TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 303, Number 2, October 1987

K_{l+1}-FREE GRAPHS: ASYMPTOTIC STRUCTURE AND A 0-1 LAW

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ABSTRACT. The structure of labeled K_{l+1}-free graphs is investigated asymptotically. Through a series of stages of successive refinement the structure of "almost all" such graphs is found sufficiently precisely to prove that they are in fact l-colorable (l-partite). With the asymptotic information obtained it is shown also that in the class of K_{l+1}-free graphs there is a first-order labeled 0-1 law. With this result, and those cases already known, we can say that any infinite class of finite undirected graphs with amalgamations, induced subgraphs and isomorphisms has a 0-1 law.

Introduction. In this paper we investigate the asymptotic behavior of K_{l+1}-free graphs, i.e., of finite undirected graphs which do not contain as a subgraph the complete graph K_{l+1} on $l+1$ vertices. In the first part asymptotic results about the number and the structure of labeled K_{l+1}-free graphs are obtained. These results are applied in the second part in order to study the labeled asymptotic probabilities of first-order sentences on the class $\mathscr{S}(l)$ of all K_{l+1}-free graphs. We now describe briefly the main theorems of this paper.

Let $l \geqslant 2$. If G is an l-colorable graph, then obviously G cannot contain as a subgraph the complete graph K_{l+1} on $l+1$ vertices. But it is well known (see e.g. Bollobás [**1980**]) that there are K_{l+1}-free graphs with arbitrarily large chromatic number. Hence, for n large enough, the number of K_{l+1}-free graphs on $\{1,\dots,n\}$ is strictly greater than the number of l-colorable graphs. In contrast to this we show that "almost all" K_{l+1}-free graphs are l-colorable. More precisely, we establish in the first part of this paper:

THEOREM 1. *Let $S_n(l)$ be the number of labeled K_{l+1}-free graphs on $\{1,2,\dots,n\}$ and let $L_n(l)$ be the number of labeled l-colorable graphs on $\{1,2,\dots,n\}$. Then for any polynomial $p(n)$ there is a constant C such that for all n*

$$S_n(l) \leqslant L_n(l)(1 + C/p(n)),$$

and hence

$$\lim_{n\to\infty}\left(\frac{L_n(l)}{S_n(l)}\right) = 1.$$

Received by the editors May 7, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03C13, 05C30; Secondary 05A15.

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The special case of Theorem 1 for $l = 2$ and $p(n) = n$ was proved in Erdös, Kleitman, Rothschild [**1976**], who also showed that

$$\lim_{n\to\infty}\left(\frac{\log L_n(l)}{\log S_n(l)}\right) = 1 \quad \text{for any } l \geqslant 2.$$

In addition to the asymptotic characterization given by Theorem 1, we obtain detailed information about the structure of almost all K_{l+1}-free graphs. For example, it turns out that almost every K_{l+1}-free graph is uniquely l-colorable and each color-class has size at least $n/l - n/\log q(n)$, where $q(n)$ is some function with $\lim_{n\to\infty} q(n) = \infty$.

Let $\mathscr{K}$ be a class of finite structures with universes initial segments $\{1,\ldots,n\}$ of natural numbers and assume that $\mathscr{K}$ is closed under isomorphisms. If ϕ is a sentence of first-order logic, then *the labeled asymptotic probability* $\mu(\phi)$ *of* ϕ *on* $\mathscr{K}$ is given by $\mu(\phi) = \lim_{n\to\infty}\mu_n(\phi)$ (provided this limit exists), where $\mu_n(\phi)$ is the fraction of labeled structures of cardinality n in $\mathscr{K}$ satisfying ϕ. The *almost sure theory* $\Pi(\mathscr{K})$ of the class $\mathscr{K}$ is the set

$$\Pi(\mathscr{K}) = \{\phi\colon \text{is a first-order sentence and } \mu(\phi) = 1 \text{ on } \mathscr{K}\}.$$

We say that the class $\mathscr{K}$ has a *first-order labeled* 0-1 *law* if $\mu(\phi) = 0$ or $\mu(\phi) = 1$ on $\mathscr{K}$ for every first-order sentence ϕ. This is equivalent to asserting that $\Pi(\mathscr{K})$ is a complete theory.

In the second part of this paper we use Theorem 1 and the additional structural information obtained in its proof, to study the labeled asymptotic probabilities of first-order sentences on the class of K_{l+1}-free graphs. We prove

THEOREM 2. *Let* $\mathscr{S}(l)$ *be the class of labeled* K_{l+1}*-free graphs. Then the labeled asymptotic probability* $\mu(\phi)$ *on* $\mathscr{S}(l)$ *of any sentence* ϕ *of first-order logic is either* 0 *or* 1.

Thus the class $S(l)$, $l \geqslant 2$, has a first-order labeled 0-1 law. Moreover we show that the almost sure theory $\Pi(S(l))$ is an ω-categorical, decidable but not finitely axiomatizable complete theory. The countable model $\mathbf{D}(l)$ of the almost sure theory $\Pi(\mathscr{S}(l))$ is an l-colorable graph with uniquely determined parts and has the property that its finite submodels are exactly the finite l-colorable graphs.

Fagin [**1976**] showed that a labeled 0-1 law holds for the class $\mathscr{G}$ of all finite undirected graphs. Moreover, the almost sure theory $\Pi(\mathscr{G})$ is ω-categorical and it turns out that its unique countable model is Rado's graph (Rado [**1964**]). Compton [**1984**] investigated 0-1 laws for various classes of finite structures which arise naturally in combinatorics. In particular he showed that a labeled 0-1 law holds for the class $\mathscr{E}$ of finite equivalence relations. Each of the classes $\mathscr{S}(l)$ ($l \geqslant 2$), $\mathscr{G}$ and $\mathscr{E}$ is a family of finite undirected graphs which has the amalgamation property and is closed under submodels. A complete classification of all such families of undirected

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graphs is given in Lachlan-Woodrow [**1980**]. Using this classification together with Theorem 2 and the 0-1 laws obtained by Fagin and Compton we establish

THEOREM 3. *Let $\mathscr{K}$ be any infinite class of finite undirected graphs having the amalgamation property and closed under induced subgraphs and isomorphisms. Then the labeled asymptotic probability $\mu(\phi)$ on K of any sentence ϕ of first-order logic is either* 0 *or* 1.

The results of this paper have been announced in Kolaitis-Prömel-Rothschild [**1985**].

1. The asymptotic structure of K_{l+1}-free graphs. By a graph we mean a set equipped with an irreflexive and symmetric binary relation. All graphs we consider are labeled. For standard definitions and basic facts about graphs see, for example, Bollobás [**1980**].

Let V be a set of m vertices, e.g. $V = \{1, \dots, m\}$. Then the set of all graphs on V containing no K_{l+1} as a subgraph is denoted by $\mathscr{S}_m(l)$. The number of graphs in $\mathscr{S}_m(l)$ is denoted by $S_m(l)$, i.e. $S_m(l) = |\mathscr{S}_m(l)|$. Furthermore, let $\mathscr{L}_m(l)$ be the set of l-partite (or equivalently l-colorable) graphs on V and $L_m(l) = |\mathscr{L}_m(l)|$. The main theorem proved in this section is

THEOREM 1. *Let $l \geqslant 2$. Then for any polynomial $p(n)$ with positive leading coefficient there is a constant $C = C(p(n), l)$ such that for all n*

$$S_n(l) \leqslant L_n(l)(1 + C/p(n)).$$

We fix $l \geqslant 2$ throughout the proof of Theorem 1. Hence, it will not be confusing to write S_m and $\mathscr{S}_m$, L_m and $\mathscr{L}_m$ resp., instead of $S_m(l)$ and $\mathscr{S}_m(l)$, $L_m(l)$ and $\mathscr{L}_m(l)$ resp.

Convention: All logarithms which occur in this paper are logarithms to the base 2.

Obviously, $\mathscr{L}_m \subseteq \mathscr{S}_m$ and therefore $L_m \leqslant S_m$. In the sequel we will consider $3l - 1$ subclasses of $\mathscr{S}_m$ which exhaust $\mathscr{S}_m \setminus \mathscr{L}_m$ completely and, additionally, each of them has only a 'small' intersection with $\mathscr{L}_m$. This will provide for the estimate for $S_n(l)$ by showing $S_n(l) - L_n(l)$ is small relative to $L_n(l)$.

In the first $2l - 3$ of these classes are all those graphs which have a vertex v having among its neighbors no complete $(l - 1)$-partite graph with parts of size at least $q(m) = q_{l-2}(m)$, where $q(m)$ satisfies $\lim_{m\to\infty} q(m) = \infty$. More precisely: the class $\mathscr{B}_0$ will contain all those graphs in $\mathscr{S}_n$ which have a vertex having only a few neighbors, say less than $q_0(m)$. In the classes $\mathscr{A}_j \cup \mathscr{B}_j$, $j = 1, \dots, l - 2$, are all graphs not in $\mathscr{B}_0 \cup (\bigcup_{i=1}^{j-1} \mathscr{A}_i \cup \mathscr{B}_i)$ having a vertex whose neighborhood does not contain a complete $(j + 1)$-partite graph with each part having size at least $q_j(m)$. These classes will be discussed, step by step, in the Lemmas 1.3 to 1.7.

Then for every graph in $\mathscr{S}_m \setminus \mathscr{A} \cup \mathscr{B}$, where $\mathscr{A} = \bigcup \mathscr{A}_i$ and $\mathscr{B} = \bigcup \mathscr{B}_i$, every vertex has a Q-set in its neighborhood, i.e., a complete $(l - 1)$-partite graph $Q = Q_1 \cup \cdots \cup Q_{l-1}$, where each part Q_i has size $q(m)$. For such a Q-set $Q = Q(v)$ of v we define the R-set $R = R(Q)$ to be the set of all vertices which have in each of the $l - 1$ parts of Q at least one neighbor. The classes $\mathscr{C}$, $\mathscr{D} \subseteq \mathscr{S}_m \setminus (\mathscr{A} \cup \mathscr{B})$ contain all graphs which have a vertex v with Q-set Q and R-set R such that R does

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not contain about $1/l$ of all vertices, i.e., either $|R| \geqslant m/l + \sqrt{m}$ or $|R| \leqslant m/l - m/\log q(m)$ (provided that m was chosen sufficiently large). Intuitively, R will serve as one part of an l-partite graph, viz. the part to which v belongs. The other $l-1$ parts are represented by the $l-1$ parts of Q. The classes $\mathscr{C}$ and $\mathscr{D}$ will be investigated in Lemma 1.8, and Lemma 1.9.

Then, in the class $\mathscr{E}_k$, for $k = 1, \ldots, l-1$, we put all graphs not contained in one of the former classes and having a k-clique which is not contained in some $(k+1)$-clique. Finally, the class $\mathscr{E}_l$ contains all graphs which have an l-clique, say $K = \{v_1, \ldots, v_l\}$ such that for each v_i there is a Q-set with a corresponding R-set R_i so that the R_i's do not cover almost all vertices, i.e., $|\{x \in V \setminus K | x \notin \bigcup R_i\}| \geqslant m/\log^{(2)} q$, for m large enough. From this immediately follows that every graph not in $\mathscr{E} = \bigcup_{i=1}^{l} \mathscr{E}_i$ has the property that whenever v_1 and v_2 are adjacent vertices with R-sets R_1 and R_2, then R_1 and R_2 almost do not intersect, i.e., $|R_1 \cap R_2| \leqslant 2m/\log^{(2)} q$.

The graphs in $\mathscr{S}_m \setminus \mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$, are called EQR-graphs. We show in a series of steps (Lemma 1.15-Corollary 1.20) that every EQR-graph is already l-colorable, i.e., the $3l-1$ classes given above cover $\mathscr{S}_m \setminus \mathscr{L}_m$ completely. In fact, every EQR-graph is l-colorable, where the size of each color class is at least $m/l - m/\log q(m)$.

In every step we show that the class under consideration is small, i.e., we estimate its size relative to S_{m-i} for some $i \geqslant 1$. To prove Theorem 1 we combine these estimates with the growth rate of L_m and with induction on n to show that each class has fewer than $(C/(3l-1)p(n))L_n$ elements and therefore is negligible relative to $\mathscr{L}_n$.

A similar method of proof was used by Kleitman and Rothschild [**1975**] and Erdös, Kleitman and Rothschild [**1976**] in the asymptotic enumeration of partial orders and K_j-free graphs, respectively.

We start the proof with giving an estimate for the growth rate of L_m:

LEMMA 1.1. *For m sufficiently large we have the following bounds on* L_m:

$$-l\log m + m\log l + \binom{l}{2}\frac{m^2}{l^2} \leqslant \log L_m \leqslant m\log l + \binom{l}{2}\frac{m^2}{l^2}.$$

PROOF OF LEMMA 1.1. For the upper bound we observe that we can construct all l-partite graphs with m vertices by partitioning the set V into l parts (at most l^m ways) and then connecting the parts (at most $2^{\sum_{i<j} x_i x_j}$ ways, where x_i is the size of the ith part, $1 \leqslant i \leqslant l$). Since $\sum_{i<j} x_i x_j \leqslant \binom{l}{2} m^2/l^2$, we have

$$\log L_m \leqslant m\log l + \binom{l}{2}\frac{m^2}{l^2}.$$

For the lower bound on L_m we consider a subclass of the l-partite graphs, which we call special l-partite graphs. These are l-partite graphs with $l-1$ parts of size $(m-\varepsilon)/l$, where ε is the remainder when m is divided by l, i.e., $\varepsilon = l(m/l - [m/l])$, and the remaining part of size $(m + (l-1)\varepsilon)/l$. Notice that the size of ε is bounded by l.

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We derive a lower bound for the number of special l-partite graphs. This is done by first counting all special l-partite graphs, but with some possible duplications, and then subtracting an overestimate of the number of duplications.

All the special l-partite graphs are constructed as follows: first the vertex set V is partitioned into l classes. This can be done in

$$\binom{m}{\frac{m-\varepsilon}{l}, \dots, \frac{m-\varepsilon}{l}, \frac{m+(l-1)\varepsilon}{l}}$$

ways. Then the connections between the parts are made. This can be done in

$$2^{\left(\frac{m-\varepsilon}{l}\right)^2\binom{l-1}{2} + \left(\frac{m+(l-1)\varepsilon}{l}\right)\left(\frac{m-\varepsilon}{l}\right)(l-1)} = 2^{(l-1)(m^2-\varepsilon^2)/2l}$$

ways. Hence, we get in total at most

$$\binom{m}{\frac{m-\varepsilon}{l}, \dots, \frac{m-\varepsilon}{l}, \frac{m+(l-1)\varepsilon}{l}} 2^{(l-1)(m^2-\varepsilon^2)/2l}$$

special l-partite graphs. Of course, any special l-partite graph is counted multiply, at least $l!$ times due just to relabeling the parts.

Now we consider those graphs which are counted even more often. The only way this can happen is if a graph can be obtained from two different partitions of V into $l-1$ parts of size $(m-\varepsilon)/l$ and one part of size $(m-\varepsilon)/l+\varepsilon$. We overestimate the number of these graphs by forming all possible pairs of distinct partitions, and for every such pair finding an upper bound for the number of graphs consistent with both partitions, that is, graphs such that no part of either partition contains two vertices joined by an edge.

Let $A_1, \dots, A_l$ and $B_1, \dots, B_l$ be two distinct partitions with $|A_i| = |B_j| = (m-\varepsilon)/l$ for $i, j \neq l$ and $|A_l| = |B_l| = (m-\varepsilon)/l + \varepsilon$. Furthermore, let $X_{ij} = A_i \cap B_j$ and $x_{ij} = |X_{ij}|$ for all $1 \leqslant i,\ j \leqslant l$. Then we have the following two immediate observations:

Since the partitions $A_1, \dots, A_l$ and $B_1, \dots, B_l$ are distinct, there exist i, j such that

(1) either $i, j \neq l$, and $1 \leqslant x_{ij} < (m-\varepsilon)/l$ or at least one of i, j equals l, and $1 \leqslant x_{ij} < (m-\varepsilon)/l + \varepsilon$.

From the definition of the x_{ij} we have that

$$\sum_{i=1}^{l} x_{ij} = \begin{cases} \dfrac{m-\varepsilon}{l}, & \text{if } j < l, \\ \dfrac{m-\varepsilon}{l} + \varepsilon, & \text{if } j = l. \end{cases} \tag{2}$$

$$\sum_{j=1}^{l} x_{ij} = \begin{cases} \dfrac{m-\varepsilon}{l}, & \text{if } i < l, \\ \dfrac{m-\varepsilon}{l} + \varepsilon, & \text{if } i = l. \end{cases}$$

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Since an edge cannot join two vertices of A_i or B_j, the number of graphs consistent with both partitions is 2^α, where

$$\alpha = \frac{1}{2} \sum_{\substack{i \neq h \\ j \neq k}} x_{ij} x_{hk}.$$

We claim that

$$\alpha = \frac{l-2}{2l} m^2 - \frac{l-1}{l} \varepsilon^2 + \frac{1}{4} \left[\sum_{i=1}^{l} \left(\sum_{j=1}^{l} x_{ij}^2 \right) + \sum_{j=1}^{l} \left(\sum_{i=1}^{l} x_{ij}^2 \right) \right]. \tag{3}$$

To see this we rewrite 2α as follows:

$$\begin{aligned}
2\alpha &= \sum_{\substack{i \neq h \\ j \neq k}} x_{ij} x_{hk} \\
&= \left(\sum_{i,j} x_{ij} \right) \left(\sum_{h,k} x_{hk} \right) - \sum_{i} \sum_{j,k} x_{ij} x_{ik} - \sum_{j} \sum_{i,h} x_{ij} x_{hj} + \sum_{i,j} x_{ij} x_{ij} \\
&= m^2 - \sum_{i} \left(\sum_{j} x_{ij} \right)^2 - \sum_{j} \left(\sum_{i} x_{ij} \right)^2 + \sum_{i,j} x_{ij}^2 \\
&= m^2 - 2(l-1) \left(\frac{m-\varepsilon}{l} \right)^2 - 2 \left(\frac{m-\varepsilon}{l} + \varepsilon \right)^2 + \sum_{i,j} x_{ij}^2, \quad \text{by observation (2)} \\
&= \frac{l-2}{l} m^2 - 2 \left(\frac{l-1}{l} \right) \varepsilon^2 + \sum_{i,j} x_{ij}^2 \\
&= \frac{l-2}{l} m^2 - 2 \left(\frac{l-1}{l} \right) \varepsilon^2 + \frac{1}{2} \left[\sum_{i} \left(\sum_{j} x_{ij}^2 \right) + \sum_{j} \left(\sum_{i} x_{ij}^2 \right) \right],
\end{aligned}$$

which proves (3).

The pair of partitions $A_1, \ldots, A_l$ and $B_1, \ldots, B_l$ can be chosen by picking the X_{ij} in an appropriate way.

We now distinguish two cases.

(4) All x_{ij} satisfy either $x_{ij} < l^2$ or $x_{ij} > m/l - l^2$.

(5) Some x_{ij} satisfies $l^2 \leqslant x_{ij} \leqslant m/l - l^2$

First we consider (4). Here we must have exactly l of the x_{ij} which satisfy $x_{ij} > m/l - l^2$. This follows immediately from (2) for m sufficiently large, say $m > l^5$. Hence, the number of ways of choosing the x_{ij} is in this case at most

$$l^2! \sum_{\substack{l_i < l^2 \\ h_j > m/l - l^2}} \binom{m}{l_1, l_2, \ldots, l_{l^2-l}, h_1, \ldots, h_l}, \quad \text{where } \sum_{i=1}^{l^2-l} l_i + \sum_{j=1}^{l} h_j = m.$$

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This number is obviously smaller than

$$l^{2l^2} \sum_{\substack{l_i < l^2 \\ h_l > m/l - l^2}} \binom{m}{l_1} \cdots \binom{m}{l_{l^2-l}} \binom{m'}{h_1, \ldots, h_l}, \quad \text{where } m' = m - \sum_{i=1}^{l^2-l} l_i$$

$$\leqslant l^{2l^2} l^{2l^2} \binom{m}{l^2}^{l^2} \max_{\substack{h_1 + \cdots + h_l = m' \\ l_i < l^2}} \binom{m'}{h_1, \ldots, h_l}$$

$$\leqslant m^{l^5} l^m, \quad \text{for } m \text{ sufficiently large.}$$

We get that in case (4) the number of graphs which are consistent with the partition induced by the X_{ij} is at most $m^{l^5} l^m 2^\alpha$, where by (3)

$$\alpha = \frac{l-2}{2l} m^2 - \frac{l-1}{l} \varepsilon^2 + \frac{1}{4} \left[\sum_{i=1}^{l} \left(\sum_{j=1}^{l} x_{ij}^2 \right) + \sum_{j=1}^{l} \left(\sum_{i=1}^{l} x_{ij}^2 \right) \right].$$

By (2), each summand $\sum_{j=1}^{l} x_{ij}^2$ and $\sum_{i=1}^{l} x_{ij}^2$ is maximized when exactly one of the terms in it is nonzero. By the requirement (1) that at least one x_{ij} satisfies $1 \leqslant x_{ij} < (m - \varepsilon)/l$ ($1 \leqslant x_{ij} < (m - \varepsilon)/l + \varepsilon$ respectively) we know that there exists i, $1 \leqslant i \leqslant l$, such that either

$$i \neq l, \quad \text{and} \quad \sum_{j=1}^{l} x_{ij}^2 \leqslant \left(\frac{m - \varepsilon}{l} - 1 \right)^2 + 1^2$$

or

$$i = l, \quad \text{and} \quad \sum_{j=1}^{l} x_{ij}^2 \leqslant \left(\frac{m - \varepsilon}{l} + \varepsilon - 1 \right)^2 + 1^2.$$

Hence we get, again using (2):

$$\sum_{i=1}^{l} \left(\sum_{j=1}^{l} x_{ij}^2 \right) \leqslant (l-1) \left(\frac{m - \varepsilon}{l} \right)^2 + \left(\frac{m - \varepsilon}{l} + \varepsilon \right)^2 - 2 \left(\frac{m - \varepsilon}{l} \right) + 2$$

$$\leqslant \frac{m^2}{l} - \frac{2m}{l} + \frac{l-1}{l} \varepsilon^2 + \frac{2}{l} \varepsilon + 2 \leqslant \frac{m^2}{l} - \frac{m}{l},$$

for m sufficiently large.

Thus the number of graphs in case (4) is less than

$$m^{l^5} l^m 2^{(l-2)m^2/2l - (l-1)\varepsilon^2/l + m^2/2l - m/2l} \leqslant 2^{(l-1)m^2/2l + m \log l - m/4l}$$

for m sufficiently large.

Now we consider the case (5). Since the X_{ij}'s partition the vertex set V there are at most l^{2m} many choices for them. Hence the number of graphs in case (5) is at most $l^{2m} 2^\alpha$, where α is as in (3). From (5) we see that there exists i, $1 \leqslant i \leqslant l$, such that either

$$i \neq l, \quad \text{and} \quad \sum_{j=1}^{l} x_{ij}^2 \leqslant \left(\frac{m - \varepsilon}{l} - l^2 \right)^2 + (l^2)^2$$

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or

$$i = l, \quad \text{and} \quad \sum_{j=1}^{l} x_{ij}^2 \leqslant \left(\frac{m-\varepsilon}{l} + \varepsilon - l^2\right)^2 + (l^2)^2.$$

So we get in this case

$$\sum_{i=1}^{l}\left(\sum_{j=1}^{l} x_{ij}^2\right) \leqslant (l-1)\left(\frac{m-\varepsilon}{l}\right)^2 + \left(\frac{m-\varepsilon}{l} + \varepsilon\right)^2 - 2l(m-\varepsilon) + l^4 + l^4$$

$$\leqslant \frac{m^2}{l} - 2lm + \frac{l-1}{l}\varepsilon^2 + 2l\varepsilon + 2l^4 \leqslant \frac{m^2}{l} - lm,$$

for m sufficiently large.

Hence the number of graphs in case (5) is less than

$$l^{2m} 2^{(l-2)m^2/2l - (l-1)\varepsilon^2/l + m^2/2l - lm/2} \leqslant 2^{(l-1)m^2/2l - lm/4}$$

for m sufficiently large.

Putting (4) and (5) together we get an upper bound for the number of (nontrivial) duplicates, i.e. of graphs, which can be obtained from two different partitions of V into $l-1$ parts of size $(m-\varepsilon)/l$ and one part of size $(m-\varepsilon)/l + \varepsilon$.

This upper bound is

$$2^{(l-1)m^2/2l + m\log l - m/4l} + 2^{(l-1)m^2/2l - lm/4} \leqslant 2^{(l-1)m^2/2l + m\log l - m/5l}.$$

A lower bound for the number of special l-partite graphs is therefore

$$\frac{1}{l!}\left[\binom{m}{\frac{m-\varepsilon}{l}, \ldots, \frac{m-\varepsilon}{l}, \frac{m-\varepsilon}{l} + \varepsilon} 2^{(l-1)m^2/2l - (l-1)\varepsilon^2/2l} - 2^{(l-1)m^2/2l + m\log l - m/5l}\right].$$

Using Stirling's formula, we can bound the multinomial coefficient from below by $l^m m^{-l/2}$. Hence, we have at least

$$2^{m\log l - (l/2)\log m - l\log l + (l-1)m^2/2l - (l-1)\varepsilon^2/2l} - 2^{(l-1)m^2/2l + m\log l - m/5l}$$

$$\geqslant 2^{(l-1)m^2/2l + m\log l - l\log m}$$

special l-partite graphs, provided that m is sufficiently large. This proves Lemma 1.1.

COROLLARY 1.2.

$$\log(L_m/L_{m+1}) \leqslant -(l-1)m/l + (l+1)\log m$$

for m sufficiently large.

Now we consider, step by step, five different families of subclasses of $\mathscr{S}_{m+1}$ and estimate the size of each subclass relative to some $\mathscr{S}_{m+1-i}$ where $i \geqslant 1$. But first recall the following estimate which can easily be derived from Stirling's formula:

$$\log\binom{m}{\varepsilon m} < -(\varepsilon\log\varepsilon + (1-\varepsilon)\log(1-\varepsilon))m,$$

where $0 < d \leqslant \varepsilon \leqslant 1/2$, for m sufficiently large depending on d. We put $b(\varepsilon) = -(\varepsilon\log\varepsilon + (1-\varepsilon)\log(1-\varepsilon))$. Observe, that $b(\varepsilon)$ has its maximum at $\varepsilon = 1/2$ and its minimum at d.

For the remainder of the proof of Theorem 1 let U be a set of $m+1$ vertices.

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LEMMA 1.3. *Let k be an integer such that $1 \leqslant k \leqslant l-2$ and let $c = c(m)$ be an integer such that $c = o(m)$ and $1/c = o(1)$. Furthermore let $\varepsilon > 0$ be such that $b(\varepsilon) = 1/4l$. Then let $\mathscr{A}(U, k, c)$ be the set of all graphs in $\mathscr{S}_{m+1}$ which contain pairwise disjoint sets $P_1, \ldots, P_k$ each of size c such that the set*

$$T_k = \left\{ x \in U \setminus \bigcup_{i=1}^{k} P_i \colon x \text{ is connected to at least } \varepsilon c \text{ vertices in each } P_i \right\}$$

satisfies

$$|T_k| \leqslant \frac{5}{4l} \left| U \setminus \bigcup_{i=1}^{k} P_i \right|.$$

Then

$$\log \frac{|\mathscr{A}(U, k, c)|}{S_{m+1-kc}} \leqslant \frac{l-1}{l} kcm - \frac{1}{2l} cm.$$

PROOF OF LEMMA 1.3. Every graph in $\mathscr{A}(U, k, c)$ can be constructed as follows. First choose the k sets $P_1, \ldots, P_k$ (at most $\binom{m}{c}^k < m^{ck} = 2^{ck \log m}$ ways). Then choose edges in $\bigcup_{i=1}^{k} P_i$ (at most $2^{k^2c^2}$ ways). Then choose a K_{l+1}-free graph on $U \setminus \bigcup_{i=1}^{k} P_i$ (S_{m+1-kc} ways). Now connect $\bigcup_{i=1}^{k} P_i$ to T_k (at most $2^{ck|T_k|}$ ways).

Finally, connect $\bigcup_{i=1}^{k} P_i$ to $U \setminus (\bigcup_{i=1}^{k} P_i \cup T_k)$: Do this by first choosing for each $x \in U \setminus (\bigcup_{i=1}^{k} P_i \cup T_k)$ one of the P_i's with at most εc edges to x. There are k choices for the P_i and at most $\binom{c}{\varepsilon c} c$ ways to connect x to this P_i. Furthermore, there are at most $2^{(k-1)c}$ ways to connect x to the other P_j's. The number of elements x in $U \setminus (\bigcup_{i=1}^{k} P_i \cup T_k)$ is $m + 1 - kc - |T_k|$. Thus, the total number of ways to connect $\bigcup_{i=1}^{k} P_i$ to $U \setminus (\bigcup_{i=1}^{k} P_i \cup T_k)$ is at most

$$\left[k \binom{c}{\varepsilon c} c 2^{(k-1)c} \right]^{m+1-kc-|T_k|} \leqslant 2^{(b(\varepsilon)c + \log c + \log k + (k-1)c)(m+1-kc-|T_k|)}.$$

This gives

$$\begin{aligned} \log \frac{|\mathscr{A}(U, k, c)|}{S_{m+1-kc}} &\leqslant ck \log m + k^2c^2 + ck|T_k| \\ &\qquad + (b(\varepsilon)c + \log k + \log c + (k-1)c)(m + 1 - kc - |T_k|) \\ &= ck \log m + k^2c^2 + (m + 1 - kc)(b(\varepsilon)c + \log k + (k-1)c) \\ &\qquad + (c - b(\varepsilon)c - \log c - \log k)|T_k|. \end{aligned}$$

Since $|T_k| \leqslant 5(m + 1 - kc)/4l$ and $|T_k|$ is multiplied by a positive factor in the expression above, for m sufficiently large, then we have

$$\begin{aligned} \log \frac{|A(U, k, c)|}{S_{m+1-kc}} &\leqslant \left[b(\varepsilon) + (k-1) + (1 - b(\varepsilon)) \frac{5}{4l} \right] mc + o(mc) \\ &= \left(\frac{4l-5}{4l} b(\varepsilon) + k - \frac{4l-5}{4l} \right) cm + o(cm) \\ &\leqslant \left(k - \frac{4l-5}{4l} + b(\varepsilon) \right) cm \leqslant \left(\frac{l-1}{l} k + \frac{4k - 4l + 5 + 1}{4l} \right) cm \\ &\leqslant \frac{l-1}{l} kcm - \frac{1}{2l} cm, \quad \text{for } m \text{ sufficiently large.} \end{aligned}$$

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LEMMA 1.4. *Let* $0 \leqslant k \leqslant l - 2$ *and let* $c = c(m)$ *be an integer such that* $c = o(m)$ *and* $1/c = o(1)$. *Furthermore, let* $\varepsilon > 0$ *be such that* $b(\varepsilon) = 1/4l$. *Then let* $\mathscr{B}(U, k, c)$ *be the set of all graphs in* $\mathscr{S}_{m+1}$ *which have a vertex* v *and* k *disjoint sets* $P_1, \ldots, P_k$ *each of size* c *with the following properties*:

(a) *All vertices in* $\bigcup_{i=1}^{k} P_i$ *are adjacent to* v.

(b) *The set* $T_k = \{x \in U \setminus \bigcup_{i=1}^{k} P_i$: *x is connected to at least* εc *vertices in each* $P_i\}$ *satisfies*

$$(1) \qquad |T_k| > \frac{5}{4l}\left|U \setminus \bigcup_{i=1}^{k} P_i\right|$$

and

$$(2) \qquad |T_k \cap \Gamma(v)| \leqslant c$$

where $\Gamma(v)$ *is the set of all vertices which are connected to* v.

Then

$$\log \frac{|\mathscr{B}(U, k, c)|}{S_m} \leqslant \frac{l-1}{l} m - \frac{1}{5l} m,$$

for m sufficiently large.

PROOF OF LEMMA 1.4. Every graph in $\mathscr{B}(U, k, c)$ can be constructed as follows: First choose v ($m + 1$ ways). Then choose an appropriate K_{l+1}-free graph on $U \setminus \{v\}$, i.e., a graph satisfying the conditions above on the P_i and T_k (at most S_m ways). Now connect v to $U \setminus \{v\}$: First choose $P_1, \ldots, P_k$ such that T_k satisfies (1) (at most $\binom{m}{c}^k$ ways). By (2), v can be connected to T_k in at most $\binom{m}{c}c$ ways. Finally, connect v to $U \setminus (\bigcup_{i=1}^{k} P_i \cup T_k)$. By (1) there are at most

$$2^{(4l-5)(m-kc)/4l-5/4l} \leqslant 2^{((l-1)/l-1/4l)m}$$

possibilities for this. This gives

$$\begin{aligned} \log \frac{|\mathscr{B}(U, k, c)|}{S_m} &\leqslant \log(m+1) + (k+1)\log\binom{m}{c} + \log c + \left(\frac{l-1}{l} - \frac{1}{4l}\right)m \\ &\leqslant \frac{l-1}{l} m - \frac{1}{5l} m, \end{aligned}$$

for m sufficiently large.

Next we combine Lemma 1.3 and Lemma 1.4 to get some structural information about those graphs in $\mathscr{S}_{m+1}$ which are not in $\bigcup_{k=1}^{l-2} \mathscr{A}(U, k, c) \cup \bigcup_{k=0}^{l-2} \mathscr{B}(U, k, c)$, where c is chosen appropriately. For this purpose we need an immediate consequence of the following theorem due to Bollobás and Erdös [**1973**] (cf. Bollobás [**1980**, p. 328]).

LEMMA 1.5. *For any* ε, $0 < \varepsilon < 1$, *there exists an* N *such that if* B *is a bipartite graph with classes* B_1 *and* B_2, *with* $N = |B_1| = |B_2|$, *and if* B *has at least* εN^2 *edges, then* B *contains as a subgraph a complete bipartite graph with classes of size* $t \geqslant \delta \log N$, *where* $1/\delta = 1 - \log \varepsilon$.

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Let $\log^{(0)}(x) = x$ and let $\log^{(s)}(x) = \log(\log^{(s-1)}(x))$ denote the sth iteration of $\log x$. Notice that, for N sufficiently large, the size t of the classes guaranteed in Lemma 1.5 is larger than $\log^{(2)}(N)$. From Lemma 1.5 we immediately derive

COROLLARY 1.6. *Let $0 < \varepsilon < 1$ and $h \geqslant 0$ be given. Then there exists N such that the following is true: Let G be a graph with vertex-set $Q_0 \cup Q_1 \cup \cdots \cup Q_h$, where the Q_i are mutually disjoint (but not necessarily independent sets) and $|Q_0| = |Q_1| = \cdots = |Q_h| = N$. Suppose that for each $u \in Q_0$ and each $i = 1, \ldots, h$ the number of edges from u to Q_1 is at least εN. Then there are subsets $Q_0', \ldots, Q_h'$ of $Q_0, \ldots, Q_h$ respectively, such that $|Q_i'| \geqslant \lceil \log^{(2h)}(N) \rceil$ for every $i = 0, \ldots, h$, and such that Q_0 is completely connected to Q_i'; that is to say all vertices of Q_0' are adjacent to all vertices of Q_i' for $i = 1, 2, \ldots, h$.*

PROOF OF COROLLARY 1.6. Fix $0 < \varepsilon < 1$. For $h = 0$ the statement is trivially true. Choose N such that Lemma 1.5 can be applied to bipartite graphs with parts of size $\lceil \log^{(2h-2)}(N) \rceil$ and let G be as required in Corollary 1.6. Assume that there are subsets $Q_0'' \subseteq Q_0$, and $Q_i' \subseteq Q_i$ for $i = 1, \ldots, h-1$ such that each of these subsets has cardinality at least $\lceil \log^{(2h-2)}(N) \rceil$, and Q_0'' is completely connected to Q_i' for each $i = 1, \ldots, h-1$.

Note that there are at least $\varepsilon |Q_0''| |Q_h|$ edges between Q_0'' and Q_h. Hence, the average number of edges between Q_0'' and a subset of Q_h of size $|Q_0''|$ is $\varepsilon |Q_0''|^2$. Then by Lemma 1.5 there are subsets $Q_0' \subseteq Q_0''$ and $Q_h' \subseteq Q_h$ with

$$|Q_0'| = |Q_h'| \geqslant \left\lceil \log^{(2)} |Q_0''| \right\rceil \geqslant \left\lceil \log^{(2h)}(N) \right\rceil$$

such that Q_0' is completely connected to Q_h'. Thus $Q_0', \ldots, Q_h'$ fulfill the requirements of Corollary 1.6.

LEMMA 1.7. *For m sufficiently large (depending on l), any graph G in*

$$\mathscr{S}_{m+1} \setminus \left[\left(\bigcup_{k=0}^{l-2} \mathscr{A}\left(U, k, \left\lceil \log^{(k^2-k+2)}(m) \right\rceil\right) \right) \cup \left(\bigcup_{k=0}^{l-2} \mathscr{B}\left(U, k \left\lceil \log^{(k^2-k+2)}(m) \right\rceil\right) \right) \right]$$

has the following property.

For every vertex v in G the set $\Gamma(v)$ of neighbors of v contains a set $Q = \bigcup_{i=1}^{l-1} Q_i$, which induces a complete $(l-1)$-partite subgraph with parts $Q_1, \ldots, Q_{l-1}$, such that $|Q_i| \geqslant \left\lceil \log^{(l^2-3l+4)}(m) \right\rceil$ for $i = 1, \ldots, l-1$.

PROOF OF LEMMA 1.7. Let v be an arbitrary vertex of G. By the definitions of the classes $\mathscr{A}(U, k, c)$ and $\mathscr{B}(U, k, c)$ for $k = 0$ and $c = \lceil \log^{(2)}(m) \rceil$ we know there exists a set P_{11} of size $\lceil \log^{(2)}(m) \rceil$ in $\Gamma(v)$.

We continue by induction on k. Suppose for some k, $1 \leqslant k \leqslant l-2$, we have sets $P_{1k}, P_{2k}, \ldots, P_{kk}$ with the following properties:

(1) the P_{ik}'s, $1 \leqslant i \leqslant k$, are pairwise disjoint,

(2) $P_{ik} \subseteq \Gamma(v)$ for every $i = 1, \ldots, k$,

(3) $|P_{ik}| = \left\lceil \log^{(k^2-k+2)}(m) \right\rceil$ for every $i = 1, \ldots, k$,

(4) for every pair $i \neq j$, P_{ik} is completely connected to P_{jk}.

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Since G is not in $\mathscr{A}(U, k, \lceil \log^{(k^2-k+2)}(m) \rceil)$, we know that for

$$T_k = \left\{ x \in U \setminus \bigcup_{i=1}^{k} P_{ik} : |\Gamma(x) \cap P_{ik}| \geqslant \varepsilon |P_{ik}|, \text{ for each } i, 1 \leqslant i \leqslant k \right\},$$

where $\varepsilon > 0$ is such that $b(\varepsilon) = 1/4l$, we have

$$|T_k| > \frac{5}{4l} \left| U \setminus \bigcup_{i=1}^{k} P_{ik} \right|.$$

Since, furthermore, G is also not in $\mathscr{B}(U, k, \lceil \log^{(k^2-k+2)}(m) \rceil)$ we have that

$$|T_k \cap \Gamma(v)| > \lceil \log^{(k^2-k+2)}(m) \rceil.$$

Now let P_{k+1} be any subset of $T_k \cap \Gamma(v)$ of size $\lceil \log^{(k^2-k+2)}(m) \rceil$. Applying Corollary 1.6 with $Q_0 = P_{k+1}$ and $Q_i = P_{ik}$ for $i = 1, \ldots, k$ we get subsets $P_{k+1,k+1} \subseteq Q_0$ and $P_{i,k+1} \subseteq Q_i$ for $i = 1, \ldots, k$ so that for $i = 1, \ldots, k$ we have

$$\begin{aligned} |P_{i,k+1}| &\geqslant \lceil \log^{(2k)}(|Q_0|) \rceil \geqslant \lceil \log^{(2k)}(\log^{(k^2-k+2)}(m)) \rceil \\ &= \lceil \log^{((k+1)^2-(k+1)+2)}(m) \rceil, \end{aligned}$$

and, in addition, $P_{k+1,k+1}$ is completely connected to $P_{i,k+1}$ for $i = 1, \ldots, k$.

This gives us the step from k to $k+1$. Continuing until $k+1 = l-1$ we obtain a subgraph with vertex set $Q = \bigcup_{i=1}^{l-1} Q_i$ where $Q_i = P_{i,k+1}$, $i = 1, \ldots, l-1$, and Q_i is completely connected to Q_j, $i \neq j$. We now observe that any such subgraph is an induced complete $(l-1)$-partite subgraph of G, because any edge in one of the parts Q_i would create, together with the vertex v, a K_{l+1}, which is excluded for $\mathscr{S}_{m+1}$.

For a vertex v, a *Q-set* in $\Gamma(v)$ is a set which is the disjoint union of sets Q_i, $1 \leqslant i \leqslant l-1$, where $|Q_i| = \lceil \log^{(l^2-3l+5)}(m) \rceil$, for every i and Q induces a complete $(l-1)$-partite graph with parts Q_i. The structural information given by Lemma 1.7 is that whenever we consider a graph G in

$$\mathscr{S}_{m+1} \setminus \left[\sum_{k=1}^{l-2} \mathscr{A}(U, k, \lceil \log^{(k^2-k+2)}(m) \rceil) \cup \bigcup_{k=0}^{l-2} \mathscr{B}(U, k, \lceil \log^{(k^2-k+2)}(m) \rceil) \right]$$

then every vertex of G has a Q-set. In fact, since in a Q-set the size of each part need only to have size $\lceil \log^{(l^2-3l+5)}(m) \rceil$, we know moreover that the property of having a Q-set is quite robust. More precisely: Let $q = q(l, m) = \lceil \log^{(l^2-3l+5)}(m) \rceil$. Then Lemma 1.7 assures that there are at least $\binom{2^q}{q}$ many Q-sets for every v in G.

For a Q-set Q of v define the *R-set* $R(Q)$ by

$$R = R(Q) = \{ x \in U \setminus \{v\} \setminus Q \mid \forall_i \exists y \in Q_i \text{ such that } y \in \Gamma(x) \}.$$

Observe that $R \subseteq U \setminus \Gamma(v)$, because an edge from v to R creates a K_{l+1}.

In the next two lemmas we give (relative) estimates for the size of those classes of graphs which contain a vertex v and a Q-set Q such that the size of the corresponding R-set is significantly different from m/l.

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LEMMA 1.8. *Let $\mathscr{C}(U)$ denote the set of graphs in $\mathscr{S}_{m+1}$ containing a vertex v with a Q-set $Q = \bigcup_{i=1}^{l-1} Q_i$ such that $|R(Q)| \geqslant m/l + m^{1/2}$. Then*

$$\log \frac{|\mathscr{C}(U)|}{S_m} \leqslant \frac{l-1}{l} m - \frac{m^{1/2}}{2},$$

for m sufficiently large.

PROOF OF LEMMA 1.8. Construct all graphs in $\mathscr{C}(U)$ as follows: Choose v ($m+1$ ways); then choose a suitable graph on $U \setminus \{v\}$, i.e., a graph which can be made to a graph in $\mathscr{C}(U)$ by adding v in an appropriate way (such a graph can be chosen in at most S_m ways); then find a Q-set $Q = \bigcup_{i=1}^{l-1} Q_i$ (at most $\binom{m}{q}^{l-1}$ ways); then connect v to the remainder of U. Since v cannot be connected to R, there are at most $2^{(l-1)m/l - m^{1/2}}$ choices. This all gives

$$\begin{aligned}\log \frac{|\mathscr{C}(U)|}{S_m} &\leqslant \log(m+1) + (l-1) q \log m + \frac{l-1}{l} m - m^{1/2} \\ &\leqslant \frac{l-1}{l} m - \frac{m^{1/2}}{2}.\end{aligned}$$

LEMMA 1.9. *Let $\mathscr{D}(U)$ denote the set of graphs in $\mathscr{S}_{m+1}$ containing a vertex v with a Q-set $Q = \bigcup_{i=1}^{l-1} Q_i$ such that $|R(Q)| \leqslant m/l - m/\log q$. Then*

$$\log \frac{|\mathscr{D}(U)|}{S_{m-(l-1)q}} \leqslant \frac{l-1}{l}((l-1)q+1)m - \frac{qm}{2\log q},$$

for m sufficiently large.

PROOF OF LEMMA 1.9. Construct all graphs in $\mathscr{D}(U)$ as follows: Choose v ($m+1$ ways); then choose $Q = \bigcup_{i=1}^{l-1} Q_i$ (at most $2^{(l-1)m}$ ways); then choose a graph in $\mathscr{S}_{m-(l-1)q}$ on $U \setminus (\{v\} \cup Q)$ (at most $S_{m-(l-1)q}$ ways); then find an R-set $R(Q)$ (at most 2^m ways). Now connect v to $U - (R \cup Q)$ (at most 2^m ways); and finally connect Q to $U - (\{v\} \cup Q)$. There are at most $2^{|R|(l-1)q}$ ways to connect R to Q. To connect an element $x \in U \setminus (\{v\} \cup Q \cup R)$ to Q choose at most $l-2$ of the Q_i (at most 2^l ways) and connect x to these Q_i (at most $2^{(l-2)q}$ ways). There are at most $(2^l \cdot 2^{(l-2)q})^{(m-|R|-(l-1)q)}$ ways to connect Q and $U \setminus (\{v\} \cup Q \cup R)$. Together this gives, for m sufficiently large,

$$\begin{aligned}\log \frac{|\mathscr{D}(U)|}{S_{m-(l-1)q}} &\leqslant \log(m+1) + (l-1)m + 2m + (l-1)q|R| \\ &\quad + (l + (l-2)q)(m - |R| - (l-1)q) \\ &\leqslant 2(l+1)m + (l-2)qm + q|R| \\ &\leqslant 2(l+1)m + (l-2)qm + q\left(\frac{m}{l} - \frac{m}{\log q}\right) \\ &\leqslant \frac{(l-1)^2}{l} qm - \frac{qm}{\log q} + 2(l+1)m \\ &\leqslant \frac{l-1}{l}(l-1)qm - \frac{qm}{2\log q} < \left(\frac{l-1}{l}\right)((l-1)q+1)m - \frac{qm}{2\log q}.\end{aligned}$$

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LEMMA 1.10. *Let $\mathscr{E}_k(U)$, $1 \leqslant k \leqslant l$, denote the set of graphs in $\mathscr{S}_{m+1}$ having k vertices $V = \{v_1, \dots, v_k\}$ forming a K_k such that*

$$\text{(1)} \qquad V \text{ is not contained in any } (k+1)\text{-clique } K_{k+1}$$

and such that for each v_i there is a Q-set $Q(i)$ for which the corresponding R-set $R_i = R(Q(i))$ satisfies $m/l - m/\log q \leqslant |R_i|$ and moreover for these R_i, the set $T = \{x \in U \setminus V \mid x \notin \bigcup_{i=1}^k R_i\}$ satisfies

$$\text{(2)} \qquad |T| \geqslant m/\log^{(2)} q.$$

Then

$$\log \frac{|\mathscr{E}_k(U)|}{S_{m-k+1}} \leqslant \frac{l-1}{l} mk - \frac{m}{\log q},$$

for m sufficiently large.

PROOF OF LEMMA 1.10. Construct all graphs in $\mathscr{E}_k(U)$ as follows. First choose $V = \{v_1, \dots, v_k\}$ (at most $(m+1)^k$ ways). Then choose a graph on $U \setminus V$, which can be made into a graph in $\mathscr{E}_k(U)$ by connecting the k-clique V in an appropriate way. Such a graph can be chosen in at most S_{m-k+1} ways.

Now choose appropriate $Q(i)$'s in $U \setminus V$ (in at most $m^{(l-1)qk}$ ways), and finally connect V to $U \setminus V$. For each $1 \leqslant h \leqslant k$ and each choice of $1 \leqslant i_1 < \cdots < i_h \leqslant k$ let $T_{i_1 \cdots i_h} = R_{i_1} \cap \cdots \cap R_{i_h}$. Note that

$$T = U \setminus \left(\bigcup_h \bigcup_{i_1, \dots, i_h} T_{i_1, \dots, i_h} \cup V \right).$$

The number of ways to connect V to $T_{i_1, \dots, i_h}$ is at most $2^{(k-h)|T_{i_1, \dots, i_h}|}$. T can be connected to V in at most $(2^k - 1)^{|T|}$ ways, by assumption (1).

Thus V can be connected to $U \setminus V$ in at most

$$(2^k - 1)^{|T|} 2^{\sum_{h=1}^k \sum_{i_1, \dots, i_h} |T_{i_1, \dots, i_h}|(k-h)}$$

ways. For the sum above we have

$$\begin{aligned} \sum_{h=1}^k \sum_{i_1, \dots, i_h} \left|T_{i_1, \dots, i_h}\right|(k-h) &= k \left(\sum_{h=1}^k \sum_{i_1, \dots, i_h} \left|T_{i_1, \dots, i_h}\right| \right) - \sum_{h=1}^k \sum_{i_1, \dots, i_h} \left|T_{i_1, \dots, i_h}\right| h \\ &= k|U - (T \cup V)| - \sum_{h=1}^k |R_h| \\ &\leqslant k(m+1) - k|T| - k\left(\frac{m}{l} - \frac{m}{\log q} \right) \\ &= \frac{l-1}{l} mk - k(|T| - 1) + \frac{km}{\log q}. \end{aligned}$$

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Thus the number of connections is at most

$$2^{|T|\log(2^k-1)+(l-1)mk/l-k(|T|-1-m/\log q)}.$$

These inequalities give

$$\log\frac{|\mathscr{E}_k(U)|}{S_{m-k+1}} \leqslant k\log(m+1)+(l-1)kq\log m+|T|\log(2^k-1)$$
$$+\frac{l-1}{l}mk-k|T|+k+\frac{km}{\log q}$$
$$\leqslant \frac{l-1}{l}mk-|T|\big(k-\log(2^k-1)\big)+\frac{2km}{\log q}\leqslant\frac{l-1}{l}mk-\frac{m}{\log q},$$

for m sufficiently large. □

Let G be a graph on U such that for every vertex v there is a Q-set Q and an R-set $R=R(Q)$ satisfying

$$\frac{m}{l}-\frac{m}{\log q}\leqslant|R|\leqslant\frac{m}{l}+m^{1/2}.$$

Note that if $k<l$, then

$$|T|\geqslant m-\frac{k}{l}m-km^{1/2}>\frac{m}{\log^{(2)}q},$$

for any k-clique of vertices V and m sufficiently large. So assumption (2) of Lemma 1.10 is automatically satisfied for such a G and V. Hence

(1.11) If $k<l$, and if G is not in $\mathscr{E}_k(U)$, and if m is sufficiently large, then assumption (1) of Lemma 1.10 must fail for every k-clique of vertices V and, therefore, every k-clique must be contained in an $(k+1)$-clique.

(1.12) If $k=l$, then assumption (1) is automatically satisfied for every l-clique of vertices V. Hence if G is not in $\mathscr{E}_l(U)$ we have $|T|\leqslant m/\log^{(2)}q$, for every l-clique V.

Moreover $\bigcup_{i=1}^{l}R_i\cup T=U\setminus V$ and

$$\sum_{i=1}^{l}|R_i|+|T|\leqslant l\Big(\frac{m}{l}+m^{1/2}\Big)+\frac{m}{\log^{(2)}q},$$

the numbers of elements in more than one of the R_i's is at most

$$(k-1)+lm^{1/2}+\frac{m}{\log^{(2)}q}\leqslant\frac{2m}{\log^{(2)}q}\quad\text{for } m \text{ sufficiently large.}$$

In particular, in this case $(k=l)$ we have for $i\neq j$: $|R_i\cap R_j|\leqslant 2m/\log^{(2)}q$.

Definition 1.13. We call a graph G in

$$\mathscr{S}_{m+1}\setminus\Bigg[\bigcup_{k=1}^{l-2}\mathscr{A}\big(U,k,\lceil\log^{(k2-k+2)}(m)\rceil\big)$$
$$\cup\Bigg(\bigcup_{k=0}^{l-2}\mathscr{B}\big(u,k\lceil\log^{(k2-k+2)}(m)\rceil\big)\Bigg)$$
$$\cup\mathscr{C}(U)\cup\mathscr{D}(U)\cup\Bigg(\bigcup_{k=1}^{l}\mathscr{E}_k(U)\Bigg)\Bigg]$$

an *EQR-graph*.

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In particular, for every vertex v in an EQR-graph G there are sets $Q'_1, \ldots, Q'_{l-1}$ each of size at least 2^{q-1} so that all choices of $Q_1, \ldots, Q_l$, where $Q_i \subseteq Q'_i$ and $|Q_i| = q = \left\lceil \log^{(l^2-3l+5)}(m) \right\rceil$, $i = 1, \ldots, l-1$, form a Q-set Q for which the corresponding R-set $R = R(Q)$ satisfies:

$$\frac{m}{l} - \frac{m}{\log q} \leqslant |R| \leqslant \frac{m}{l} + m^{1/2}.$$

LEMMA 1.14. *Let G be an EQR-graph. Let v, v' be adjacent vertices in G, let Q, Q' be Q-sets of v, v' respectively and R, R' be the corresponding R-sets. Then*

$$|R \cap R'| \leqslant 2m/\log^{(2)} q \quad \textit{for } m \textit{ sufficiently large}.$$

PROOF OF LEMMA 1.14. From (1.11) we see that v, v' must be part of a K_3 (choosing $k = 2$) and thus part of a K_4 (choosing $k = 3$ in (1.11)), and so on, until we know that v, v' are part of a K_l for m large enough. Then, by (1.12), $|R \cap R'| \leqslant 2m/\log^{(2)} q$.

Next we derive some properties of EQR-graphs, which enable us to show that EQR-graphs l-colorable.

LEMMA 1.15. *Let G be an EQR-graph. Then G has the following property for m sufficiently large.*

(A3) *For every $x_1, \ldots, x_l$ forming a K_l and every w there is an i, $1 \leqslant i \leqslant l$, and there are vertices $y_1, \ldots, y_{l-1}, u, z_1, \ldots, z_{l-1}$ such that u is different from the y_k's and from the z_k's, the y_k's and the z_k's each form a K_{l-1}, x_i and u are connected to each y_k and u and w are connected to each z_k. (See Figure 1.)*

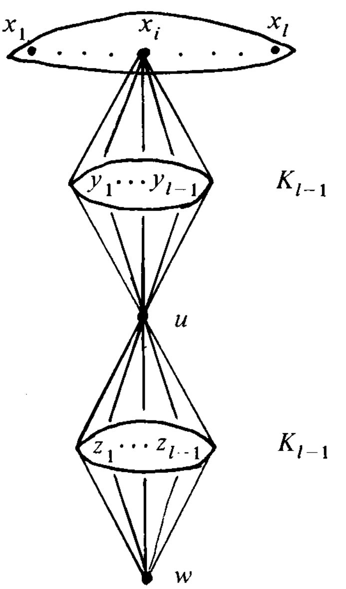

FIGURE 1

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In this case we say that x_i and w are connected via a double spindle through u. Note that if $w = u$ and the y_k's are the same vertices as the z_k's then the double spindle degenerates to a spindle.

PROOF OF LEMMA 1.15. Let $x_1, \dots, x_l$ be a K_l-subgraph of G and let w be an arbitrary vertex in G. Let $Q(0)$, $Q(i)$, $i = 1, \dots, l$, be Q-sets of w, $x_1, \dots, x_l$ resp., and $R(0)$, $R(i)$, $i = 1, \dots, l$, be the corresponding R-sets.

Since G is an EQR-graph it follows from Lemma 1.14 that

$$\left| \bigcup_{i=1}^{l} R(i) \right| \geqslant m - \frac{lm}{\log q} - \binom{l}{2} \frac{2m}{\log^{(2)} q}.$$

Furthermore, we have

$$|R(0)| \geqslant \frac{m}{l} - \frac{m}{\log q}.$$

Hence, there exists an i, $1 \leqslant i \leqslant l$, such that $|R(0) \cap R(i)| \geqslant m/l^2 - 3ml/\log^{(2)} q$ for m sufficiently large. Then for every $u \in R(0) \cap R(i)$ the vertices x_i and w are connected via a double spindle through u. □

For later use we derive the following corollary.

COROLLARY 1.16. *Let $k > 0$ be an integer. Let G be an EQR-graph and $v_1, \dots, v_k$ be vertices of G. Then the subgraph of G induced by $U \setminus \{v_1, \dots, v_k\}$ still has the property* (A3), *provided that m is sufficiently large depending on k.*

PROOF OF COROLLARY 1.16. Let $x_1, \dots, x_l$ and w be vertices in $U \setminus \{v_1, \dots, v_k\}$ such that $x_1, \dots, x_l$ form a K_l. From Definition 1.13 of EQR-graphs it follows that there are Q-sets $Q(0)$, $Q(1), \dots, Q(l)$, associated with $w, x, \dots, x_l$ respectively, which are all disjoint from $v_1, \dots, v_k$. Since we know that there exists i, $1 \leqslant i \leqslant l$, such that

$$|R(0) \cap R(i)| \geqslant \frac{m}{l^2} - \frac{3ml}{\log^{(2)} q}$$

in G, we conclude that there is a vertex $u \in (R(0) \cap R(i)) \setminus \{v_1, \dots, v_k\}$ and thus x_i is connected to w via a double spindle through u. □

LEMMA 1.17. *Let G be an EQR-graph. Let $x, u, w \in U$, let $Q(x), Q(u), Q(w)$ be Q-sets of x, u, w respectively and let $R(x), R(u)$ and $R(w)$ be the corresponding R-sets. Assume that x and w are connected via a double-spindle through u. Then*

$$|R(x) \cap R(w)| \geqslant \frac{m}{l} - \frac{5l^2 m}{\log^{(2)} q} \quad \textit{for } m \textit{ sufficiently large}.$$

PROOF OF LEMMA 1.17. The argument is basically that the R-sets associated with x (with w respectively) and with u must be almost the same, since they are both almost disjoint from the R-sets associated to the $l - 1$ vertices in the spindle joining them. Transitivity then gives that the R-sets of x and w are almost the same.

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More precisely: Let $y_1, \dots, y_{l-1}$ form a K_{l-1} and let x and u be connected to each of the y_k. Let $Q(i)$ be a Q-set of y_i and $R(i)$ be the corresponding R-set, for $i = 1, \dots, l-1$. From Lemma 1.14 we know that in $R(x)$ there are at least

$$\frac{m}{l} - \frac{m}{\log q} - (l-1)\frac{2m}{\log^{(2)} q} \geqslant \frac{m}{l} - \frac{2lm}{\log^{(2)} q}$$

vertices which are not in $\bigcup_{i=1}^{l-1} R(i)$. Similarly $R(u)$ contains at least $m/l - 2lm/\log^{(2)} q$ vertices not in $\bigcup_{i=1}^{l-1} R(i)$.

But

$$\left| U - \bigcup_{i=1}^{l-1} R(i) \right| \leqslant m + 1 - \left[(l-1)\left[\frac{m}{l} - \frac{m}{\log q}\right] - \binom{l-1}{2}\frac{2m}{\log^{(2)} q} \right]$$

$$\leqslant \frac{m}{l} + \frac{l^2 m}{\log^{(2)} q} \quad \text{for } m \text{ sufficiently large.}$$

Therefore

$$|R(x) \cap R(u)| \geqslant \frac{m}{l} - \frac{4lm}{\log^{(2)} q} - \frac{l^2 m}{\log^{(2)} q} \geqslant \frac{m}{l} - \frac{2l^2 m}{\log^{(2)} q}.$$

Analogously,

$$|R(u) \cap R(w)| \geqslant \frac{m}{l} - \frac{2l^2 m}{\log^{(2)} q}.$$

Combining this and $|R(u)| \leqslant m/l + m^{1/2}$ we obtain

$$|R(x) \cap R(w)| \geqslant \frac{m}{l} - \frac{4l^2 m}{\log^{(2)} q} - m^{1/2}$$

$$\geqslant \frac{m}{l} - \frac{5l^2 m}{\log^{(2)} q} \quad \text{for } m \text{ sufficiently large.} \quad \square$$

Lemma 1.17 yields immediately

COROLLARY 1.18. *Let G be an EQR-graph. Then G has the following property for m sufficiently large*:

(A4) *If H is a graph of the following type, then H is not a subgraph of G: H has vertices* $t, u, u', v, v', y_1, \dots, y_{l-1}, y'_1, \dots, y'_{l-1}, z_1, \dots, z_{l-1}, z'_1, \dots, z'_{l-1}$, *where* v *is different from* v' *and* t, u, u', v, v', *are different from all the* y_k, y'_k, z_k *and* z'_k. *The edges of H are as follows*: t *and* u *are connected to each* y_k, t *and* u' *are connected to each* y'_k, u *and* v *are connected to each* z_k, *and* u' *and* v' *are connected to each* z'_k. *Moreover, all the* y_k *form an* $(l-1)$-*clique as do the* y'_k, *the* z_k *and the* z'_k, *respectively. Finally,* v *is connected to* v'.

Notice that any graph H as in Figure 2 has chromatic number $l+1$. Also notice that we allow the spindles to overlap, e.g., in the most extreme case $z_i = z'_i = y_i = y'_i$ for every $i = 1, \dots, l-1$ and $v = u = u' = t$. Hence K_{l+1} is one of these graphs.

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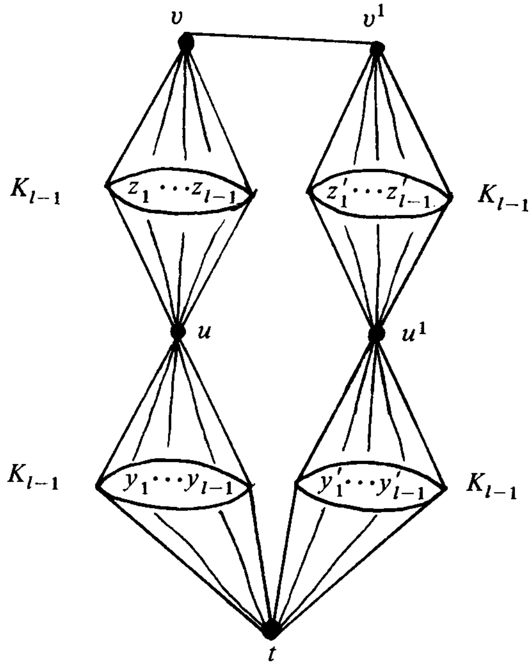

FIGURE 2

We refer to these graphs as the *l-Moser graphs*. We should mention that the 2-Moser graphs include the odd cycles of length 3, 5, 7 and 9 and that the Moser graph M itself is a special 3-Moser graph. (See Figure 3.)

PROOF OF COROLLARY 1.18. Let $Q(t), Q(v)$ and $Q(v')$ be Q-sets of t, v, v' resp. and let $R(t), R(v)$ and $R(v')$ be the corresponding R-sets. From Lemma 1.17 it follows that

$$|R(t) \cap R(v)| \geqslant \frac{m}{l} - \frac{5l^2m}{\log^{(2)} q}$$

and

$$|R(t) \cap R(v')| \geqslant \frac{m}{l} - \frac{5l^2m}{\log^{(2)} q}.$$

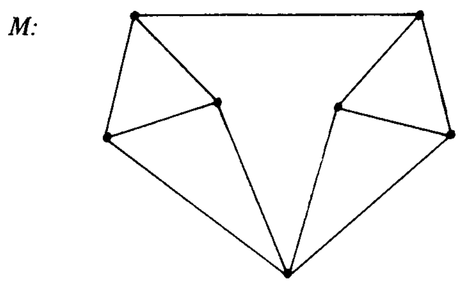

FIGURE 3

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Hence

$$|R(v)\cap R(v')|\geqslant|R(t)\cap R(v)\cap R(v')|\geqslant\frac{m}{l}-\frac{10l^2m}{\log^{(2)}q}-m^{1/2},$$

which contradicts Lemma 1.14, for m sufficiently large. □

LEMMA 1.19. *Let G be a graph defined on U satisfying the properties*:

(A1) *G is K_{l+1}-free*,

(A2) *G contains a K_l*,

(A3) *and* (A4) *as given in Lemma* 1.15, *Corollary* 1.18 *respectively.*

Let $x_1,\ldots,x_l$ be vertices in G forming a K_l and for each i, $1\leqslant i\leqslant l$, let

$$P(x_i)=\{w\in G\colon w \text{ is connected to } x_i \text{ via a double spindle}\}.$$

Then G is l-colorable with parts $P(x_1),\ldots,P(x_l)$.

PROOF OF LEMMA 1.19. From property (A3) it follows that $\bigcup_{i=1}^{l}P(x_i)=U$. Using property (A4) twice it follows easily that each $P(x_i)$ is an independent set and that $P(x_i)\cap P(x_j)=\varnothing$ for every $i\neq j$. □

COROLLARY 1.20. *Let G be an EQR-graph. Then G is l-colorable and the size of each color-class is at least $m/l-m/\log q$, provided that m is sufficiently large.* □

PROOF OF THEOREM 1. Let $p(n)$ be an arbitrary polynomial with positive leading coefficient. We want to show that for some C

$$S_n\leqslant(1+C/p(n))L_n \quad \text{for all } n.$$

As indicated, we use induction on n.

Let n_0 be large enough so that for $m\geqslant n_0$ all the previous lemmas, corollaries, statements and the inequalities given below hold. Then we assume that $C>1$ is so large that

$$S_n\leqslant(1+C/p(n))L_n \quad \text{for all } n\leqslant n_0.$$

Now assume that the inequality is valid for some $n\geqslant n_0$. We wish to show that

$$S_{n+1}\leqslant(1+C/p(n+1))L_{n+1},$$

where the graphs in $\mathscr{S}_{n+1}$ (recall that $S_{n+1}=|\mathscr{S}_{n+1}|$) are assumed to be defined on a set U, with $|U|=n+1$, e.g., $U=\{1,\ldots,n+1\}$.

By Corollary 1.20 we have that

$$\begin{aligned}|\mathscr{S}_{n+1}|\leqslant{}&\sum_{k=1}^{l-2}\left|\mathscr{A}\left(U,k,\lceil\log^{(k^2-k+2)}(n)\rceil\right)\right|\\&+\sum_{k=0}^{l-2}\left|\mathscr{B}\left(U,k,\lceil\log^{(k^2-k+2)}(n)\rceil\right)\right|\\&+|\mathscr{C}(U)|+|\mathscr{D}(U)|+\sum_{k=1}^{l}|\mathscr{E}_k(U)|+L_{n+1}.\end{aligned}$$

Thus it is sufficient to show that each of the $3l-1$ terms on the right side of this inequality different from L_{n+1} is at most $(C/(3l-1)p(n+1))L_{n+1}$.

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These inequalities are verified below:

(A) Let k, $1 \leqslant k \leqslant l-2$, be fixed. Let $x(k,n) = \lceil \log^{(k^2-k+2)}(n) \rceil$.

$$\frac{|\mathscr{A}(U,k,x(k,n))|}{L_{n+1}} = \frac{|\mathscr{A}(U,k,x(k,n))|}{S_{n+1-kx(k,n)}} \cdot \frac{S_{n+1-kx(k,n)}}{L_{n+1-kx(k,n)}} \cdot \prod_{i=1}^{kx(k,n)} \frac{L_{n+1-i}}{L_{n+1-(i-1)}}.$$

We have upper bounds for the first factor by Lemma 1.3, for the second factor by the inductive hypothesis and for the remaining product by Corollary 1.2. Hence we get

$$\frac{|\mathscr{A}(U,k,x(k,n))|}{L_{n+1}} \leqslant 2^{(l-1)knx(k,n)/l - nx(k,n)/2l}\left(1 + \frac{C}{p(n+1-kx(k,n))}\right)$$
$$\cdot 2^{-(l-1)(n+1-kx(k,n))kx(k,n)/l} 2^{(l+1)kx(k,n)\log n}$$
$$\leqslant 2^{-nx(k,n)/3l}\left(1 + \frac{C}{p(n+1-kx(k,n))}\right) \leqslant \frac{C}{(3l-1)p(n+1)},$$

for n sufficiently large (independent of C).

(B) Let k, $0 \leqslant k \leqslant l-2$, be fixed. Then

$$\frac{|\mathscr{B}(U,k,x(k,n))|}{L_{n+1}} \leqslant \frac{|\mathscr{B}(U,k,x(k,n))|}{S_n} \cdot \frac{S_n}{L_n} \cdot \frac{L_n}{L_{n+1}}.$$

Here we have upper bounds from Lemma 1.4, the induction hypothesis and Corollary 1.2.

Hence

$$\frac{|\mathscr{B}(U,k,x(k,n))|}{L_{n+1}} \leqslant 2^{(l-1)n/l - n/5l}\left(1 + \frac{C}{p(n)}\right) 2^{-(l-1)n/l + (l+1)\log n}$$
$$\leqslant 2^{-n/6l}\left(1 + \frac{C}{p(n)}\right) \leqslant \frac{C}{(3l-1)p(n+1)},$$

for n sufficiently large (independent of C).

(C)
$$\frac{|\mathscr{C}(U)|}{L_{n+1}} = \frac{|\mathscr{C}(U)|}{S_n} \cdot \frac{S_n}{L_n} \cdot \frac{L_n}{L_{n+1}}.$$

Here we get upper bounds from Lemma 1.8, the induction hypothesis and Corollary 1.2.

$$\frac{|\mathscr{C}(U)|}{L_{n+1}} \leqslant 2^{(l-1)n/l - n^{1/2}/2}\left(1 + \frac{C}{p(n)}\right) 2^{-(l-1)n/l + (l+1)\log n}$$
$$\leqslant 2^{-n^{1/2}/3}\left(1 + \frac{C}{p(n)}\right) \leqslant \frac{C}{(3l-1)p(n+1)}$$

for n sufficiently large (independent of C).

(D) Let $x = \log(x(l-1,n)) = \lceil \log^{(l^2-3l+5)}(n) \rceil$.

$$\frac{|\mathscr{D}(U)|}{L_{n+1}} = \frac{|\mathscr{D}(U)|}{S_{n-(l-1)x}} \frac{S_{n-(l-1)x}}{L_{n-(l-1)x}} \cdot \prod_{i=1}^{(l-1)x+1} \frac{L_{n+1-i}}{L_{n+1-(i-1)}}.$$

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Using Lemma 1.9, induction hypothesis and Corollary 1.2 we get

$$\frac{|\mathscr{D}(U)|}{L_{n+1}} \leqslant 2^{((l-1)/l)((l-1)x+1)n-xn/2\log x}\left(1+\frac{C}{p(n-(l-1)x)}\right)$$
$$\cdot 2^{-((l-1)/l)(n+1-(l-1)x-1)((l-1)x+1)} \cdot 2^{(l+1)((l-1)x+1)\log n}$$
$$\leqslant 2^{-xn/3\log x}\left(1+\frac{C}{p(n-(l-1)x)}\right) \leqslant \frac{C}{(3l-1)p(n+1)},$$

for n sufficiently large (independent of C).

(E) Let k, $1 \leqslant k \leqslant l$, be fixed.

$$\frac{|\mathscr{E}_k(U)|}{L_{n+1}} = \frac{|\mathscr{E}_k(U)|}{S_{n-k+1}} \frac{S_{n-k+1}}{L_{n-k+1}} \prod_{i=1}^{k} \frac{L_{n+1-i}}{L_{n+1-(i-1)}}.$$

By Lemma 1.10 and again the induction hypothesis and Corollary 1.2 we have

$$\frac{|\mathscr{E}_k(U)|}{L_{n+1}} \leqslant 2^{(l-1)nk/l-n/\log q}\left(1+\frac{C}{p(n+1-k)}\right) \cdot 2^{-(l-1)(n+1-k)k/l} 2^{(l+1)k\log n}$$
$$\leqslant 2^{-n/2\log q}\left(1+\frac{C}{p(n+1-k)}\right) \leqslant \frac{C}{(3l-1)p(n+1)},$$

for n sufficiently large (independent of C). This completes the proof of Theorem 1. □

It is worthwhile to note that in fact the proof of Theorem 1 yields stronger results than stated in the theorem.

COROLLARY 1.21. *Almost every K_{l+1}-free graph is an EQR-graph* (*in particular it has the properties* (A1)–(A4) (*cf. Lemma* 1.19). □

LEMMA 1.22. *Every K_{l+1}-free graph* (*not necessary finite*) *which has properties* (A2), (A3) *and* (A4) *is uniquely l-colorable.*

PROOF OF LEMMA 1.22. Let $G = (U, E)$ be a K_{l+1}-free graph satisfying (A2), (A3) and (A4). First notice that the same arguments as given in Lemma 1.19. yields that G is l-colorable. Thus let any l-coloring Δ of G be given.

Let w, w' be vertices in the same color with respect to Δ. We claim, that there exists x such that both w and w' are connected via a double spindle to x. From this follows immediately that w and w' must be in the same colorclass with respect to every l-coloring of G.

Let $\{x_1, \ldots, x_l\}$ be an l-clique in G (guaranteed by property (A2)). By property (A3) there exists x_i, x_j such that w is connected via a double spindle to x_k and w' is connected via a double spindle to x_j. Hence, w must have the same color as x_i and w' must have the same color as x_j. Therefore $x_i = x_j = x$, which completes the proof of the lemma. □

Combining Corollary 1.21 with Lemma 1.22 we get

COROLLARY 1.23. *Almost every K_{l+1}-free graph is uniquely l-colorable.* □

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2. A labeled 0-1 law for K_{l+1}-free graphs. A *similarity type* s is a finite sequence $s = (R_1, \ldots, R_m)$ of relational symbols $R_1, \ldots, R_m$. It **A** is any structure of similarity type s, then the *first-order theory* $T(\mathbf{A})$ of **A** is the set

$$T(\mathbf{A}) = \{\phi\colon \phi \text{ is a first-order sentence and } \mathbf{A} \models \phi\}.$$

Let $\mathscr{K}$ be an infinite class of finite structures such that $\mathscr{K}$ is closed under isomorphisms and each structure in $\mathscr{K}$ is of similarity type s and has universe an initial segment $\{1, 2, \ldots, n\}$ of the natural numbers.

If ϕ is a first-order sentence, then the *labeled asymptotic probabiliy* $\mu(\phi)$ *of* ϕ *on* $\mathscr{K}$ is given by the equation $\mu(\phi) = \lim_{n\to\infty} \mu_n(\phi)$ (provided this limit exists), where $\mu_n(\phi)$ is the fraction of members of $\mathscr{K}$ having cardinality n and satisfying ϕ.

The *first-order almost sure theory* $\Pi(\mathscr{K})$ of $\mathscr{K}$ is the set $\Pi(\mathscr{K}) = \{\phi\colon \phi$ is a first-order sentence and $\mu(\phi) = 1$ on $\mathscr{K}\}$. Using the compactness theorem it is easy to see that the almost sure theory $\Pi(\mathscr{K})$ is always consistent.

We say that the class $\mathscr{K}$ has a *first-order labeled* 0-1 *law* if $\mu(\phi)$ exists and $\mu(\phi) = 0$ or $\mu(\phi) = 1$ on $\mathscr{K}$ for every first-order sentence ϕ.

Notice that a class $\mathscr{K}$ has a first-order labeled 0-1 law if and only if the almost sure theory $\Pi(\mathscr{K})$ is complete. In this case for any model **A** of $\Pi(\mathscr{K})$ we have that

$$\mathbf{A} \models \phi \text{ if and only if } \mu(\phi) = 1 \text{ on } \mathscr{K},$$

for every first-order sentence ϕ.

The following straightforward and well-known proposition gives a sufficient condition for a class $\mathscr{K}$ to have a first-order labeled 0-1 law.

PROPOSITION 2.1. *Let $\mathscr{K}$ be an infinite class of finite structures with universe initial segments $\{1, 2, \ldots, n\}$ of the natural numbers such that $\mathscr{K}$ is closed under isomorphisms. Assume that Σ is a set of first-order sentences with the following properties*:

(i) $\mu(\psi) = 1$ *on $\mathscr{K}$ for every sentence ψ in Σ.*

(ii) Σ *has a unique* (*up to isomorphism*) *countable model* **A** (*in this case we say that Σ is ω-categorical*).

Then $\mathscr{K}$ has a first-order labeled 0-1 *law and moreover the almost sure theory $\Pi(\mathscr{K})$ of $\mathscr{K}$ coincides with the theory $T(\mathbf{A})$ of* **A**. *If in addition Σ is a recursive set of sentences, then $\Pi(\mathscr{K})$ is a decidable theory.* □

At this point we remind the reader of our blanket assumption that all graphs considered here are undirected and without loops. More precisely, a graph is a structure $G = \langle V, E\rangle$ such that E is a binary irreflexive and symmetric relation on V.

Fagin [**1976**] proved that if $\mathscr{G}$ is the class of all finite graphs, then $\mathscr{G}$ has a first-order labeled 0-1 law. This is done by considering the set of first-order sentences

$$\Sigma = \{\tau_0\} \cup \{\psi_k\colon k \geqslant 1\},$$

where τ_0 is the axiom

$$(\forall x)(\forall y)(xEy \to yEx) \wedge (\forall x)(\neg x\exists x)$$

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(E is a symmetric, irreflexive relation) and ψ_k is the "extension axiom"

$$(\forall y_1 \cdots \forall y_k)(\forall z_1 \cdots \forall z_k)\left(\bigwedge_{\substack{i,j \\ i \neq j}} \left((y_i \neq y_j) \wedge (z_i \neq z_j)\right) \wedge \bigwedge_{i,j} (y_i \neq z_j) \right.$$

$$\left. \rightarrow (\exists w)\left(\bigwedge_i (w \neq y_i) \wedge \bigwedge_i (wEz_i) \wedge \bigwedge_i (\neg(wEy_i))\right)\right).$$

Σ is shown to be an ω-categorical theory using a back and forth argument. Finally by estimating the number of graphs on which ψ_k fails, Fagin showed that $\mu(\neg\psi_k) = 0$. It turns out that the unique countable model of Σ is Rado's graph (Rado [**1964**]).

Compton [**1984**] investigated asymptotic probabilities on classes of finite relational structures and showed that in many cases the existence of a first-order labeled 0-1 law for such a class $\mathscr{C}$ is closely related to the asymptotic behavior of the exponential generating series $a(x)$ of $\mathscr{C}$, where

$$a(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n,$$

and a_n is the number of labeled structures of cardinality n in $\mathscr{C}$.

Let $\mathscr{C}$ be a class of finite structures closed under disjoint unions and connected substructures and assume that $a(x)$ has positive radius of convergence. One of Compton's main results states that for any such class $\mathscr{C}$ a labeled first-order 0-1 law holds for $\mathscr{C}$ if and only if the coefficients $a_n/n!$, $n \geqslant 1$, of $a(x)$ satisfy a certain growth condition. From this theorem Compton [**1984**] derives 0-1 laws for many classes of finite structures which arise naturally in combinatorics, including the collection of equivalence relations. Notice however that the class $\mathscr{G}$ of all finite graphs is not covered by this result, since in this case $a(x)$ converges only for $x = 0$. For the same reason Compton's theorem has no implication for the class $\mathscr{S}(l)$ of K_{l+1}-free graphs, $l \geqslant 2$.

We prove next that for any l the class $\mathscr{S}(l)$ of K_{l+1}-free graphs has a first-order labeled 0-1 law. We introduce a set $\Sigma(l)$ of first-order axioms and we show that they form an ω-categorical theory and that each axiom has labeled asymptotic probability equal to 1 on $\mathscr{S}(l)$. The motivation for the axioms in $\Sigma(l)$ comes from the structural information for almost all K_{l+1}-free graphs obtained in the first part of this paper.

Let

$$\Sigma(l) = \{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4,\} \cup \{\sigma_k : k \geqslant 1\}$$

be the set of the first-order sentences τ_i and σ_k ($1 \leqslant i \leqslant 4$, $k \geqslant 1$) over the similarity type with only one binary relation symbol E, where these sentences are given as follows:

τ_0 is the axiom $(\forall x)(\forall y)(xEy \rightarrow yEx) \wedge (\forall x)(\neg(xEx))$ and $\tau_1, \tau_2, \tau_3, \tau_4$ express the properties (A1), (A2), (A3), (A4) respectively in Lemma 1.19. This means that if G is a graph satisfying $\tau_1, \tau_2, \tau_3, \tau_4$, then

1. G is K_{l+1}-free,
2. G contains a K_l as a subgraph,

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3. if the vertices $x_1, \ldots, x_l$ form a K_l, then every vertex w is connected to one of the x_i's via a double spindle, and

4. G does not contain any of the l-Moser graphs as a subgraph.

Lemma 1.19 implies that any such graph G is l-colorable and that any l-clique $x_1, \ldots, x_l$ in G gives rise to the parts $P(x_1), \ldots, P(x_l)$ defined by the first-order formula $\psi(x_i, w)$ which asserts that w is connected to x_i via a double spindle.

The axioms σ_k are "extension principles for l-colorable graphs". For each $k \geqslant 1$ we let σ_k be the first-order sentence

$$(\forall x_1 \cdots \forall x_l)(\forall y_1 \cdots \forall y_k)(\forall z_1 \cdots \forall z_k)(\forall w_1 \cdots \forall w_k)$$

$$\Bigg[\Bigg[\bigwedge_{\substack{i,j\\ i \neq j}} (x_i E x_j) \wedge \bigwedge_{\substack{i,j\\ i \neq j}} \big((y_i \neq y_j) \wedge (z_i \neq z_j) \wedge (w_i \neq w_j)\big)$$

$$\wedge \bigwedge_{\substack{i,j\\ i \neq j}} (y_i \neq z_j) \wedge \bigwedge_{i} \big(\neg\psi(x_1, y_i) \wedge \neg\psi(x_1, z_i) \wedge \psi(x_1, w_i)\big)\Bigg]$$

$$\to (\exists w)\Big(\psi(x_1, w) \wedge \bigwedge_{i} \big((w \neq w_i) \wedge (wEy_i) \wedge (\neg(wEz_i))\big)\Big)\Bigg].$$

If a graph G satisfies $\tau_1, \tau_2, \tau_3, \tau_4$, and σ_k, then for any l-clique $x_1, \ldots, x_l$, any two sets $\{y_1, \ldots, y_k\}$, $\{z_1, \ldots, z_k\}$ disjoint from $P(x_1)$, and any subset $\{w_1, \ldots, w_k\}$ of $P(x_1)$ there is a vertex w in $P(x_1)$ which is different from all the w_i and such that it connects to all the y_i and to none of the z_i.

First we prove that each axiom in $\Sigma(l)$ has labeled asymptotic probability equal to 1 on the class $\mathscr{S}(l)$ of K_{l+1}-free graphs.

It is obvious that $\mu(\tau_0) = \mu(\tau_1) = 1$ on $\mathscr{S}(l)$. By Corollary 1.21 almost every K_{l+1}-free graph is an EQR-graph and $\mu(\tau_2) = \mu(\tau_3) = \mu(\tau_4) = 1$ on $\mathscr{S}(l)$. We now show

Lemma 2.2. *Let $I_n(k)$ be the number of graphs in $\mathscr{S}(l)$ with n vertices which do not satisfy the axiom σ_k. Then*

$$\lim_{n \to \infty} \frac{I_n(k)}{S_n(l)} = 0$$

and thus $\lim_{n \to \infty} \mu_n(\neg\sigma_k) = 0$ *and* $\mu(\sigma_k) = 1$ *on* $\mathscr{S}(l)$.

Proof. Let $I_n'(k)$ be the number of graphs in $\mathscr{S}(l)$ with n vertices which are EQR and satisfy $\neg\sigma_k$, and let $I_n''(k) = I_n(k) - I_n'(k)$. Corollary 1.21 implies immediately that

$$\lim_{n \to \infty} \frac{I_n''(k)}{S_n(l)} = 0.$$

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We show that $\lim_{n\to\infty} I_n'(k)/S_n(l) = 0$ by finding an upper bound for $I_n'(k)$ and using Theorem 1 together with the lower bound for $L_n(l)$ from Lemma 1.1.

If G is any graph which satisfies $\tau_1, \tau_2, \tau_3, \tau_4$, and $\neg\sigma_k$, then there are vertices $y_1, \dots, y_k, z_1, \dots, z_k, w_1, \dots, w_k, x_1, \dots, x_l$ in G such that the x_i's form a K_l $\{y_1, \dots, y_k, z_1, \dots, z_k\}$ is disjoint from $P(x_1)$, $\{w_1, \dots, w_k\}$ is a subset of $P(x_1)$ and for every w in $P(x_1)$ which is different from all the w_i's either w does not connect to one of y_i's or it connects to one of the z_i's. Therefore we can construct every graph G in $\mathscr{S}(l)$ with n vertices which is EQR and satisfies $\neg\sigma_k$ as follows:

First we choose vertices $y_1, \dots, y_k, z_1, \dots, z_k$ (at most $\binom{n}{k}^2$ ways). Then we impose a graph on these $2k$ elements (at most $2^{\binom{2k}{2}}$ ways). Next we choose a graph H on the remaining $n - 2k$ elements with the prcperty that $y_1, \dots, y_k, z_1, \dots, z_k$ can be connected to the vertices of H in such a way that the resulting graph G is EQR, and thus, by Corollary 1.20, l-colorable with parts of size at least $(n-1)/l - (n-1)/\log q(n-1)$ for all sufficiently large n (there are at most $L_{n-2k}(l)$ ways to choose such a graph H). Corollary 1.16 and Lemma 1.22 imply that H must be uniquely l-colorable and therefore for every part P of H and all sufficiently large n we must have

$$\frac{n-1}{l} - \frac{n-1}{\log q(n-1)} - 2k \leqslant |P|.$$

As a consequence

$$|P| \leqslant \frac{n}{l} + \frac{(l-1)n}{\log q(n-1)} + 1 - \frac{1}{l} + 2k(l-1) \leqslant \frac{n}{l} + \frac{(l-1)n}{\log^{(2)} q(n-1)}$$

for every part P of H and all sufficiently large n.

In order to connect H to $\{y_1, \dots, y_k, z_1, \dots, z_k\}$ in such a way that the y_i's and the z_i's witness $\neg\sigma_k$ in G we first choose a part P_1 (l-ways) and for each of the y_i's and each of the z_i's we choose a part different from P_1 (at most $(l-1)^{2k}$ ways). We choose next the remaining witnesses $w_1, \dots, w_k$ of $\neg\sigma_k$ from the part P_1 (at most $\binom{n}{k}$ ways). For each w in P_1 which is different from the w_k's we have $2^{2k} - 1$ ways to connect it to the y_i's and the z_i's (in order to avoid connecting w to all the y_k's and to none of the z_i's). It follows that there are at most $(2^{2k} - 1)^{n/l + (l-1)n/\log^{(2)} q(n-1)}$ ways to connect $P_1 - \{w_1, \dots, w_k\}$ to $\{y_1, \dots, y_k, z_1, \dots, z_k\}$. Also there are 2^{2k^2} ways to connect $\{w_1, \dots, w_k\}$ to $\{y_1, \dots, y_k, z_1, \dots, z_k\}$. It only remains to connect the y_i's and the z_i's to vertices in the $l-1$ parts of H which are disjoint from P_1. Each y_k and each z_i must connect to vertices in at most $l-2$ parts because it must avoid its own part. Therefore there are at most

$$2^{2k(l-2)(n/l + n(l-1)/\log^{(2)} q(n-1))}$$

ways to make these connections. Finally combining these estimates we obtain that for all sufficiently large n

$$I_n'(k) \leqslant \binom{n}{k}^2 2^{\binom{2k}{2}} L_{n-2k}(l) l (l-1)^{2k} \binom{n}{k} (2^{2k-1})^{n/l + (l-1)n/\log^{(2)} q(n-1)}$$
$$\cdot 2^{2k^2} 2^{2k(l-2)(n/l + n(l-1)/\log^{(2)} q(n-1))}$$

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and hence

$$I_n'(k) \leqslant L_{n-2k}(l) 2^{2k(l-2)n/l}(2^{2k}-1)^{n/l} \cdot 2^{o(n)}.$$

By Lemma 1.1 we have that for all sufficiently large n

$$\frac{I_n'(k)}{L_n(l)} \leqslant \frac{L_{n-2k}(l)}{L_n(l)} 2^{2k(l-1)n/l}(2^{2k-1})^{n/l} \cdot 2^{o(n)}$$

$$\leqslant \frac{2^{\binom{l}{2}(n-2k)^2/l^2+(n-2k)\log l} \cdot 2^{2k(l-2)n/l}(2^{2k-1})^{n/l} \cdot 2^{o(n)}}{2^{\binom{l}{2}n^2/l^2+(\log l)n - l\log n}}$$

$$\leqslant 2^{-\binom{l}{2}4kn/l^2+2k(l-2)n/l}(2^{2k-1})^{n/l} \cdot 2^{o(n)}$$

$$= 2^{-2kn/l}(2^{2k}-1)^{n/l} \cdot 2^{o(n)} = (1-1/2^{2k})^{n/l} \cdot 2^{o(n)}$$

and hence $\lim_{n\to\infty} I_n'(k)/L_n(l) = 0$. Since, by Theorem 1, $\lim_{n\to\infty} L_n(l)/S_n(l) = 1$ we obtain immediately that $\lim_{n\to\infty} I_n'(k)/S_n(l) = 0$ and consequently $\mu(\neg\sigma_k) = 0$ on $\mathscr{S}(l)$.

This completes the proof that the axioms in $\Sigma(l)$ have labeled asymptotic probability equal to 1 on $\mathscr{S}(l)$. We can now use this fact to derive further structural information about almost all K_{l+1}-free graphs.

COROLLARY 2.3. (i) *Let G be any finite l-colorable graph. Then almost all K_{l+1}-free graphs contain G as an induced subgraph.*

(ii) *Almost all K_{l+1}-free graphs are uniquely l-colorable and have the property that any two vertices in the same part are connected via a spindle.*

PROOF. Both (i) and (ii) are proved by iterated applications of appropriate extension axioms σ_k. □

LEMMA 2.4. *For each $l \geqslant 2$ the set of axioms*

$$\Sigma(l) = \{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4\} \cup \{\sigma_k\colon k \geqslant 1\}$$

forms an ω-categorical theory, that is to say it has a unique (up to isomorphism) countable model $\mathbf{D}(l)$.

PROOF. From Lemma 2.2 it follows that every finite subset of $\Sigma(l)$ has a model (actually a finite model) and hence the compactness theorem guarantees the existence of a countable model of $\Sigma(l)$.

Assume now that G and G' are two countable models of $\Sigma(l)$. Since both G and G' satisfy $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4$, we have, by Lemma 1.22, that they are uniquely l-colorable graphs with parts $P_1, \ldots, P_l$ and $P_1', \ldots, P_l'$ respectively. We claim that we can easily construct an isomorphism $f\colon G \to G'$ such that $f(P_i) = P_i'$ for $i = 1, 2, \ldots, l$. The extension axioms σ_k, $1 \leqslant k < \omega$, ensure that we can obtain this isomorphism using a back and forth argument. □

Combining Proposition 2.1 with Lemmas 2.2 and 2.4 we establish

THEOREM 2. *The class $\mathscr{S}(l)$ of K_{l+1}-free graphs has a first-order labeled* 0-1 *law.*

COROLLARY 2.5 *The class $\mathscr{L}(l)$ of l-colorable graphs has a first-order labeled* 0-1 *law.*

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The proof of Theorem 2 actually yields the following facts about the almost sure theory $\Pi(\mathscr{S}(l))$:

COROLLARY 2.6. *Let*

$$\Pi(\mathscr{S}(l)) = \{\phi\colon \phi \textit{ is a first-order sentence and } \mu(\phi) = 1 \textit{ on } \mathscr{S}(l)\}$$

be the almost sure theory of K_{l+1}-free graphs. Then

(i) $\Pi(\mathscr{S}(l))$ *is an ω-categorical, decidable theory*

(ii) $\Pi(\mathscr{S}(l))$ *has the finite substructure property, that is to say if ψ is a first-order sentence true in a model* **B** *of* $\Pi(\mathscr{S}(l))$, *then ψ is true in a finite substructure of* **B**. *As a consequence*, $\Pi(\mathscr{S}(l))$ *is not a finitely axiomatitable theory.*

(iii) *The countable model* $\mathbf{D}(l)$ *of* $\Pi(S(l))$ *is an l-colorable graph with uniquely determined parts. Moreover, a finite graph H is a submodel of* $\mathbf{D}(l)$ *if and only if H is l-colorable.*

3. Labeled 0-1 laws and the amalgamation property. The classes $\mathscr{G}$ of all graphs and $\mathscr{S}(l)$ of K_{l+1}-free graphs, where $l \geqslant 2$, are both examples of families of finite graphs which are closed under isomorphisms and substructures (induced subgraphs) and have the *amalgamation property* given below.

A class $\mathscr{K}$ of relational structures has the *amalgamation property* if for any structures **A**, **B**, **C**, in $\mathscr{K}$ and any embeddings $f\colon \mathbf{C} \to \mathbf{A}$ and $g\colon \mathbf{C} \to \mathbf{B}$ there is a structure **D** in $\mathscr{K}$ and embeddings $f'\colon \mathbf{A} \to \mathbf{D}$ and $g'\colon \mathbf{B} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$. (See Figure 4.)

We say that a relational structure **A** is *homogeneous* if any isomorphism between two finite substructures of **A** can be extended to an automorphism of **A**. Homogeneous structures were first considered by Fraissé [**1954**]. The following result from Woodrow [**1976**] relates classes having the amalgamation property to homogeneous structures.

LEMMA 3.1. (i) *Let $\mathscr{K}$ be an infinite class of finite relational structures having the amalgamation property and closed under isomorphisms and substructures. Then there is a unique* (*up to isomorphism*) *countable homogeneous structure $\mathscr{F}(\mathscr{K})$ such that the class of finite substructures of $\mathscr{F}(\mathscr{K})$ coincides with $\mathscr{K}$.*

(ii) *If* **A** *is a countable homogeneous relational structure, then the class $\mathscr{K}(\mathbf{A})$ of finite relational structures which are embeddable in* **A** *has the amalgamation property* (*and is obviously closed under isomorphisms and submodels*).

$$\begin{tikzcd}
& \mathbf{D} & \\
\mathbf{A} \arrow[ur, "f'"] & & \mathbf{B} \arrow[ul, "g'"'] \\
& \mathbf{C} \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

FIGURE 4

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If $\mathscr{K}$ is a class as in (i), then we refer to $\mathscr{F}(\mathscr{K})$ as the *Fraissé structure* of $\mathscr{K}$.

Notice that if $\mathscr{O}$ is the class of finite linear orderings, then $\mathscr{F}(\mathscr{O})$ is isomorphic to the ordering of the rational, that is to say the countable dense linear ordering without endpoints. Also if $\mathscr{G}$ is the class of all finite graphs, then it is easy to see that $\mathscr{F}(\mathscr{G})$ is Rado's graph.

We say that a graph H is an *equivalence graph* if it is the disjoint union of complete graphs. Notice that if $\mathscr{E}$ is the family of all finite equivalence graphs, then $\mathscr{F}(\mathscr{E})$ is the countable equivalence graph which has as components infinitely many infinite complete graphs.

Let $\mathbf{D}(l) = \langle V, E\rangle$ be the (unique) countable model of the almost sure theory of the K_{l+1}-free and the l-colorable graphs, where $l \geqslant 2$. From Corollary 2.6 we have that $\mathbf{D}(l)$ is an l-colorable graph with uniquely determined parts $P_1, \ldots, P_l$. It is quite clear that $\mathbf{D}(l)$ is not a homogeneous structure and, as a result, the class $\mathscr{L}(l) = \mathscr{K}(\mathbf{D}(l))$ of all finite l-colorable graphs does not have the amalgamation property. In contrast to this, it can be easily verified that the expanded structure $\mathbf{D}^*(l) = \langle V, E, P_1, \ldots, P_l\rangle$ is homogeneous. As a matter of fact $\mathbf{D}^*(l)$ is the Fraissé structure of the class $\mathscr{L}^*(l)$ of all finite *l-colored graphs*, that is to say relational structures of the form $\langle V', E', P_1', \ldots, P_l'\rangle$, where $\langle V', E'\rangle$ is a graph and all the P_i' are independent subsets of V' which partition V'.

Lachlan and Woodrow [**1980**] have given a complete classification of all infinite families $\mathscr{K}$ of finite graphs having the amalgamation property and closed under isomorphisms and substructures. By Lemma 3.1 this yields a classification of all countable homogeneous graphs.

If $G = \langle V, E\rangle$ is a graph, then the *complementary graph* $\overline{G}$ is defined as $\overline{G} = \langle V, \overline{E}\rangle$, where $\overline{E} = \{(x, y)\colon x \neq y \text{ and } (x, y) \notin E\}$.

If $\mathscr{K}$ is a class of graphs, then the *complementary class* $\overline{\mathscr{K}}$ consists of all graphs $\overline{G}$ such that G is in $\mathscr{K}$. Lemma 3.1 implies that $\overline{\mathscr{F}(\mathscr{K})} = \mathscr{F}(\overline{\mathscr{K}})$ for any infinite class $\mathscr{K}$ of finite graphs having the amalgamation property and closed under isomorphisms and substructures.

Let $T = \langle\{x, y, z\}, \{(x, y), (y, z)\}\rangle$ be the graph whose diagram is

and let $\overline{T}$ be the complementary graph of T. The diagram of $\overline{T}$ is

Notice that for each $l \geqslant 1$ the complementary graph $\overline{K_l}$ of the l-clique K_l is an independent set of size l.

With these preliminaries at hand, we state now the main theorem of Lachlan and Woodrow [**1980**].

THEOREM 3.2. *Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two infinite classes of finite graphs having the amalgamation property and closed under isomorphisms and induced subgraphs. Then*

$$\mathscr{K}_1 = \mathscr{K}_2 \quad \textit{if and only if} \quad \mathscr{K}_1 \cap \mathscr{B} = \mathscr{K}_2 \cap \mathscr{B},$$

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where

$$\mathscr{B} = \{T, \overline{T}\} \cup \{K_m\colon m \geqslant 1\} \cup \{\overline{K}_m\colon m \geqslant 1\}.$$

This theorem implies that there are only countably many infinite classes $\mathscr{K}$ of finite graphs having the amalgamation property and closed under induced subgraphs and isomorphisms. The assumption that the graphs under consideration are undirected is essential, since Henson [**1972**] has proved that there exist $2^{\aleph_0}$ nonisomorphic countable homogeneous directed graphs.

We investigate next first-order labeled 0-1 laws on each class $\mathscr{K}$ of finite graphs which arises from Theorem 3.2. We distinguish the following cases for $\mathscr{K}$ and the corresponding Fraissé structure $\mathscr{F}(\mathscr{K})$.

Case 1. $\{T, \overline{T}\} \subseteq \mathscr{K} \cap \mathscr{B}$.

Case 1a. $\mathscr{K} \cap \mathscr{B} = \mathscr{B}$.

Here we have that $\mathscr{K} = \mathscr{G}$, the class of all finite graphs, and the Fraissé structure $\mathscr{F}(\mathscr{G})$ is Rado's graph. As mentioned before, $\mathscr{F}(\mathscr{G})$ is the unique countable model of $\Sigma = \{\tau_0\} \cup \{\psi_k\colon k \geqslant 1\}$, where ψ_k, $k \geqslant 1$, are the extension axioms for graphs.

Case 1b. There is an $l \geqslant 2$ such that

$$\mathscr{K} \cap \mathscr{B} = \{T, \overline{T}\} \cup \{K_1, \ldots, K_l\} \cup \{\overline{K}_m\colon m \geqslant 1\}.$$

Here we have that $\mathscr{K} = \mathscr{S}(l)$, the class of all K_{l+1}-free graphs. The Fraissé structure $\mathscr{F}((\mathscr{S}(l))$ has been studied by Henson [**1971**]. It is shown there that $\mathscr{F}(\mathscr{S}(l))$ is the unique countable model of a set of axioms which assert that the graph is K_{l+1}-free and satisfies certain extension principles for K_{l+1}-free graphs. These principles state that any finite subgraph can be extended by one vertex in any way provided the extension does not contain a K_{l+1}.

Case 1c. There is an $l \geqslant 2$ such that

$$\mathscr{K} \cap \mathscr{B} = \{T, \overline{T}\} \cup \{K_m\colon m \geqslant 1\} \cup \{\overline{K}_1, \ldots, \overline{K}_l\}.$$

Here we have that $\mathscr{K} = \overline{\mathscr{S}(l)}$, the class of all graphs in which the largest independent set has at most l vertices. For the Fraissé structure of $\overline{\mathscr{S}(l)}$ we have that $\mathscr{F}(\overline{\mathscr{S}(l)}) = \overline{\mathscr{F}(\mathscr{S}(l))}$.

Case 2. $T \notin \mathscr{K} \cap \mathscr{B}$ and $\overline{T} \in \mathscr{K} \cap \mathscr{B}$.

Case 2a. $\mathscr{K} \cap \mathscr{B} = \{\overline{T}\} \cup \{K_m\colon m \geqslant 1\} \cup \{\overline{K}_m\colon m \geqslant 1\}$.

Here we have that $\mathscr{K} = \mathscr{E}$, the class of all finite equivalence graphs, and $\mathscr{F}(\mathscr{E})$ is the countable equivalence graph with infinitely many infinite complete graphs as components.

Case 2b. There is an $l \geqslant 2$ such that

$$\mathscr{K} \cap \mathscr{B} = \{\overline{T}\} \cup \{K_1, \ldots, K_l\} \cup \{\overline{K}_m\colon m \geqslant 1\}.$$

Here we have that $\mathscr{K} = \mathscr{E}'(l)$, the class of all finite equivalence graphs with each component a complete graph of size at most l. The Fraissé structure $\mathscr{F}(\mathscr{E}'(l))$ is the countable equivalence graph consisting of infinitely many components each of which is the complete graph K_l of size exactly l.

Case 2c. There is an $l \geqslant 2$ such that

$$\mathscr{K} \cap \mathscr{B} = \{\overline{T}\} \cup \{K_m\colon m \geqslant 1\} \cup \{\overline{K}_1, \ldots, \overline{K}_l\}.$$

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Here we have that $\mathcal{K} = \mathcal{E}''(l)$, the class of all finite equivalence graphs with at most l components. The Fraissé structure $\mathcal{F}(\mathcal{E}''(l))$ is the countable equivalence graph consisting of exactly l components each of which is the infinite complete graph.

Case 3. $T \in \mathcal{K} \cap \mathcal{B}$ and $\overline{T} \notin \mathcal{K} \cap \mathcal{B}$.

In this case we get the complementary classes to those in Case 2.

Case 3a. $\mathcal{K} \cap \mathcal{B} = \{T\} \cup \{\overline{K}_m: m \geqslant 1\} \cup \{K_m: m \geqslant 1\}$.

Here $\mathcal{K} = \bar{\mathcal{E}}$, the class of finite complete partite graphs and $\mathcal{F}(\bar{\mathcal{E}})$ is the complete ω-partite graph with parts of size ω.

Case 3b. There is an $l \geqslant 2$ such that

$$\mathcal{K} \cap \mathcal{B} = \{T\} \cup \{\overline{K}_1, \ldots, \overline{K}_l\} \cup \{K_m: m \geqslant 1\}.$$

Here $\mathcal{K} = \overline{\mathcal{E}'(l)}$, the class of all finite complete partite graphs with parts of size at most l. The Fraissé structure $\mathcal{F}(\overline{\mathcal{E}'(l)})$ is the complete ω-partite graph with parts of size exactly l.

Case 3c. There is an $l \geqslant 2$ such that

$$\mathcal{K} \cap \mathcal{B} = \{T\} \cup \{\overline{K}_m: m \geqslant 1\} \cup \{K_1, \ldots, K_l\}.$$

Here $\mathcal{K} = \overline{\mathcal{E}''(l)}$, the class of all finite complete m-partite graphs for some $m \leqslant l$. The Fraissé structure $\mathcal{F}(\overline{\mathcal{E}''(l)})$, is the complete l-partite graph with parts of size ω.

Case 4. T, $\overline{T} \notin \mathcal{K} \cap \mathcal{B}$.

Case 4a. $\mathcal{K} \cap \mathcal{B} = \{K_m: m \geqslant 1\}$.

Here $\mathcal{K} = \{K_m: m \geqslant 1\} = \mathcal{E}''(1)$ and the Fraissé structure is the countable complete graph.

Case 4b. $\mathcal{K} \cap \mathcal{B} = \{\overline{K}_m: m \geqslant 1\}$.

Here $\mathcal{K} = \{\overline{K}_m: m \geqslant 1\} = \overline{\mathcal{E}''(1)}$ and the Fraissé structure is the countable graph with no edges.

These cases exhaust all classes given by the Lachlan-Woodrow Theorem 2.8, because in Cases 1, 2 and 3 Ramsey's Theorem guarantees that $\{K_m: m \geqslant 1\} \subseteq \mathcal{K}$ or $\{\overline{K}_m: m \geqslant 1\} \subseteq \mathcal{K}$, while in Case 4 the amalgamation property rules out the possibility that $\{K_2, \overline{K}_2\} \subseteq \mathcal{K}$.

We verify now that in each of the above cases a first-order labeled 0-1 law holds for the corresponding class $\mathcal{K}$. To facilitate this we use the following

LEMMA 3.3. *Let $\mathcal{K}$ be a class of finite graphs with universes initial segments $\{1, 2, \ldots, n\}$ of the natural numbers and such that $\mathcal{K}$ is closed under isomorphisms. Then $\mathcal{K}$ has a first-order labeled* 0-1 *law if and only if the complementary class $\overline{\mathcal{K}}$ does. Moreover if* $\mathbf{A}$ *is a model of the almost sure theory $\Pi(\mathcal{K})$ of $\mathcal{K}$, the $\overline{\mathbf{A}}$ is a model of $\Pi(\overline{\mathcal{K}})$.*

PROOF. For any formula $\phi(x_1, \ldots, x_m)$ over the similarity type with only one binary symbol E, we define the *complementary formula* $\bar{\phi}(x_1, \ldots, x_n)$ by induction on the construction of ϕ as follows:

(1) if ϕ is $x_i E x_j$, then $\bar{\phi}$ is $\neg(x_i E x_j)$;

(2) if ϕ is one of the three formulas $x_i E x_i$, $x_i = x_j$, $x_i = x_i$, then $\bar{\phi}$ is ϕ;

(3) if ϕ is $\psi_1 \wedge \psi_2$, then $\bar{\phi}$ is $\bar{\psi}_1 \wedge \bar{\psi}_2$;

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(4) if ϕ is $\neg\psi$, then $\bar{\phi}$ is $\neg\bar{\psi}$;

(5) if ϕ is $\exists x_i \psi$, then $\bar{\phi}$ is $\exists x_i \bar{\psi}$.

A straightforward induction shows that for any graph G, any sequence $a_1, \ldots, a_m$ of vertices in G and any first-order formula $\phi(x_1, \ldots, x_m)$ we have that

$$G, a_1, \ldots, a_m \models \phi \quad \text{if and only if} \quad \bar{G}, a_1, \ldots, a_m \models \bar{\phi}.$$

As a result, if ϕ is any first-order sentence, then the labeled asymptotic probability of ϕ on $\mathscr{K}$ is equal to the labeled asymptotic probability of $\bar{\phi}$ on $\bar{\mathscr{K}}$. The conclusion of the lemma follows immediately. □

In Case 1a, as we have already mentioned, the class $\mathscr{G}$ of all finite graphs has a first-order labeled 0-1 law (Fagin [**1976**]). The almost sure theory $\Pi(\mathscr{G})$ is ω-categorical and coincides with the theory $T(\mathscr{F}(\mathscr{G}))$ of the Fraissé structure of $\mathscr{G}$ which in this case is Rado's graph.

In Case 1b the first-order labeled 0-1 law for the class $\mathscr{S}(l)$ of K_{l+1}-free graphs ($l \geqslant 2$) was obtained in Theorem 2. The almost sure theory $\Pi(\mathscr{S}(l))$ is ω-categorical and coincides with the theory $T(\mathbf{D}(l))$ of the unique countable model $\mathbf{D}(l)$ of the axioms $\Sigma(l)$. Notice that the graph $\mathbf{D}(l)$ is not isomorphic to the Fraissé structure $\mathscr{F}(\mathscr{S}(l))$, because $\mathscr{F}(\mathscr{S}(l))$ contains all K_{l+1}-free graphs as induced subgraphs and in particular it is not l-colorable. As a consequence we have that $\Pi(\mathscr{S}(l)) \neq T(\mathscr{F}(\mathscr{S}(l))$.

By Lemma 3.3 the first-order labeled 0-1 law for $\overline{\mathscr{S}(l)}$ in Case 1c follows from Case 1b. In particular,

$$\Pi\left(\overline{\mathscr{S}(l)}\right) = T\left(\overline{\mathbf{D}(l)}\right) \neq T\left(F\left(\overline{\mathscr{S}(l)}\right)\right).$$

In Case 2a the first-order labeled 0-1 law for the class $\mathscr{E}$ of all finite equivalence graphs is an easy consequence of the results of Compton [**1984**]. The argument is identical to that for the finite equivalence relations in Compton [**1984**], since $\mathscr{E}$ and the class of finite equivalence relations have the same exponential generating series. Moreover, $\Pi(\mathscr{E})$ coincides with the theory of the countable equivalence graph $\mathscr{E}$ which has as components infinitely many complete graphs of each finite cardinality. As a result, $\Pi(\mathscr{E})$ is not ω-categorical and $\Pi(\mathscr{E}) \neq T(\mathscr{F}(\mathscr{E}))$.

In Case 2b the first-order labeled 0-1 law for the class $\mathscr{E}'(l)$ of equivalence graphs with components of size at most l ($l \geqslant 2$) follows again from Compton [**1984**]. The almost sure theory $\Pi(\mathscr{E}'(l))$ is the theory of the countable equivalence graph $\mathbf{E}(l)$ which has as components infinitely many complete graphs of every cardinality less than or equal to l. In particular, $\Pi(\mathscr{E}'(l))$ is ω-categorical, but $\Pi(\mathscr{E}'(l)) \neq T(\mathscr{F}(\mathscr{E}'(l))$.

In Case 2c, let $\mathscr{E}''(l)$ be the class of equivalence graphs with at most l components ($l \geqslant 2$). We claim that a first-order labeled 0-1 law holds for $\mathscr{E}''(l)$ and that $\Pi(\mathscr{E}''(l)) = T(\mathscr{F}(\mathscr{E}''(l)))$.

Notice that, since $\mathscr{F}(\mathscr{E}''(l))$ is the countable equivalence graph consisting of exactly l infinite components, the theory $T(\mathscr{F}(\mathscr{E}''(l)))$ is axiomatizable by the set of first-order axioms $\{\phi_1, \phi_2, \phi_3\} \cup \{\chi_m : m \geqslant 1\}$, where

(1) ϕ_1 asserts that E is an irreflexive, symmetric relation such that if xEy, yEz and $x \neq z$, then xEz;

(2) ϕ_2 asserts that there is no independent set of size $l + 1$;

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(3) ϕ_3 asserts that there is an independent set of size l;

(4) For each $m \geqslant 1$, χ_m asserts that for every vertex there are at least m different vertices connected to it.

To prove the claim it is enough to show that

$$\mu(\phi_1) = \mu(\phi_2) = \mu(\phi_3) = \mu(\chi_m) = 1 \quad \text{for all } m \geqslant 1 \text{ on } \mathscr{E}''(l).$$

It is obvious that $\mu(\phi_1) = \mu(\phi_2) = 1$ on $\mathscr{E}''(l)$. The number of equivalence graphs on n vertices with exactly k components is the Stirling number $s(n,k)$ of the second kind. It is well known (Jordan [**1939**]) that $\lim_{n\to\infty}(s(n,k)/(k^n/k!)) = 1$. This asymptotic result implies immediately that on $\mathscr{E}''(l)$

$$\mu(\neg\phi_3) = \lim_{n\to\infty}\left(\sum_{k=0}^{l-1}\frac{s(n,k)}{s(n,l)}\right) = 0$$

and so $\mu(\phi_3) = 1$. The number of equivalence graphs on $\mathscr{E}''(l)$ with n vertices which satisfy $\neg\chi_m$ is at most $m\binom{n}{m}(s(n,l-1) + \cdots + s(n,1))$ and thus $\mu(\neg\chi_m) = 0$ on $\mathscr{E}''(l)$.

Since the classes in Case 3 are complementary to those in Case 2, it follows from Lemma 3.3 that first-order labeled 0-1 laws holds for these classes. Moreover, in each case we have conclusions about the relation between the almost sure theory and the theory of the corresponding Fraissé graph similar to those in Case 2.

Finally, it is obvious that in both Case 4a and Case 4b a first-order labeled 0-1 law holds and that the Fraissé structure is the unique countable model of the almost sure theory.

We have thus established the following

THEOREM 3. *Let $\mathscr{K}$ be an infinite class of finite graphs with universes initial segments $\{1,\ldots,n\}$ of the natural numbers such that $\mathscr{K}$ has the amalgamation property and is closed under isomorphisms and induced subgraphs. Then the labeled asymptotic probability $\mu(\phi)$ on K of any sentence ϕ of first-order logic is either* 0 *or* 1.

From the proof of this theorem we can obtain additional information about the almost sure theory $\Pi(\mathscr{K})$ of each such class $\mathscr{K}$. In particular, we have

COROLLARY 3.4. *Let $\mathscr{K}$ be an infinite class of finite graphs satisfying the hypotheses of Theorem* 3. *Let*

$$\Pi(\mathscr{K}) = \{\phi\colon \phi \textit{ is a first-order sentence and } \mu(\phi) = 1 \textit{ on } \mathscr{K}\}$$

be the almost sure theory of K. Then

(i) $\Pi(\mathscr{K})$ *is a decidable theory,*

(ii) $\Pi(\mathscr{K})$ *has the finite substructure property and thus it is not finitely axiomatizable.*

PROOF. For each such class $\mathscr{K}$ we have that $\Pi(\mathscr{K})$ is decidable, because it is complete (by Theorem 3) and has a recursive set of axioms which is given for each case in the proof of Theorem 3. The finite substructure property follows from the fact that if ϕ is true in some model **B** of $\Pi(\mathscr{K})$, then $\mu(\phi) = 1$ on $\mathscr{K}$ and thus ϕ must be true in some member of $\mathscr{K}$ which is a finite substructure of **B**.

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Class $\mathscr{K}$	Countable model of $\Pi(\mathscr{K})$	Is $\Pi(\mathscr{K})$ ω-categorical?	Is $\Pi(\mathscr{K}) = T(\mathscr{F}(\mathscr{K}))$?
$\mathscr{G}$ = all graphs	Rado's graph	Yes	Yes
$\mathscr{S}(l) = \mathscr{K}_{l+1}$-free graphs ($l \geqslant 2$)	$\mathbf{D}(l)$, the universal graph for l-colorable graphs	Yes	No
$\mathscr{E}$ = equivalence graphs	equivalence graph which is the disjoint union of infinitely many complete graphs of each finite cardinality	No	No
$\mathscr{E}'(l)$ = equivalence graphs with components of size at most l ($l \geqslant 2$)	equivalence graph which is the disjoint union of finitely many complete graphs of each cardinality $\leqslant l$	Yes	No
$\mathscr{E}''(l)$ = equivalence graphs with at most l components ($l \geqslant 1$)	equivalence graph which is the disjoint union of exactly l infinite complete graphs	Yes	Yes

TABLE 1

We have also found the classes $\mathscr{K}$ for which $\Pi(\mathscr{K})$ is ω-categorical and have determined when the almost sure theory $\Pi(\mathscr{K})$ coincides with the theory $T(\mathscr{F}(\mathscr{K}))$ of the Fraissé structure $\mathscr{F}(\mathscr{K})$. We summarize this information in Table 1, in which only one of the classes $\mathscr{K}$ and $\bar{\mathscr{K}}$ appears.

We conclude by pointing out that if $\mathscr{K}$ is an infinite class of finite relational structures having the amalgamation property and closed under isomorphisms and substructures, then $\mathscr{K}$ does not necessarily have a first-order labeled 0-1 law. For this consider the countable partial ordering $\mathbf{A} = \langle A, \leqslant \rangle$ which has infinitely many components each isomorphic to the ordering of the rationals. It is clear that $\mathbf{A}$ is a homogeneous structure and therefore the class $\mathscr{K}(\mathbf{A})$ of the finite substructures of $\mathbf{A}$ has the amalgamation property. However, $\mathscr{K}(\mathbf{A})$ is the class of finite linear forests and Compton [**1984**] has proved that the first-order labeled 0-1 law does not hold for this class.

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