

## 1/2-Transitive Graphs of Order $3p$

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**Abstract.** A graph  $X$  is called *vertex-transitive*, *edge-transitive*, or *arc-transitive*, if the automorphism group of  $X$  acts transitively on the set of vertices, edges, or arcs of  $X$ , respectively.  $X$  is said to be *1/2-transitive*, if it is vertex-transitive, edge-transitive, but not arc-transitive.

In this paper we determine all 1/2-transitive graphs with  $3p$  vertices, where  $p$  is an odd prime. (See Theorem 3.4.)

**Keywords:** 1/2-transitive graph, metacirculant, factor graph

### 1. Introduction

Let  $X = (V(X), E(X))$  be a graph (that is, no multiple edges or loops). We call an ordered pair of adjacent vertices an *arc* of  $X$ . The set of all arcs associated with a graph  $X$  is denoted by  $A(X)$ . Thus,  $|A(X)| = 2|E(X)|$ . If  $G$  is a subgroup of  $\text{Aut}X$  and  $G$  acts transitively on the set of vertices, edges, or arcs of  $X$ , then  $X$  is said to be  *$G$ -vertex-transitive*,  *$G$ -edge-transitive*, or  *$G$ -arc-transitive*, respectively. The graph  $X$  is said to be *vertex-transitive*, *edge-transitive*, or *arc-transitive*, if it is  $\text{Aut}X$ -vertex-transitive,  $\text{Aut}X$ -edge-transitive, or  $\text{Aut}X$ -arc-transitive, respectively. We call a graph  $X$  *1/2-transitive*, if it is vertex-transitive, edge-transitive, but not arc-transitive.

D. Marušič, L. Nowitz and the first author of this paper studied 1/2-transitive graphs [1] and found several infinite families of such graphs. In [7], R.J. Wang and the second author gave a classification of arc-transitive graphs of order  $3p$ , where  $p$  is a prime. The purpose of this paper is to determine all 1/2-transitive graphs of order  $3p$ .

We use standard terminology and notation for the most part and refer the reader to [5, 6, 8] if necessary. For  $v \in V(X)$ ,  $X_1(v)$  denotes the neighborhood of  $v$  in  $X$ , that is, the set of vertices adjacent to  $v$  in  $X$ . If  $X$  is a graph and  $A$  and  $B$  are two vertex-disjoint subsets of the vertex-set  $V(X)$  of  $X$ , we let  $\langle A \rangle$  and  $\langle A, B \rangle$  denote the subgraph induced on  $A$  and the bipartite subgraph, with bipartition sets  $A$  and  $B$ , induced on  $A \cup B$  by  $X$ , respectively. We remind the reader that two representations of a group  $G$  as transitive permutation groups are said to be *equivalent* if the pointwise stabilizers of one representation are conjugate in  $G$  to the pointwise stabilizers of the other representation.

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Assume that  $G$  acts on  $X$  imprimitively and that  $B_0$  is a nontrivial block of  $G$ . Let  $\overline{X} = \{B_0, B_1, \dots, B_{n-1}\}$  be the complete block system of  $G$  containing  $B_0$ . We define the *factor graph* of  $X$  corresponding to  $\overline{X}$ , which is still denoted by  $\overline{X}$ , by

$$\begin{aligned} V(\overline{X}) &= \overline{X}, \\ E(\overline{X}) &= \{B_i B_j : \text{there exist } v_i \in B_i, v_j \in B_j \text{ such that } v_i v_j \in E(X)\}. \end{aligned} \quad (1)$$

The group  $G$  induces an action on  $\overline{X}$ . Assume that the kernel of this action is  $K$ . Set  $\overline{G} = G/K$ . Then  $\overline{G}$  acts on  $\overline{X}$  faithfully. The proof of the following result is immediate and is omitted.

**Proposition 1.1** *Let  $X$  be a 1/2-transitive graph such that  $\text{Aut}X$  acts imprimitively on  $X$ . Then*

- (1)  $\overline{X}$  is 1/2-transitive or arc-transitive;
- (2) if  $X$  is connected, then so is  $\overline{X}$ ; and
- (3) if  $\langle B_i \rangle$  has an edge, then  $B_i$  is a union of several connected components of  $X$ .

Given two graphs  $X$  and  $Y$  the *wreath product*  $X \wr Y$  is defined as the graph with vertex set  $V(X) \times V(Y)$  such that  $(x, y)(x', y')$  is an edge if and only if either  $xx'$  is an edge of  $X$ , or  $x = x'$  and  $yy'$  is an edge of  $Y$ .

Informally  $X \wr Y$  is obtained by taking  $|V(X)|$  copies of  $Y$ , labelling these copies with the vertices of  $X$ , and, whenever  $xx'$  is an edge of  $X$ , joining each vertex in the copy of  $Y$  labelled  $x$  to each vertex in the copy of  $Y$  labelled  $x'$ . The automorphism group of  $X \wr Y$  contains the *wreath product*  $\text{Aut}Y \text{ wr } \text{Aut}X$  (but may be larger).

The next obvious proposition gives a method of constructing larger 1/2-transitive graphs from smaller ones.

**Proposition 1.2** *If  $\overline{X}$  is a 1/2-transitive graph of order  $n$ , then the wreath product  $\overline{X} \wr mK_1$  of  $\overline{X}$  by  $mK_1$  is a 1/2-transitive graph of order  $nm$ .*

Next we quote two propositions from [1].

**Proposition 1.3** *Every vertex- and edge-transitive Cayley graph on an abelian group is also arc-transitive.*

**Proposition 1.4** *Every vertex- and edge-transitive graph with  $p$  or  $2p$  vertices,  $p$  a prime, is also arc-transitive.*

Finally we quote a result from [7].

**Proposition 1.5** *Let  $X$  be an arc-transitive graph of order  $3p$  and  $A = \text{Aut}X$ . If  $A$  has a block of imprimitivity of length  $p$ , then  $X$  is a Cayley graph on a cyclic group  $Z_{3p}$ .*

## 2. Reductions

In this section we eliminate some possible types of 1/2-transitive graphs of order  $3p$ . There is no 1/2-transitive graph of order less than 27 as observed in [1]. All suborbits of a primitive group of degree  $3p$  are self-paired [7], which implies there are no vertex-primitive 1/2-transitive graphs of order  $3p$ . So we may assume that  $p \geq 11$  in what follows, and we only need consider those 1/2-transitive graphs with an imprimitive automorphism group.

Let  $X$  be a 1/2-transitive graph and  $A = \text{Aut}X$ . We have two cases: (1)  $A$  has a block of length  $p$ , and (2)  $A$  has a block of length 3, but no blocks of length  $p$ . We shall show in the next section that the latter case cannot occur.

Assume that  $\overline{X} = \{B_i \mid i \in Z_3\}$  is a complete block system of  $A$ , and that  $K$  is the kernel of the action of  $A$  on  $\overline{X}$ . Set  $\overline{A} = A/K$ . We also use  $\overline{X}$  to denote the corresponding factor graph. Then  $\overline{X} \cong K_3$ . Since  $X$  is 1/2-transitive,  $X$  is not isomorphic to  $K_{p,p,p}$ .

**Lemma 2.1** *The kernel  $K$  acts faithfully on each block  $B_i$ .*

**Proof:** We use  $K_{B_i}$  to denote the pointwise-stabilizer of  $B_i$  in  $K$ . If  $K$  acts unfaithfully on  $B_i$ , we have  $K_{B_i}^{B_j}$  is nontrivial for some block  $B_j$  adjacent to  $B_i$ . Since  $K_{B_i} \triangleleft K$ ,  $K_{B_i}^{B_j} \triangleleft K^{B_j}$ . Since  $|B_j| = p$ ,  $K^{B_j}$  is primitive which implies  $K_{B_i}^{B_j}$  is transitive. It follows that the induced subgraph  $\langle B_i, B_j \rangle \cong K_{p,p}$ . The edge-transitivity of  $\overline{X}$  implies that  $X \cong K_{p,p,p}$ . The latter is impossible since  $K_{p,p,p}$  is arc-transitive.  $\square$

Using the same method as above we can prove the following more general result, which will be used in the next section.

**Lemma 2.2** *Suppose that  $X$  is a connected 1/2-transitive graph and  $\overline{X} = \{B_1, B_2, \dots, B_n\}$  is a complete block system of  $A = \text{Aut}X$ . If  $K$ , the kernel of the action of  $A$  on  $\overline{X}$ , acts on  $B_i$  unfaithfully and  $K^{B_i}$  is primitive, then  $X$  is isomorphic to the wreath product  $\overline{X} \wr mK_1$  of  $\overline{X}$  with  $mK_1$ , where  $m = |B_i|$ .*

By virtue of Lemma 2.1, we may assume that  $K$  acts on each  $B_i$  faithfully in what follows. Since  $|B_i| = p$ , we have two cases: (1)  $K$  acts on  $B_i$  doubly-transitively, and (2)  $K$  acts on  $B_i$  simply primitively. We shall treat these two subcases next.

**Lemma 2.3** *Let  $X$  be a connected vertex- and edge-transitive graph of order  $3p$ , and  $\overline{X} = \{B_0, B_1, B_2\}$  be a complete block system of  $A = \text{Aut}X$ . If  $K$ , the kernel of the action of  $A$  on  $\overline{X}$ , acts doubly transitively on a block, then  $X$  is arc-transitive. That is, there are no 1/2-transitive graphs of order  $3p$  for which  $K$  acts doubly transitively on a block.*

**Proof:** By the classification of 2-transitive groups (see [4], for example), it is easy to check that every 2-transitive group has at most two non-equivalent 2-transitive representations. So, without loss of generality, we may assume that  $K^{B_0}$  and  $K^{B_1}$  are equivalent. Since  $\overline{X}$  is edge-transitive, all three groups  $K^{B_0}$ ,  $K^{B_1}$  and  $K^{B_2}$  are equivalent to each other. Hence, for any vertex  $v_0 \in B_0$ , the stabilizer  $K_{v_0}$  must fix a vertex  $v_1$  in  $B_1$ , and a vertex

$v_2$  in  $B_2$ . By the transitivity of  $K_{v_1}$  on  $B_1 - \{v_1\}$ , either  $v_0$  is adjacent to every vertex in  $B_1$ , every vertex in  $B_1 - \{v_1\}$ , or only  $v_1$ . Thus, the induced bipartite graph  $\langle B_0, B_1 \rangle$  is either  $K_{p,p}$ ,  $K_{p,p}$  minus a 1-factor, or a 1-factor. By the edge-transitivity of  $\overline{X}$ ,  $X$  is either  $K_{p,p,p}$ ,  $K_{p,p,p}$  minus  $pK_3$ , or of degree 2. In all these cases,  $X$  is arc-transitive.  $\square$

We now consider the case that  $K$  acts on  $B_i$  simply primitively. Then  $K^{B_i} < AGL(1, p)$  is solvable. So  $K$  has only one transitive representation of degree  $p$ , and for any  $v \in B_i$ ,  $K_v < Z_{p-1}$  is semiregular on  $B_i - \{v\}$ . Furthermore, the Sylow  $p$ -subgroup of  $K^{B_i}$  is normal in  $K^{B_i}$ . Since  $K$  acts on  $B_i$  faithfully, the Sylow  $p$ -subgroup  $P$  of  $K$  is normal, and then is characteristic, in  $K$ , implying  $P \trianglelefteq A$ . Since  $|P| = p$ ,  $P$  is cyclic. We will use this information later.

In the next proposition, we determine the factor group  $\overline{A} = A/K$ .

**Proposition 2.4** *If  $X$  is a 1/2-transitive graph of order  $3p$  with three blocks of length  $p$ , then  $\overline{A} \cong Z_3$ .*

**Proof:** By Proposition 1.4, there is no 1/2-transitive graph of prime order, which implies that  $\overline{X}$  is connected. By Proposition 1.1,  $\overline{X} \cong K_3$  so that  $\overline{A} \cong S_3$  or  $Z_3$ . Assume that  $\overline{A} \cong S_3$ . Let  $P$  be the Sylow  $p$ -subgroup of  $K$ . Then  $P$  is normal in  $A$  and is cyclic of order  $p$  by the information above. Put  $C = C_A(P)$ . We have  $A/C$  is isomorphic to a subgroup of  $\text{Aut}P = Z_{p-1}$ , so that  $A' \leq C$ . Since  $A/K \cong S_3$ ,  $A'K/K \cong Z_3$ . Hence 3 divides  $|A'|$ , and then 3 divides  $|C|$ . Assume that  $P = \langle g \rangle$ . Take  $h \in C$  with  $o(h) = 3$ . Set  $H = \langle g, h \rangle$ . Since  $h \in C = C_A(P)$ ,  $h$  and  $g$  commute. Then  $H \cong Z_3 \times Z_p \cong Z_{3p}$  is a regular subgroup of  $A$ . By [3, Lemma 16.3],  $X$  is a Cayley graph of  $Z_{3p}$ . Finally, by Proposition 1.3,  $X$  is arc-transitive, a contradiction.  $\square$

Now we give an example via the next theorem. We need the concept of metacirculant defined in [2].

Let  $n \geq 2$ . A permutation on a finite set is said to be  $(m, n)$ -semiregular if it has  $m$  cycles of length  $n$  in its disjoint cycle decomposition. We shall be sloppy and refer to the orbits of the group  $\langle \alpha \rangle$  generated by  $\alpha$  as the orbits of  $\alpha$ . A graph  $X$  is an  $(m, n)$ -metacirculant if it has an  $(m, n)$ -semiregular automorphism  $\alpha$  together with another automorphism  $\beta$  normalizing  $\alpha$  and cyclically permuting the orbits of  $\alpha$ . Therefore, we may partition the vertex-set of an  $(m, n)$ -metacirculant into the orbits  $B_0, B_1, \dots, B_{m-1}$  of  $\alpha$ , where  $B_i^\beta = B_{i+1}$  for all  $i \in Z_m$ . We shall refer to the orbits of  $\alpha$  as the blocks of the metacirculant graph. It should be pointed out that the blocks of a metacirculant graph need not be blocks of imprimitivity of the automorphism group of the graph.

Recall that a circulant graph is a Cayley graph on a cyclic group. Using additive notation for the underlying cyclic group, the symbol  $S$  of a circulant is defined by  $S = \{j: u_0 u_j \text{ is an edge of the circulant graph}\}$ . If  $S_0 \subseteq Z_n \setminus \{0\}$  is the symbol of the subcirculant  $\langle B_0 \rangle$  and, for all  $i \in Z_m \setminus \{0\}$ ,  $T_i \subseteq Z_n$  is the symbol of the bipartite subgraph  $\langle B_0, B_i \rangle$ , then there exists an  $r \in Z_n^*$ , where  $Z_n^*$  denotes the multiplicative group of units in  $Z_n$ , such that for all  $j \in Z_m$ , the symbol of  $\langle B_j \rangle$  is  $r^j S_0$  and the symbol of the bipartite graph  $\langle B_j, B_{j+i} \rangle$ ,  $i \in Z_m$ , is  $r^j T_i$ . Moreover, for all  $i \in Z_m$ , we have  $T_{m-i} = r^{m-i}(-T_i)$ . Thus, the metacirculant

graph  $X$  is completely determined by the  $\lfloor (m + 4)/2 \rfloor$ -tuple  $(r; S_0, T_1, T_2, \dots, T_{\lfloor m/2 \rfloor})$  which is called a *symbol* of  $X$ . (For a more detailed discussion of metacirculants, the reader is referred to [2].)

Now let  $p$  be a prime and  $p \equiv 1 \pmod{3}$ . Assume that  $u$  is an element of order 3 in  $Z_p^*$ . We use  $H_{3p}$  to denote the unique non-abelian group of order  $3p$ , that is,

$$H_{3p} = \langle \alpha, \beta \mid \alpha^p = 1, \beta^3 = 1, \alpha^\beta = \alpha^u \rangle.$$

**Definition** Let  $p$  be a prime with  $p \equiv 1 \pmod{3}$ . Let  $d > 1$  be a divisor of  $(p - 1)/3$ . Let  $T = \langle t \rangle$  be the subgroup of  $Z_p^*$  of order  $d$ . Let  $r \in Z_p^* \setminus T$  be a 3-element with  $r^3 \in T$ . We use  $M(d; 3, p)$  to denote the  $(3, p)$ -metacirculant graph with symbol  $(r; \emptyset, T)$ .

**Theorem 2.5** *If  $(d, p) \neq (2, 7)$  or  $(3, 19)$ , then the graph  $M(d; 3, p)$  is a 1/2-transitive graph of order  $3p$  and of degree  $2d$ . This graph is independent of the choice of  $r$ . The automorphism group  $A = \text{Aut}M(d; 3, p)$  is isomorphic to a semidirect product of  $Z_p$  and  $Z_{3d}$ , and  $A$  acts regularly on the edge set of  $M(d; 3, p)$ .*

**Proof:** Checking the vertex-primitive graphs of order  $3p$  listed in [7], we know that  $M(2; 3, 7)$  and  $M(3; 3, 19)$  are the only vertex-primitive  $(3, p)$ -metacirculants and both of them are arc-transitive. Suppose now that  $p \geq 11$  and  $d \neq 3$  if  $p = 19$ .

Assume that  $B_i = \{x_j^i \mid j = 0, 1, \dots, p - 1\}$ ,  $i = 0, 1, 2$ , are the three blocks of  $X = M(d; 3, p)$  as a metacirculant. It is easy to see that the following mappings  $\alpha$ ,  $\beta$  and  $\gamma$  are automorphisms of  $X$ :

$$\begin{aligned} \alpha: x_j^i &\mapsto x_{j+1}^i \\ \beta: x_j^i &\mapsto x_{rj}^{i+1} \\ \gamma: x_j^i &\mapsto x_{tj}^i. \end{aligned}$$

Assume that  $3^e \parallel d$ . Then  $o(\alpha) = p$ ,  $o(\gamma) = d$  and  $o(\beta) = 3^{e+1}$ . Set  $P = \langle \alpha \rangle$ ,  $L = \langle \alpha, \gamma \rangle$ ,  $M = \langle \beta, \gamma \rangle$  and  $G = \langle \alpha, \gamma, \beta \rangle$ . We can see that  $P \triangleleft G$ ,  $G$  is a semidirect product of  $P$  and  $M$ , and the centralizer of  $P$  in  $G$  is  $P$  itself. Thus,  $M \cong G/P$  is isomorphic to a subgroup of  $\text{Aut}P \cong Z_{p-1}$ , so that in particular,  $M = \langle \beta\gamma \rangle$  is cyclic. Also it is easy to see that  $X$  is  $G$ -vertex-transitive and  $G$ -edge-transitive.

To prove that  $X$  is not arc-transitive, first we claim that  $A$  has a block of length  $p$ . If not,  $A$  is either primitive, or imprimitive but only has blocks of length 3 on the vertex set of  $X$ . By the reason mentioned at the beginning of the proof, assuming that  $p \geq 11$  and  $d \neq 3$  for  $p = 19$ , we have that  $X$  is vertex-imprimitve and  $A$  has only blocks of length 3. By a result in the next section, there are no 1/2-transitive graphs having this property, so that  $X$  must be arc-transitive. By a result in [7] the only arc-transitive graphs, which are not vertex-primitive and whose automorphism groups do not have a block of length  $p$ , have automorphism groups  $A$ , with  $PSL(2, 2^{2^s}) \leq A \leq P\Gamma L(2, 2^{2^s})$ , and  $p = 2^{2^s} + 1$  being a Fermat prime. In this case 3 does not divide  $p - 1 = 2^{2^s}$ , so this case cannot occur.

We have proved that  $A$  has a block of length  $p$ . Since the only blocks of length  $p$  of  $G$ , which is a subgroup of  $A$ , are  $B_i$ ,  $i = 0, 1, 2$ , they must be blocks of  $A$  too. Let  $K$  be

the kernel of  $A$  acting on  $\overline{X} = \{B_0, B_1, B_2\}$ . By the same argument as in the proof of Lemma 2.3, we know that  $K$  is not doubly-transitive on  $B_i$ . This implies that the Sylow  $p$ -subgroup of  $K$ , which is  $P$  defined above and generated by  $\alpha$ , is normal in  $A$ . Assume that  $X$  is arc-transitive. Noting that  $X$  is not isomorphic to the multipartite complete graph  $K_{p,p,p}$ , by Theorem 3 in [7],  $X \cong G(3p, d)$  defined in [7]. By Example 3.4 in [7],  $A = \text{Aut}G(3p, d) \cong (Z_p \cdot Z_d) \cdot S_3$ , where  $G.H$  denotes an extension of  $G$  by  $H$ , and  $A$  contains a cyclic subgroup of order  $3p$ . It follows that the order of a Sylow 3-subgroup of  $A$  is  $3^{e+1}$ , where  $3^e \parallel d$ , and that the centralizer of the Sylow  $p$ -subgroup  $P$  contains an element of order 3. Since  $o(\beta) = 3^{e+1}$ ,  $\langle \beta \rangle$  is the Sylow 3-subgroup of  $A$ . It follows that  $\beta^{3^e}$  and  $\alpha$  commute. However, it is not the case, a contradiction. This shows that  $X$  is 1/2-transitive as required.

It is not difficult to show that different choices of  $r$  correspond to isomorphic graphs. We leave this as an exercise for the reader.

Now we determine the automorphism group  $A = \text{Aut}M(d, 3, p)$ . Since  $A$  is an extension of the kernel  $K$  by  $Z_3$ , and  $K$  is an extension of  $P$  by the stabilizer  $K_v$  of  $v = x_0^0$  in  $K$ , it is easy to see that  $K_v \cong T$ . This shows that  $K \cong L$  defined before. Note that  $\beta$  is not in  $K$  and is a 3-element, implying that  $A = \langle K, \beta \rangle = \langle L, \beta \rangle = G$ , as desired. It follows that  $A = G \cong Z_p \cdot Z_{3d}$  acts regularly on the edge set of  $X$ .

(Note that if  $3 \nmid d$ , then  $M(d, 3, p)$  is a Cayley graph on  $H_{3p}$  with respect to  $S = \{\beta\alpha^i \mid i \in T\} \cup \{\beta^2\alpha^{-u^2i} \mid i \in T\}$ , while if  $3 \mid d$ ,  $M(d, 3, p)$  is not a Cayley graph.)  $\square$

**Theorem 2.6** *Let  $X$  be a 1/2-transitive graph of order  $3p$ . If  $\text{Aut}X$  acts imprimitively on  $V(X)$  and has a block of length  $p$ , then  $X$  is isomorphic to  $M(d, 3, p)$  for some divisor  $d$  of  $\frac{p-1}{3}$ , where  $(d, p) \neq (2, 7)$  or  $(3, 19)$ .*

**Proof:** Assume that  $\overline{X} = \{B_0, B_1, B_2\}$  is a complete block system of  $A = \text{Aut}X$  and that  $K$  is the kernel of  $A$  acting on  $\overline{X}$ .

(1) We claim that  $X$  is connected. If not, every connected component has either  $p$  or 3 vertices, and is also 1/2-transitive. But by Proposition 1.4, there are no 1/2-transitive graphs with a prime number of vertices.

(2) It follows from Proposition 1.1 and Proposition 2.4 that there are no edges in any induced subgraph  $\langle B_i \rangle$ , the factor graph  $\overline{X}$  is a triangle, and the factor group  $\overline{A} = A/K$  is isomorphic to  $Z_3$ .

(3) By the information preceding Proposition 2.4 we have that the Sylow  $p$ -subgroup  $P$  of  $K$  is cyclic and normal in  $A$ .

(4) We claim that  $X$  is a  $(3, p)$ -metacirculant. Let  $P = \langle \alpha \rangle$ . Then  $\alpha$  is a  $(3, p)$ -semiregular automorphism of  $X$ . Since  $A/K \cong Z_3$ , any element  $\beta \in A \setminus K$  permutes  $\overline{X} = \{B_0, B_1, B_2\}$  cyclically. Replacing it by its suitable power, we may assume that  $\beta$  is a 3-element. Hence, by definition,  $X$  is a  $(3, p)$ -metacirculant. Assume that  $\alpha^\beta = \alpha^r$ . Then  $r$  is a 3-element in  $Z_p^* \cong Z_{p-1}$ .

Now we may label the vertices of  $X$  as follows: for  $i = 0, 1, 2$ , let  $B_i = \{x_0^i, x_1^i, \dots, x_{p-1}^i\}$ , and we may assume that  $x_j^{i\alpha} = x_{j+1}^i$  and  $x_j^{i\beta} = x_{rj}^{i+1}$  for all  $i$  and  $j$ .

(5) Finally, we claim that  $X \cong M(d; 3, p)$  for a divisor  $d > 1$  of  $\frac{p-1}{3}$ . Since there are no edges in  $\langle B_i \rangle$  for any  $i$ ,  $X$  has a symbol of the form  $(r; \emptyset, S)$ . Since  $X$  has an odd number of vertices, the degree of  $X$  is even, say  $2d$ . Fix a vertex  $v = x_0^0$ . The neighborhood of  $v$  in  $X$  is  $X_1(v) = X_1^{B_1}(v) \cup X_1^{B_2}(v)$ , where  $X_1^{B_i}(v) = X_1(v) \cap B_i$ .

Consider the stabilizer  $A_v$  of  $v$  in  $A$ . Since  $A/K \cong Z_3$ ,  $A_v$  fixes  $B_i$  setwise for each  $i$ . So  $A_v = K_v$ . Since  $K$  is solvable,  $K$  has only one permutation representation of degree  $p$ , and  $K$  is a Frobenius group or  $K = P$ . So  $K_v$  must fix one vertex in  $B_1$  and one vertex in  $B_2$ . Without loss of generality, we may assume that  $K_v$  fixes  $v_1 = x_0^1$  in  $B_1$  and  $v_2 = x_0^2$  in  $B_2$ . By the edge-transitivity of  $X$ ,  $A_v$  has two orbits in  $X_1(v)$ , which must be  $X_1^{B_1}(v) = \{x_j^1 \mid j \in S\}$  and  $X_1^{B_2}(v) = \{x_j^2 \mid j \in -r^2S\}$ . Since  $A_v = K_v$ , the action of  $A_v$  on  $B_1$  is equivalent to the action of  $K_v$  on  $B_1$ , and then to the action of  $K_{v_1}$  on  $B_1$ . Since  $K^{B_1}$  is a Frobenius group, the subscripts of the vertices in  $X_1^{B_1}(v)$ , which is an orbit of  $K_{v_1}$ , is a coset of a subgroup of  $Z_{p-1}$  of order  $d$ , say  $aT$ , where  $T \leq Z_{p-1}$ ,  $|T| = d$  and  $a \neq 0$ . So we have proved  $d$  is a divisor of  $p - 1$ . If  $d = 1$ , then  $X$  has degree 2, contradicting the fact that  $X$  is not arc-transitive. So  $d > 1$ . If  $d$  does not divide  $\frac{p-1}{3}$ , then  $r \in T$ . Set  $\nu: x_j^i \mapsto x_{r^{-1}j}^i$  for all  $i$  and  $j$ . Then  $\nu \in A$ , and  $\beta\nu$  maps  $x_j^i$  to  $x_j^{i+1}$ . Thus  $\beta\nu$  is an automorphism of  $X$  of order 3 which commutes with  $\alpha$ . This implies that  $\langle \alpha\beta\nu \rangle$  is a regular subgroup of  $A$  which is isomorphic to  $Z_{3p}$ . By Proposition 1.5,  $X$  is arc-transitive which is a contradiction. So we have  $d \mid \frac{p-1}{3}$ . Finally, noticing that two metacirculants with symbols  $(r; \emptyset, T)$  and  $(r; \emptyset, aT)$  are isomorphic, we have the desired result.  $\square$

**3. A has a block of length 3**

The results of the previous section leave us with the case where  $A$  has a block of length 3 and no blocks of length  $p$ . We assume that  $\overline{X} = \{B_i \mid i \in Z_p\}$  is a complete block system of  $A$ , and that  $K$  is the kernel of the action of  $A$  on  $\overline{X}$ . Set  $\overline{A} = A/K$ . We also use  $\overline{X}$  to denote the corresponding block graph.

By Lemma 2.2, if  $K$  acts on  $B_i$  unfaithfully, then  $X \cong \overline{X} \wr 3K_1$ , where  $\overline{X}$  is an arc-transitive graph of order  $p$ . Thus,  $X$  is also arc-transitive, so that we may assume that  $K$  acts on each  $B_i$  faithfully. Thus, we have  $K \cong S_3$ , or  $K \cong Z_3$ , or  $K = 1$ . We also know that there are no edges inside any  $B_i$ .

**Lemma 3.1** *The group  $\overline{A}$  is insolvable, and the graph  $\overline{X}$  is isomorphic to  $K_p$ .*

**Proof:** If  $\overline{A}$  is solvable, then  $\overline{A}$  has a normal subgroup  $\overline{H} = H/K$  of order  $p$  since  $\overline{A}$  is of degree  $p$ . Let  $P \in \text{Syl}_p(H)$ . Then it is easy to check that  $P \triangleleft H$ , and hence  $P \triangleleft A$ . So  $A$  has a block of length  $p$ , contradicting our assumption.

Since  $\overline{A}$  is insolvable and transitive of degree  $p$ , the well known theorem of Burnside implies that it is doubly transitive. Since there is no 1/2-transitive graphs of order 3,  $\overline{X}$  is connected. Hence,  $\overline{X}$  must be isomorphic to  $K_p$ .  $\square$

**Lemma 3.2** *The group  $K = 1$ .*

**Proof:** If  $K \neq 1$ , either  $K \cong S_3$  or  $K \cong Z_3$ . Hence,  $K$  has a characteristic subgroup  $N$  of order 3 which is normal in  $A$ . Put  $C = C_A(N)$ . Then  $A/C$  is isomorphic to a subgroup of  $\text{Aut}K \cong Z_2$ . This implies that every element of order  $p$  in  $A$  is contained in  $C$ . It follows that  $A$  has a subgroup isomorphic to  $Z_{3p}$ , and this subgroup must be regular. Therefore,  $X$  is a Cayley graph on an abelian group. By Proposition 1.3,  $X$  is arc-transitive, which is a contradiction.  $\square$

We may now assume that  $K = 1$ . In this case,  $A \cong \bar{A}$  as abstract groups. But as permutation groups,  $\bar{A}$  is a group of degree  $p$  and  $A$  is of degree  $3p$ . Since  $\bar{A}$  is insolvable, it is doubly-transitive as observed above. Then  $\bar{A}$  is known by the finite simple group classification.

If  $G$  is a doubly-transitive group of degree  $p$ , one necessary condition for  $G$  to be the automorphism group of a 1/2-transitive graph of order  $3p$  (as abstract groups) is that the point stabilizer  $G_\alpha$  has a subgroup of index 3. A table of 2-transitive groups of degree  $p$  with simple socle is given in [7], and after checking all (insolvable) doubly-transitive groups of degree  $p$  listed there, the only possible groups have socle either  $PSL(3, 2)$ ,  $p = 7$ , or  $PSL(2, 2^{2^s})$ , where  $s > 0$  and  $p = 2^{2^s} + 1$  is a Fermat prime.

There are no 1/2-transitive graphs with fewer than 27 vertices [1], so we only need to consider the latter case, where the socle is  $PSL(2, 2^{2^s})$  and  $p = 2^{2^s} + 1$  is a Fermat prime. In this case, noting that  $|P\Gamma L(2, 2^{2^s}) : PSL(2, 2^{2^s})| = 2^s$ , the stabilizer of  $\bar{A}$  having a subgroup of index 3 implies that the stabilizer of  $PSL(2, 2^{2^s})$  also has such a subgroup. This is true since  $2^{2^s} - 1$  is divisible by 3. Hence  $PSL(2, 2^{2^s})$  is vertex-transitive on  $X$ .

**Lemma 3.3** *If  $PSL(2, 2^{2^s}) \leq A \leq P\Gamma L(2, 2^{2^s})$ ,  $A$  is not the automorphism group of any 1/2-transitive graph.*

**Proof:** As noted above  $PSL(2, 2^{2^s})$  is vertex-transitive. Then since all orbitals are self-paired (see [7]) it follows that any edge-transitive graph  $X$  admitting the group is arc-transitive.  $\square$

Summarizing the result of Section 2 and Section 3, we get the main theorem of this paper.

**Theorem 3.4** *A graph of order  $3p$  is 1/2-transitive if and only if it is a  $(3, p)$ -metacirculant graph of the form  $M(d; 3, p)$ , where  $(d, p) \neq (2, 7)$  or  $(3, 19)$ .*

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