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1-(2-) Prime Ideals in Semirings

Pandarinathan Nandakumar
Department of Mathematics, Perunthalaivar Kamarajar Institute of Engineering and Technology, (Government of Puducherry Institution), Nedungadu, Karaikal - 609 603 (U. T. of Puducherry), India
e-mail : drpnandakumar@gmail.com

Abstract. In this paper, we introduce the concepts of 1-prime ideals and 2-prime ideals in semirings. We have also introduced m_1-system and m_2-system in semiring. We have shown that if Q is an ideal in the semiring R and if M is an m_2-system of R such that $\overline{Q} \bigcap M = \emptyset$ then there exists as 2-prime ideal P of R such that $Q \subseteq P$ with $P \bigcap M = \emptyset$.

1. Introduction

A semiring is a non-empty set R equipped with two binary opertions, called addition, $+$, and multiplication (denoted by juxtaposition), such that R is multiplicatively a semigroup and additively a commutative semigroup and that the multiplication is distributed across the addition both from the left and from the right. An element denoted by 0 is called the zero of R if $a + 0 = a$ and $0a = a0 = 0$ for all $a \in R$. A non-empty subset of a semiring R is called an ideal of R iff $a+b \in I$, $ra \in I$, $ar \in I$ hold for all $a, b \in I$ and for all $r \in R$. The notions of left, right and two-sided ideals, as well as sums and products of such ideals are defined as usual. The word ideal will always mean a two-sided ideal. An ideal I of R is called a k-ideal if $a, a + b \in I$ implies $b \in I$ for any elements of $a, b \in R$. If A is an ideal of a semiring R then $\overline{A} = \{a \in R/a + x \in A$, for some $x \in A\}$ is called a k-closure of A. It can be easily verified that $\overline{A}$ is a k-ideal (see [6]). If $A \subseteq R$, then the ideal (k-ideal) of R generated by A will be denoted $< A >$ $(< A >_k)$. If $A = a$, we write $< a >$ instead of $< \{a\} >$ for convenience. In this paper we introduce the concepts of $1-(2-)$ prime ideals as well as $1-(2-)$ semiprime ideals. If R is semiring and P is an ideal of R then P is $0-(2-)$ prime ideal if A and B are ideals (k-ideals) of R such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. If R is a semiring and P is an ideal of R then P is 1-prime ideal if A is a k-ideal of R and B is an ideal of R such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. If R is a semiring and Q is an ideal of R,then Q is $0-(2-)$ semiprime ideal if A is an ideal (k-ideal) of R such that $A^2 \subseteq Q$ implies $A \subseteq Q$. If R is a semiring and Q is an ideal of R then Q is 1-semiprime if A is an ideal of R such that $\overline{A}A \subseteq Q$ implies $A \subseteq Q$. A semiring R is called fully idempotent if $I^2 = I$ for every ideal I of R.

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If $A, B \subseteq R$, then we define $(A : B)_l = \{r \in R/rB \subseteq A\}$, $(A : B)_r = \{r \in R/Br \subseteq A\}$, $(A : B) = \{r \in R/rB \subseteq A$ and $Br \subseteq A\}$, As easily seen, 0-prime ideal of R is 2-prime ideal of R, but not conversely. Clearly 0-prime $\Longrightarrow$ 1-prime $\Longrightarrow$ 2-prime, 0-semiprime $\Longrightarrow$ 1-semiprime $\Longrightarrow$ 2-semiprime. Next we introduce the concepts of m_1-system, m_2-system, n_1-system and n_2-system. We have shown that if Q is an ideal of R and if M is an m_2-system of R such that $\overline{Q} \cap M = \emptyset$, then there exists a 2- prime ideal P of R such that $Q \subseteq P$ with $P \cap M = \emptyset$. Throughout this paper R stands for semiring.

Definition 1.1. A semiring R is an ordered triple $R = (R, +, .)$ such that (a) $< R, + >$ is a commutative monoid with identity denoted 0_R or simply 0, (b) $< R, . >$ is a semigroup, (c) For every $r, s, t \in R$, $r(s + t) = rs + rt$ and $(s + t)r = sr + tr$, (d) For every $r \in R, r0 = 0r = 0$.

Definition 1.2. Following Alarcon and Polkowska [2], we have the following definition for B(n, i) semirings.

Let $n \geq 2 \in N$ and $0 \leq i < n$ and $m = n - i$. Let $B(n, i)$ be the following semirings :

$B(n, i) = \{0, 1, 2, ..., n - 1\}$ and the operations in $B(n, i)$ are:

$$x +_{B(n,i)} y = \begin{cases} x + y & \text{if } x + y \leq n - 1 \\ l & \text{if } x + y \geq n \\ with & l = \text{(x +y) mod m and } i \leq l \leq n - 1 \end{cases}$$

$$x \cdot_{B(n,i)} y = \begin{cases} xy & \text{if } xy \leq n - 1 \\ l & \text{if } xy \geq n \\ with & l = \text{(xy) mod m and } i \leq l \leq n - 1 \end{cases}$$

Definition 1.3. If R is a semiring and P is an ideal of R, then P is $0-(2-)$ prime if A and B are ideals (k-ideals) of R such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. If R is a semiring and P is an ideal of R then P is 1-prime if A is a k-ideal of R and B is an ideal of R such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Remark 1.4. As in rings, if P is an ideal in a semiring R, then P is a 0-prime ideal iff $a, b \in R$ such that $aRb \subseteq P$ then $a \in P$ or $b \in P$.

As in rings, if Q is an ideal in a semiring R, then Q is a 0-semiprime ideal iff $a \in R$ such that $aRa \subseteq Q$ then $a \in Q$.

Lemma 1.5. *If A and B are left ideals of R then $(A : B)_l$ is an ideal.*

Lemma 1.6. *If A is a left k-ideal of R and B is a left ideal then $(A : B)_l$ is a* *k-ideal.*

Lemma 1.7. *If A and B are right ideals of R then $(A : B)_r$ is an ideal.*

Lemma 1.8. *If A is a right k-ideal of R and B is a right ideal then $(A : B)_r$ is a k-ideal.*

Lemma 1.9. *If P is a 0-prime ideal of R then P is a 2-prime ideal (1-prime ideal) of R.*

But the converse need not be true as the folloiwng example shows.

Example 1.10. Consider the semiring B(4, 3) = {0, 1, 2, 3}, where + and . are defined as follows.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	3
2	2	3	3	3
3	3	3	3	3

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

Since the ideals are {0}, {0, 3}, {0, 2, 3},{0, 1, 2, 3} and k-ideals are {0} and {0, 1, 2, 3} the ideal {0, 3} is 2-prime but not 0-prime.

Theorem 1.11. *If P is a k-ideal of R then P is a 0-prime ideal if and only if P is 2-prime ideal.*

Proof. Let us assume that P is 2-prime and P is a k-ideal of R. Let us assume that A and B are ideals of R such that $AB \subseteq P$. If $AB \subseteq P$, then $A \subseteq (P : B)_l$. By Lemma 1.6, $(P : B)_l$ is a k-ideal of R. Therefore $< A >_k \subseteq (P : B)_l$. It follows that $< A >_k B \subseteq P$. Hence $B \subseteq (P :< A >_k)_r$. By Lemma 1.8, $(P :< A >_k)_r$ is a k-ideal of R. Therefore $< B >_k \subseteq (P :< A >_k)_r$. It follows that $< A >_k < B >_k \subseteq P$ Since P is 2-prime, we have $< A >_k \subseteq P$ or $< B >_k \subseteq P$. Hence $A \subseteq P$ or $B \subseteq P$. □

Definition 1.12. $M \subseteq R$ is called an m_0-system if for every $a, b \in M$ there exists $x \in R$ such that $axb \in M$. $M \subseteq R$ is called an m_1-system if for every $a, b \in M$ there exists $a_1 \in < a >_k$ and there exists $b_1 \in < b >$ such that $a_1 b_1 \in M$. $M \subseteq R$ is called an m_2 system if for every $a, b \in M$ there exists $a_1 \in < a >_k$ and there exists $b_1 \in < b >_k$ such that $a_1 b_1 \in M$.

Lemma 1.13. *Every m_0 system is an m_2 (m_1-system). But the converse need not* *be true as the following example shows.*

Example 1.14. Consider the semiring B(4, 3) in Example 1.10. Clearly $M = \{1,2\}$ is an m_2 system, but not an m_0-system.

Lemma 1.15. *If Pis an ideal of R, P is a 2-prime ideal (1-prime ideal, 0-prime ideal) of R iff $R \setminus P$ is an m_2-system (m_1 system, m_0-system) of R.*

Theorem 1.16. *Let Q be an ideal of R, and let M be an m_2-system (m_1-system) of R such that $\overline{Q} \bigcap M = \emptyset$. Then there exists a 2-prime ideal (1-prime ideal) P of R such that $Q \subseteq P$ with $P \cap M = \emptyset$.*

Proof. Let Q be an ideal of R and let M be an m_2-system of R such that $\overline{Q} \bigcap M = \emptyset$. Now we consider the set $\mathcal{M}= \{I/$(i) I is an ideal of R such that $Q \subseteq I$, (ii) $\overline{I} \bigcap M = \emptyset\}$. Clearly Q is in M. Let $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq ...$ be any chain of ideals of $\mathcal{M}$. Let $A = \bigcup Q_i$. Clearly A is an ideal of R. We claim that $\overline{A} \bigcap M = \emptyset$. Suppose not, let $a \in \overline{A} \bigcap M$ implies $a \in \overline{A}$ and $a \in M$. Hence $a + x \in A$ for some $x \in A$. Since $a + x \in A = \bigcup Q_i$ implies $a + x \in Q_i$ for some i, since $x \in A = \cup Q_i$ implies $x \in Q_k$ for some k. Without loss of generality let us assume that $k < i$. Then $a + x, x \in Q_i$ implies $a \in \overline{Q_i}$. Therefore $a \in \overline{Q_i} \bigcap M$ which is a contradiction, since $\overline{Q_i} \bigcap M = \emptyset$. Thus $\overline{A} \bigcap M = \emptyset$. Then by Zorn's Lemma, there exists an ideal P of R that is maximal with respect to above properties. Let A be a k-ideal of R such that $A \nsubseteq P$. We calim that $(\overline{P} : A)_r = (\overline{P} : A)_l = P$.Since P is an ideal we have $AP \subseteq P \subseteq \overline{P}$ and so $P \subseteq (\overline{P} : A)_r$. Similarly $P \subseteq (\overline{P} : A)_l$. By Lemma 1.8, $(\overline{P} : A)_r$ is a k-ideal and by Lemma 1.6 $(\overline{P} : A)_l$ is a k -ideal. Thus $(\overline{P} : A)_l$ and $(\overline{P} : A)_r$ are k-ideals of R containing P. By maximality of P either $P = (\overline{P} : A)_l$ or $\overline{(\overline{P} : A)_l} \bigcap M = (\overline{P} : A)_l \bigcap M \neq \emptyset$ as $(\overline{P} : A)_l$ is a k-ideal. Similarly $P = (\overline{P} : A)_r$ or $\overline{(\overline{P} : A)_r} \bigcap M = (\overline{P} : A)_r \bigcap M \neq \emptyset$ as $(\overline{P} : A)_r$ is a k-ideal. Suppose that $(\overline{P} : A)_r \bigcap M \neq \emptyset$. Let $x \in (\overline{P} : A)_r \bigcap M$. We consider two separate cases.

Case 1: $A \bigcap M \neq \emptyset$. Let $a \in A \cap M$. Since $x \in (\overline{P} : A)_r$ we have $< x >_k \subseteq (\overline{P} : A)_r$,since $(\overline{P} : A)_r$ is a k-ideal. Hence $A < x >_k \subseteq \overline{P}$. Since $a, x \in M$ and M is an m_2-system of R there exists $a_1 \in < a >_k$ and there exists $x_1 \in < x >_k$ such that $a_1 x_1 \in M$. Since $A < x >_k \subseteq \overline{P}$ implies $a_1 x_1 \in \overline{P}$.Therefore $a_1 x_1 \in \overline{P} \bigcap M$, which is impossible since $\overline{P} \bigcap M = \emptyset$. Thus $P = (\overline{P} : A)_r$ in this case. Similarly if $y \in (\overline{P} : A)_l \bigcap M$ then it follows that $P = (\overline{P} : A)_l$. Thus if $A \bigcap M \neq \emptyset$ then $(\overline{P} : A)_r = (\overline{P} : A)_l = P$.

Case 2: $A \bigcap M = \emptyset$. Again we have $A < x >_k \subseteq \overline{P}$. This implies that $A \subseteq (\overline{P} :< x >_k)_l = P$ by case 1. This contradicts our assumption that $A \nsubseteq P$. Thus $(\overline{P} : A)_l = P$ in this case. Similarly $(\overline{P} : A)_r = P$. Finally we show that P is 2-prime. Let A and B are k-ideals of R such that $AB \subseteq P$ and $A \nsubseteq P$. Clearly $P \subseteq \overline{P}$. Hence $AB \subseteq \overline{P}$. It follows that $B \subseteq (\overline{P} : A)_r = P$. Therefore $B \subseteq P$. □

Theorem 1.17. *Let Q be an ideal of R and let M be an m_0-system of R such that $Q \bigcap M = \emptyset$. then there exists a 0-prime ideal P of R such that $Q \subseteq P$ with $P \bigcap M = \emptyset$.*

The proof is similar to Theorem 1.16.

Definition 1.18. If R is a semiring and Q is an ideal of R then Q is 0-(2-) semiprime ideal if A is an ideal (k-ideal) of R such that $A^2 \subseteq Q$ implies $A \subseteq Q$. If R is semiring and Q is an ideal of R then Q is 1-semirprime if A is an ideal of R such that $\overline{A}A \subseteq Q$ implies $A \subseteq Q$.

Definition 1.19. $N \subseteq R$ is called an n_0-system if for every $a \in N$ there exists $x \in R$ such that $axa \in N$. $N \subseteq R$ is called an n_1-system if for every $a \in N$ there exists $a_1 \in < a >_k$ and there exists $a_2 \in < a >$ such that $a_1a_2 \in N.N \subseteq R$ is called an n_2-system if for every $a \in N$ there exists $a_1, a_2 \in < a >_k$ such that $a_1a_2 \in N$.

Lemma 1.20. *If S is a 0-semiprime ideal of R then S is a 2-semiprime ideal(1-semiprime ideal) of R.*

But the converse need not be true as the following an example shows.

Example 1.21. Consider the semiring in Example 1.10 Clearly $S = \{0, 3\}$ is 2-semiprime ideal but not 0-semiprime ideal. Since if $A = \{0, 2, 3\}$, then A is an ideal and $A^2 = \{0, 3\} \subseteq S$ but $A \nsubseteq S$.

Theorem 1.22. *If S is a k-ideal of R, then S is 0-semiprime ideal iff S is 2-semirpime ideal.*

The proof is similar to Theorem 1.11.

Lemma 1.23. *Let A be an n_1-system and $a \in A$. Then there is some m_1-system M with $a \in M \subseteq A$.*

Proof. Let $a \in A$. Hence there exists $a_1 \in < a >_k$ and $a_2 \in < a >$ such that $a_1a_2 \in A$. Since $a_1a_2 \in A$ there exists $a_1^{'} \in < a_1a_2 >_k$ and $a_2^{'} \in < a_1a_2 >$ such that $a_1^{'}a_2^{'} \in A$. Continuing this process, we get a sequence $\{a, a_1a_2, a_1^{'}a_2^{'}, a_1^{''}a_2^{''}, ...\}$ such that for every positive integer k, $a_1^k a_2^k \in A$ with $< a >_k \supseteq < a_1a_2 >_k \supseteq < a_1^{'}a_2^{'} >_k \supseteq < a_1^{''}a_2^{''} >_k \supseteq$ ... and $< a > \supseteq < a_1a_2 > \supseteq < a_1^{'}a_2^{'} > \supseteq$ .... Take $M = \{a, a_1a_2, a_1^{'}a_2^{'}, a_1^{''}a_2^{''}, ...\}$. We show that M is a desired m_1-system. If $a_1^la_2^l, a_1^ka_2^k \in M$ (w.l.o.g.,let $k \leq l$) then $< a_1^ka_2^k >_k \supseteq < a_1^la_2^l >_k$ and $< a_1^ka_2^k > \supseteq < a_1^la_2^l >$. Now there exists $a_1^{l+1} \in < a_1^la_2^l >_k$ and there exists $a_2^{l+1} \in < a_1^la_2^l > \subseteq < a_1^ka_2^k >$ such that $a_1^{l+1}a_2^{l+1} \in M$. This implies that M is a desired m_1-system. □

Lemma 1.24. *Let A be an n_0-system and $a \in A$. Then there is some m_0-system M with $a \in M \subseteq A$.*

Lemma 1.25. *Let A be an n_2-system and $a \in A$. Then there is some m_2-system M with $a \in M \subseteq A$.*

Definition 1.26. If A is an ideal of R, then we define $\mathcal{B}_0(A) = \bigcap \{P$ is a 0-prime ideal of R and $A \subseteq P\}$. Similarly we define $\mathcal{B}_1(A)$ and $\mathcal{B}_2(A)$.

Theorem 1.27. *Let Q be an ideal of R. (i) $\overline{Q}$ is a 2-semiprime ideal in R iff $\mathcal{B}_2(\overline{Q})$*

$= \overline{Q}$. *(ii)* $\overline{Q}$ *is a* 1*-semiprime ideal in* R *iff* $\mathcal{B}_1(\overline{Q}) = \overline{Q}$. *(iii)* Q *is a* 0*-semiprime ideal in* R *iff* $\mathcal{B}_0(Q) = Q$.

Proof. Suppose $\mathcal{B}_2(\overline{Q}) = \overline{Q}$. Then $\overline{Q}$ is the intersection of the 2-prime ideals of R which contain Q from which it follows easily that $\overline{Q}$ is 2-semiprime. Conversely, let $\overline{Q}$ be 2-semiprime. Clearly $\overline{Q} \subseteq \mathcal{B}_2(\overline{Q})$. Let $a \in R \setminus \overline{Q}$. Since $\overline{Q}$ is 2-semiprime, we have $R \setminus \overline{Q}$ is an n_2-systme of R. By Lemma 1.25 there exists an m_2-system M in$R$ such that $a \in M \subseteq R \setminus \overline{Q}$. By Theorem 1.16 there exists a 2-prime ideal P of R such that $Q \subseteq P$ and $P \bigcap M = \emptyset$. Then $a \notin P$ and so $a \notin \mathcal{B}_2(\overline{Q})$. Thus $\overline{Q} = \mathcal{B}_2(\overline{Q})$ and the proof is complete. □

The next is a direct consequence of the above theorem.

Corollary 1.28([1], Theorem 1)**.** *Let* R *be a semiring. Then the following assertions are equivalent:* 1. *R is fully idempotent.* 2. *Each proper ideal of R is the intersection of prime ideals which contain it.*

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