\mathcal{S}$ are in the boundary of the manifold.

Theorem 5.9. *Given a closed, orientable 3-manifold M, there is an algorithm to construct a finite family of 3-manifolds, $M_1, \ldots, M_n$, so that $M = M_1 \# \ldots \# M_n$, where $M_i, i = 1, \ldots, n$, either has a 0-efficient triangulation or can be shown to be homeomorphic with one of $S^3, S^2 \times S^1, \mathbb{R}\mathrm{P}^3$, or the lens space, $L(3,1)$.*

Proof. We shall first outline our approach. If $\mathcal{T}$ is not 0-efficient, then there is a non-vertex-linking normal 2-sphere and by Proposition 5.7, we have an algorithm to construct one. Our next step is to construct a punctured 3-sphere, say P, which contains the vertices of $\mathcal{T}$ and has non-vertex-linking normal 2-spheres in its boundary or we can conclude that $M = S^3$. So, suppose we have such a punctured 3-sphere P. If X is the closure of a component of the complement of P, then $\hat{X}$, the 3-manifold X with its 2-sphere boundary components "capped-off" with 3-cells, is a factor in a connected sum decomposition of M; however, if the boundary of X is not connected, then we must take care not to miss copies of $S^2 \times S^1$, which will also be factors in such a connected sum decomposition. If we do have some component of the complement of P whose closure does not have connected boundary, then we can use Lemma 5.8 to enlarge P and obtain a connected sum of a punctured 3-sphere and, possibly, some copies of $S^2 \times S^1$ so that each component of the boundary of our new connected sum is a non vertex-linking normal 2-sphere and the closure of each component of the complement of this new connected sum has connected boundary. We will continue to use P to denote this new punctured 3-sphere connected sum with (possibly) some copies of $S^2 \times S^1$. Next, we consider the closure of a component of the complement of our new P, say X. The 3-manifold X has connected boundary and has a nice cell-decomposition, say $\mathcal{C}$, induced by $\mathcal{T}$ (all the vertices of $\mathcal{T}$ are in P). We wish to crush the triangulation along the 2-sphere in ∂X. To do this we must establish the three conditions in the hypothesis of Theorem 4.1. The first condition is that $\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle. To establish this condition for components of the complement of P, we may need to enlarge P, constructing a connected sum of a punctured 3-sphere with (possibly) some copies of $S^2 \times S^1$ and (possibly) some copies of $\mathbb{R}\mathrm{P}^3$. However, we will retain the property that the closure of each component of the complement of our new connected sum has connected boundary and we will continue to call this new connected sum P. The second condition in the hypothesis of Theorem 4.1 is that for the closure of each component of the complement of P, we can construct a trivial product region. To establish this, we again may need to enlarge P; again, we maintain that P is a connected sum of a punctured 3-sphere with (possibly) some copies of $S^2 \times S^1$ and (possibly) some copies of $\mathbb{R}\mathrm{P}^3$. The third condition in the hypothesis of Theorem 4.1 is that there are no cycles of truncated prisms in the induced cell decomposition $\mathcal{C}$ on X, which are not in $\mathbb{P}(X)$. To establish this condition, we may need to enlarge P again but now we may need to add some copies of $L(3,1)$ as connected summands of P.

Finally, we arrive at the situation where we have constructed in M a connected sum of a punctured 3-sphere with, possibly, some copies of $S^2 \times S^1$, possibly, some copies of $\mathbb{R}\mathrm{P}^3$ and, possibly, some copies of $L(3,1)$, say P, so that either $M = P$ or if X is the closure of a component of the complement of P and if $\mathcal{C}$ is its induced cell-decomposition, then X has connected boundary, $\mathbb{P}(\mathcal{C}) \neq X$, each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle, $\mathbb{P}(X)$ is a trivial product region for X and there are no cycles of truncated prisms in $\mathcal{C}$, which are not in $\mathbb{P}(X)$. Thus, we can crush the triangulation $\mathcal{T}$ along S. We obtain a triangulation, $\mathcal{T}_X$, of $\hat{X}$. The 3-manifold $\hat{X}$ is a factor (and so is $\hat{P}$) in a connected sum decomposition of M; furthermore, the total number of tetrahedra in the triangulations $\mathcal{T}_X$ over all components X in the complement of P is less than the number of tetrahedra in $\mathcal{T}$. Now, for each $\hat{X}$ we consider if the triangulation $\mathcal{T}_X$ is 0-efficient. If it is, we set it aside as one of the desired factors in a connected sum decomposition of M; if it is not, then we go through the above routine, now with $\hat{X}$ replacing M. After a finite number of steps, we have the desired connected sum decomposition of M.

So, this is the idea; we follow with the details.

The 3-manifold M is given via some triangulation, say $\mathcal{T}$. By Proposition 5.7, we can decide if $\mathcal{T}$ is 0-efficient; if it is not, then our algorithm will construct a non-vertex-linking normal 2-sphere. If $\mathcal{T}$ is 0-efficient, there is nothing to prove; so, we may assume $\mathcal{T}$ is not 0-efficient and we have a non-vertex-linking, normal 2-sphere, S.

Claim 1. *We can construct a punctured* 3*-sphere connected sum with* (*possibly*) *some copies of* $S^2 \times S^1$*, say* P*, embedded in* M*, so that* P *contains all the vertices of* $\mathcal{T}$*, each* 2*-sphere component in* ∂P *is normal and not vertex-linking and if* X *is the closure of a component of the complement of* P*, then* X *has precisely one normal* 2*-sphere in its boundary or* $P = M$*.*

Proof. By Proposition 3.4 there are compression bodies H and H' embedded in M, $H \cap H' = \partial_+ H = S = \partial_+ H'$, each component of $\partial_- H \cup \partial_- H'$ is a normal 2-sphere and $H \cup H'$ contains the vertices of $\mathcal{T}$. Let $P = H \cup H'$. Then P is a punctured 3-sphere containing all the vertices of $\mathcal{T}$ and each 2-sphere in ∂P is normal and not vertex-linking.

If P has no boundary components, then M is S^3. So, we may assume that $\partial P \neq \emptyset$. If the closure of each component X of the complement of P has precisely one boundary component, then we are done. So, suppose X is the closure of a component of the complement of P and X has more than one boundary component. Hence, by Lemma 5.8 there is a punctured 3-cell, P_X, embedded in X, each component of ∂P_X is a normal 2-sphere, $\partial X \subset \partial P_X$ and no component of the frontier of P_X in X is normally isotopic to a component of ∂X. It is possible, of course, that there are no components in the frontier of P_X ($P_X = X$) and so we have X is itself a punctured 3-sphere. In any case, we add the punctured 3-sphere P_X to P along their intersection, ∂X. We now may have introduced some copies of $S^2 \times S^1$ as factors in a connected sum decomposition. We continue to call the resulting 3-manifold P. We consider the closure of a component of the complement of this new P. If such a component does not have connected boundary, we repeat this procedure. The only difference from the first stage is that we are adding punctured 3-spheres to a connected sum of a punctured 3-sphere with some number of copies of $S^2 \times S^1$ along 2-sphere boundary components; however, the result is still a punctured 3-sphere connected sum some number of copies of $S^2 \times S^1$. By Kneser's Finiteness Theorem, Theorem 2.3, the procedure must stop and we end up having constructed a connected sum of a punctured 3-sphere with some number of copies of $S^2 \times S^1$, again say P, satisfying the conclusions of our claim. Namely, the closure of a component of the complement of P has precisely one normal, non-vertex-linking 2-sphere boundary component. This completes the proof of Claim 1.

If the triangulation $\mathcal{T}$ has more than one vertex, there are other ways to achieve this claim without using the algorithm of Proposition 5.7. For example, we could take a vertex-linking normal 2-sphere and then use Proposition 3.3 to obtain a punctured 3-sphere containing all the vertices of $\mathcal{T}$ with each 2-sphere in its boundary normal and not vertex-linking. Similarly, if $\mathcal{T}$ has more than one vertex, then we can take a collection of pairwise disjoint vertex-linking normal 2-spheres, one for each vertex of $\mathcal{T}$ and apply Lemma 5.8 where we take for X of that lemma the closure of the component of the complement of our vertex-linking normal 2-spheres, which does not meet any vertex. Now, we use the punctured 3-sphere from Lemma 5.8, which has each of our vertex-linking normal 2-spheres in its boundary and add each 3-cell about the vertices to get a punctured 3-sphere containing all the vertices of $\mathcal{T}$ with each 2-sphere in its boundary normal and not vertex-linking.

If we split M along the 2-spheres in ∂P, from the previous claim, and cap off the boundary components with 3-cells, than M is a connected sum of the resulting 3-manifolds. Furthermore, from our construction we know precisely how many copies of $S^2 \times S^1$ we have in P. Let X be the closure of a component of the complement of P. Then X has only one 2-sphere in its boundary; denote it by S.

We wish to crush the triangulation $\mathcal{T}$ along S. If we can do this, then we will have an ideal triangulation of $\overset{\circ}{X}$; however, since S is a single 2-sphere in ∂X, we will have a triangulation of $\hat{X}$ the manifold obtained by capping off the boundary 2-sphere of X with a 3-cell, which we have observed is a factor of M in a connected sum decomposition.

As in the proof of Theorem 5.5, no vertices of $\mathcal{T}$ are in X; so, X has a nice induced cell decomposition consisting of truncated tetrahedra, truncated prisms, and product triangular and quadrilateral pieces (See Figure 10). Now, following the above constructions where we crush a triangulation along a normal surface, we need to establish the conditions in the hypothesis of Theorem 4.1. Let $\mathcal{C}$ denote the cell-decomposition on X induced from $\mathcal{T}$ and let $\mathbb{P}(\mathcal{C})$ denote the I-bundle consisting of all edges in $\mathcal{C}$, all cells of Type III and IV, along with all trapezoids in the faces of the cells of Types II, III, and IV in $\mathcal{C}$.

Claim 2. *We can construct a connected sum of a punctured* 3*-sphere with, possibly, copies of* $S^2 \times S^1$ *and, possibly, copies of* $\mathbb{R}\mathrm{P}^3$*, embedded in* M*, say* P*, so that either* P *contains all the vertices of* $\mathcal{T}$*, each* 2*-sphere component in* ∂P *is normal and not vertex-linking, the closure of each component of the complement of* P *has connected boundary, and if* X *is the closure of a component of the complement of* P *and* $\mathcal{C}$ *is its induced cell-decomposition, then* $\mathbb{P}(\mathcal{C}) \neq X$ *and each component of* $\mathbb{P}(\mathcal{C})$ *is a product* I*-bundle or* $P = M$*.*

Proof. By our earlier claim, there is a connected sum of a punctured 3-sphere with, possibly, copies of $S^2 \times S^1$, say P, embedded in M, so that P contains all the vertices of $\mathcal{T}$, each 2-sphere component in ∂P is normal and not vertex-linking and the closure of each component of the complement of P, has precisely one normal 2-sphere in its boundary. Let X be the closure of a component of the complement of P and let $\mathcal{C}$ denote the induced cell-decomposition of X.

If $\mathbb{P}(\mathcal{C}) = X$, then since $\partial X = S$ is connected, X is a twisted I-bundle over $\mathbb{R}\mathrm{P}^2$; so, we add X to P and we have a connected sum of a punctured 3-sphere with, possibly, some copies of $S^2 \times S^1$, along with a copy of $\mathbb{R}\mathrm{P}^3$.

If $\mathbb{P}(\mathcal{C}) \neq X$ and is not a product I-bundle, then there is a Möbius band, say A, embedded in X with its boundary in S. We may assume that A is normal in the induced cell-decomposition $\mathcal{C}$ on X. A small regular neighborhood $N(S\cup A)$ of $S \cup A$ is a punctured $\mathbb{RP}^3$ and its frontier, say S', is a 2-sphere, which, by Theorem 3.2, Item 2, is a barrier surface in the component of its complement not meeting $S \cup A$. We shrink the 2-sphere S'. We get a punctured $\mathbb{RP}^3$, say P_X, which contains $N(S \cup A)$, and each component of the boundary of P_X is either a normal 2-sphere or a 2-sphere contained entirely in the interior of a cell of $\mathcal{C}$. We fill in each of the 0-weight 2-spheres contained entirely in the interiors of cells of $\mathcal{C}$ with 3-cells missing $S \cup A$. We will continue to denote the resulting punctured $\mathbb{RP}^3$ by P_X. If there are no normal 2-spheres beside S in the boundary of P_X, then $X = P_X$ is a punctured $\mathbb{RP}^3$ and we can again add X to P.

So, we assume there are normal 2-spheres other than S in the boundary of P_X. In this situation, if the closure of every component of the complement of P_X in X has precisely one 2-sphere in its boundary, then we add P_X to P. If there is a component of the complement of P_X in X whose closure has more than one 2-sphere in its boundary, then we apply Lemma 5.8. Specifically, if X' is the closure of a component of the complement of P_X in X and X' has more than one boundary component, then by, possibly, repeated applications of Lemma 5.8 and Kneser's Finiteness Theorem, there is a punctured 3-sphere, $P_{X'}$, embedded in X' so that $\partial X' \subset \partial P_{X'}$ and each component of the complement of $P_{X'}$ in X' has precisely one normal 2-sphere boundary component. We add $P_{X'}$ to P_X and still have a connected sum of a punctured $\mathbb{RP}^3$ with, possibly, some factors which are copies of $S^2 \times S^1$. We do this for each component of the complement of P_X in X, which does not have connected boundary.

So, we have shown that if $\mathbb{P}(\mathcal{C})$ is not a product I-bundle, then either we have that $\hat{X}$ is an $\mathbb{RP}^3$ or there is a connected sum of a punctured $\mathbb{RP}^3$ with, possibly, some copies of $S^2 \times S^1$, say P_X, embedded in X so that $S \subset \partial P_X$, each component of the boundary of P_X distinct from S is a non vertex-linking normal 2-sphere not normally isotopic to S, and the closure of a component of the complement of P_X in X has precisely one boundary component. This latter situation adds 2-spheres to our collection, none of which are normally isotopic to ones we have; hence, again by Kneser's Finiteness Theorem, this situation can occur only finitely many times. If $P_X = X$ for every component X of the complement of P, then we have that M is itself a connected sum of a punctured 3-sphere with possibly some copies of $S^2 \times S^1$ and possibly some copies of $\mathbb{RP}^3$. This completes the proof of Claim 2.

Claim 3. *We can construct a connected sum of a punctured* 3*-sphere with, possibly, copies of* $S^2 \times S^1$ *and, possibly, copies of* $\mathbb{RP}^3$*, embedded in* M*, say* P*, so that either* P *contains all the vertices of* $\mathcal{T}$*, each* 2*-sphere component in* ∂P *is normal and not vertex-linking, the closure of each component of the complement of* P *has connected boundary, and if* X *is the closure of a component of the complement of* P*, then we can construct a trivial induced product region* $\mathbb{P}(X)$ *for* X *or* $P = M$*.*

Proof. We start with the conclusion of Claim 2 and let P be the connected sum of a punctured 3-sphere with, possibly, copies of $S^2 \times S^1$ and, possibly, copies of $\mathbb{RP}^3$ in the conclusion of that Claim. Then P contains all the vertices of $\mathcal{T}$, each 2-sphere component in ∂P is normal and not vertex-linking, the closure of each component of the complement of P has connected boundary, and if X is the closure of a component of the complement of P and $\mathcal{C}$ is its induced cell-decomposition, then $\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle.

Let X be the closure of a component of the complement of P and let $\mathcal{C}$ be its induced cell-decomposition. Let $K_1 \times [0,1], \dots, K_i \times [0,1], \dots, K_n \times [0,1]$ be some ordering of the components of $\mathbb{P}(\mathcal{C})$. We will, just as in the proof of Theorem 5.5, attempt to construct a trivial product region for X.

Again, the issue is to show that each plug for an appropriate $K_i \times [0,1]$ is a 3-cell. So, let us consider an arbitrary component $K_i \times [0,1]$ of $\mathbb{P}(\mathcal{C})$.

If K_i is simply connected, then let $D_i = K_i$. If K_i is not simply connected, let D_i^0 and D_i^1 be as defined above. Then, using the above notation, we let N_i be a small regular neighborhood of $K_i \times [0,1] \cup D_i^0 \cup D_i^1$. Now, we have a possibly different situation from that above. We still have that the boundary components of N_i consist of one properly embedded annulus and some 2-spheres (there are some 2-spheres as we are assuming K_i is not simply connected) and they form barrier surfaces in the components of their complements not meeting $K_i \times [0,1] \cup S$. However, if we let $S_{i,j}, j = 1, \dots, n_i$ denote the 2-sphere boundary components of N_i, then a component of the complement of N_i may have more than one $S_{i,j}$ in its boundary (an $S_{i,j}$ need not separate X) and the closure of a component of the complement of N_i in X may be reducible and so has no hope of being a 3-cell plug.

So, as in the proof of Theorem 5.5, suppose for the components $K_1 \times [0,1], \dots, K_k \times [0,1]$ of $\mathbb{P}(\mathcal{C})$, we have simply connected planar complexes $D_1, \dots, D_{k'}, k' \leq k$ and embeddings $D_j \times [0,1], 1 \leq j \leq k'$ into X so that $D_j \times \varepsilon = D_j^\varepsilon, \varepsilon = 0, 1$. Furthermore, suppose for $j \neq j'$, either $(D_j \times [0,1]) \cap (D_{j'} \times [0,1]) = \emptyset$ or $D_j \times [0,1] \subset D_{j'} \times [0,1]$ or vice versa; $\bigcup_1^k (K_i \times [0,1]) \subset \bigcup_1^{k'} (D_j \times [0,1])$ and the frontier of $\bigcup_1^{k'} (D_j \times [0,1])$ is contained in the frontier of $\bigcup_1^k (K_i \times [0,1])$. So, we have replaced a number of the $K_i \times [0,1]$ with trivial products. See Figures 12 and 23.

If $k < n$, consider the component $K_{k+1} \times [0,1]$ of $\mathbb{P}(\mathcal{C})$. If $K_{k+1} \times [0,1] \subset \bigcup_1^{k'} (D_j \times [0,1])$, then $\bigcup_1^{k+1} (K_i \times [0,1]) \subset \bigcup_1^{k'} (D_j \times [0,1])$ and the frontier of $\bigcup_1^{k'} (D_j \times [0,1])$ is contained in the frontier of $\bigcup_1^{k+1} (K_i \times [0,1])$ and there is nothing to do. So, suppose $K_{k+1} \times [0,1] \not\subset \bigcup_1^{k'} (D_j \times [0,1])$. If K_{k+1} is simply connected, we set $D_{k'+1} = K_{k+1}$; $D_{k'+1} \times [0,1]$ is disjoint from each $D_j \times [0,1], 1 \leq j \leq k'$. So, suppose K_{k+1} is not simply connected. We let $D_{k+1}^\varepsilon, \varepsilon = 0, 1$ be defined as above and let $N_{k+1} = N((K_{k+1} \times [0,1]) \cup D_{k+1}^0 \cup D_{k+1}^1)$ denote a small regular neighborhood of $(K_{k+1} \times [0,1]) \cup D_{k+1}^0 \cup D_{k+1}^1$ and let $N = N(\bigcup_1^{k'} (D_j \times [0,1])$ be a small regular neighborhood of $\bigcup_1^{k'} (D_j \times [0,1])$. The frontier of N consists of annuli and is a barrier surface in the component of its complement in X not containing $\bigcup_1^{k'}(D_j \times [0,1])$. If we denote the 2-spheres in the frontier of N_{k+1} by $S_{k+1,j}, j = 1, \ldots, n_{k+1}$, then we can shrink the $S_{k+1,j}$ in the complement of $(K_{k+1} \times [0,1]) \cup S$ and $\bigcup_1^{k'}(D_j \times [0,1])$. After shrinking the $S_{k+1,j}$, we have a collection of 0-weight 2-spheres entirely contained in the interior of cells of $\mathcal{C}$ and, possibly, some normal 2-spheres. We can fill in any of the 0-weight 2-spheres with 3-cells missing $(K_{k+1} \times [0,1]) \cup S \cup \bigcup_1^{k'}(D_j \times [0,1])$. Suppose we have done this. If there are no normal 2-spheres, then it follows that each $S_{k+1,j}$ separates X and each plug for N_{k+1} is a 3-cell. However, if there are normal 2-spheres, then we let $\mathcal{S}$ denote this collection of normal 2-spheres along with the 2-sphere S. We apply Lemma 5.8 using the collection $\mathcal{S}$. It follows, there is a punctured 3-sphere, P_X, embedded in X, each component of ∂P_X is a normal 2-sphere, $\partial X \subset \partial P_X$ and no component of the frontier of P_X in X is normally isotopic to a component of ∂X. It is possible that there is a component of the complement of P_X in X whose closure does not have connected boundary, again, we can enlarge P_X in a finite number of steps (Kneser's Finiteness Theorem) so that we have a punctured 3-sphere connected sum with, possibly, some copies of $S^2 \times S^1$, which we continue to call P_X. Thus we have P_X embedded in X, S is in the boundary of P_X and the closure of each component of the complement of P_X has connected boundary. We add P_X to P and continue to use P for the resulting connected sum of a punctured 3-sphere with, possibly, some copies of $S^2 \times S^1$ and, possibly, some copies of $\mathbb{R}\mathrm{P}^3$.

Now, it is possible that our new P does not satisfy the conclusions of Claim 2; namely, we no longer know that for X the closure of a component of the complement of P and $\mathcal{C}$ its induced cell-decomposition that $\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle. However, we can go back and fix this and by Kneser's Finiteness Theorem, we will only need to go back a finite number of times. Thus, we eventually are able to construct a P satisfying the conclusions of Claim 3.

Claim 4. *We can construct a connected sum of a punctured* 3*-sphere with, possibly, copies of* $S^2 \times S^1$*, possibly, copies of* $\mathbb{R}\mathrm{P}^3$*, and, possibly, copies of the lens space* $L(3,1)$ *embedded in* M*, say* P*, so that* P *contains all the vertices of* $\mathcal{T}$*, each* 2*-sphere component in* ∂P *is normal and not vertex-linking, the closure of each component of the complement of* P *has connected boundary, and if* X *is the closure of a component of the complement of* P *and* $\mathcal{C}$ *is the induced cell-decomposition on* X*, then all the conditions in the hypothesis of Theorem* 4.1 *are satisfied to crush the triangulation along the normal* 2*-sphere in the boundary of* X *or* $P = M$*.*

Proof. We have, from the above Claims, constructed a connected sum of a punctured 3-sphere with, possibly, copies of $S^2 \times S^1$ and, possibly, copies of $\mathbb{R}\mathrm{P}^3$, say P, which contains all the vertices of $\mathcal{T}$, each 2-sphere component in ∂P is normal and not vertex-linking, the closure of each component of the complement of P has connected boundary, and if X is the closure of a component of the complement of P and $\mathcal{C}$ is its induced cell-decomposition, then we can construct a trivial induced product region $\mathbb{P}(X)$ for X (in particular, $\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a trivial product I-bundle). We need to show that P can be constructed so that, in addition, if X is the closure of a component of the complement of P and $\mathcal{C}$ is its induced cell-decomposition, then there are no cycles of truncated prisms in $\mathcal{C}$, which are not in $\mathbb{P}(X)$.

As in the proof of Theorem 5.5, there are two types of cycles of truncated prisms: one is a cycle about an edge e of $\mathcal{C}$ (see Figure 24(A)) and the other cycles about more than one edge of $\mathcal{C}$ (see Figure 24(B)).

If there is a complete cycle about a single edge e as in Figure 24(A), then a surgery on S at D gives two *normal* 2-spheres S_0 and S_1, neither of which is vertex-linking. See Figure 25. Furthermore, S along with S_0 and S_1 bound a punctured 3-cell, say P_X. As before, it is possible that the closure of a component of the complement of P_X in X does not have connected boundary, neither S_0 nor S_1 separates. We apply Lemma 5.8 to the closure of the complement of P_X and if necessary, we repeatedly apply Lemma 5.8 until we have that there is a connected sum of a punctured 3-sphere, possibly, with some copies of $S^2 \times S^1$ embedded in X, which we continue to denote P_X, S is in the boundary of P_X along with possibly other normal 2-spheres distinct from S and the closure of each component of the complement of P_X in X has connected boundary. We add P_X to P. Now, we may need to go back and construct an even larger P so that the hypotheses of this Claim are still satisfied; however, again, we will need to do this at most a finite number of times by Theorem 2.3.

If there is a complete cycle about more than one edge (see Figure 24(B)), then, as above, the collection of cells of Type II (truncated prisms), form a solid torus with, possibly, some self identifications in its boundary. Using the same notation as earlier, we use $\hat{\tau}$ to denote the cycle of truncated prisms and τ to denote the cycle of truncated prisms minus the bands of trapezoids. If $\tau \cap S$ is three open annuli, then, as above, it follows that $\hat{\tau}$ is in the induced product region for X, $\mathbb{P}(X)$.

So, we suppose that $\tau \cap S$ is a single open annulus, meeting a meridional disk of τ three times. See Figure 26. Our argument here is a bit different from that above in the similar situation, as we do not have that the manifold M is irreducible. We now consider the cycle of truncated prisms with the trapezoidal faces slightly pulled into the truncated prisms. We get an embedded solid torus, which we will denote by τ', which meets S in a single annulus A and A meets the meridional disk of τ' three times. Let A' denote the closure of the annulus in $\partial\tau'$ complementary to A.

Now, let N be a small regular neighborhood of $\tau' \cup S$. N is the punctured lens space $L(3,1)$ and its frontier is a 2-sphere S', which is a barrier surface in the component of its complement not meeting $\tau' \cup S$ (S' is parallel to the 2-sphere $(S \setminus A) \cup A'$). We shrink S' and obtain a punctured $L(3,1)$ having S in its boundary along with, possibly, some normal 2-spheres and some 0-weight 2-spheres contained entirely in the interiors of tetrahedra. This is a familiar situation and after using Lemma 5.8, Kneser's Finiteness Theorem

(Theorem 2.3), and, if necessary, returning to the previous steps, we obtain a connected sum of a punctured $L(3,1)$ along with, possibly, some copies of $S^2 \times S^1$ and $\mathbb{R}\mathrm{P}^3$, say P_X. We add P_X to P. This is what adds the possibility that we now have copies of $L(3,1)$ in our connected sum.

This completes the proof of Claim 4.

As we remarked earlier, if X is the closure of a component of the complement of P, then $\hat{X}$ and $\hat{P}$ are factors in a connected sum decomposition of M. The 3-manifold $\hat{P}$ is a connected sum of S^3 and (possibly) some copies of $S^2 \times S^1$, $\mathbb{R}\mathrm{P}^3$ and $L(3,1)$. If X is the closure of a component of P, then we can crush the triangulation along the 2-sphere in the boundary of X; hence, we construct from the truncated tetrahedra in $\mathcal{C}$ which are not in $\mathbb{P}(X)$, an ideal triangulation $\mathcal{T}_X^*$ of $\mathring{X}$. However, since boundary X is a single 2-sphere, $|\mathcal{T}_X^*|$ is homeomorphic to $\widehat{X}$ and $\mathcal{T}_X^*$ is a triangulation of $\widehat{X}$. Since each 2-sphere in the boundary of P is non-vertex-linking, there is at least one truncated prism in X and so there are strictly fewer truncated tetrahedra in $\mathcal{C}$ than there are in $\mathcal{T}$. It follows that the triangulation $\mathcal{T}_X^*$ of $\hat{X}$ has fewer tetrahedra than the triangulation $\mathcal{T}$. In fact, the entire number of tetrahedra in the triangulations obtained by crushing $\mathcal{T}$ along each 2-sphere in the boundary of P is less than the number of tetrahedra in $\mathcal{T}$.

Now, we consider each $\hat{X}$ and triangulation $\mathcal{T}_X^*$ of $\hat{X}$. If $\mathcal{T}_X^*$ is 0-efficient, then we are satisfied; if not, then we construct in $\hat{X}$ a punctured 3-sphere connected sum with, possibly, copies of $\mathcal{S}^2 \times S^1$, copies of $\mathbb{R}\mathrm{P}^3$ and copies of $L(3,1)$ embedded in $\hat{X}$, say P_X, so that P_X contains all the vertices of $\mathcal{T}_X^*$, each 2-sphere component in ∂P_X is normal and not vertex-linking, the closure of each component of the complement of P_X has connected boundary, and if X' is the closure of a component of the complement of P_X and $\mathcal{C}'$ is the induced cell-decomposition on X', then all the conditions in the hypothesis of Theorem 4.1 are satisfied to crush the triangulation along the normal 2-sphere in the boundary of X'. Since the number of tetrahedra in each of our triangulations obtained after crushing is strictly decreasing, and any factors in a connected sum decomposition of $\hat{X}$ are factors in a connected sum decomposition of M, the process must terminate in the desired connected sum decomposition of M.

This completes the proof of Theorem 5.9. q.e.d.

Next we show that given a triangulation of a closed, orientable 3-manifold M, there is a construction of a maximal, pairwise disjoint collection of distinct normal 2-spheres. As an immediate corollary of this construction, we get an irreducible decomposition of M. This can also be used as an alternate approach to Theorem 5.9. We are using that a pairwise disjoint collection of distinct normal 2-spheres $\mathcal{S}$ is *maximal* if and only if for $\mathcal{S}'$ a pairwise disjoint collection of distinct normal 2-sphere with $\mathcal{S} \subset \mathcal{S}'$, then $\mathcal{S}' = \mathcal{S}$.

Theorem 5.10. *Suppose $\mathcal{T}$ is a triangulation of the closed, orientable 3-manifold M. There is an algorithm to construct a maximal, pairwise disjoint collection of distinct normal 2-spheres in M.*

Proof. Let $\mathcal{S}_1$ be the (pairwise disjoint) collection of distinct, normal, vertex-linking 2-spheres in M. Now, we want to know if there is a normal 2-sphere disjoint and distinct from those in the collection $\mathcal{S}_1$. To do this we use a re-writing of the normal surfaces in the complement of the collection $\mathcal{S}_1$.

Let M_1 denote the manifold obtained by splitting M along the 2-spheres in $\mathcal{S}_1$ and let $\mathcal{C}_1$ be the induced cell-decomposition on M_1. Suppose there is a normal 2-sphere in M, which is disjoint and distinct from the 2-spheres in $\mathcal{S}_1$. Then there is a normal 2-sphere in M_1, which is disjoint from the boundary of M_1 and is not normally isotopic into a 2-sphere in the boundary of M_1. Recall from Proposition 5.7 we proved that if there is a non-vertex-linking normal 2-sphere at all, then there is one at a vertex of the projective solution space. We shall do the same here but we replace the projective solution space $\mathcal{P}(M, \mathcal{T})$ by the projective solution space of the sub-cone of parameterizations of normal surfaces in $\mathcal{C}_1$ obtained by setting all variables corresponding to normal disks types which meet the boundary of M_1 equal to zero, say $\overline{\mathcal{P}}(M_1, \mathcal{C}_1)$. To this end suppose S_1 is a normal 2-sphere in M_1, which is disjoint and distinct from the 2-spheres in ∂M_1; suppose $\mathcal{C}(S_1)$ is the carrier of S_1; and S_1 has been chosen so that among all normal 2-spheres in M_1, which are disjoint and distinct from the 2-spheres in ∂M_1, the dimension of $\mathcal{C}(S_1)$ is a minimum. If $\mathcal{C}(S_1)$ is not a vertex of the projective solution space, $\overline{\mathcal{P}}(M_1, \mathcal{C}_1)$, then there are normal surfaces X and Y carried by proper faces of $\mathcal{C}(S_1)$ and nonnegative integers k, n, and m with $kS_1 = nX + mY$. Since, $\chi(S_1) > 0$, we must have either $\chi(X) > 0$ or $\chi(Y) > 0$ and so a component of one has positive Euler characteristic, say a component X' of X. The carrier of X' is also a proper face of $\mathcal{C}(S_1)$. If X' is a projective plane, then its double is a normal 2-sphere carried by a proper face of $\mathcal{C}(S_1)$ and so, as a proper face of $\mathcal{C}(S_1)$, has dimension less than that of $\mathcal{C}(S_1)$. The normal surface $2X'$ is disjoint from the 2-spheres in ∂M_1 and so, the only possibility by our choice of S_1 is that $2X'$ is normally isotopic into a component of ∂M_1. But then $2X'$ would have to be a component of the sum $kS_1 = nX + my$, which contradicts our choice of S. We get a similar contradiction if X' is, itself, a 2-sphere. Thus S_1 has to be carried by a vertex of the projective solution space $\overline{\mathcal{P}}(M_1, \mathcal{C}_1)$. The vertices of this projective solution space can be constructed and we can decide if a normal surface is a 2-sphere distinct from a 2-sphere in ∂M_1. We let $\mathcal{S}_2 = \mathcal{S}_1 \cup S_1$.

Now, suppose we have constructed a collection $\mathcal{S}_n$ of pairwise disjoint, distinct normal 2-spheres, $n \geq 2$. Let M_{n+1} denote the manifold obtained by splitting M along the 2-spheres in the collection $\mathcal{S}_n$. Then just as above, there is any normal 2-sphere in M, which is disjoint and distinct from the 2-spheres in the collection $\mathcal{S}_n$, if and only if in re-writing of the normal surfaces, there is a normal 2-sphere which is a vertex solution. However, by Kneser's Finiteness Theorem, we eventually have a collection $\mathcal{S}$ so there are no normal 2-spheres which are disjoint and distinct from the 2-spheres in this collection. q.e.d.

The preceding proof uses the re-writing process. Each 2-sphere in our collection is normal in M. However, we find these 2-spheres by re-writing normal surfaces in M as normal surfaces in the induced cell-decomposition of M split along a pairwise disjoint collection of normal 2-spheres. In this re-writing, we show that if there is a normal 2-sphere disjoint and distinct from our collection, then there is one at a vertex in a new projective solution space determined by re-writing the parametrization of the normal surfaces in M, which are disjoint from our collection of 2-spheres. Hence, we can decide if there is such a 2-sphere and we know when we have indeed constructed a maximal collection. In [8], it is shown that a maximal pairwise disjoint collection of distinct normal 2-spheres actually exists at the vertices of $\mathcal{P}(M, \mathcal{T})$; and in fact, the 2-spheres in the collection have parameterizations which project to vertices of a face of $\mathcal{P}(M, \mathcal{T})$, which is a simplex. However, in Theorem 5.10, we do not necessarily have that the 2-spheres in our collection have projective representatives which are vertices (or for that matter, even fundamental). It is not clear which of these algorithms might be the most efficient in constructing such a maximal collection.

There is a straight forward observation about the way a maximal collection of normal 2-spheres decompose a 3-manifold, which was used by the authors in 1988 and is well-known. It also was used by both Rubinstein and Thompson in their proofs of the 3-sphere recognition algorithm. This observation follows immediately from Lemma 5.8.

Remark 5.1. Suppose $\mathcal{T}$ is a triangulation of the closed, orientable 3-manifold M and $\mathcal{S}$ is a maximal, pairwise disjoint collection of distinct normal 2-spheres in M. Then if X is a component of M split along $\mathcal{S}$, X satisfies one of the following:

(1) X has connected boundary, which is a vertex-linking normal 2-sphere, and X is 3-cell about a vertex of $\mathcal{T}$,

(2) X has disconnected boundary and is a punctured 3-sphere, or

(3) X has connected boundary but is not of the first type.

It seems possible that if X satisfies Item 2 (does not have connected boundary), then there can be at most three 2-spheres in boundary X. This does not seem to be important but is a curiosity. Below we also leave open a (seemingly) related question about 0-efficient triangulations of the 3-cell.

We remarked earlier that Theorem 5.10 gives an alternate proof of the decomposition obtained in Theorem 5.9. To see this, suppose $\mathcal{T}$ is a triangulation of the closed, orientable 3-manifold M and $\mathcal{S}$ is a maximal, pairwise disjoint collection of distinct normal 2-spheres in M. Now, if X is a component of M split along the 2-spheres in the collection $\mathcal{S}$ and X is a punctured 3-cell, then X determines a factor in a connected sum decomposition of M which is either S^3 or a connected sum of some number of copies of $S^2 \times S^1$. On the other hand, if X is as in Item 3 (Remark 5.1), then $\hat{X}$, the manifold obtained from X by filling in its boundary 2-sphere with a 3-cell, is a factor in a connected sum decomposition of M. We attempt to crush the triangulation $\mathcal{T}$ along the 2-sphere in ∂X. If there are obstructions to crushing the triangulation, then since the collection $\mathcal{S}$ is maximal, we have that $\hat{X}$ is either $S^3, \mathbb{RP}^3$ or $L(3,1)$. Otherwise, we can crush the triangulation $\mathcal{T}$ along ∂X. In this way we get a one-vertex triangulation $\mathcal{T}_X$ of $\hat{X}$. We observe that the number of tetrahedra in $\mathcal{T}_X$ is strictly smaller than the number of tetrahedra in $\mathcal{T}$ if $\mathcal{S}$ has at least one 2-sphere which is not vertex-linking. So, we can use our argument on $\hat{X}$, constructing a maximal collection of pairwise disjoint, distinct normal 2-spheres in $\hat{X}$. Since the number of tetrahedra is strictly decreasing, the process must stop with a decomposition of M into factors which are either $S^3, S^2 \times S^1, \mathbb{RP}^3, L(3,1)$ or have triangulations, which do not have any non vertex-linking normal 2-spheres; hence, these latter factors have 0-efficient triangulations.

We suspect that if we have a maximal collection of pairwise disjoint, distinct, normal 2-spheres embedded in M, say $\mathcal{S}$, and X is a component of M split along $\mathcal{S}$ which is as in Item 3 of Remark 5.1, then if the triangulation $\mathcal{T}$ can be crushed along ∂X, then the resulting triangulation is immediately 0-efficient. However, we have not been able to establish this. Certainly if $\mathcal{T}_X$ is the resulting one-vertex triangulation of $\hat{X}$ and if S is a non-vertex-linking normal 2-sphere in $\hat{X}$, it seems likely S would give rise to a normal 2-sphere embedded in X which is distinct from ∂X and thereby contradict the maximality of the collection $\mathcal{S}$. Such a 2-sphere is the image of a 2-sphere in X (crushing is a cellular map) but it may not be normal and in shrinking, it may become equivalent to ∂X. We have not spent much time considering this. Recall that our basic philosophy in this work is to crush a triangulation along a normal 2-sphere as soon and as often as we can. The construction of a maximal collection of pairwise disjoint, distinct, normal 2-spheres seems to be computationally quite expensive, whether using the method above or the method in [8]. On the other hand, crushing a triangulation along an embedded normal surface seems to quite sharply reduce the complexity of any further computations.

Recall the solution to the 3-sphere recognition problem in both [18, 22] uses almost normal 2-spheres. In one direction, it is argued that in any triangulation of S^3 there is an embedded almost normal 2-sphere. Conversely, under special circumstances, namely, in each component of the complement of a maximal collection of embedded, normal 2-spheres, it is argued that one can decide if there is an almost normal 2-sphere and if there is one, then that component is a 3-cell. One reason for investigating 0-efficient triangulations is that they provide a constructible and natural environment for this latter direction in the 3-sphere recognition algorithm. This is discussed in detail in [5, 6]. Here we give only this latter direction in the 3-sphere recognition algorithm. We note, however, the subtleness for getting an almost normal, *octagonal* 2-sphere in the 3-sphere recognition algorithm. There are triangulations of S^3 where the only almost normal 2-spheres are tubed; for example, the triangulation of the 3-sphere coming from the boundary of the 4-simplex. We do, however, observe in the proof of Proposition 5.12 below, that in a one-vertex, 0-efficient triangulation an almost normal 2-sphere must be octagonal.

Theorem 5.11 (3-Sphere Recognition Problem [18, 22])**.** *Given a* 3*-manifold M, it can be decided if M is homeomorphic with the* 3*-sphere.*

Suppose we are given a 3-manifold M via a triangulation $\mathcal{T}$. By Theorem 5.9, we can construct a connected sum decomposition of M, where we know each factor is either S^3, $S^2 \times S^1, \mathbb{R}\mathrm{P}^3, L(3,1)$ or has a 0-efficient triangulation. Of course, if any of the factors are $S^2 \times S^1$, $\mathbb{R}\mathrm{P}^3$ or $L(3,1)$, then M is not S^3; in practice, we compute homology invariants directly from $\mathcal{T}$, which indicate if any factors having nontrivial homology will show up in a connected sum decomposition. So, we may assume that in our connected sum decomposition, we have only factors which are either known to be S^3 or ones with 0-efficient triangulations and about which we know nothing else (we know they are homology 3-spheres). We now employ the next proposition, which exhibits a basic feature of 0-efficient triangulations and is one of our motivating reasons for constructing them.

Proposition 5.12. *Suppose the* 3*-manifold M has a* 0*-efficient triangulation. Then it can be decided if M has an almost normal* 2*-sphere. Furthermore, if M has a* 0*-efficient triangulation and an almost normal* 2*-sphere, then $M = S^3$.*

Proof. Suppose $\mathcal{T}$ is a 0-efficient triangulation of M. If $\mathcal{T}$ has more than one vertex, then by Proposition 5.1, M is S^3. Furthermore, M has a tubed almost normal 2-sphere determined by taking a copy of each vertex-linking normal 2-sphere and a tube along an edge joining them. So, if $\mathcal{T}$ has two vertices, it is easy to find an almost normal (tubed) 2-sphere and M is the 3-sphere. Hence, we may assume $\mathcal{T}$ has only one vertex. Now, with $\mathcal{T}$ having only one vertex and being 0-efficient, an almost normal 2-sphere must be octagonal. For if there is an almost normal tubed, 2-sphere, then a compression of the tube gives two normal 2-spheres. Since $\mathcal{T}$ is 0-efficient both are vertex-linking and so must be the same 2-sphere. But in this case, the only possibility is that they are tubed through the normal product region. A tube through the product region between two copies of a normal surface does not give an almost normal surface.

Hence, if there is an almost normal 2-sphere, there is an octagonal one. Let Σ be an octagonal almost normal 2-sphere so that $\mathrm{wt}(\Sigma)$, the weight of Σ, is a minimum among all such almost normal 2-spheres. We claim, Σ is fundamental. For, if not, then there are normal and almost normal surfaces X and Y so that $\Sigma = X + Y$ is a nontrivial Schubert sum; that is, over all possible ways to express $\Sigma = X + Y$ as a geometric sum, we have chosen one with $X \cap Y$ having a minimal number of components. But $\chi(\Sigma) > 0$; so, since, $\chi(\Sigma) = \chi(X) + \chi(Y)$, we have $\chi(X) > 0$, say. X can not be a normal $\mathbb{R}\mathrm{P}^2$, since $\mathcal{T}$ is 0-efficient; if X were an almost normal $\mathbb{R}\mathrm{P}^2$, then Y would need to be a normal $\mathbb{R}\mathrm{P}^2$. Again, this leads to a contradiction. (Note: There can not be a normal or almost normal $\mathbb{R}\mathrm{P}^2$ in a 0-efficient triangulation. An almost normal one shrinks to a normal one and a normal $\mathbb{R}\mathrm{P}^2$ has a double which is a non vertex-linking normal 2-sphere.) So, the only possibility is for X to be a normal or an almost normal 2-sphere. X is not normal, since the only normal 2-spheres are vertex-linking and hence are a component in any geometric sum. And by our choice of Σ, X is not almost normal, since $\mathrm{wt}(X) < \mathrm{wt}(\Sigma)$. We conclude, Σ must be fundamental.

Now, if we have an almost normal 2-sphere Σ in a 0-efficient triangulation of M, then since Σ is unstable to each side, we have that Σ shrinks in each component of its complement to a collection of 2-spheres, which are either normal or are 0-weight and embedded entirely in the interior of a tetrahedron. But in a 0-efficient triangulation any normal 2-sphere is vertex-linking. Hence, in either case, it follows that each component of the complement of an almost normal 2-sphere is a 3-cell. It follows in this situation that $M = S^3$. q.e.d.

Our argument in Proposition 5.12 does not conclude that if in a 0-efficient triangulation there is an almost normal 2-sphere, there is an almost normal 2-sphere which is a vertex solution in the projective solution space. However, the following argument, while very similar, provides a clever variant to the standard argument for showing a solution is a vertex solution and is from A. Casson. Recall if there is an embedded normal or almost normal surface Σ, then every surface carried in a face of the carrier of Σ is embedded and all the surfaces in the cone over the carrier of Σ are compatible; i.e., their geometric sum is defined. Also, as we observed in the proof of the previous lemma, in a one-vertex, 0-efficient triangulation, there are no almost normal tubed 2-spheres. Hence, we need only consider octagonal almost normal surfaces. Instead of the weight functional on normal and almost normal surfaces, a different functional is used. In the cone, over the carrier of an almost normal surface a solution may have multiple copies of an almost normal octagon or almost normal tube; in our situation we are only concerned with octagonal surfaces. If F is a solution over the carrier of Σ, we let $O(F)$ denote the number of octagons in F. If $\chi(F)$ is the Euler characteristic of F, we use the linear functional $L(F) = \chi(F) - O(F)$. Note that in a closed 3-manifold whenever we have Σ a connected normal or almost normal surface and $L(\Sigma) > 0$, then Σ is a normal or an almost normal 2-sphere or a normal $\mathbb{R}\mathrm{P}^2$; and as we have noted, in a 0-efficient triangulation, there is no normal $\mathbb{R}\mathrm{P}^2$.

Now, suppose we have a 0-efficient triangulation and there is an embedded almost normal 2-sphere. Let Σ be one that has the lowest dimensional carrier; we will show that it is a vertex of the projective solution space. For, if not, then there are nonnegative integer solutions X and Y to the normal equations and nonnegative integers k, n and m so that $k\Sigma = nX + mY$. It follows that either $nL(X) > 0$ or $mL(Y) > 0$; say $nL(X) > 0$. We then have $L(X) > 0$ and hence a component of X, say X', is a normal 2-sphere or an almost normal 2-sphere. However, in the former case, since we have a 0-efficient triangulation, we have that X' is a vertex-linking normal 2-sphere and so, Σ would have a component a vertex-linking normal 2-sphere, which is a contradiction. In the latter case, it follows there is an almost normal 2-sphere carried by a proper face of the carrier of Σ. However, this situation contradicts the choice of Σ.

Hence, Σ must be itself a vertex solution. This completes this argument.

So, a special feature of a 0-efficient triangulation is that in such a triangulation, it can be decided if there is an embedded almost normal 2-sphere. Furthermore, if there is one, then the 3-manifold must be S^3.

We comment that in seeking almost normal surfaces, one can consider the subspace of solutions obtained by setting all normal octagon types equal to zero except for one in some tetrahedron; similarly for that tetrahedron, we can also set all normal quad types in it equal to zero. In this way we have at most 3^t such almost normal solution spaces to consider when searching for almost normal surfaces.

To complete the 3-sphere recognition algorithm, after arriving at a connected sum decomposition of M into factors, which are known to be either S^3 or to have 0-efficient triangulations, we can check each factor with a 0-efficient triangulation to see if it has an almost normal 2-sphere. By our above argument, this can be decided in a 0-efficient triangulation. If the answer is yes, then, again by the preceding arguments, that factor is S^3. Of course, now comes a hard part of the 3-sphere recognition algorithm. We use the work of [18, 22] at this point for the case when the answer is no. As pointed out above, from [18, 22] it is known that if a factor is S^3, then for any triangulation, it must have an almost normal 2-sphere; so, this would be true for our 0-efficient triangulations. Hence, if the answer is no for a factor, then that factor is not S^3; and so, $M \neq S^3$.

An algorithm can be deduced from [18] to decide if there is an almost normal 2-sphere in an arbitrary triangulation of a 3-manifold. However, notice that in an arbitrary triangulation, the shrinking of an almost normal 2-sphere may get hung up on a normal 2-sphere which we may know nothing about; hence, the manifold may not be S^3. There are easily constructed one-vertex triangulations of lens spaces, for example, which have almost normal octagonal 2-spheres. Also, any triangulation of a 3-manifold with more than one vertex has a tubed almost normal 2-sphere.

We have as an easy corollary to Theorem 5.9 and the 3-sphere recognition algorithm, recognition algorithms for $\mathbb{R}\mathrm{P}^3$ and $S^2 \times S^1$. This was observed earlier by the second author in [18] and by M. Stocking in [21].

Corollary 5.13. *Given a closed, orientable* 3*-manifold* M*, it can be decided if* M *is homeomorphic to* $\mathbb{R}\mathrm{P}^3$ *or to* $S^2 \times S^1$*. In particular, it can be decided if* M *is a connected sum of some number of copies of* $\mathbb{R}\mathrm{P}^3$ *along with some number of copies of* $S^2 \times S^1$*.*

Proof. By Theorem 5.9, we can construct a connected sum decomposition of M into copies of $\mathbb{R}\mathrm{P}^3, S^2 \times S^1, L(3,1)$ along with some number of closed 3-manifolds each having a 0-efficient triangulation. If there are no copies of $\mathbb{R}\mathrm{P}^3$ or $S^2 \times S^1$ or if there is a copy of $L(3,1)$, then we know M is not a connected sum of some number of copies of $\mathbb{R}\mathrm{P}^3$ and $S^2 \times S^1$. So, we may assume there are some copies of $\mathbb{R}\mathrm{P}^3$ or $S^2 \times S^1$ and no copies of $L(3,1)$. Each of the other factors in our connected sum decomposition of M has a 0-efficient triangulation; hence, we can decide, using the 3-sphere recognition algorithm in these 0-efficient triangulations, if such factors are S^3. If some factor with a 0-efficient triangulation is not S^3, then M is not a connected sum of some number of copies of $\mathbb{R}\mathrm{P}^3$ or $\mathcal{S}^2 \times S^1$. On the other hand, if all factors with a 0-efficient triangulation are S^3, then M has been decomposed into a connected sum of some number of copies of $\mathbb{R}\mathrm{P}^3$ and $S^2 \times S^1$ and we have the desired result. q.e.d.

Note that we are very close in the previous Corollary to being able to decide if M is $L(3,1)$. However, there are 0-efficient triangulations of $L(3,1)$; so, one of the factors with 0-efficient triangulation may be $L(3,1)$. In [11], we give an algorithm to decide if a 3-manifold is a lens space and in particular, an algorithm which decides precisely which lens space. Similar results have been obtained by [21].

Our next result is a constructive method to alter a given triangulation to a 0-efficient one. It uses the 3-sphere recognition algorithm and so it is not the most preferred in practice. Also, it follows from Theorem 5.9 (and the 3-sphere recognition algorithm) as we will point out in its proof. However, we give here a proof, which implements our basic philosophy “crush first and ask questions later.”

Theorem 5.14. *Suppose M is a closed, orientable, irreducible 3-manifold. Then any triangulation of M can be modified to a 0-efficient triangulation or it can be shown that M is one of $S^3, \mathbb{R}\mathrm{P}^3$ or $L(3,1)$.*

Proof. First, we point out how this Theorem follows from Theorem 5.9. We can construct 3-manifolds $M_1, \ldots, M_n$ such that $M = M_1\#, \ldots, \#M_n$ and each M_i is either $S^3, \mathbb{R}\mathrm{P}^3, S^2 \times S^1, L(3,1)$ or has a 0-efficient triangulation. However, since M is irreducible, we can not have $S^2 \times S^1$ and if we have $\mathbb{R}\mathrm{P}^3$ or $L(3,1)$ as a factor, then M is itself $\mathbb{R}\mathrm{P}^3$ or $L(3,1)$. Thus, the only possibility is that we have M is S^3 or there are only factors which are known to be S^3 and some factors which have 0-efficient triangulations. We employ the 3-sphere recognition algorithm on each of the 0-efficient factors. If any one is not S^3, then that factor must be our manifold M and it has a 0-efficient triangulation. If all factors are S^3, then M is S^3. So, this completes the argument using Theorem 5.9.

So, more directly, and the method employed by Letscher and Burton in their computer program, REGINA.

Suppose $\mathcal{T}$ is the given triangulation of the 3-manifold M. By the Proposition 5.7, we can decide if $\mathcal{T}$ is 0-efficient. If it is, then there is nothing to prove. So, we may suppose that $\mathcal{T}$ is not 0-efficient and by Proposition 5.7 our algorithm has constructed a non-vertex-linking, normal 2-sphere, say S. Again, if $\mathcal{T}$ has more than one vertex, we can construct directly a non-vertex-linking normal 2-sphere or we know that $M = S^3$. Since M is irreducible, S separates and, in fact, bounds a 3-cell; however, we do not know, *a priori*, which side of S is a 3-cell.

Now, suppose we have constructed a non-vertex-linking normal 2-sphere S.

By Proposition 3.4, we can construct a punctured 3-sphere P, which contains S and all the vertices of $\mathcal{T}$ or M is S^3. Furthermore, since M is irreducible, the closure of each component of the complement of P has connected boundary (a single 2-sphere).

So, if X is a component of the complement of P, then there are no vertices of $\mathcal{T}$ in X and as above, we can crush the triangulation along the 2-sphere in ∂X or we have that M is $\mathbb{R}\mathrm{P}^3$, or M is $L(3,1)$ or there is a punctured 3-cell P_X embedded in X, each component of ∂P_X is a normal 2-sphere and the closure of each component of the complement of P_X in X has connected boundary. In the last case, it is possible that $P_X = X$ and so X is a 3-cell. If $P_X \neq X$, we add P_X to P getting a larger punctured 3-sphere, which contains all vertices of $\mathcal{T}$ and the closure of each component of its complement has connected boundary or we have that M is S^3. We use this larger punctured 3-sphere and observe that the number of times we encounter this possibility is limited by Kneser's Finiteness Theorem. Hence, we have that M is $S^3, \mathbb{R}\mathrm{P}^3, L(3,1)$ or we can crush the triangulation $\mathcal{T}$ along each 2-sphere in ∂P. Having crushed the triangulation along each of the 2-spheres in ∂P, we have a number of 3-manifolds each has a one-vertex triangulation with fewer tetrahedra than $\mathcal{T}$ (in fact, the number of tetrahedra in all the triangulations sums to some number less than the number of tetrahedra in $\mathcal{T}$). Furthermore, one of these manifolds is homeomorphic to M and each of the others is a 3-sphere.

By Proposition 5.7, we can decide if any of these has a 0-efficient triangulation. Those that have 0-efficient triangulations, set aside. Those that do not, then we can construct for each a non-vertex-linking normal 2-sphere. We go through the previous argument, we used for M. Since the number of tetrahedra in each new triangulation we consider is monotonically decreasing, having begun with the number of tetrahedra in $\mathcal{T}$, we eventually have (and in practice quite quickly) either that M is one of $S^3, \mathbb{R}\mathrm{P}^3$ or $L(3,1)$ or we have a finite collection of 3-manifolds, each with a 0-efficient triangulation and one is M and the others are S^3.

Now, we run the 3-sphere recognition algorithm on each of these 3-manifolds with 0-efficient triangulation. q.e.d.

5.2 0-efficient triangulations for bounded 3-manifolds

It is well-known [3] that a compact 3-manifold with boundary can be triangulated with all the vertices in the boundary. Furthermore, a triangulation with all the vertices in the boundary can be modified by "closing-the-book" to a triangulation having precisely one vertex in each boundary component, which is not a 2-sphere, and precisely three vertices in each boundary component, which is a 2-sphere (see [5]). So, one can achieve a triangulation having all vertices in the boundary and having the property that the triangulation restricted to a boundary component is a minimal triangulation of that boundary component. In [5], it also is shown that if M is a compact, orientable 3-manifold with boundary, then any triangulation on ∂M can be extended to a triangulation of

M without adding vertices. Our methods here are different and give minimal triangulations of boundary components while restricting normal spheres and disks in the manifold.

A triangulation of a compact 3-manifold with nonempty boundary is 0-*efficient* if and only if all normal disks are vertex-linking (recall that a normal surface with boundary must be properly embedded). It turns out that certain technical nuisances appear if our manifolds have boundary components which are 2-spheres. So, we will restrict our investigations to 3-manifolds with boundary, where no boundary component is a 2-sphere. In this subsection we also need some terminology for specially embedded punctured 3-cells. We say that a punctured 3-cell embedded in a 3-manifold M so that it meets ∂M in a connected (planar) subset of its boundary is a *relative* punctured 3-cell in M. The frontier of a relative punctured 3-cell consists of properly embedded disks and 2-spheres.

Our first result is analogous to Proposition 5.1.

Proposition 5.15 ([5]). *Suppose M is a compact, orientable 3-manifold with nonempty boundary, no component of which is a 2-sphere. If M has a 0-efficient triangulation, then there are no normal 2-spheres and M is irreducible and ∂-irreducible. Furthermore, the triangulation has all its vertices in ∂M and has precisely one vertex in each boundary component.*

Proof. Suppose $\mathcal{T}$ is a 0-efficient triangulation of M. First, we show there are no normal 2-spheres in M. To this end, suppose S is a normal 2-sphere. Let X be the closure of a component of the complement of S in M, which also contains a component of ∂M. Let $\mathcal{C}$ be the induced cell-decomposition on X. There is an arc, say Λ, in the 1-skeleton of $\mathcal{C}$ so that Λ meets ∂M in a vertex of $\mathcal{C}$ (also a vertex of $\mathcal{T}$) and meets S in precisely one point (a vertex in the induced cell structure on S). The frontier of a small regular neighborhood of $S \cup \Lambda$ consists of a copy S' of S and a properly embedded disk E, both of which are barrier surfaces in the components of their complements not containing $S \cup \Lambda$. Notice that $E \cup S'$ is the frontier of a relative punctured 3-cell containing $S \cup \Lambda$ and meeting ∂M in a disk E' where $\partial E' = \partial E$. We proceed by shrinking E in the component of its complement not meeting $S \cup \Lambda$. We have a new phenomenon that we did not see above; namely, in the shrinking of E, we may need to make ∂-compressions, as well as compressions and isotopies. A ∂-compression may be to the inside of our relative punctured 3-cell, in which case it is clear that we split the relative punctured 3-cell into two relative punctured 3-cells with one containing $S \cup \Lambda$. On the other hand, we may have a ∂-compression to the exterior of a relative punctured 3-cell; however, the result of this is still a relative punctured 3-cell (see Figure 30). As a result of shrinking E, we have a number of relative punctured 3-cells, one containing $S \cup \Lambda$, and the components of their frontiers are either normal 2-spheres, normal disks, or 0-weight 2-spheres or disks properly embedded in some of the tetrahedra of $\mathcal{T}$. We may ignore all of the relative punctured 3-cells except that one containing $S \cup \Lambda$, say P. Furthermore, we may fill in any 0-weight components of the frontier with 3-cells, missing $S \cup \Lambda$. Since $S \cup \Lambda \subset P$ and Λ meets ∂M, any normal disk (a vertex-linking normal disk) in the frontier of P is parallel into ∂M away from $\Lambda \cap \partial M$. However, this is only possible if a component of ∂M is a 2-sphere, a contradiction. So, this is where we use that there are no 2-spheres in ∂M and conclude that there are no normal 2-spheres in M.

If M were not irreducible, then by Theorem 2.2 there would necessarily be an essential, embedded 2-sphere and thus an essential embedded, normal 2-sphere; so, M is irreducible. Similarly, if there were an essential, properly embedded disk, then by Theorem 2.1, there would be an essential, embedded normal disk. But such a disk is not vertex-linking; so M is ∂-irreducible.

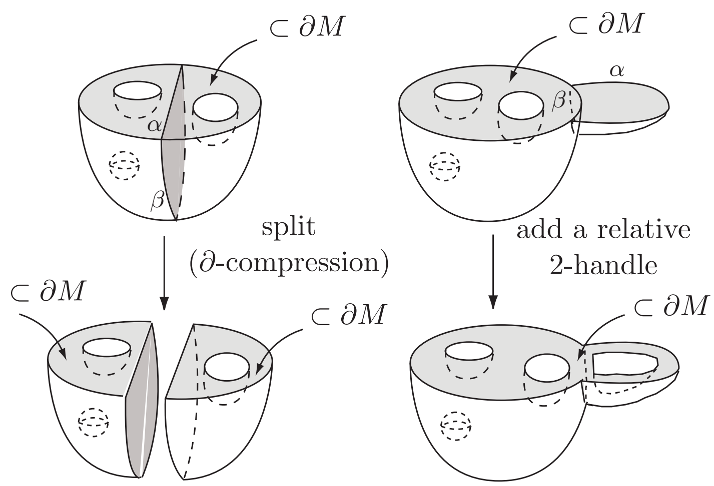


Figure 30: A relative punctured 3-cell.

Now, a vertex in $\mathring{M}$, the interior of M, would have a normal vertex-linking 2-sphere; so, again it follows that all vertices are in ∂M.

Suppose there is more than one vertex in a component of ∂M. Then there is an edge, e, in a component of ∂M, which is also an arc (has distinct vertices). The frontier of a small regular neighborhood of e is a properly embedded disk and forms a barrier surface in the component of its complement not meeting e; hence, it can be shrunk in the complement of e. As above, we arrive at a relative punctured 3-cell containing e, where each component of its frontier is either a normal 2-sphere, a normal disk, or a 0-weight 2-sphere or disk properly embedded in a tetrahedron of $\mathcal{T}$. But then we conclude that not only would ∂M be a 2-sphere but M would itself be a 3-cell. This completes the proof. q.e.d.

In the preceding, we considered 3-manifolds having no 2-sphere boundary components. A 2-sphere in the boundary of a 3-manifold is a nuisance that just doesn't fit well with 0-efficient triangulations. On the other hand, one might try to avoid the nuisance of 2-spheres in the boundary by considering triangulations of 3-manifolds with nonempty boundary, where the only normal disks are vertex-linking and there are no normal 2-spheres. It seems likely that if a 3-cell has a 0-efficient triangulation and has no normal 2-spheres, then the situation is extremely limited, as the only possibility is the single tetrahedron triangulation of the 3-cell with three vertices (see Figure 2(2)). However, there is a curiosity associated with triangulations of the 3-cell, which we have not resolved and seems to exhibit features similar to the problem posed above about the number of 2-sphere boundary components of the closure of the complement of a maximal collection of normal 2-spheres. Namely, if we assume the only normal disks are vertex-linking, then all vertices must be in the boundary and there must be at least three vertices in the boundary. Furthermore, there can not be an edge in the boundary which is disjoint from any other edge. It follows that every edge in the boundary must have a vertex in common; and, if there are more than three vertices in the boundary, then every edge in the triangulation must have a vertex in common. We have examples, constructed by Ben Burton, of triangulations of a 3-cell so that every edge has a vertex in common and there are more than three vertices, all in the boundary; however, every one we know is not 0-efficient but has no normal 2-spheres. An answer to this problem is not important to this work, but it is a curiosity we would like to resolve.

In this section, as in the preceding work, in order to obtain 0-efficient triangulations, we "crush" a given triangulation along an appropriate normal surface. Here, in the bounded case, we need a relative version of our concept of crushing used above; for this, the normal surface along which we crush has each component an embedded normal disk.

Suppose $\mathcal{T}$ is a triangulation of a compact, orientable, irreducible 3-manifold with nonempty boundary. Suppose $\mathcal{E}$ is pairwise disjoint collection of embedded normal disks in M and X is the closure of a component of the complement of the disks in $\mathcal{E}$; furthermore, suppose X does not contain any vertices of $\mathcal{T}$. In this situation, we give sufficient conditions for constructing a triangulation $\mathcal{T}'$ of X, having, in particular, the property that the number of tetrahedra in $\mathcal{T}'$ is strictly less than the number of tetrahedra in $\mathcal{T}$. The ideas are modelled on those used in Section 4. We will assume the reader is now familiar with our techniques and so, we will be brief in our descriptions at this point. However, we will organize our construction into a theorem at the end of our discussion.

Just as above, the manifold X has a nicely described cell-decomposition, say $\mathcal{C}$, induced from the triangulation $\mathcal{T}$. The induced cells are truncated tetrahedra, truncated prisms, triangular parallel regions, and quadrilateral parallel regions. See Figure 10. We will now show that under the right conditions, one can construct a triangulation of X by essentially replacing the truncated tetrahedra in $\mathcal{C}$ by tetrahedra.

We follow the notation used above. Let $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in the disks in $\mathcal{E}\} \cup \{$cells of Type III and Type IV in $\mathcal{C}\} \cup \{$all trapezoidal faces of $\mathcal{C}\}$. Each component of $\mathbb{P}(\mathcal{C})$ is an I-bundle. As above, we suppose $\mathbb{P}(\mathcal{C}) \neq X$ and all of the components of $\mathbb{P}(\mathcal{C})$ are product I-bundles; of course, just as above, we will need to confirm this in practice. We continue to use $K_i \times [0,1], 1 \leq i \leq k$, to denote the components of $\mathbb{P}(\mathcal{C})$, where $K_i^\varepsilon = K_i \times \varepsilon, \varepsilon = 0, 1$ are isomorphic subcomplexes of the induced normal cell structure on the disks in $\mathcal{E}$. Notice that the vertices of all edges of $\mathcal{C}$ not in $\mathcal{E}$ are themselves in $\mathcal{E}$; similarly, the boundaries of the trapezoidal regions are in $\mathcal{E}$. While the techniques above (in the case of crushing a triangulation along a closed normal surface) work perfectly well for when the surface is not connected, we applied the method in cases where the normal surface was connected. We remark that now it is most likely that the collection of disks $\mathcal{E}$ has a number of components; in practice, there is one disk for each boundary component of the 3-manifold M. We assume there is a pairwise disjoint collection of contractible planar complexes $D_j, 1 \leq j \leq k' \leq k$, along with pairwise disjoint embeddings $D_j \times [0,1]$ into X so that $D_j \times [0,1]$ is a subcomplex of X and for $D_i^\varepsilon = D_i \times \varepsilon, \varepsilon = 0, 1$, we have D_i^ε is embedded as a subcomplex of the induced cell structure on the disks in $\mathcal{E}$; and, $\bigcup(K_i \times [0,1]) \subset \bigcup(D_j \times [0,1])$ and the frontier of $\bigcup(D_j \times [0,1])$ is contained in the frontier of $\bigcup(K_i \times [0,1])$. As above, we have K_i^0 is isomorphic to K_i^1 for every i, but it is not necessarily the case that D_j^0 is isomorphic to D_j^1 for every j. See Figure 12 keeping in mind now that instead of a single surface S, we might have a number of surfaces $E_1, \ldots, E_n$.

We let $\mathbb{P}(X)$ be the collection of components of $\bigcup(D_j \times [0,1])$ and again call $\mathbb{P}(X)$ the induced product region for X. Here we have that the induced product region $\mathbb{P}(X)$ for X is a trivial product region for X.

Essentially, the induced product region is just as in the closed case but since the components of $\mathcal{E}$ are all disks, we have that $\mathbb{P}(X)$ is a trivial product region. On the other hand, we do have a new twist for chains of truncated prisms. Recall that a cell of X of Type II, a truncated prism, has two hexagonal faces. For closed 3-manifolds, above, these hexagonal faces in the cell decomposition of X can be identified to either a hexagonal face of a cell of Type I (truncated tetrahedron) or a hexagonal face of another cell of Type II. This is also true here but there is another possibility; namely, a hexagonal face may have no identifications because it is in ∂M.

If we follow a sequence of such identifications through cells of Type II, we again trace out a well-defined arc which now can terminate either at the identification of a hexagonal face of a cell of Type II with one of Type I or in the boundary of M. As above, it is also possible that these identifications form a simple closed curve through cells of Type II, forming a complete cycle. We continue to call the collection of truncated prisms identified in this way a *chain*. If at some point the identification of the hexagonal faces end in a truncated tetrahedron or in ∂M, we say the chain *terminates*. Note, a chain may terminate with truncated tetrahedrons at both ends or a truncated tetrahedron only at one end and the other in ∂M or with both ends in ∂M. If the chain begins and ends in ∂M, we call it a *relative cycle* and, as above, we call a complete chain a *cycle* (see Figure 24 above and Figure 31 below).

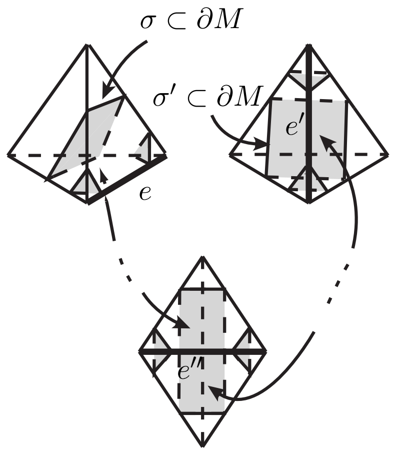


Figure 31: Relative cycles of truncated prisms — boundary case.

We will assume there are no cycles or relative cycles of truncated prisms in the induced cell structure on X, which are not in $\mathbb{P}(X)$. Later, of course, we will have the burden of proof to establish this in applications of crushing along a collection of embedded normal disks.

Again, if $\{\overline{\Delta}_1, \ldots, \overline{\Delta}_n\}$ is the collection of truncated tetrahedra in X, which are not in $\mathbb{P}(X)$, and $\overline{\sigma}_i$ is a face of $\overline{\Delta}_i$, then there are four possibilities for $\overline{\sigma}_i$. We have $\overline{\sigma}_i$ is identified with the face $\overline{\sigma}_j$ of $\overline{\Delta}_j$; we have $\overline{\sigma}_i$ is in ∂M and is not identified with any other face; we have that $\overline{\sigma}_i$ is at one end of a chain of truncated prisms and the face $\overline{\sigma}_j$ of $\overline{\Delta}_j$ is at the other end and so there is an induced identification of $\overline{\sigma}_i$ with $\overline{\sigma}_j$ through this chain of truncated prisms; and, lastly, we have $\overline{\sigma}_i$ is at one end of a chain of truncated prisms and the face $\overline{\sigma}_j$ is at the other end of this chain and also in ∂M and so there is no identification of $\overline{\sigma}_i$. So, as before, the faces of the truncated tetrahedra in $\{\overline{\Delta}_1, \ldots, \overline{\Delta}_n\}$ have an induced pairing but now, not only is it the case that faces which are in ∂M don't have any pairings, faces which are at the end of a chain of truncated prisms which start in ∂M do not have a pairing. Each truncated tetrahedron in X has its triangular faces in S. We can identify each such triangular face to a point (distinct points for each triangular face) and we get tetrahedra. We will now use the notation $\widetilde{\Delta}'_i$ for the tetrahedron coming from the truncated tetrahedron $\overline{\Delta}_i$ by identifying each of the triangular faces of $\overline{\Delta}_i$ to a point (distinct points for each triangular face) and use $\widetilde{\sigma}_i$ for the triangle coming from the hexagonal face $\overline{\sigma}_i$. Then $\mathbf{\Delta}'(X) = \{\widetilde{\Delta}'_1, \ldots, \widetilde{\Delta}'_n\}$ is a collection of tetrahedra with orientation induced by that on $\mathcal{T}$ and the induced pairings described above is a family $\mathbf{\Phi}'$ of orientation reversing affine isomorphisms. Hence, we get a 3-complex $\mathbf{\Delta}'(X)/\mathbf{\Phi}'$, which has its underlying point set a 3-manifold, which is homeomorphic to X. We will denote the associated triangulation by $\mathcal{T}'$. We call $\mathcal{T}'$ the triangulation obtained by *crushing the triangulation* $\mathcal{T}$ *along* $\mathcal{E}$. We denote the image of a tetrahedron $\widetilde{\Delta}'_i$ by Δ'_i and, as above, call $\widetilde{\Delta}'_i$ the lift of Δ'_i; we denote the image of the face $\widetilde{\sigma}_i$ of $\widetilde{\Delta}'_i$ by σ_i. (See Figure 14, replacing in that Figure the $\widetilde{\Delta}^*_k, k = i, j$ by $\widetilde{\Delta}'_k, k = i, j$, respectively.)

We summarize in the following theorem.

Theorem 5.16. *Suppose* $\mathcal{T}$ *is a triangulation of the compact, orientable, irreducible* 3*-manifold* M *with nonempty boundary. Suppose* $\mathcal{E}$ *is a pairwise disjoint collection of embedded normal disks in* M*,* X *is the closure of a component of the complement of* $\mathcal{E}$ *and* X *does not contain any vertices of* $\mathcal{T}$*. Suppose there is an induced product region,* $\mathbb{P}(X)$*, for* X*. If:*

i) $X \neq \mathbb{P}(X)$*,*

ii) $\mathbb{P}(X)$ *is a trivial product region for* X*, and*

iii) *there are no cycles or relative cycles of truncated prisms in* X*, which are not in* $\mathbb{P}(X)$*,*

then the triangulation $\mathcal{T}$ *can be crushed along* $\mathcal{E}$ *and* $\mathcal{T}'$ *is a triangulation of* X*.*

If the disks in $\mathcal{E}$ are not all vertex-linking and t, t' are the number of tetrahedra of $\mathcal{T}$ and $\mathcal{T}'$, respectively, then $t > t'$; in fact, $\mathcal{T}' = \mathcal{T}$ if an only if all the disks in $\mathcal{E}$ are vertex-linking.

The following is an existence theorem and is the version of Theorem 5.5 for bounded 3-manifolds. We have organized this study of 0-efficient triangulations for bounded 3-manifolds analogously to that above for closed 3-manifolds; however, we point out here and below that if one is given a compact, orientable 3-manifold with boundary, which is known to be irreducible and ∂-irreducible, then any triangulation can be modified to a 0-efficient triangulation or the 3-manifold is the 3-cell and (here is the major point) we do *not* need to employ the 3-sphere recognition algorithm. See Theorem 5.14 above and Theorem 5.20 below.

Theorem 5.17. *A compact, orientable, irreducible,* ∂*-irreducible* 3*-manifold with nonempty boundary has a* 0*-efficient triangulation.*

Proof. Suppose M is a compact, orientable, irreducible and ∂-irreducible 3-manifold with nonempty boundary. Let $\mathcal{T}$ be a triangulation of M. We wish to mimic, with appropriate modification, the proof of Theorem 5.5.

Let $S_1, \ldots, S_n$ denote the boundary components of M. Let $\mathcal{T}_i$ denote the triangulation of S_i induced by $\mathcal{T}$. If a component of the boundary is a 2-sphere, then since M is irreducible, M is a 3-cell. If M is the 3-cell, we can give M a one tetrahedron, three vertex triangulation, which is a 0-efficient triangulation of a 3-cell. Hence, we will assume M is not a 3-cell and, so, no component of ∂M is a 2-sphere.

If there is a normal 2-sphere, there is a normal disk which is not vertex-linking or, following the arguments from above, ∂M would be a 2-sphere. So, in our situation, to show $\mathcal{T}$ is 0-efficient is completely dependent on whether every normal disk is vertex-linking.

If every normal disk is vertex-linking, we are done. So, we may assume there is a normal disk, which is not vertex-linking. Notice that if E is any properly embedded disk, then there is a disk $E' \subset \partial M$, $E' \cap E = \partial E = \partial E'$ and $E \cup E'$ bounds a 3-cell in M. Furthermore, if E is a normal non-vertex-linking disk and F is a properly embedded normal disk so that E is contained in the 3-cell that F co-bounds with a disk $F' \subset \partial M$, then F is not vertex-linking (M is not a 3-cell). So, under the assumption that there is a non-vertex-linking disk, we consider a collection $\mathcal{E}$ of pairwise disjoint, properly embedded, normal disks, $E_1, \ldots, E_n$ where $\partial E_i \subset S_i, i = 1, \ldots, n$ and E_1, say, is not vertex-linking; furthermore, we may take the disks in this collection *maximal* in the sense that if F is a properly embedded, normal disk in M, F is disjoint from all the disks in $\mathcal{E}$, F co-bounds a 3-cell B with a disk in ∂M and some $E_i \subset B$, then $F = E_i$.

For each disk $E_i \in \mathcal{E}$, let E_i' denote the disk in S_i so $E_i \cap E_i' = \partial E_i = \partial E_i'$ and $E_i \cup E_i'$ bounds a 3-cell. Denote these 3-cells bounded by $E_i \cup E_i'$, $B_i, i = 1, \ldots, n$.

We claim that all the vertices of $\mathcal{T}$ are in $\bigcup_{i=1}^n B_i$. For suppose there is a vertex v of $\mathcal{T}$ not in $\bigcup_{i=1}^n B_i$. Then there is an arc Λ in the 1-skeleton of $\mathcal{T}$, Λ has one end at v and meets $\bigcup_{i=1}^n B_i$ in a single point in, say E_j, for some $j, 1 \le j \le n$. Furthermore, if v is in $\mathcal{T}_j$ we can take Λ in the 1-skeleton of $\mathcal{T}_j$. We consider a small regular neighborhood, N, of $B_j \cup \Lambda$. The frontier of N consists of a properly embedded disk F_j, which may not be normal. However, the frontier of N along with the collection of normal disks $E_i, i \neq j$ is a barrier surface in the component of their complement not meeting $\bigcup_{i=1}^n B_i \cup \Lambda$. Furthermore, there is a disk $F_j' \subset S_j, \partial F_j' = \partial F_j$ and $F_j' \cup F_j$ bounds a relative 3-cell in M. Note that $F_j' = E_j'$ (normally isotopic with the same normal curve as a boundary) if Λ is not in S_j and, in general, $F_j' \supset E_j'$. We shrink F_j, using Theorem 3.2, obtaining a relative punctured 3-cell, say P, embedded in M, which contains $B_j \cup \Lambda$, $P \cap \partial M \subset S_j$, $P \cap E_i = \emptyset$ and each component of the frontier of P in M is either a normal disk, a normal 2-sphere or a 0-weight 2-sphere or disk properly embedded in a tetrahedron of $\mathcal{T}$. Each 0-weight 2-sphere in the frontier of P bounds a 3-cell, whose interior misses P; similarly, for each 0-weight disk in the frontier of P there is a disk in ∂M, which together with the disk in the frontier of P bounds a 3-cell whose interior misses P. We fill in these 0-weight frontiers with these 3-cells. We still have a relative punctured 3-cell, which we continue to call P, $B_j \cup \Lambda \subset P$, $P \cap \partial M \subset S_j$, and $P \cap E_i = \emptyset, i \neq j$. If there is a normal 2-sphere in ∂P, then, since M is irreducible, such a 2-sphere bounds a 3-cell in M, which can not meet ∂M. We add all such 3-cells to P. We get a relative punctured 3-cell, we continue to call it P, $B_j \cup \Lambda \subset P$, $P \cap \partial M \subset S_j$, and each component of the frontier of P is a normal disk. Now, suppose F is a normal disk in the frontier of P. Then there is a disk $F' \subset S_j, \partial F = \partial F'$ and $F \cup F'$ bounds a 3-cell in M. If this 3-cell contains P and thus E_j, then we contradict the maximality of the disks in the collection $\mathcal{E}$; so, we may assume that the interior of the 3-cell is disjoint from P. However, if this happens for every normal disk in the frontier of P, then S_j would be a 2-sphere, which leads to a contradiction. Thus assuming that a vertex of $\mathcal{T}$ is not in $\bigcup_{i=1}^n B_i$ leads to a contradiction.

Let X be the closure of the component of $M \setminus \bigcup_{i=1}^n B_i$ that does not contain the vertices of $\mathcal{T}$. First we notice that X is homeomorphic with M. As in the proof of Theorem 5.5, the manifold X has a nicely described cell decomposition induced from the triangulation $\mathcal{T}$ and the way in which the normal disks $E_1, \ldots, E_n$ sit in $\mathcal{T}$ (all the vertices are in the 3-cells complementary to X).

The cells of X are the same four types as above: truncated tetrahedra, truncated prisms, triangular parallel regions, and quadrilateral parallel regions (see Figure 10), except that some of the truncated faces (those faces also in faces of tetrahedra of $\mathcal{T}$) may now be in ∂M, including some trapezoidal faces.

We need to show the conditions of Theorem 5.16 are satisfied and hence, we can crush the triangulation $\mathcal{T}$ along the collection of normal disks $\mathcal{E}$.

We let $\mathcal{C}$ denote the induced cell-decomposition of X and, as above, let $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in the disks in $\mathcal{E}\} \cup \{$cells of Type III and Type IV in $\mathcal{C}\} \cup \{$all trapezoidal faces of $\mathcal{C}\}$. Each component of $\mathbb{P}(\mathcal{C})$ is an I-bundle.

Claim 1. *$\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle.*

Proof. If $\mathbb{P}(\mathcal{C}) = X$, then X is an I-bundle with corresponding ∂I-bundle a collection of disks. Hence, X is the product of a disk and an interval; but then M would be the 3-cell, a contradiction.

If some component of $\mathbb{P}(\mathcal{C})$ is not a product I-bundle, then there is a Möbius band properly embedded in X with its boundary in one of the disks D_i. However, then we would have a projective plane embedded in M; this is impossible since we have assumed $\partial M \neq \emptyset$ and M is irreducible. This completes the proof of Claim 1.

Claim 2. *There is a trivial induced product region $\mathbb{P}(X)$ for X.*

Proof. Essentially, we follow the argument as in Claim 2 in the proof of Theorem 5.5. However, here in the bounded case, we have two additional and new phenomena from what we had above, in the case of M closed.

We have established that each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle. As above, we write each I-bundle component of $\mathbb{P}(\mathcal{C})$ as $K_i \times I, i = 1, \ldots, k$, where k is the number of components of $\mathbb{P}(\mathcal{C})$ and again, we set $K_i^\varepsilon = K_i \times \varepsilon, \varepsilon = 0, 1$. The subcomplexes K_i^0 and K_i^1 are isomorphic subcomplexes in the induced normal cell structures on the disks in our collection $\mathcal{E}$. Here, we may have K_i^0 and K_i^1 in the same or in distinct disks in $\mathcal{E}$.

We would like to have for each of the I-bundles $K_i \times [0,1]$ that K_i is simply connected; but just as above, this is not necessarily the case. So, let's consider such a $K_i \times [0,1]$. We may as well assume that K_i is not simply connected.

Thus if K_i^ε is in the disk E_j in the collection $\mathcal{E}$, then K_i^ε separates E_j; and if K_i^ε meets ∂E_j, then every component of the complement of K_i^ε is simply connected. This is just as in the closed case; however, if K_i^ε does not meet ∂E_j, then the component of the complement of K_i^ε in E_j, which contains ∂E_j, is not simply connected. Of course, all the other components of the complement of K_i^ε in E_j are simply connected. Because of the boundary of E_j, we have a new consideration.

Suppose K_i is not simply connected, $K_i^\varepsilon \subset E_j, \varepsilon = 0, 1$ and $K_i \times [0,1]$ does not meet ∂E_j. We claim that by having chosen the collection $\mathcal{E}$ maximal, we must have that K_i^0 is in the same component of the complement of K_i^1 as ∂E_j and K_i^1 is in the same component of the complement of K_i^0 as ∂E_j. For suppose K_i^1 is not in the component of the complement of K_i^0 that meets ∂E_j. Define D_i^0 to be K_i^0 along with all the components of its complement not meeting ∂E_j. Then we have that D_i^0 is simply connected and also we have $K_i^1 \subset D_i^0$. See Figure 32. Let $N_i = N(D_i^0 \cup (K_i \times [0,1]))$ be a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1])$. Since $K_i^1 \subset D_i^0$, the frontier of N_i consists of an annulus, possibly some 2-spheres and a disk F_i, properly embedded in X and having its boundary in E_j. There is a disk F_i' in E_j so that $\partial F_i' = \partial F_i$ and $F_i \cup F_i'$ bounds a 3-cell, say B, in X; actually, we want to think of B as a relative (punctured) 3-cell. Furthermore, B contains N_i and F_i along with $\cup E_i, i \neq j$, form a barrier surface in the component of the complement of F_i not meeting N_i. We shrink F_i in the component of the complement of F_i not meeting N_i. Since M (and hence, X) is not a 3-cell, our standard arguments give that in this situation we would have a *normal* disk E in X and a disk E' in ∂X so that $\partial E' = \partial E$ and $E \cup E'$ bounds a 3-cell containing B. But this is a contradiction to our choice of the collection $\mathcal{E}$ being maximal.

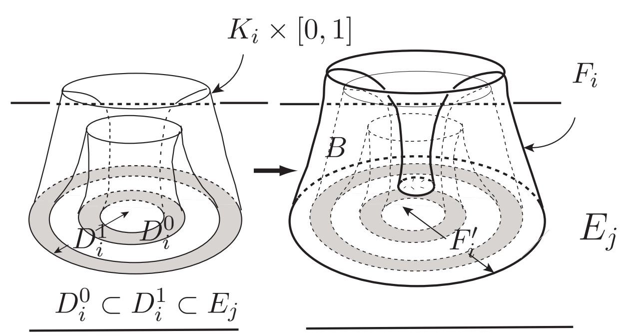


Figure 32: K_i^1 is not in the same component of the complement of K_i^0 as ∂E_j.

So, having made this observation, let's consider the possibilities for $K_i \times [0,1]$. One possibility is that K_i^0 and K_i^1 are in distinct disks of our collection $\mathcal{E}$, say $K_i^0 \subset E_j$ and $K_i^1 \subset E_{j'}, j \neq j'$. In this case, K_i^0 does not meet ∂E_j

and K_i^1 does not meet $\partial E_{j'}$. So, we let D_i^0 be the union of K_i^0 along with the components of the complement of K_i^0 in E_j, which do not meet ∂E_j; similarly, we let D_i^1 be the union of K_i^1 along with the components of the complement of K_i^1 in $E_{j'}$, which do not meet $\partial E_{j'}$. Then $D_i^\varepsilon, \varepsilon = 0, 1$ is simply connected. (If K_i is simply connected, then $K_i^\varepsilon, \varepsilon = 0, 1$ does not separate and $D_i^\varepsilon = K_i^\varepsilon$.) We let $N_i = N(D_i^0 \cup (K_i \times [0,1]) \cup D_i^1)$ be a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1]) \cup D_i^1$. Then the frontier of N_i consists of a single annulus and possibly some 2-spheres. Furthermore, since M is irreducible and $\partial M \neq \emptyset$, we have immediately that any such 2-sphere bounds a 3-cell not meeting $D_i^0 \cup (K_i \times [0,1]) \cup D_i^1$ and so each plug for N_i is a 3-cell. It follows that there is a simply connected planar complex D_i and an embedding of $D_i \times [0,1]$ into X so that $D_i \times 0 = D_i^0, D_i \times 1 = D_i^1, K_i \times [0,1] \subset D_i \times [0,1]$, and the frontier of $D_i \times [0,1]$ is contained in the frontier of $K_i \times [0,1]$. See Figure 12.

Another possibility is that both K_i^0 and K_i^1 are in the same disk E_j and K_i^0 (and hence, K_i^1) does not meet ∂E_j. By our observation above, we have that K_i^0 (K_i^1, respectively) is in the component of the complement of K_i^1 (K_i^0, respectively) that meets ∂E_j. In this situation, we let D_i^0 (respectively D_i^1) be the union of K_i^0 (respectively K_i^1) along with all components of the complement of K_i^0 (respectively K_i^1) in E_j not meeting K_i^1 (respectively K_i^0). Then $D_i^\varepsilon, \varepsilon = 0, 1$ is simply connected. Here again, we get straight away that for $N_i = N(D_i^0 \cup (K_i \times [0,1]) \cup D_i^1)$, a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1]) \cup D_i^1$, then the plugs for N_i are 3-cells. Hence, in this case as in the previous case, it follows that there is a simply connected planar complex D_i and an embedding of $D_i \times [0,1]$ into X so that $D_i \times 0 = D_i^0, D_i \times 1 = D_i^1, K_i \times [0,1] \subset D_i \times [0,1]$, and the frontier of $D_i \times [0,1]$ is contained in the frontier of $K_i \times [0,1]$. Again, see Figure 12.

The only remaining possibility is that both K_i^0 and K_i^1 are in the same disk E_j and now K_i^0 (and hence, K_i^1) does meet ∂E_j. As we observed above, if we let D_i^0 (respectively D_i^1) be the union of K_i^0 (respectively K_i^1) along with all components of the complement of K_i^0 (respectively K_i^1) in E_j not meeting K_i^1 (respectively K_i^0), then $D_i^\varepsilon, \varepsilon = 0, 1$, is simply connected. Just as in all the similar situations above, we let $N_i = N(D_i^0 \cup (K_i \times [0,1]) \cup D_i^1)$ be a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1]) \cup D_i^1$. However, now we have a new situation. The components of the frontier of N_i still might contain some 2-spheres but now must contain properly embedded disks, whose boundaries are in ∂M (see Figure 33). Just as before, any 2-sphere frontier bounds a 3-cell in M which misses N_i and we have a 3-cell plug. So, let's consider what happens for a component of the frontier of N_i which is a disk, say E. Then there is a disk E' contained in ∂X, $\partial E' = \partial E$, and $E \cup E'$ bounds a 3-cell B in X. If B does not contain N_i (E' does not contain E'_j), then we have a 3-cell plug and are satisfied. However, *a priori*, it might be possible that the 3-cell B contains N_i. We shall show that this is impossible, having chosen the collection $\mathcal{E}$ to be maximal. First, note that E along with the normal disks in $\mathcal{E}$, distinct from E_j, form a barrier surface in the component of the complement of E not meeting N_i; so, we can shrink E. However, just as above, since M is irreducible, ∂-irreducible and not itself a 3-cell, this situation leads to a normal disk, say F, properly embedded in M and a disk $F' \subset S_j$ so that $\partial F' = \partial F$ and $F \cup F'$ bounds a 3-cell containing E_j. This contradicts our choice of $\mathcal{E}$ maximal. Hence, all plugs for N_i must be 3-cells and again we have that there is a simply connected planar complex D_i and an embedding of $D_i \times [0,1]$ into X so that $D_i \times 0 = D_i^0, D_i \times 1 = D_i^1, K_i \times [0,1] \subset D_i \times [0,1]$, and the frontier of $D_i \times [0,1]$ is contained in the frontier of $K_i \times [0,1]$ (see Figure 33).

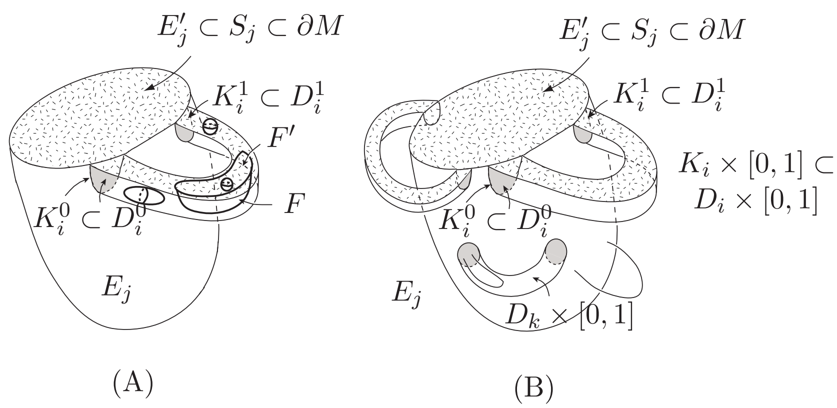


Figure 33: (A) K_i^1 and K_i^0 both meet ∂E_j, leading to a 3-cell plug or a new normal disk F; (B) induced product region, $\mathbb{P}(X)$.

Now, having made these observations, we can choose some order for the components of $\mathbb{P}(\mathcal{C})$, say $K_1 \times [0,1], \ldots, K_k \times [0,1]$, and from this, just as in the proof of Theorem 5.5, derive that there is a trivial induced product region $\mathbb{P}(X)$ for X. Again, see Figure 33.

This completes the proof of Claim 2.

Claim 3. *There is no cycle* (*complete or relative*) *of cells of Type* II *in* X, *which are not in* $\mathbb{P}(X)$.

Proof. In this situation we have four possibilities: a complete cycle about a single edge, a complete cycle about more than one edge, a relative cycle about a single edge and a relative cycle about more than one edge.

Suppose there were a complete cycle about a single edge e as in Figure 24, Part A. As in the similar situation in the proof of Theorem 5.5, there would be a disk D meeting e in precisely one point and meeting a non-vertex-linking disk, say E_j, of the collection $\mathcal{E}$ in ∂D. A surgery on E_j at D (see Figure 25), which is the same as adding a 2-handle to E_j along the curve ∂D, gives a *normal* 2-sphere, S_j, and a new *normal* disk, F_j. Furthermore, $\partial F_j = \partial E_j$ and $F_j \cup E_j'$ bounds a relative punctured 3-cell with frontier $F_j \cup S_j$; this relative punctured 3-cell contains the 3-cell bounded by $E_j \cup E_j'$ (the disk D is not in the 3-cell bounded by $E_j \cup E'_j$). Since M is irreducible, S_j bounds a 3-cell whose interior does not meet the 3-cell bounded by $E_j \cup E'_j$. But then F_j co-bounds a 3-cell with E'_j and this contradicts maximality of the disks in the collection $\mathcal{E}$. Hence, if the collection $\mathcal{E}$ is maximal, there is no complete cycle of truncated prisms about a single edge.

Now, suppose there is a complete cycle of truncated prisms about more than one edge (see Figure 24(B)), then as in the consideration for closed manifolds, the collection of cells of Type II (truncated prisms), form a solid torus with, possibly, some self identifications in its boundary (possibly, some of the trapezoidal faces in the boundary are identified); furthermore, each hexagonal face of a truncated prism in the cycle of truncated prisms is a meridional disk for the solid torus. As above, we distinguish between the cycle of truncated prisms, which we denote $\hat{\tau}$, and the cycle of truncated prisms minus the bands of trapezoids, which we denote by τ. We have that τ meets the collection $\mathcal{E}$ in either three open annuli, each meeting a meridional disk of τ precisely once, or a single open annulus, meeting a meridional disk of τ three times. See Figure 26.

First, we consider the case we have three annuli in τ, say A_1, A_2 and A_3, each meeting a hexagonal face of a truncated prism in $\hat{\tau}$ precisely once. The frontier of an $A_i, i = 1, 2, 3$ in a disk in the collection $\mathcal{E}$ is in the collection of trapezoids in the faces of the cycle of truncated prisms $\hat{\tau}$; thus in the induced I-bundle region $\mathbb{P}(\mathcal{C})$. Now, just as above, there are two components of the frontier of each A_i and each component of the frontier of A_i separates the disk of $\mathcal{E}$, which it is in. Consider A_1 and suppose A_1 is in E_j and denote the two components of the frontier of A_1 by a_1 and a'_1. Then both a_1 and a'_1 are in trapezoids in $\hat{\tau}$ and so in $\mathbb{P}(\mathcal{C})$. By our above construction of the induced product region for X, $\mathbb{P}(X)$, it follows that a_1 and a'_1 are each in a simply connected region of E_j, which does not meet ∂E_j. But this is possible only if A_1 is also in such a simply connected region of E_j. Therefore, A_1 and hence $\hat{\tau}$ is in $\mathbb{P}(X)$, the induced product region for X. See Figure 27.

While it can be shown directly, we have, in particular, from the conclusion of this argument that the torus τ can meet at most two distinct components of $\mathcal{E}$.

Suppose now we have just one annulus, say A_1, common to τ and the collection $\mathcal{E}$. We have that A_1 meets each hexagonal face in the chain of truncated prisms, $\hat{\tau}$, three times. We can wipe this situation away quite easily by noting that this would imply that there is a lens space $L(3, 1)$ embedded in M as a connected summand but this is impossible as M is irreducible. On the other hand, the reader may wonder why this differs from the previous case (where there were three annuli common to τ and $\mathcal{E}$), since we only used that there was a single annulus in the disk $E_j \in \mathcal{E}$). The same argument could be used to conclude that $\hat{\tau}$ is in the induced product region $\mathbb{P}(X)$; however, this skirts the fact that above we used quite strongly that M is irreducible to get the 3-cell plugs in order to get a trivial induced product region. This completes the proof of the Claim and there is no complete cycle of truncated prisms, which is not in $\mathbb{P}(X)$.

Finally, we have the completely new possibility in the case when M has boundary; this is the possibility of a relative cycle of truncated prisms. In this case, we also have two possible situations: a relative cycle about a single edge or a relative cycle about different edges.

In the first situation, a relative cycle about a single edge e, then necessarily the edge e is in ∂M. There is a disk D which meets the edge e in a single point, meets some E_i in a spanning arc β and meets $S_i \subset \partial M$ in a spanning arc α. See Figure 34. Furthermore, α meets E_i' only in its end points, which are in $\partial E_i'$. The disk D, which lies outside the 3-cell B_i, gives a "∂-compression" (adds a relative 2-handle, see Figure 30) to the 3-cell B_i along the disk E_i, resulting in a relative punctured 3-cell with two (new) *normal*, non-vertex-linking disks in its frontier. Consider the disks in the frontier of this relative punctured 3-cell. Since M is irreducible and ∂-irreducible, each co-bounds a 3-cell in M with a disk in S_i and since S_i is not a 2-sphere at least one must co-bound a 3-cell with a disk in S_i, which also contains B_i. However, this contradicts the maximality of the disks in the collection $\mathcal{E}$.

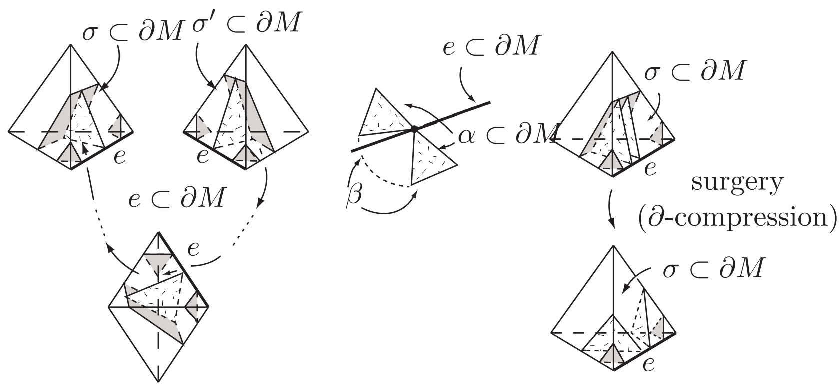


Figure 34: Relative cycle about a single edge gives a new normal disk.

Now, suppose we have a relative cycle about more than one edge (see Figure 31), then the collection of cells of Type II (truncated prisms), is a 3-cell having the form of a product I-bundle, $D^2 \times I$, with, possibly, some self identifications in its vertical boundary $\partial D^2 \times I$ (possibly, some of the trapezoidal faces in the boundary are identified). Furthermore, each hexagonal face of a truncated prism in the relative cycle of truncated prisms is a section in the I-bundle structure for the 3-cell and there is a component of ∂M, say S_j, which contains $D^2 \times \varepsilon, \varepsilon = 0, 1$. As above in the case of a complete cycle of truncated prisms, and because of the possible singularities, we distinguish between the relative cycle of truncated prisms, which we denote $\hat{\delta}$, and the cycle of truncated prisms minus the bands of trapezoids, which we denote by

δ. We have that δ meets the collection $\mathcal{E}$ only in E_j ($\hat{\delta}$ meets only S_j) and then in three vertical bands of the form $A_i = \alpha_i \times I$ for the pairwise disjoint open intervals $\alpha_i, i = 1, 2, 3$ in ∂D^2. See Figure 35.

Now, as in the complete cycle case, the frontier of each $A_i, i = 1, 2, 3$ in E_j is in the collection of trapezoids in the faces of the relative cycle of truncated prisms $\hat{\delta}$; thus, the frontier of each A_i is in the induced I-bundle region $\mathbb{P}(\mathcal{C})$. Furthermore, there are two components of the frontier of each A_i and each component of the frontier of A_i separates E_j. Consider A_1 and denote the two components of the frontier of A_1 by a_1 and a_1'. Then both a_1 and a_1' are in trapezoids in $\hat{\delta}$ and so in $\mathbb{P}(\mathcal{C})$. By our construction of the induced product region for X, $\mathbb{P}(X)$, in the case that a component of $\mathbb{P}(\mathcal{C})$ meets the boundary of a disk E_j, we have that a_1 and a_1' are each in a simply connected region of E_j, which is also in $\mathbb{P}(X)$. Hence, we either have A_1, and therefore $\hat{\delta}$ in $\mathbb{P}(X)$ or $\mathbb{P}(X)$ meets E_j in the complement of A_1. But the latter is impossible, since A_2 and A_3 are in regions of E_j complementary to A_1. So, the only possibility is that A_1 is in $\mathbb{P}(X)$; that is, such a relative cycle of truncated prisms, $\hat{\tau}$, is in the induced product region for X, $\mathbb{P}(X)$. See Figure 35.

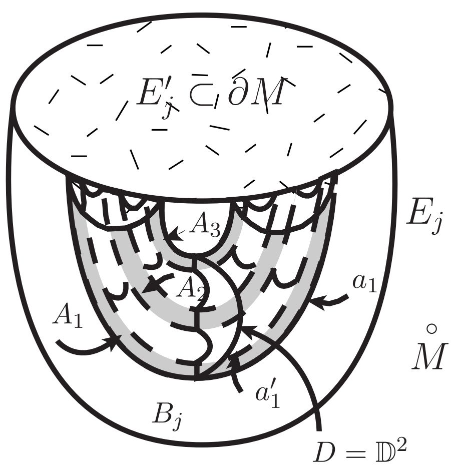


Figure 35: Relative cycle about more than one edge.

This completes the proof of Claim 3.

Hence, we have established that all the conditions of Theorem 5.16 (namely, $\mathbb{P}(X)$ exists and $\mathbb{P}(X) \neq X$, $\mathbb{P}(X)$ is a trivial product region for X, and there are no cycles or relative cycles of truncated prisms in X, which are not in $\mathbb{P}(X)$). Thus by Theorem 5.16, there is a triangulation $\mathcal{T}'$ of X, where the tetrahedra of $\mathcal{T}'$ are in one-one correspondence with the truncated tetrahedra of $\mathcal{C}$ not in $\mathbb{P}(X)$. Since, we had that at least one of the disks in the collection $\mathcal{E}$ is not vertex-linking, then the number of tetrahedra of $\mathcal{T}'$ is smaller than the number of tetrahedra in the original triangulation $\mathcal{T}$. By repeated applications of the above techniques, it follows that we must eventually get the desired triangulation of M. q.e.d.

As we pointed out at the beginning of this section, there are various ways to prove that a compact, orientable 3-manifold with nonempty boundary, no component of which is a 2-sphere, admits a triangulation having all vertices in the boundary and precisely one vertex in each boundary component. The methods of the previous theorem give a "natural method of identifying maximal trees to points" to achieve such triangulations for bounded manifolds, which are also irreducible and ∂-irreducible. We have the following corollary to Theorem 5.17 and Proposition 5.15. After the Corollary, we give an algorithm for constructing a 0-efficient triangulation, which also provides a construction of triangulations having all vertices in the boundary and just one vertex in each boundary component.

Corollary 5.18. *Suppose M is a compact, orientable, irreducible, ∂-irreducible 3-manifold with nonempty boundary, no component of which is a 2-sphere. Then M admits a triangulation having all vertices in the boundary and precisely one vertex in each boundary component.*

We need the following proposition. It is analogous to Proposition 5.7. An alternate proof appears in [8].

Proposition 5.19. *Given a triangulation $\mathcal{T}$ of a compact, orientable 3-manifold with nonempty boundary, no component of which is a 2-sphere, there is an algorithm to decide if $\mathcal{T}$ is 0-efficient; furthermore, the algorithm will construct a non-vertex-linking normal disk, if one exists.*

Proof. First, we observe that if we are given a normal 2-sphere, we can construct a non-vertex-linking normal disk. This uses the techniques from above; namely if S is a normal 2-sphere, then there is an arc in the 1-skeleton of $\mathcal{T}$, meeting S in a single point and meeting the boundary of M in a vertex of the triangulation. A small regular neighborhood of this arc and S has boundary a normal 2-sphere and a properly embedded disk, possibly not normal; however, they are barrier surfaces in the components of their complement not containing S (and the arc). We shrink the disk. We get a non-vertex-linking, normal disk. This is where we use that no component of the boundary of M is a 2-sphere.

So, we may assume there are no interior vertices and hence, no vertex-linking normal 2-spheres. Among all (non-vertex-linking) normal 2-spheres and all non-vertex-linking normal disks, suppose Σ is one for which the dimension of its carrier in the projective solution space of $\mathcal{T}$ is a minimum. We claim Σ is carried by a vertex.

If Σ is not carried by a vertex of the projective solution space, then there are nonnegative integers k, n, and m and normal surfaces X and Y so that $k\Sigma = nX + mY$, where X and Y are carried by *proper* faces of the carrier of Σ. It follows as above that either $\chi(X) \geq 0$ or $\chi(Y) \geq 0$. Suppose $\chi(X) \geq 0$. Then a component of X, say X', is either a normal 2-sphere, projective plane or disk and is carried by a proper face of the carrier of Σ. Now, by our observations, X' is not a vertex-linking normal 2-sphere (this could only come from an interior vertex). We may also assume that X' is not a vertex-linking normal disk; for then, it would be, itself, a factor of $k\Sigma$. Hence, X' is either a non-vertex-linking disk, a non vertex-linking 2-sphere or it is a projective plane and therefore, its double is a non-vertex-linking 2-sphere. But this contradicts our choice of Σ. q.e.d.

We could have begun this section with the following theorem which has as an immediate corollary that a compact, orientable 3-manifold with nonempty boundary, which is known to be irreducible and ∂-irreducible, admits a 0-efficient triangulation. It does *not* use the 3-sphere recognition algorithm as the analogous theorem, Theorem 5.14, does in the closed case. This is because, for a bounded 3-manifold, anytime we have a 2-sphere and we know it bounds a 3-cell, then we know for which side of the 2-sphere we have the 3-cell.

Theorem 5.20. *Suppose M is a compact, orientable, irreducible and ∂-irreducible 3-manifold with nonempty boundary. Then any triangulation of M can be modified to a 0-efficient triangulation or M is a 3-cell.*

Proof. Suppose M is given as in the hypothesis and $\mathcal{T}$ is a triangulation of M. We can decide if ∂M is a 2-sphere. If it is, then M is a 3-cell and there is nothing to prove; so, we may assume no component of ∂M is a 2-sphere.

Let $S_1, \ldots, S_n$ denote the boundary components of M and let $\mathcal{T}_i$ denote the triangulation $\mathcal{T}$ restricted to $\mathcal{S}_i$. We want to use the techniques of the proof of Theorem 5.17; there we had a pairwise disjoint collection of normal disks, one for each component of ∂M to guide a crushing of the triangulation. Here we can proceed in several ways. For example, if there is more than one vertex in a component of ∂M or if there is an interior vertex, then we can begin straight away to construct the desired collection of normal disks. On the other hand, if there is only one vertex in each boundary component of M and no vertices in $\mathring{M}$, then we need to use the algorithm in Proposition 5.19 to decide if $\mathcal{T}$ is 0-efficient and if it is not, then to find a non vertex-linking disk. It turns out that this latter approach works in both situations but is a bit over the top in the case it is obvious that $\mathcal{T}$ is not 0-efficient.

So, we apply the algorithm of Proposition 5.19. If $\mathcal{T}$ is 0-efficient there is nothing to prove. If it is not then there is a non-vertex-linking normal disk and the algorithm will construct one for us. Say E_1 is such a disk and notation has been chosen so that $\partial E_1 \subset S_1$. Since M is irreducible and ∂-irreducible, there is a disk $E_1' \subset S_1$, $\partial E_1 = \partial E_1'$ and $E_1 \cup E_1'$ bounds a 3-cell B_1. Since no component of ∂M is a 2-sphere, we know precisely which component of the complement of $E_1 \cup E_1'$ meets B_1. Let E_i be a vertex-linking disk in $S_i, i \neq 1$; then for each i we have a disk $E_i' \subset S_i, \partial E_i' = \partial E_i$ and $E_i \cup E_i'$ bound a 3-cell, say B_i. Thus we have a pairwise disjoint collection of normal disks $\mathcal{E} = \{E_1, \ldots, E_n\}$, so that $\partial E_i \subset S_i$, for each i there is a disk $E_i' \subset S_i, \partial E_i = \partial E_i'$, $E_i \cup E_i'$ bounds a 3-cell B_i and E_1 is not vertex-linking.

However, we want to have all the vertices of $\mathcal{T}$ contained in $\bigcup B_i$.

So, suppose not all the vertices of $\mathcal{T}$ are in $\bigcup B_i$. First, we want to make sure all the vertices of $\mathcal{T}$ in ∂M are in some B_i. If this is not the case, then there is a vertex v of $\mathcal{T}$, v is not in $\bigcup B_i$ and v is in some $\mathcal{T}_j$ ($v \in S_j$). In this case, there is an arc α_j in the 1-skeleton of $\mathcal{T}_j$ having one end point v and the other in $\partial E'_j$ and otherwise, missing E'_j. Let N_j be a small regular neighborhood of $B_j \cup \alpha_j$. Then since $\alpha_j \subset S_j$, we have N_j is a 3-cell (a relative punctured 3-cell); and if F_j is the frontier of N_j, then F_j is a properly embedded disk and F_j along with the $E_i, i \neq j$, form a barrier surface in the component of the complement of F_j not meeting $B_j \cup \alpha_j$. We shrink F_j. Then just as so many times above, in this way, we get a normal disk $\hat{E}_j$, $\partial \hat{E}_j \subset S_j$ and there is a disk $\hat{E'}_j \subset S_j$ so that $\partial \hat{E}_j = \partial \hat{E'}_j$ and $\hat{E}_j \cup \hat{E'}_j$ bounds a 3-cell $\hat{B}_j$ which contains $B_j \cup \alpha_j$. Hence, $\hat{B}_j$ contains v. Furthermore, $\hat{E}_j$ (as well as E_1) is not vertex-linking. It is possible that $j = 1$, but still $\hat{E}_j$ is not vertex-linking.

Thus, if we stick with our original notation in order to keep things simpler, we now may assume that we have constructed a pairwise disjoint collection of normal disks $\{E_1, \ldots, E_n\}$, so that $\partial E_i \subset S_i$, for each i there is a disk $E'_i \subset S_i, \partial E_i = \partial E'_i$, $E_i \cup E'_i$ bounds a 3-cell B_i, all the vertices of $\mathcal{T}$ in ∂M are in $\bigcup B_i$ and E_1, at least, is not vertex-linking.

If all the vertices of $\mathcal{T}$ are not in $\bigcup B_i$, then there is a vertex $v \in \overset{\circ}{M}$ not in some B_i. Thus there is an arc α in the 1-skeleton of $\mathcal{T}$ so that v is at one end of α and the other end of α is in some E_j and otherwise α misses $\bigcup B_i$; furthermore, since all the vertices in ∂M are in some B_i, the entire arc $\alpha \subset \overset{\circ}{M}$. Let N_j be a small regular neighborhood of $B_j \cup \alpha$. Then N_j is a 3-cell (a relative punctured 3-cell); and if F_j is the frontier of N_j, then F_j is a properly embedded disk and F_j along with the $E_i, i \neq j$, form a barrier surface in the component of the complement of F_j not meeting $B_j \cup \alpha$. From here the argument is just as above.

So, we can construct a pairwise disjoint collection of normal disks $\mathcal{E} = \{E_1, \ldots, E_n\}$ so that $\partial E_i \subset S_i$, for each i there is a disk $E'_i \subset S_i, \partial E_i = \partial E'_i$, $E_i \cup E'_i$ bounds a 3-cell B_i, all the vertices of $\mathcal{T}$ are contained in $\bigcup B_i$ and E_1, at least, is not vertex-linking.

We are now set up to follow the steps in the proof of Theorem 5.17; however, the difference is that we do not have here that the normal disks in the collection $\mathcal{E}$ are maximal. (Recall we are using maximal here in the sense that if F is a normal disk in M, F is disjoint from the disks in the collection $\mathcal{E}$ and there is a disk $F' \subset S_j$, $\partial F' = \partial F$, $F' \cup F$ bounds a 3-cell B, $B_j \subset B$, then we must have $F = E_j$.) We will follow the steps of the proof of Theorem 5.17; while we will be very brief in doing this, we do need some terminology for the situations where we are not able to use that the disks in the collection $\mathcal{E}$ are maximal.

So, suppose $\mathcal{E}$ is as above and there is a pairwise disjoint collection of normal disks $\mathcal{F} = \{F_1, \ldots, F_n\}$ so that $\partial F_i \subset S_i$, for each i there is a disk $F'_i \subset S_i, \partial F_i = \partial F'_i$, $F_i \cup F'_i$ bounds a 3-cell B'_i, and for all $i, B_i \subseteq B'_i$. In this case we say that $\mathcal{F}$ is *larger* than $\mathcal{E}$ and if for some j, F_j is not equivalent to

E_j, we say $\mathcal{F}$ is *strictly larger* than $\mathcal{E}$.

Let X be the closure of the component of the complement of $\bigcup B_i$ in M. Then X is homeomorphic to M and if $\mathcal{C}$ is the induced cell decomposition on X, the cells of $\mathcal{C}$ are of Type I, II, III, and IV (all the vertices of $\mathcal{T}$ are in $\bigcup B_i$).

Claim 1. *$\mathbb{P}(\mathcal{C}) \neq X$ and each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle.*

Proof. This follows just as in the proof of Theorem 5.17.

Claim 2. *There is a trivial induced product region $\mathbb{P}(X)$ for X or we can construct a strictly larger collection of normal disks than $\mathcal{E}$.*

Proof. We have established that each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle. As above, we write each I-bundle component of $\mathbb{P}(\mathcal{C})$ as $K_i \times I, i = 1, \ldots, k$, where k is the number of components of $\mathbb{P}(\mathcal{C})$ and again, we set $K_i^\varepsilon = K_i \times ve, ve = 0, 1$. The subcomplexes K_i^0 and K_i^1 are isomorphic subcomplexes in the induced normal cell structures on the disks in our collection $\mathcal{E}$. Here, we may have K_i^0 and K_i^1 in the same or in distinct disks in $\mathcal{E}$.

We may as well assume that K_i is not simply connected. Recall that if K_i^ε is in the disk E_j in the collection $\mathcal{E}$, then K_i^ε separates E_j. If K_i^ε meets ∂E_j, then every component of the complement of K_i^ε is simply connected; however, if K_i^ε does not meet ∂E_j, then the component of the complement of K_i^ε in E_j, which contains ∂E_j is not simply connected.

As above, we consider the possibilities for $K_i \times [0, 1]$.

If K_i^0 and K_i^1 are in distinct disks of our collection $\mathcal{E}$, say $K_i^0 \subset E_j$ and $K_i^1 \subset E_{j'}, j \neq j'$. Then we let D_i^0 be the union of K_i^0 along with the components of the complement of K_i^0 in E_j, which do not meet ∂E_j; similarly, we let D_i^1 be the union of K_i^1 along with the components of the complement of K_i^1 in $E_{j'}$, which do not meet $\partial E_{j'}$. Then $D_i^\varepsilon, \varepsilon = 0, 1$ is simply connected. So, as in the proof of Theorem 5.17, it follows that there is a simply connected planar complex D_i and an embedding of $D_i \times [0, 1]$ into X so that $D_i \times 0 = D_i^0, D_i \times 1 = D_i^1, K_i \times [0, 1] \subset D_i \times [0, 1]$, and the frontier of $D_i \times [0, 1]$ is contained in the frontier of $K_i \times [0, 1]$.

If both K_i^0 and K_i^1 are in the same disk E_j and K_i^0 (and hence, K_i^1) does not meet ∂E_j, then there are two possibilities.

One possibility is that K_i^0 is in the same component of the complement of K_i^1 as ∂E_j and K_i^1 is in the same component of the complement of K_i^0 as ∂E_j. Then we let D_i^0 be the union of K_i^0 along with the components of the complement of K_i^0 in E_j, which do not meet ∂E_j; similarly, we let D_i^1 be the union of K_i^1 along with the components of the complement of K_i^1 in E_j, which do not meet ∂E_j. Then $D_i^\varepsilon, \varepsilon = 0, 1$ is simply connected. It follows, just as above and as in the proof of Theorem 5.17, that there is a simply connected planar complex D_i and an embedding of $D_i \times [0, 1]$ into X so that $D_i \times 0 = D_i^0, D_i \times 1 = D_i^1, K_i \times [0, 1] \subset D_i \times [0, 1]$, and the frontier of $D_i \times [0, 1]$ is contained in the frontier of $K_i \times [0, 1]$.

So, suppose K_i^1 is not in the component of the complement of K_i^0 that meets ∂E_j. Define D_i^0 to be K_i^0 along with all the components of its complement not meeting ∂E_j. Then we have that D_i^0 is simply connected and also we have $K_i^1 \subset D_i^0$. See Figure 33. Let $N_i = N(D_i^0 \cup (K_i \times [0,1]))$ be a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1])$. Since $K_i^1 \subset D_i^0$, the frontier of N_i consists of an annulus, possibly some 2-spheres and a disk F_i, properly embedded in X and having its boundary in E_j. There is a disk F_i' in E_j so that $\partial F_i' = \partial F_i$ and $F_i \cup F_i'$ bounds a 3-cell, say B, in X; actually, we want to think of B as a relative (punctured) 3-cell. Furthermore, B contains N_i and F_i along with $E_k, k \neq j$ form a barrier surface in the component of the complement of F_i not meeting N_i. We shrink F_i in the component of the complement of B not meeting N_i. Since M (and hence, X) is not a 3-cell, our standard arguments give that in this situation we have a *normal* disk E in X and a disk E' in ∂X so that $\partial E' = \partial E$ and $E \cup E'$ bounds a 3-cell containing P. We replace E_j in the collection $\mathcal{E}$ by the disk E and thus construct a strictly larger collection of normal disks than $\mathcal{E}$. We now begin our considerations over again, using this larger collection of normal disks. This can only happen a finite number of times by Theorem 2.3.

The only remaining possibility is that both K_i^0 and K_i^1 are in the same disk E_j and now K_i^0 (and hence, K_i^1) does meet ∂E_j. We can go through the steps in this case just as in the proof of Theorem 5.17 and we find that we can either construct an appropriate product $D_i \times [0,1]$ or we construct a strictly larger collection of normal disk than $\mathcal{E}$. In the latter situation, as before, we go back to the earlier steps and use this new collection of normal disks; again, this phenomenon can happen at most a finite number of times.

Now, having made these observations, we can choose some order for the components of $\mathbb{P}(\mathcal{C})$, say $K_1 \times [0,1], \ldots, K_k \times [0,1]$, and from this, just as in the proof of Theorem 5.5 derive that there is a trivial induced product region $\mathbb{P}(X)$ for X.

This completes the proof of Claim 2.

Claim 3. *There is no cycle* (*complete or relative*) *of cells of Type* II, *which are not in* $\mathbb{P}(X)$.

Proof. In this situation we have four possibilities: a complete cycle about a single edge, a complete cycle about more than one edge, a relative cycle about a single edge and a relative cycle about more than one edge.

Suppose there were a complete cycle about a single edge e as in Figure 24, Part A. Just as in the proof of Theorem 5.17, there would be a disk D meeting e in precisely one point and meeting a non-vertex-linking disk, say E_j, of the collection $\mathcal{E}$ in ∂D. A surgery on E_j at D (see Figure 25), which is the same as adding a 2-handle to E_j along the curve ∂D, gives a *normal* 2-sphere, S_j, and a new *normal* disk, F_j. But here the existence of the disk F_j, which co-bounds a 3-cell with E_j' gives us a strictly larger collection of disks than the collection $\mathcal{E}$. We use this new collection and go back to the beginning.

Now, suppose there is a complete cycle of truncated prisms about more than one edge (see Figure 24(B)), then as in the consideration in the proof of Theorem 5.17, the collection of cells of Type II (truncated prisms), form a solid torus with, possibly, some self identifications in its boundary. As above, we distinguish between the cycle of truncated prisms, which we denote $\hat{\tau}$, and the cycle of truncated prisms minus the bands of trapezoids, which we denote by τ. We have that τ meets the collection $\mathcal{E}$ in either three open annuli, each meeting a meridional disk of τ precisely once, or a single open annulus, meeting a meridional disk of τ three times. See Figure 26.

In the case we have three annuli in τ meeting the collection $\mathcal{E}$, we have just as above that the cycle of truncated prisms $\hat{\tau}$ is in $\mathbb{P}(X)$, the induced product region for X.

Suppose we have just one annulus common to τ and the collection $\mathcal{E}$. Just as in the proof of Theorem 5.17, we have that there is a lens space $L(3,1)$ embedded in M as a connected summand but this is impossible as M is irreducible.

Now, we consider the possibility of a relative cycle of truncated prisms. If there is a relative cycle about a single edge, then we can find a strictly larger collection of normal disk than our collection $\mathcal{E}$. If there is a relative cycle about more than one edge, then the argument in the proof of Theorem 5.17 can be used to show that such a relative cycle of truncated prisms is in the induced product region for X, $\mathbb{P}(X)$.

This completes the proof of Claim 3.

Hence, as above, we have established all the conditions of Theorem 5.16 and so there is a triangulation $\mathcal{T}'$ of X, where the tetrahedra of $\mathcal{T}'$ are in one-one correspondence with the truncated tetrahedra of $\mathcal{C}$ not in $\mathbb{P}(X)$. Since, we had that at least one of the disks in the collection $\mathcal{E}$ is not vertex-linking, then the number of tetrahedra of $\mathcal{T}'$ is smaller than the number of tetrahedra in the original triangulation $\mathcal{T}$. By repeated applications of the above techniques, it follows that we start with $\mathcal{T}$ and through a sequence of such modifications get a 0-efficient triangulation of M. q.e.d.

In the previous subsection our methods could be used to give a connected sum decomposition of any 3-manifold into known factors and factors with 0-efficient triangulations. A similar result in the bounded case for disk sum decompositions is possible but is not too interesting. On the other hand, there are a couple of related and curious questions, which our methods do not answer and for which we do not know the answer. Recall that we have defined inessential 2-spheres and inessential properly embedded disks. We will say a properly embedded disk D in the 3-manifold M ($\partial M \neq \emptyset$) is *trivial* if there is a disk $D' \subset \partial M, D \cap D' = \partial D = \partial D'$ and $D \cup D'$ bounds a 3-cell (D is *parallel into* ∂M). An inessential disk is trivial in an irreducible 3-manifold. Now, does any closed, orientable 3-manifold have a triangulation in which every inessential, normal 2-sphere is vertex-linking? Does any compact, orientable 3-manifold with nonempty boundary have a triangulation in which there are no inessential, normal 2-spheres and every properly embedded, trivial, normal disk is vertex-linking? We know from [5] that there are one-vertex triangulations of handlebodies, which have no normal 2-spheres and every trivial, normal disk is vertex-linking.

6. 0-efficient and minimal triangulations

A triangulation $\mathcal{T}$ of a 3-manifold M is said to be a *minimal* triangulation if for any other triangulation $\mathcal{T}'$ of M, the number of tetrahedra of $\mathcal{T}$ is no greater than the number for $\mathcal{T}'$. Similarly one defines a *minimal* triangulation for a surface in terms of number of triangles.

Recall for a closed surface S, we have $f = 2(\nu - \chi(S))$, where f is the number of triangles and ν the number of vertices in a triangulation of S. Hence, a minimal triangulation occurs when the triangulation has precisely one vertex, except for S^2 and $\mathbb{RP}^2$, which have minimal triangulations with three and two vertices, respectively. There does not seem to be any simple way, such as this, to determine much about a minimal triangulation of a 3-manifold. However, directly from the proof of Theorem 5.5, we have the following theorem.

Theorem 6.1. *A minimal triangulation of a closed, orientable, irreducible* 3*-manifold is* 0*-efficient or the manifold is homeomorphic with either* $\mathbb{RP}^3$ *or* $L(3,1)$*; hence, a minimal triangulation of a closed, orientable, irreducible* 3*-manifold has one-vertex unless the* 3*-manifold is one of* $S^3, \mathbb{RP}^3$ *or* $L(3,1)$.

There are two one-tetrahedron (minimal) triangulations of S^3; one has one vertex and one has two vertices (see Figure 2 (4) and (5)). Both are 0-efficient. There are two distinct minimal triangulations of $\mathbb{RP}^3$, each having precisely two tetrahedra. One of these triangulations has one vertex and the other has two vertices. For the example with two vertices, see Figure 17; and for the one with one vertex, see Figure 29(A). Of course, as observed above, neither is 0-efficient. There are four distinct minimal triangulations of $L(3,1)$, each having two tetrahedra. Two of these triangulations have one vertex (see Figure 16) and two have two vertices (see Figure 17). Of the two triangulations of $L(3,1)$ with one vertex, only one is 0-efficient. Neither of the two tetrahedra, two-vertex triangulations of $L(3,1)$ are 0-efficient (see Proposition 5.1).

In the case of manifolds with boundary, we have the following, again, directly from the proof of Theorem 5.17.

Theorem 6.2. *A minimal triangulation of a compact, orientable, irreducible and* ∂*-irreducible* 3*-manifold with nonempty boundary is* 0*-efficient, or it is the underlying point set of a tetrahedron* (*the* 3*-cell triangulated by a single tetrahedron with no identifications*)*. Hence, a minimal triangulation of such a* 3*-manifold has all vertices in the boundary and just one vertex in each boundary component or is a* 3*-cell and is triangulated by just one tetrahedron and has either three or four vertices, all in the boundary.*

As pointed out in the previous Theorem, there are two (minimal) one-tetrahedron triangulations of the 3-cell. One is the tetrahedron with no identifications, see Figure 2 (1), and is not 0-efficient. The other is a single tetrahedron with two faces identified, see Figure 2 (2); it has three vertices and is 0-efficient.

We do not know if a minimal triangulation of a closed, reducible and orientable 3-manifold needs to be a one-vertex triangulation. However, we can use Euler characteristic to show that a minimal triangulation of a compact, orientable 3-manifold with nonempty boundary, no component of which is a 2-sphere, has precisely one vertex in each boundary component and the number of tetrahedra can be expressed as

$$t = \sum_{j=1}^{b}(3g_j - 2) + e_{\text{int}} - v_{\text{int}},$$

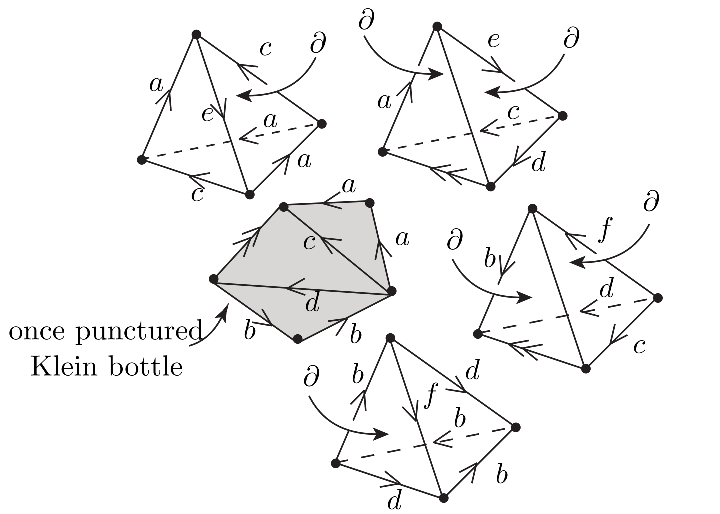
where b is the number of boundary components, g_j is the genus of the j^{th} boundary component, and e_{int} and v_{int} are the number of interior edges and interior vertices of the triangulation, respectively. Since $e_{\text{int}} - v_{\text{int}} \geq 0$, if there are no interior edges, then we would have the number of tetrahedra in a minimal possible triangulation. Of course, if there are no interior edges, then if the manifold is also irreducible, it is a handlebody. There are triangulations of a genus g handlebody, which have just one vertex (of course, in the boundary) and realize this minimum number of tetrahedra, $t = 3g - 2$. See Figure 2 (3) for $g = 1$ and Figure 36 for $g = 2$. A study of "layered triangulations" of handlebodies is given in [5]. The minimal triangulation of $S^2 \times S^1$, also, is one-vertex (see Figure 29).

We complete this section on 0-efficient and minimal triangulations with some observations about the order of edges in these triangulations. We observed above, Corollary 5.4, that a 0-efficient triangulation has no edge of order one and no faces which are cones unless the manifold is S^3.

Proposition 6.3. *A minimal and* 0*-efficient triangulation* $\mathcal{T}$ *of the closed, orientable and irreducible* 3*-manifold* M *has:*

- *no edge of order one unless* $M = S^3$*,*
- *no edge of order two unless* $M = L(3,1)$ *or* $L(4,1)$*, and*
- *no edge of order three unless either* $M = L(5,2)$ *or* $\mathcal{T}$ *contains, as a subcomplex, the two-tetrahedron, geodesic layered triangulation* $\{4,3,1\}$ *of the solid torus.*

Before giving the proof, we recall that, with just few exceptions, a minimal triangulation is 0-efficient; however, we have combined the two notions to reduce the number of exceptions in the statement of the theorem. Also, a $\{4,3,1\}$ geodesic layered triangulation of the solid torus is given as a subcomplex in Figure 37. This is just one in a family of triangulations of the solid torus having a central role in our study of triangulations of 3-manifolds, see [11, 5, 6, 13, 7]. The geodesic, layered triangulation of the lens space $L(6,1)$, shown in Figure 37, is minimal and 0-efficient and has two edges, those labeled



genus 2 handlbody — layered triangulation

Figure 36: A minimal triangulation of the genus 2 handlebody, having $t = 3g - 2 = 4$ tetrahedra and one vertex. Four tetrahedra "layered" on a once-punctured Klein bottle.

"2" and "4", each of order three. There are many examples of 0-efficient triangulations of lens spaces, which have edges of order three; however, while we suspect the geodesic, layered triangulations of lens spaces are minimal, we do not have a proof of this.

Proof. Let $\mathcal{T}$ be the given minimal and 0-efficient triangulation. If there is an edge of order one, then we saw above that $M = S^3$. So, suppose we have an edge, e, of order two. If it is in just one tetrahedron, then the triangulation $\mathcal{T}$ has just one tetrahedron. In this case, we see in Figure 2(6) that we must have $L(4,1)$. So, we may suppose there are two tetrahedra $\widetilde{\Delta}'$ and $\widetilde{\Delta}''$ in our triangulation $\mathcal{T}$, which are identified along two faces from each and the faces have the edge e in common. Let $e' \subset \widetilde{\Delta}'$ and $e'' \subset \widetilde{\Delta}''$ be edges disjoint from e in $\widetilde{\Delta}'$ and $\widetilde{\Delta}''$, respectively, so that $\widetilde{\Delta}' = e * e'$ and $\widetilde{\Delta}'' = e * e''$. See Figure 38. Now, $e' \cup e''$ is a loop in the 1-skeleton of our triangulation. There are three possibilities for identifications of e' and e'' we must consider: e' and e'' are not identified, e' and e'' are identified with "opposite" orientations, and e' and e'' are identified with the "same" orientation. In the last situation, there would be an embedded $\mathbb{R}\mathrm{P}^2$ in the manifold M; but this contradicts the triangulation

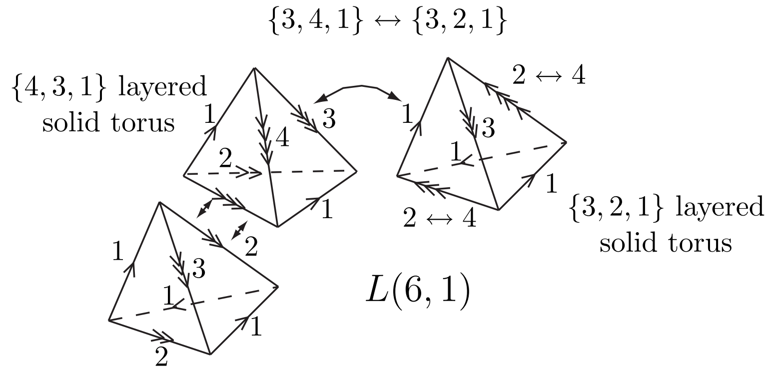


Figure 37: Layered triangulation of $L(6,1)$, which is a minimal layered triangulation, is 0-efficient, has two edges of order three and contains the layered solid torus $\{4,3,1\}$ as a subcomplex.

being 0-efficient. So, we may assume the last possibility does not occur. Hence, if we are to collapse the two tetrahedra $\widetilde{\Delta}'$ and $\widetilde{\Delta}''$, there is no obstruction to identifying the edges e' and e''.

Let σ' and β' denote the two faces of $\widetilde{\Delta}'$ containing $e' = \overline{b'c'}$ and let σ'' and β'' denote the two faces of $\widetilde{\Delta}''$ containing $e'' = \overline{b''c''}$. Again, see Figure 38. In general, we should find that we can collapse the two tetrahedra $\widetilde{\Delta}'$ and $\widetilde{\Delta}''$ identifying the faces σ' with σ'' to get a single face σ and similarly identifying the faces β' and β'' to get a single face β. However, if we could collapse, then the triangulation would not be minimal; so, in a minimal triangulation, if we have an edge, e, of order two, there is an obstruction to such a collapse. The possible obstruction are that we already have the faces σ' and σ'' (or the faces β' and β'') identified in some way or we have a two tetrahedron triangulation.

Suppose σ' is identified with σ'' in some way. The possibilities are $a'b'c' \leftrightarrow a''b''c''$, $a'b'c' \leftrightarrow b''c''a''$ or $a'b'c' \leftrightarrow c''a''b''$. On the other hand, we already have $\overline{a'b'} = \overline{ab} = \overline{a''b''}$ and $\overline{a'c'} = \overline{ac} = \overline{a''c''}$. So, for either of the latter two identifications we have that $M = L(3,1)$; hence, if this is the case and $\mathcal{T}$ is minimal and 0-efficient, we have $\mathcal{T}$ the triangulation given in Figure 16(A). If we have $a'b'c' \leftrightarrow a''b''c''$ and we are not in a two-tetrahedron triangulation, then we have a cone and we can eliminate the face σ, identify $d'b'c'$ with $d''b''c''$ and reduce the number of tetrahedra. Again this is a contradiction. So, the only remaining possibility is that we have a two tetrahedron triangulation with an edge of order two. Since, it is 0-efficient, it is $L(3,1)$. Both $\mathbb{R}\mathrm{P}^3$ and $\mathcal{S}^2 \times S^1$ have minimal two-tetrahedra triangulations with edges of order two (see Figure 17 and 29(B)); however, these are not 0-efficient. This resolves the issue of an edge of order two.

So, suppose we have an edge of order three. If there are three distinct

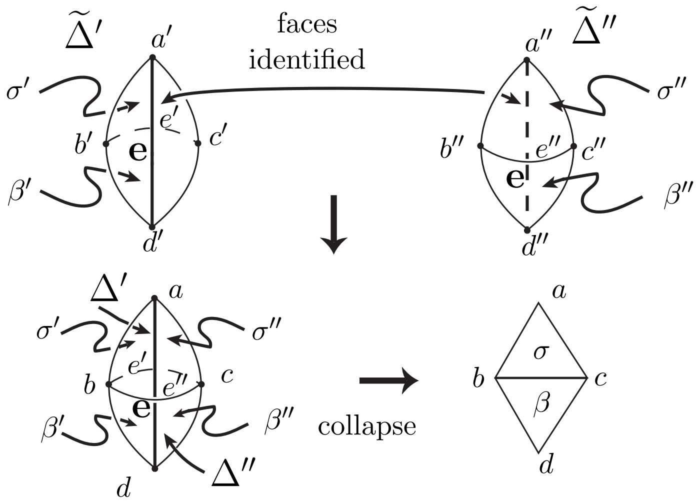


Figure 38: An edge e of order two.

tetrahedra around the edge, then we could make a $3 \to 2$ move and reduce the number of tetrahedra. So, if we assume $\mathcal{T}$ is minimal, we must have that the edge of order three is in just one or in just two tetrahedra.

If we have an edge e of order three in just one tetrahedron, then we must have two faces, having e as a common edge identified. Since the edge e is order three, this is only possible if the triangulation has just one tetrahedron. Hence, we must have $L(5, 2)$; see Figure 2(7). So, we assume we have the edge of order three is in two tetrahedra. So, again, we have a tetrahedron with two faces identified but the edge is not that edge common to both faces. See Figure 2, Parts(2) and (3). However, we can not have an edge of order one; so, we have a one-tetrahedron solid torus, Figure 2(3). Since the edge is of order two in this single tetrahedron solid torus, and we are assuming it is of order three, we have a tetrahedron layered on the faces adjacent to these edges. It follows we have the "layered," two tetrahedron, $\{4, 3, 1\}$ solid torus as a subcomplex.

q.e.d.

7. 0-efficient and minimal ideal triangulations

Suppose M is a compact 3-manifold with boundary, no component of which is a 2-sphere, and $\mathcal{T}$ is an ideal triangulation of the interior of M. Recall that in this work, we do not allow any "regular" vertices in an ideal triangulation; i.e., vertices with vertex-linking surface a 2-sphere. Hence, all vertices are ideal and the index of a vertex is ≥ 1. We say the ideal triangulation $\mathcal{T}$ of $\overset{\circ}{M}$ is a 0-*efficient ideal triangulation* if and only if there are no normal 2-spheres. We say an ideal triangulation of $\overset{\circ}{M}$ is a *minimal ideal triangulation* if and only if for any ideal triangulation $\mathcal{T}'$ of $\overset{\circ}{M}$, $\mathcal{T}$ has no more tetrahedra than $\mathcal{T}'$. Clearly, if M has a 0-efficient ideal triangulation, then M is irreducible.

Our first result in this section is an existence theorem for ideal triangulations. Typically, one incurs ideal triangulations in methods to construct a complete hyperbolic metric on a link complement in the 3-sphere. There are several algorithms starting with a link projection to construct an ideal triangulation of the link complement. The ones we know all give at some point in the construction an ideal triangulation of the link complement as a combinatorial object, just as we have defined. However, beyond ideal triangulations of link complements in the 3-sphere, we have not seen any results on existence of ideal triangulations. Our methods provide useful generalizations as in [11], where we construct ideal triangulations of link complements in 3-manifolds other than S^3, and in the next theorem, where we obtain ideal triangulations of the interior of a compact 3-manifold, allowing genus > 1 boundary components. These methods also allow us to modify any ideal triangulation of these 3-manifolds to a 0-efficient ideal triangulation and show that a minimal ideal triangulation is 0-efficient.

We will say the 3-manifold M is *anannular* if every properly embedded annulus in M is parallel into ∂M.

Theorem 7.1. *Suppose M is a compact, irreducible, ∂-irreducible, anannular* 3*-manifold. Then $\overset{\circ}{M}$ admits an ideal triangulation.*

Proof. Let $\mathcal{T}$ be a triangulation of the 3-manifold M so that each vertex of $\mathcal{T}$ is in ∂M. Recall by [2], the manifold M has a triangulation with all vertices in the boundary; of course, under our hypothesis and Corollary 5.18, there is a triangulation of M that not only has all vertices in ∂M but has precisely one vertex in each component of ∂M. Let $S_1, \ldots, S_n$ denote the components of ∂M; let N denote a collared neighborhood of ∂M; and let E_i denote the component of the frontier of N that is isotopic to $S_i, i = 1, \ldots, n$. The frontier of N is a barrier surface in the component of its complement not meeting ∂M. Hence, we can shrink each E_i. It may be the case that some of the E_i are normal and thus already stable and the shrinking does not change these surfaces. By shrinking, we arrive at a collection of normal surfaces and possibly some 2-spheres embedded entirely in the interior of tetrahedra. However, since M is irreducible and ∂-irreducible, each E_i is incompressible; so, we have precisely one normal surface for each E_i, which is isotopic to E_i and thus isotopic to the component S_i of ∂M. We may also have some number of normal 2-spheres. We discard all the 2-spheres. We continue to denote the (now) normal surface, which is parallel to the boundary component, S_i by $E_i, 1 \leq i \leq n$; and we will call the product region determined by the isotopy between S_i and E_i, P_i. Having shown existence of the normal surfaces E_i, we want to take a maximal such collection in the following sense. We choose a collection of normal surfaces $E_1, \ldots, E_n$ so that E_i is isotopic to $S_i, 1 \leq i \leq n$, and having the property that if E_i' is a normal surface isotopic to S_i and $E_i' \cap E_j = \emptyset, j \neq i$, then E_i' is normally isotopic into P_i, the product region between E_i and S_i. This is possible by Kneser's Finiteness Theorem, Theorem 2.3.

Let X denote the closure of the complement of $\bigcup_{i=1}^n P_i$. The compact, bounded 3-manifold X is homeomorphic to M and has a nice cell structure $\mathcal{C}$ induced by the triangulation $\mathcal{T}$ and the fact that each vertex of $\mathcal{T}$ is in ∂M (see Figure 10). Now, we want to crush the triangulation $\mathcal{T}$ along the surfaces in the collection $E_1, \dots, E_n$, distinct points for each E_i.

Just as above, we need to show the conditions of Theorem 4.1 are satisfied; however, here we have that after crushing we do not have a manifold at the vertex points; however, we arrive at an ideal triangulation of $\overset{\circ}{X}$, which is homeomorphic to $\overset{\circ}{M}$.

We proceed as in the proof of Theorem 5.5 but, since the boundary components of X have genus ≥ 1, we have some new considerations.

Claim 1. *$\mathbb{P}(\mathcal{C})$ is a product I-bundle and $\mathbb{P}(\mathcal{C}) \neq X$.*

Proof. If all cells in X were of Type III and IV, $\mathbb{P}(\mathcal{C}) = X$, then X is an I-bundle over a closed surface, having nonpositive Euler characteristic (no component of ∂M is a 2-sphere). But then M also would be such an I-bundle and this would contradict that M is anannular. Similarly, if $\mathbb{P}(\mathcal{C})$ is not a product I-bundle, then there is a Möbius band properly embedded in X with its boundary in some E_i. If this were the case, then the frontier of a small regular neighborhood of this Möbius band would give an essential annulus in M.

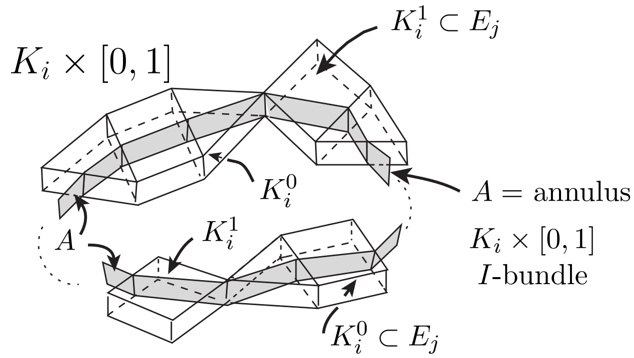
This completes the proof of Claim 1.

Claim 2. *There is a trivial induced product region $\mathbb{P}(X)$ for X.*

Proof. We will show that any obstruction to this claim leads either to a contradiction that M is anannular or to a contradiction of the collection $E_1, \dots, E_n$ being maximal.

We use the same notation as in earlier sections. From Claim 1, we have that each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle, which we write $K_i \times [0,1]$. Just as above, we have that $K_i^\varepsilon = K_i \times \varepsilon, \varepsilon = 0, 1$, has an induced cell-decomposition from $K_i^0 \subset E_j$ and $K_i^1 \subset E_{j'}$ with K_i^0 isomorphic to K_i^1.

First, we observe that $K_i^\varepsilon, \varepsilon = 0, 1$ are contained in simply connected regions of the surfaces E_j and $E_{j'}$. For otherwise, we would have, say, K_i^0 meeting E_j in a region which is not simply connected. It follows that there is a properly embedded annulus A in $K_i \times [0,1]$, A has one of its boundaries in K_i^0 and the other in K_i^1 and the boundary of A in K_i^0 is not contractible in E_j. See Figure 39. Since M is ∂-irreducible it follows that the boundary of the annulus A in K_i^1 is also not contractible in $E_{j'}$. But since M is anannular, the only possibility is that $j' = j$ and the annulus A is parallel into E_j. Let T denote the solid torus determined by this region of parallelism. Then a small regular neighborhood of $P_j \cup T$ has frontier, E_j', isotopic to E_j; furthermore, E_j', along with all the $E_k, k \neq j$, form a barrier surface in the component of the complement of E'_j not meeting $P_j \cup T$. We can shrink E'_j but then we must get a normal surface isotopic to E_j but not normally isotopic to it. This is a contradiction to the collection $E_1, \ldots, E_n$ being maximal. Thus, we have that any component of $\mathbb{P}(\mathcal{C})$ has a product structure $K_i \times [0,1]$ and its "top," K_i^1, and "bottom," K_i^0, meet the surfaces E_j in a union of triangles and quadrilaterals in the induced cell structure on E_j forming disjoint isomorphic subcomplexes each of which are contained in simply connected regions of the surfaces $E_1, \ldots, E_n$.



Cells of Type III and IV

Figure 39: A 0-weight annulus, A, in cells of Type III and IV.

This seems good; however, again we have a complication here that did not occur in the proof of Theorem 5.5; however, we did see a similar situation in the proof of Theorem 5.17. In a 2-sphere, if we have two disjoint, connected subcomplexes, then we can find disjoint simply connected subcomplexes containing them. In a surface of higher genus, even if both subcomplexes are connected and contained in simply connected subcomplexes, this may not be the case; one subcomplex may be contained in the simply connected subcomplex determined by the other.

So, suppose K_i is not simply connected, $K_i^\varepsilon \subset E_j, \varepsilon = 0,1$. We have shown that $K_i^\varepsilon, \varepsilon = 0,1$ are contained in simply connected regions of E_j. Thus there are simply connected regions of the complement of K_i^ε in S'_j so that we can add them to K_i^ε and get simply connected subcomplexes $D_i^\varepsilon \supset K_i^\varepsilon, \varepsilon = 0,1$. We claim that by having chosen the collection $E_1, \ldots, E_n$ maximal, we must have that K_i^0 is not in D_i^1 and K_i^1 is not in D_i^0. For suppose K_i^1, say, is in D_i^0. Just as in the proof of Theorem 5.17, let $N_i = N(D_i^0 \cup (K_i \times [0,1]))$ be a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1])$. Since $K_i^1 \subset D_i^0$, the frontier of N_i consists of an annulus, possibly some 2-spheres and a disk F_i, properly embedded in X and having its boundary in E_j. Since M is ∂-irreducible (E_j

is incompressible), there is a disk F_i' in E_j so that $\partial F_i' = \partial F_i$ and $F_i \cup F_i'$ bounds a 3-cell, say B, in X. Furthermore, B contains N_i. The boundary of a small regular neighborhood of $P_j \cup B$, say E_j', along with each of the $E_k, k \neq j$ form a barrier surface. Now, if we shrink E_j', we arrive at a contradiction to the collection $E_1, \ldots, E_n$ being maximal just as in the proof of Theorem 5.17. Again, see Figure 32.

To meet the conditions of Theorem 4.1, we now need to show that there are product regions $\mathbb{P}(X)$ and they are trivial. We have that each component of $\mathbb{P}(\mathcal{C})$ is a product I-bundle, $K_i \times [0,1]$, and for each $i, 1 \leq i \leq k$, where k is the number of components of $\mathbb{P}(\mathcal{C})$. Furthermore, we have shown, there are simply connected subcomplexes $D_i^\varepsilon, \varepsilon = o, 1$ in the collection $E_1, \ldots, E_n$ so that $K_i^\varepsilon \subset D_i^\varepsilon$ and $D_i^0 \cap D_i^1 = \emptyset$. So, just as so many times above, to show that the product regions $\mathbb{P}(X)$ exist and are trivial, we need to show that for $N_i = N(D_i^0 \cup (K_i \times [0,1]) \cup D_i^1)$ a small regular neighborhood of $D_i^0 \cup (K_i \times [0,1]) \cup D_i^1$, then N_i has 3-cell plugs. However, this is straight forward as M is irreducible and each of the surfaces E_j is parallel into a component of ∂M.

So, by choosing some order, say $K_1 \times [0,1], \ldots, K_k \times [0,1]$ for the components of $\mathbb{P}(\mathcal{C})$, we can construct the trivial product regions $D_j \times [0,1], 1 \leq j \leq k' \leq k$ so that $\bigcup (K_i \times [0,1]) \subset \bigcup (D_j \times [0,1])$ and the frontier of $\bigcup (D_j \times [0,1])$ is contained in the frontier of $\bigcup (K_i \times [0,1])$. We let $\mathbb{P}(X)$ be the components of $\bigcup (D_j \times [0,1])$.

This completes the proof of Claim 2.

Claim 3. *There is no cycle of truncated prisms in X, which is not in $\mathbb{P}(X)$.*

Proof. Just as above, there is the possibility of two types of cycles of truncated prisms: one is a cycle about an edge e of $\mathcal{T}$ (see Figure 24(A)) and the other cycles about more than one edge of $\mathcal{T}$ (see Figure 24(B)).

If there is a complete cycle about an edge e as in Figure 24(A), then one of the surfaces E_j contains a thin tube of elementary quads about the edge e. In this case, there is a properly embedded disk D in X meeting e in precisely one point and meeting E_j in ∂D. A surgery on E_j at D gives two *normal* surfaces. One is a normal 2-sphere, since ∂M in ∂-irreducible (E_j is incompressible), and the other is isotopic with E_j. However, this contradicts the choice of the collection $\mathcal{E} = \{E_1, \ldots, E_n\}$ being maximal. See Figure 25. It follows that, having chosen $\mathcal{E}$ maximal, there is no complete cycle of truncated prisms in the induced cell structure on X about a single edge.

If there is a complete cycle about more than one edge (see Figure 24(B)), then, as above, the collection of cells of Type II (truncated prisms), form a solid torus with, possibly, some self identifications in its boundary. Again, because of the possible singularities, we distinguish between the cycle of truncated prisms, which we denote $\hat{\tau}$, and the cycle of truncated prisms minus the bands of trapezoids, which we denote by τ. We have that τ meets the surfaces in the collection $\mathcal{E}$ either in three open annuli, each meeting a meridional disk of τ

precisely once, or a single open annulus, meeting a meridional disk of τ three times. See Figure 26.

In the case we have three annuli common to τ and the surfaces in $\mathcal{E}$, we conclude as in the argument for Theorem 5.17 that the cycle of truncated prisms must be in the induced product region $\mathbb{P}(X)$.

So, consider the case when we have just one annulus, say A_1, in $\tau \cap E_j$. Then A_1 meets each hexagonal face in the chain of truncated prisms, $\hat{\tau}$, three times. If the annulus A_1 is trivial in E_j (i.e., the core curve in A_1 is contractible in E_j), then there would be an $L(3,1)$ as a connected summand of M. But this is impossible as M is irreducible and $\partial M \neq \emptyset$. Thus, we must have that the annulus A_1 is not trivial in E_j. Again, by possibly shrinking the torus $\hat{\tau}$, we have an embedded torus, $\overline{\tau}$, $\overline{\tau}$ meets E_j in an annulus $A \subset \partial\overline{\tau}$ (A is just the, possibly singular, annulus A_1 slightly shrunk and is embedded), which meets the meridional disk of $\overline{\tau}$ exactly three times. Let A' denote the closure of the complement of A in $\partial\overline{\tau}$ (A' is the band of trapezoids, A_1', in $\hat{\tau}$ pulled slightly into the truncated prisms and is embedded). Then since M can not be a solid torus, we have that A' is an essential annulus in X, which contradicts that M is annular. This completes the proof of Claim 3.

Thus we have shown that the conditions in the hypothesis of Theorem 4.1 are satisfied and $\mathcal{T}^*$ is an ideal triangulation of $\overset{\circ}{X}$. The ideal triangulation $\mathcal{T}^*$ has a distinct ideal vertex for each component S_i of ∂M; its index is the genus of S_i. Also, notice that every tetrahedron of $\mathcal{T}$ which meets a P_i in a truncated tetrahedron or meets an E_i in a quadrilateral piece, is crushed and does not occur in the ideal triangulation.

This completes the proof of the Theorem. q.e.d.

The next theorem shows that under the same hypothesis, we can get 0-efficient ideal triangulations for the interiors of these manifolds.

Theorem 7.2. *Suppose M is a compact, irreducible, ∂-irreducible, anannular 3-manifold. Then any ideal triangulation of $\overset{\circ}{M}$ can be modified to a 0-efficient ideal triangulation of $\overset{\circ}{M}$.*

Proof. Assume $\mathcal{T}$ is an ideal triangulation of $\overset{\circ}{M}$. If $\mathcal{T}$ is 0-efficient, there is nothing to prove; so, we assume $\mathcal{T}$ is not 0-efficient. Hence, there is a normal 2-sphere Σ embedded in $\overset{\circ}{M}$. Now, for some normal vertex-linking surface S we have an arc Λ in the 1-skeleton of $\mathcal{T}$, which meets Σ in a single point, meets S in a single point and does not meet any of the other vertex-linking surfaces. Since M is irreducible, Σ bounds a 3-cell B in M.

Let N be a small regular neighborhood of $S \cup \Lambda \cup B$. Then one component of the boundary of N is a copy of S but the other is a surface S' isotopic to S and, along with other vertex-linking surfaces, a barrier surface in the component of its complement not meeting $S \cup \Lambda \cup B$. We can shrink S'. Again, as above, since M is irreducible and ∂-irreducible, we arrive at a normal surface S'', which is isotopic with S' and hence, isotopic with S, and possibly some normal 2-spheres along with some 0-weight 2-spheres embedded entirely in the interior of tetrahedra. We discard any such 2-spheres. Since Σ must contain a quadrilateral and M is not an I-bundle (M is anannular), the normal surface S'' is not vertex-linking. It follows that S'' contains a quadrilateral. The surface S'' is isotopic with the vertex-linking surface S and so S'' bounds a product $P = S \times [0,1)$ containing S and with $S'' = S \times 0$.

Using the same techniques as in the proof of the previous theorem, we crush the ideal triangulation along the normal surface S''. Actually, we can think of this as a crushing along the normal surface S'' along with all the vertex-linking surfaces distinct from S. We arrive at a new ideal triangulation $\mathcal{T}'$ of the interior of a manifold M' homeomorphic with M. However, we have that the triangulation $\mathcal{T}'$ has strictly fewer tetrahedra than the triangulation $\mathcal{T}$. It follows that after possibly some further, but finite number of steps, we have an ideal triangulation having no normal 2-spheres and so is 0-efficient. q.e.d.

Corollary 7.3. *Suppose M is a compact, irreducible, ∂-irreducible, anannular 3-manifold. Then a minimal ideal triangulation of $\mathring{M}$ is 0-efficient.*

Proof. If an ideal triangulation of $\mathring{M}$ is not 0-efficient, then by the proof of the previous theorem we can find another ideal triangulation having strictly fewer tetrahedra. However, if our ideal triangulation is minimal, this is impossible. It follows that a minimal ideal triangulation must be 0-efficient. q.e.d.

We observed above that there are several known algorithms for constructing ideal triangulations of the complements of links in the 3-sphere. We also use different techniques in [5] and get results similar to Theorem 7.1 in more general circumstances. The methods used here lead to an algorithm for constructing ideal triangulations of irreducible, ∂-irreducible, anannular 3-manifolds. Furthermore, there is an algorithm to decide if an ideal triangulation is or is not 0-efficient and, if it is not 0-efficient, the algorithm will construct a normal 2-sphere. It follows there is an algorithm to construct 0-efficient ideal triangulations for irreducible, ∂-irreducible and anannular 3-manifolds. Finally, as an aside, we note it can be decided if a 3-manifold is irreducible, or if it is ∂-irreducible, or if it is anannular, see [4, 19, 14, 8, 18, 22].

We end this section with the following observation about the ideal triangulations we have constructed.

Theorem 7.4. *Suppose M is a compact, irreducible, ∂-irreducible, anannular 3-manifold. Then $\mathring{M}$ has an ideal triangulation which is 0-efficient and the only closed normal surface in a collared neighborhood of a vertex is vertex-linking.*

Proof. We know such a 3-manifold M has an ideal triangulation, say $\mathcal{T}$ and we may as well assume this triangulation is 0-efficient.

Now, suppose S is a vertex-linking surface and in a collared neighborhood of the vertex, v_S at S, there is a closed normal surface F. Then there is a collared neighborhood N of v_S so that $F \subset \overset{\circ}{N}$ and the boundary S' of N is isotopic to S; furthermore, S' is a barrier surface in the component of its complement not meeting N. Now, we shrink S'. If F is not a vertex-linking surface, then we arrive at a normal surface S'' which is isotopic to S, is not vertex-linking (contains a quadrilateral) and bounds a product $P = S \times [0,1), S'' = S \times \{0\}$ and $F \subset P$. We wish to crush the triangulation along the surface S''. By following the steps, exactly as above, we are able to show that the conditions of Theorem 4.1 are satisfied and we arrive at a new ideal triangulation $\mathcal{T}'$ of the interior of a 3-manifold M' which is homeomorphic with M. Furthermore, since S'' contains a quadrilateral, the ideal triangulation $\mathcal{T}'$ has strictly fewer tetrahedra than $\mathcal{T}$.

Now, it might be the case we have introduced some normal 2-spheres. But if we do then by using the same methods, we can get still another ideal triangulation with even fewer tetrahedra. Since there were only a finite number of tetrahedra in $\mathcal{T}$ to start with, the process must terminate in the desired triangulation. q.e.d.

8. 0-efficient triangulations and irreducible knots

In [2] it is shown that for any closed, orientable 3-manifold M there is a knot K embedded in M so that the complement of K, $M \setminus K$, is irreducible. We shall say a knot in a closed 3-manifold M is an *irreducible knot* if its complement in M is an irreducible 3-manifold. Theorem 5.14 provides a constructive method for finding such knots in irreducible 3-manifolds.

Theorem 8.1. *Given the closed, irreducible 3-manifold M, $M \neq \mathbb{R}\mathrm{P}^3$, a one-vertex triangulation $\mathcal{T}$ of M may be constructed so that every edge of $\mathcal{T}$ (a knot in M) is an irreducible knot.*

Proof. By Theorem 5.14, given any triangulation of M, we can modify that triangulation to a 0-efficient triangulation or we can show the manifold is $S^3, \mathbb{R}\mathrm{P}^3$ or $L(3,1)$. For S^3 and $L(3,1)$, we also have one-vertex, 0-efficient triangulations. See Figures 2 (5) and Figure 16 A, respectively. Hence, except for $\mathbb{R}\mathrm{P}^3$ we can construct a 0-efficient triangulation, with just one vertex.

Now, if $\mathcal{T}$ is a one-vertex triangulation, then any edge, e, in $\mathcal{T}$ is a knot in M. Furthermore, if $\mathcal{T}$ is also 0-efficient, then for any edge e, $M \setminus \{e\}$ is irreducible. For otherwise, there would be an essential 2-sphere in $M \setminus \{e\}$; however, the boundary of a small neighborhood of e acts as a barrier surface and so such a 2-sphere would lead to a normal 2-sphere in M, rel $\mathcal{T}$, which is not vertex-linking (it misses the edge e). This contradicts $\mathcal{T}$ being 0-efficient. q.e.d.

We do not know if $\mathbb{R}\mathrm{P}^3$ has a one vertex triangulation in which every edge is an irreducible knot; however, in the two-tetrahedra, one-vertex triangulation of $\mathbb{R}\mathrm{P}^3$ in Figure 29, there are three edges, two are irreducible knots and the third is not irreducible; it misses an embedded $\mathbb{R}\mathrm{P}^2$. On the other hand, this third edge bounds a disk, is a trivial knot and so, could not be irreducible in any manifold except S^3. Anyhow, for all irreducible 3-manifolds, there is an algorithm to construct irreducible knots; our algorithm gives them as edges in nice triangulations. This is somewhat a basic theme of these methods and triangulations; namely, conditions on the triangulations give edges of the triangulations, which provide special properties for the associated knot complements.

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