Higher Order Modal Logic*

Reinhard Muskens

1 Introduction

A logic is called *higher order* if it allows for quantification (and possibly abstraction) over higher order objects, such as functions of individuals, relations between individuals, functions of functions, relations between functions, etc. Higher order logic (often also called *type theory* or the *Theory of Types*) began with Frege, was formalized in Russell [46] and Whitehead and Russell [52] early in the previous century, and received its canonical formulation in Church [14].[1] While classical type theory has since long been overshadowed by set theory as a foundation of mathematics, recent decades have shown remarkable comebacks in the fields of *mechanized reasoning* (see, e.g., Benzmüller et al. [9] and references therein) and *linguistics*. Since the late 1960's philosophers and logicians, for various reasons which we will dwell upon, have started to combine higher order logic with modal operators (Montague [35, 37, 38], Bressan [11], Gallin [22], Fitting [19]). This combination results in *higher order modal logic*, the subject of this chapter.

The chapter will be set up as follows. In the next section we will look at possible motivations behind the idea of combining modality and higher order logic. Then, in section 3, Richard Montague's system of 'Intensional Logic', by far the most influential of higher order modal logics to date, will be discussed. This logic will be shown to have some limitations. One of these is that, despite its name, the logic is not fully intensional, as it validates the axiom of Extensionality. This leads to a series of well-known problems centering around 'logical omniscience'. Another limitation is that the logic is not Church-Rosser (it matters in which order λ-conversions are carried out). These limitations can be overcome and the remaining sections of the chapter will contain an exposition of a modal type theory that is intensional in two ways: in the sense of being a modal logic and in the sense that Extensionality does not hold. The logic in itself is not strong enough to make the usual rules of λ-conversion derivable, but these rules can consistently be added as an axiomatic extension and in that case the Church-Rosser property will hold (as an alternative, the rules can be hard-

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[1] For a good survey of (non-modal) higher order logic, see van Benthem and Doets [8]; for a textbook development, Andrews [3].

wired into the theory, in which case the theory is also Church-Rosser). Section 4 will introduce the basic syntax and semantics of this logic, section 5 will give a tableau calculus, and section 6 provides some elementary model theory in the form of a model existence theorem and its usual corollaries, such as generalized completeness. We conclude with a conclusion.

2 Motivation

Why should one want to combine modality with quantification or abstraction over objects of higher type? Possible reasons come from areas as diverse as rational theology, the axiomatization of classical mechanics, the semantics of natural language, and modal logic itself. Let us look at each of these in turn.

2.1 The Ontological Argument

Anselm (1033–1109) proved the existence of God by defining him as "a being than which none greater can be thought" and by arguing that, since that definition can be understood, such a being must "exist in the understanding". But if this being exists in the understanding, one can also think of it as existing in reality and, since real existence is "greater" than mere conceptual existence, the "being than which none greater can be thought" must truly exist. Otherwise one could think of an even greater being that did truly exist. Moreover, by an analogous argument, Anselm comes to the conclusion that it is even *impossible* to think of God as nonexistent. For something that cannot be thought of as nonexisting is greater than something that can be so thought of. It follows that a "being than which none greater can be thought" cannot merely exist contingently, otherwise one could think of an even greater being with necessary existence.

Anselm's original argument was phrased in ordinary Latin and its lack of precision may be deemed a weakness by some, but increasingly more precise variants of the argument have been put forward by Descartes, Leibniz and, more recently, Gödel [24]. Gödel's argument centers around "positive" properties and being a god can be defined as having every positive property. There are axioms regulating the behaviour of the predicate "positive", stipulating e.g. that exactly one of a property or its complement is positive, that, whenever P is positive and the extension of P necessarily is a subset of that of Q, Q is also positive, that necessary existence is a positive property, etc., etc. The conclusion is identical to that of Anselm's: God necessarily exists (and is unique), but this time premises and argument are spelled out in great detail (some of the premises may be hard to swallow, even for those who are willing to accept the conclusion). The argument combines quantification over properties with the modal notions of necessity and possibility and as a consequence is naturally framed in a higher order modal logic. For a recent evaluation of Gödel's proof, its history, a precise formalization, and extensive discussion of the argument and subsequent literature, see Fitting [19].

2.2 Axiomatization of Classical Mechanics

As a second example of the use of a higher order modal logic, let us briefly mention the proposal in Bressan [11] for a logical foundation of classical mechanics in general and Mach's often criticized definition of *mass* in particular. Mach's definition has a counterfactual character and this is where modality comes in. Suppose we have a particle M, whose mass is to be established. Fix some inertial frame. If a particle M_1 with unit mass and velocity v_1 parallel to M's velocity v were to collide with M at time t, then, if the changes in the velocities of M and M_1 at t would be $\Delta v \neq 0$ and Δv_1 respectively, the mass of M would be $\Delta v_1/\Delta v$. This means that the mass of M *can* be established experimentally, but, as Bressan points out, in an axiomatic foundation of physics it is important that the axioms do not imply that the experiment actually takes place, as many physically possible situations that one wants to be able to describe are in fact incompatible with such an assumption. Thus, Bressan argues, an axiomatization based on a modal logic is required. Bressan's logic is not only modal but also higher order, as it essentially replaces set theory and concepts such as natural number and real number should therefore be definable within the logic (for example, the natural number n is defined as the property of having n elements, $\lambda F.\exists_n x\, Fx$, with $\exists_n$ the obvious abbreviation, in Frege's way).

2.3 The Semantics of Natural Language

A third illustration of the use of higher order modal logic comes from Richard Montague's [36, 37, 38] contributions to the semantics of natural language, work that truly revolutionized the subject.[2] It was Montague's aim to treat the semantics of natural language in a completely precise way and to provide a truth definition for sentences of (say) English very much in accordance with the usual Tarski truth definition for logical languages. One way to achieve this is to directly assign modeltheoretic objects to syntactic expressions. This road was taken in Montague [36], but a way that is easier to go in practice is to translate expressions of English to an interpreted logical language. The interpretation of the logic then indirectly provides a model theory for the fragment of English under consideration. This is done in Montague [37, 38]. The logic used is higher order and modal.

Why did Montague use a logic that combined modality with higher order quantification and abstraction? It is not difficult to see why one should want a *modal* logic for the treatment of natural language, as the latter abounds with phrases and constructions that have motivated modal logics in the first place (temporal operators, counterfactuals, true modals like *can, might* and *would*, propositional attitude verbs, and so on). But the reason for employing a *higher order* logic may be less clear to logicians not working in linguistics. Although natural language is able to quantify over properties and in general can express things that are not normally expressible with first order means only (think of

[2]The presentation here is inspired by Montague's work but deviates from it in many minor details.

sentences like *most men have green eyes*, for example), this is not the sole or even the primary reason for using type theory in linguistics. The main reason is that in type theory the availability of lambda-abstraction allows for a *closure of the gap between the syntactic forms of natural language expressions and those of their logical translations.*

Let us illustrate this with the help of the simple example sentence in (1a). Linguists almost universally provide this sentence with a constituent structure along the lines of (1b), i.e. the determiner *every* is thought to form a constituent (a noun phrase) with the noun *elephant* and this resulting constituent then forms another constituent (a sentence) with the verb phrase *danced*. Essentially, therefore, the linguistic analysis of such sentences follows the pre-Fregean pattern of dividing each sentence in a subject (here *every elephant*) and a predicate (*danced*).

(1) a. every elephant danced
  b. [[every elephant] danced]
  c. $\forall x\,(Ex \to Dx)$
  d. $(((\lambda P_1 \lambda P_2.\forall x\,(P_1 x \to P_2 x))E)D)$

The analysis of natural language expressions as consisting of larger and larger clusters of constituents is an important feature of modern linguistic theory, and syntacticians are in the possession of a whole battery of empirical tests to determine constituenthood, but the syntactic form that is given to any sentence is not in general congruent with its usual logical form. The structure in (1b), for example, is fundamentally different from that of the logical sentence (1c), the usual translation of (1a). While the constituents *elephant* and *danced* in (1b) reappear in (1c) as E and D respectively, there are no continuous parts of (1c) corresponding to *every* or *every elephant*. This gap between logical form and linguistic form is what logicians such as Russell and Quine had in mind when they alluded to the *misleading form* of natural language: the 'correct' form of (1a) according to this perspective is (1c); (1b) merely misleads. This point of view could never be shared by the linguistic community, as giving up the standard notion of constituenthood would greatly diminish the predictive power of syntactic theory.

Can the gap be bridged? Here lambdas come to the rescue, for in a higher order logic with lambda abstraction (1c) can alternatively be written as (1d). While (1c) is the β-normal form of (1d), the latter, but not the former, follows the syntactic pattern in (1b). Lambdas allow us to have our cake and eat it. They allow us to maintain the view that the logical form of an expression closely mirrors its syntactic form without having to give up the usual logical analysis.

In fact, with lambdas in hand, it is now possible to think of inductive translation mechanisms sending syntactic forms to logical forms. In the present case one can translate *every* as $\lambda P_1 \lambda P_2.\forall x\,(P_1 x \to P_2 x)$, a term containing two λ-abstractions over predicates, *elephant* can be translated as the predicate constant E and *dances* as D. If one lets onstituent formation correspond to application, [every elephant] translates as $(\lambda P_1 \lambda P_2.\forall x\,(P_1 x \to P_2 x))E$, which reduces to $\lambda P_2.\forall x\,(Ex \rightarrow P_2x)$ (a *generalized quantifier*), and a further step shows that (1b) translates as (1d), or, equivalently, (1c).

But now a difficulty crops up. If [every elephant] translates as $\lambda P_2.\forall x\,(Ex \rightarrow P_2x)$, how are we going to translate the verb phrase [fed [every elephant]] in (2b), the syntactic analysis of (2a)? The verb *fed* should presumably be translated as some binary relation F between individuals and this is not the kind of object that $\lambda P_2.\forall x\,(Ex \rightarrow P_2x)$ can apply to (or that can apply to that term).

Montague solved this by complicating the translations of transitive verbs like *fed*. He translated *fed* not simply as F, but as the term $\lambda Q\lambda x.Q(Fx)$[3] (with Q ranging over quantifiers and x over individuals), and if the translations for *a* and *girl* are chosen to be $\lambda P_1\lambda P_2.\exists x\,(P_1x \wedge P_2x)$ and G respectively, the translation in (2c) results, as the reader may care to verify.

(2) a. a girl fed every elephant
    b. [[a girl][fed [every elephant]]]
    c. $\exists x\,(Gx \wedge \forall y\,(Ey \rightarrow Fxy))$
    d. $\forall y\,(Ey \rightarrow \exists x\,(Gx \wedge Fxy))$

Translating an intransitive verb like *fed* as $\lambda Q\lambda x.Q(Fx)$, and not as the simpler and more intuitive binary relation symbol F, seems ad hoc, however. In fact, researchers in the Montague tradition have argued that a combination of giving simple translations with providing systematic ways of obtaining certain translations from others is not only more elegant than Montague's original approach was, but also gives a better fit with the data (Partee and Rooth [44], Hendriks [26, 27]). Discussing the calculi for 'shifting' translations that these authors have proposed would lead us too far afield here. Suffice it to say that from their considerations, in conjunction especially with those of van Benthem [7], the picture emerges that *linear combinators*[4] play an all-important role. The translation of *fed* as $\lambda Q\lambda x.Q(Fx)$, for example, can be thought to result from applying the linear combinator $\lambda R\lambda Q\lambda x.Q(Rx)$ to a basic translation F, while applying the combinator $\lambda R\lambda Q_1\lambda Q_2.Q_1(\lambda y.Q_2(\lambda x.Rxy))$ to F results in a translation that eventually leads to (2d), another possible translation of the original sentence.[5]

For more information on Montague's approach to the semantics of natural language, see the textbooks Dowty et al. [15] and Gamut [23], the survey in Partee with Hendriks [43], and the chapter on Linguistics by Moss and Tiede

[3] For the sake of exposition I am disregarding Montague's intensional operators here.

[4] A *combinator* is a closed λ-term built from variables with the help of λ-abstraction and application only. A combinator M is *linear* if each abstractor λX in M binds exactly one X in M.

[5] While linear combinators play an important role in semantic composition, just letting them apply to semantic translations without further ado results in serious overgeneration. Applying the permutation operator $\lambda R\lambda y\lambda x.Rxy$ to F above, for example, would allow the derivation of translations for *a girl fed every elephant* that are normally associated with *an elephant fed every girl*. Partee and Rooth [44] and Hendriks [26, 27] provide calculi in which permutation is not derivable, while de Groote [25] and Muskens [40, 41] base their grammars entirely on linear lambda terms but make sure that any permutation in semantics is mirrored by a permutation in syntax.

in this handbook (chapter 19). Montague's higher order modal logic **IL** will be described shortly.

2.4 Modal Logics with Propositional Quantifiers

Motives inherent in modal logic itself may also lead to a combination of modality with higher order, or at least second order, quantification. The standard definition of the truth of a formula in a frame at a world is defined with the help of a quantification over valuations and therefore essentially corresponds to universal quantification over sets of possible worlds. More precisely, the frame truth of a formula φ containing proposition letters $p_1, \ldots, p_n$ corresponds to the truth of a formula $\forall P_1 \ldots P_n \varphi'$, where φ' is the standard translation $ST(\varphi)$ of φ; see chapter 1 by Blackburn and van Benthem in this handbook. This gives *global* second order quantification, with the second order universal quantifiers taking scope over the whole formula, but one may now be inspired to add quantifiers $\forall p$ and $\exists p$ ranging over sets of possible worlds to given modal logics. This was done in Kripke [33] and modal logics with propositional quantifiers have been studied by a variety of authors since, amongst whom are Bull [12], Fine [16], Kaplan [30], Kremer [32, 31], Fitting [18], and ten Cate [13], to name but a few.

Semantically there are two lines of attack here. If one has a frame $\langle W, R \rangle$, the most obvious interpretation of quantifiers $\forall p$ and $\exists p$ in that frame lets them range over the power set $\mathcal{P}(W)$ of the set of possible worlds W. This is called the *second order* (or *primary*) *interpretation* of propositional quantifiers. If propositional quantifiers are added to a modal logic $\mathbf{L}$ in this way (where $\mathbf{L} = \mathbf{S4}$, $\mathbf{S5}$, etc.), the resulting logic is called $\mathbf{L}\pi+$. The behaviour of the logics thus obtained rather varies. $\mathbf{S5}\pi+$, on the one hand, is decidable (Fine [16], Kaplan [30]), as this logic is embeddable into monadic second order logic. (The embedding essentially is the standard translation, with clauses such as $ST(\Box\varphi) = \forall x ST(\varphi)$ and $ST(\forall p \varphi) = \forall P\, ST(\varphi)$.) Fine and Kaplan also axiomatize $\mathbf{S5}\pi+$. The logics $\mathbf{K}\pi+$, $\mathbf{T}\pi+$, $\mathbf{K4}\pi+$, $\mathbf{B4}\pi+$, $\mathbf{S4.2}\pi+$, and $\mathbf{S4}\pi+$, on the other hand, are recursively isomorphic to full second order logic (this was proved independently by Kripke and Fine; Fine [16] has a weaker result).

In order to obtain nice proof systems for modal logics with propositional quantification, one can also follow the example of Henkin [28], who, in the context of higher order logic, defined a class of models in which higher order quantifiers do not necessarily range over *all* subsets of the relevant domains, but only over designated subsets of them. In the present context such a set-up means that frames $\langle W, R \rangle$ are replaced by triples $\langle W, R, \Pi \rangle$ such that $\Pi \subseteq \mathcal{P}(W)$. Here Π must be closed under boolean operations, including arbitrary unions and intersections and it must be the case that $R[P] \in \Pi$ and $R^{-1}[P] \in \Pi$ whenever $P \in \Pi$ (see e.g. Thomason [50], who considers such structures for tense logics). Propositional quantification is now interpreted as quantification over Π. This is the so-called *first order* (or *secondary*) *interpretation* of propositional quantifiers. The resulting logics are denoted as $\mathbf{S4}\pi$, $\mathbf{S5}\pi$, etc., according to the constraints that are put on accessibility relations R. All these logics are axiomatizable with the help of reasonable axioms.

How does the axiomatization of **S5**π that one gets in this way (basically the usual **S5** axioms and rules + the usual quantification axioms and rules for propositional quantification) compare to the one obtained by Fine and Kaplan? Curiously, an axiomatization of **S5**π+ requires one additional axiom, namely

(3) $\exists p(p \wedge \forall q(q \to \Box(p \to q)))$

A little reflection shows that if this formula is evaluated in a world w in a frame $\langle W, R\rangle$ using the primary interpretation, it is true, with $\{w\}$ as a sole witness for p. On the other hand, evaluation with respect to w in a frame $\langle W, R, \Pi\rangle$ may not result in truth, as there may be no $P \in \Pi$ such that $w \in P$ and $P \subseteq P'$ for all P' such that $w \in P' \in \Pi$. A very similar situation obtains in higher order logic. In the models of Henkin [28] sets may be so sparse that there are not enough of them to distinguish between objects that are in fact not identical. Two distinct objects d_1 and d_2 may have exactly the same properties, and in particular $\{d_1\}$ may fail to exist (Andrews [2]). In a modal context definability of singleton sets $\{w\}$ can be enforced through the introduction of *nominals* (Blackburn et al. [10], Areces and ten Cate, chapter 14 of this handbook).

3 Montague's Intensional Logic

In the previous section we have explained some of Montague's ideas with the help of a non-modal logic, but Montague himself actually framed them in **IL** (Intensional Logic), a higher order modal logic that will be discussed in this section (see also Moss and Tiede's chapter 19 of this handbook). The logic is an extension of Church's [14] theory of types and inherits many, though not all, of the latter's properties. After giving a brief overview of **IL** we will point out some of the logic's limitations.

3.1 Overview of IL

In order to set up the logic, one first needs to define a simple type system.

Definition 1. *The set of* **IL** types *is the smallest set of strings such that:*

(i) e *and* t *are* **IL** *types;*

(ii) If α *and* β *are* **IL** *types, then* $(\alpha\beta)$ *is an* **IL** *type;*

(iii) If α *is an* **IL** *type, then* $(s\alpha)$ *is an* **IL** *type.*

Here the type e is the type of *entities*, while t is the type of *truth-values*. Note that, while s can be used to form complex **IL** types, it is not itself an **IL** type. The intended interpretation of the types defined here is that objects of a type $\alpha\beta$ (also written $\alpha \to \beta$) are functions from objects of type α to objects of type β and that objects of type $s\alpha$ are functions from the set of possible worlds to objects of type α.

The next step is to define the *terms* of **IL**. It will be assumed that each **IL** type α comes with a denumerably infinite set of variables and a countable set of constants. Terms are built up from these as follows.

Definition 2. *Define, for each* **IL** *type* α*, the set of* **IL** *terms* T_α *as follows.*

(i) Every constant or variable of any type α *is an element of* T_α*;*

(ii) If $A \in T_{\alpha\beta}$ *and* $B \in T_\alpha$*, then* $(AB) \in T_\beta$*;*

(iii) If $A \in T_\beta$ *and* x *is a variable of type* α *then* $(\lambda x.A) \in T_{\alpha\beta}$*;*

(iv) If $A, B \in T_\alpha$ *then* $(A \equiv B) \in T_t$*;*

(v) If $A \in T_\alpha$ *then* $(\,\hat{}A) \in T_{s\alpha}$*;*

(vi) If $A \in T_{s\alpha}$ *then* $(\,\check{}A) \in T_\alpha$*.*

So we have application and abstraction, identity, and "cap" and "cup" operators that, as we will see, are very much analogous to application and abstraction. If $A \in T_\alpha$ we will often indicate that fact by writing A_α. Terms of type t are called *formulas* and we often use metavariables φ, ψ, etc. to range over them.

Definition 2 does not seem to provide us with the expressivity that we want, as the common logical operators, including the modal $\Box$ and $\Diamond$ seem to be absent, but in fact such operators are definable from the ones just adopted (Henkin [29], Gallin [22]).

Definition 3. *Write*

$$\begin{aligned}
\top &\ \textit{for}\ (\lambda x_t.x) \equiv (\lambda x_t.x),\\
\bot &\ \textit{for}\ (\lambda x_t.x) \equiv (\lambda x_t.\top),\\
\neg\varphi &\ \textit{for}\ \bot \equiv \varphi,\\
\varphi \wedge \psi &\ \textit{for}\ (\lambda f_{tt}.f\varphi \equiv \psi) \equiv (\lambda f_{tt}.f\top),\\
\forall x_\alpha \varphi &\ \textit{for}\ (\lambda x_\alpha.\varphi) \equiv (\lambda x_\alpha.\top),\ \textit{and}\\
\Box\varphi &\ \textit{for}\ \hat{}\varphi \equiv \hat{}\top.
\end{aligned}$$

Other operators will have their usual definitions.

Whether these abbreviations make sense can be checked as soon as we are in the possession of a semantics for the language. So let us turn to that.

Definition 4. *A* (standard) model *for* **IL** *is a triple* $\langle D, W, I\rangle$ *such that* D *and* W *are non-empty sets and* I *is a function with the set of all constants as its domain, such that* $I(c) \in D_{s\alpha}$ *for each constant* c *of type* α*, where the sets* D_α *are defined using the following induction.*

$$\begin{aligned}
D_e &= D\\
D_t &= \{0, 1\}\\
D_{\alpha\beta} &= \{F \mid F : D_\alpha \to D_\beta\}\\
D_{s\alpha} &= \{F \mid F : W \to D_\alpha\}.
\end{aligned}$$

The function I *is called an* interpretation *function. Intuitively, we interpret* D *as a domain of possible individuals and* W *as a set of possible worlds.*

In order to interpret terms on models, we additionally need to define an *assignment* to $M = \langle D, W, I\rangle$ as a function a with the set of all variables as its domain, such that $a(x) \in D_\alpha$ if x is of type α. The notation $a[d/x]$ is defined as usual. Terms can now be evaluated on models with the help of a Tarski-style truth definition.

Definition 5. *The* value $\|A\|^{M,w,a}$ *of a term A on a model $M = \langle D, W, I\rangle$ in world $w \in W$ under an assignment a to M is defined in the following way:*

(i) $\|c\|^{M,w,a} = I(c)(w)$ *if c is a constant;* $\|x\|^{M,w,a} = a(x)$ *if x is a variable;*

(ii) $\|AB\|^{M,w,a} = \|A\|^{M,w,a}(\|B\|^{M,w,a})$*;*

(iii) $\|\lambda x_\beta A\|^{M,w,a} =$ *the function F with domain D_β such that* $F(d) = \|A\|^{M,w,a[d/x]}$ *for all $d \in D_\beta$;*

(iv) $\|A \equiv B\|^{M,w,a} = 1$ *iff* $\|A\|^{M,w,a} = \|B\|^{M,w,a}$*;*

(v) $\|{}^\wedge A\|^{M,w,a} =$ *the function F with domain W such that* $F(w') = \|A\|^{M,w',a}$ *for all $w' \in W$;*

(vi) $\|{}^\vee A\|^{M,w,a} = \|A\|^{M,w,a}(w)$*;*

Note the special treatment of the non-logical constants in the first clause of this definition: constants of type α are interpreted as functions of type $s\alpha$ by the interpretation function I but these functions are applied to the current world in order to get the actual value, an object which is of type α again. The second and third clauses interpret application and abstraction in a way that is to be expected. The fourth clause interprets $\equiv$ as identity *relative to a possible world*, i.e. $A \equiv B$ means that A and B have the same extension in the world of evaluation, not necessarily in all possible worlds. The last two clauses interpret the cap and cup operators in a way that is analogous to abstraction and application; cap is abstraction over possible worlds while cup is application to the current world. We leave it to the reader to verify that the abbreviations of definition 3 provide the operators defined there with their usual semantics (with $\Box$ the universal modality).

A formula φ is *true* in a model M in world w under an assignment a if $\|\varphi\|^{M,w,a} = 1$. The notion of standard entailment, or s-entailment for short, is defined accordingly.

Definition 6. *Let Γ and Δ be sets of* **IL** *formulae. We say that Γ* s-entails *Δ, $\Gamma \models_s \Delta$, if, whenever $M = \langle D, W, I\rangle$ is a model, $w \in W$, and a is an assignment to M, $\|\varphi\|^{M,w,a} = 1$ for all $\varphi \in \Gamma$ implies $\|\psi\|^{M,w,a} = 1$ for some $\psi \in \Delta$.*

While it is clear from Gödel's incompleteness theorem that the relation $\models_s$ can have no recursive axiomatization, it is possible to define a *generalized* notion of entailment $\models_g$ that can be so axiomatized. For Church's logic this was done in Henkin [28], while Gallin [22] (in general a rich source of information about Montague's logic) generalizes the completeness proof found there to the setting of **IL**. The $\models_g$ notion is obtained with the help of *generalized* (or: *Henkin*) *models*, the main difference between these and standard models being that, while for each α and β it must hold that $D_{\alpha\beta} \subseteq \{F \mid F : D_\alpha \to D_\beta\}$, the $D_{\alpha\beta}$ need not be the *entire* function spaces $\{F \mid F : D_\alpha \to D_\beta\}$. Similarly, it is only required that $D_{s\alpha} \subseteq \{F \mid F : W \to D_\beta\}$. We will not pursue the proof of Henkin (or: generalized) completeness for **IL** here, but refer to Gallin's original work. For a generalized completeness proof for a similar higher order modal logic, see section 6.

3.2 Limitations of IL

Montague's work has been a tremendous boost for natural language semantics but with the advantage of hindsight it is possible to point out some shortcomings of the logic that he used. These limitations will be reviewed here. First, let us ask ourselves the question whether the logic lives up to its name. Is **IL** really an *intensional* logic? If "intensional" merely is another word for "modal" there can be no discussion, but there is an older definition of the concept of intensionality that makes perfect sense in a higher order context and in which sense **IL** is not intensional. Whitehead and Russell's *Principia Mathematica* [52, number *20] is one place where this definition can be found. In this work a distinction between *extensional* and *intensional* functions of functions is made and Whitehead and Russell give as "the mark of an extensional function f" a condition which in their notation reads

(4) $\varphi!x. \equiv_x .\psi!x :\supset_{\varphi,\psi}: f(\varphi!\hat{z}). \equiv .f(\psi!\hat{z})$

but which in the present setting can be written as

(5) $\forall gh(\forall x(gx \equiv hx) \rightarrow fg \equiv fh)$

Thus a function of functions f is extensional if, whenever f is applied to a function g, the resulting value fg depends only on the extension of g; a function of functions is intensional if not extensional.[6]

Whitehead and Russell point out that contexts of propositional attitude such as "I believe that p" are examples of functions that are not extensional and hence intensional. However, it is immediately clear that in **IL** all functions of functions are extensional in the sense of (5) and that intensional functions are ruled out. **IL** conforms to the following form of the axiom of Extensionality:

(6) $\forall f \forall gh(\forall x(gx \equiv hx) \rightarrow fg \equiv fh)$

For an Intensional Logic this seems below par. The situation is alleviated in a sense by the fact that the following scheme (in which $\varphi\{P := F\}$ denotes the result of substituting the constant F for the variable P in φ) is not generally valid.

(7) $\forall x(Fx \equiv Hx) \rightarrow (\varphi\{P := F\} \equiv \varphi\{P := H\})$

For example, one does not have

(8) $\forall x(Fx \equiv Hx) \rightarrow (\Box(H \equiv F) \equiv \Box(H \equiv H))$,

as it is easy to construct a model in which H and F are coextensive at some point but not at another. This is desirable, since from the premise that all

[6] Whitehead and Russell only consider *propositional* functions and as a consequence their f, if it had been typed in our way, would have received a type of the form $(\alpha t)t$ (so that $\equiv$ can be read as $\leftrightarrow$). In **IL** the scheme in (5) will be valid for f of any type $(\alpha\beta)\gamma$ (with g and h of type $\alpha\beta$ and x of type α).

and only humans are featherless bipeds (to take a truly Russellian example) it should not follow that being a featherless biped necessarily is being human.

But now there is room for a second point of criticism, for how come (6) can be valid while (7) is not? Surely, one can always instantiate g as the constant F, h as H, f as $\lambda P.\,\varphi$ and from

(9) $\forall x(Fx \equiv Hx) \rightarrow ((\lambda P.\varphi)F \equiv (\lambda P.\varphi)H)$

get (7) with the help of two β-conversions? The answer is that β-conversion unfortunately is not generally valid in **IL** but is subject to side conditions additional to the usual constraint on substitutability. $(\lambda P.\Box(H \equiv P))F$, for example, is not semantically equivalent to $\Box(H \equiv F)$, as the reader may care to verify.

We will turn to the side conditions on β-conversion shortly, but first, as a third criticism, let us notice that, while the scheme in (7) is not valid, the strengthened version in (10) does hold in all models at any possible world (the proof is by induction on the complexity of φ).

(10) $\Box\forall x(Fx \equiv Hx) \rightarrow (\varphi\{P := F\} \equiv \varphi\{P := H\})$

But this is far from desirable. Read "is provable with the help of Zorn's Lemma" for F and "is provable with the help of the Axiom of Choice" for H while choosing "John believes that Zorn's Lemma is P" for φ. It is presumably a necessary fact that everything that is provable with the help of Zorn's Lemma is provable with the help of the Axiom of Choice and vice versa. But from "John believes that Zorn's Lemma is provable from Zorn's Lemma" one cannot conclude "John believes that Zorn's Lemma is provable from the Axiom of Choice". Hence (10) should in fact not be valid. This is what is usually called the problem of logical omniscience but is really a consequence of one variant of the Extensionality principle.

Let us consider the side conditions on β-conversion in **IL**. They will unfortunately lead to a fourth problem. Define a term to be *modally closed* if it is built up from variables and terms of the form $\hat{\ }A$ with the help of application, λ-abstraction and $\equiv$. The following scheme is valid.

(11) $(\lambda x_\alpha A_\beta)B_\alpha \equiv A\{x := B\}$, if

  (a) B is free for x in A, and
  (b) either no free occurrence of x in A lies within the scope of $\hat{\ }$ or B is modally closed.

This is in fact one of the six axiom schemes that are used to axiomatize generalized consequence in Gallin [22]. But, as was observed by Friedman and Warren [21], the second side condition that needs to be imposed here destroys one of the attractive properties that lambda calculi usually have. For notions of reduction $\twoheadrightarrow$ such as $\twoheadrightarrow_\beta$ or $\twoheadrightarrow_{\beta\eta}$ (see Barendrecht [4] for definitions), one can often establish that whenever $A \twoheadrightarrow A_1$ and $A \twoheadrightarrow A_2$ there is an A_3 such that $A_1 \twoheadrightarrow A_3$ and $A_2 \twoheadrightarrow A_3$, i.e. it is immaterial in which order reductions are made. This so-called *Church-Rosser* property is not retained in **IL** as Friedman and Warren show with the help of (12).

(12) $(\lambda x_\alpha(\lambda y_\alpha.\hat{}y \equiv f_{\alpha(s\alpha)}x)x)c_\alpha$

Here x, y, and f are variables, while c is a constant. One possible reduction leads to

(13) $(\lambda y.\hat{}y \equiv fc)c$,

while another reduction of (12) results in

(14) $(\lambda x.\hat{}x \equiv fx)c$.

Neither of these terms can be reduced any further (as c is not modally closed but the variable that is abstracted over lies in the scope of $\hat{}$) and hence there is no single term to which both reduce.

Gallin [22] gives a translation of **IL** into a two-sorted variant $\mathbf{TY}_2$ of Church's original logic, which has an extra type s for possible worlds. The translation proceeds by letting $\hat{}$ correspond to λ-abstraction over a fixed variable x_s, while $\check{}$ corresponds to application to x_s (the translation is related to the standard translation of modal logic into first order logic). Constants are translated as the result of application of a constant to the fixed type s variable. This translation clarifies the behaviour of **IL** in many circumstances. For example, since a term that is not modally closed will translate to a term containing a free occurrence of x_s, the side condition (ii) in (11) in a sense reduces to side condition (i) after all. Since the logic $\mathbf{TY}_2$ is just Church's logic (but two-sorted), it is Church-Rosser, but the difficulty of not being intensional is shared between **IL** and $\mathbf{TY}_2$.

4 A Modal Type Theory

In the previous section Montague's logic **IL** was described and various criticisms were levelled against it. In this and the next few sections we will propose a logic **MTT** that is compatible with the usual (α), (β) and (η) rules and that is intensional in the sense that two relations can have the same extension yet be different. In order to obtain this logic we must deviate from **IL** in two respects. First, we shall follow Bressan [11] in letting the value of an expression AB in some world w depend not only on (the value of A and) the value of B in w, but possibly on the values of B in other worlds as well. This immediately solves the problem with β-conversion, as no extra side conditions on that rule will then be necessary.[7]

For the second deviation from **IL**, and indeed from the usual semantics for Church's [14] classical type logic, a class of models will be considered that is a further generalization of the generalized models considered in Henkin [28]. These

[7] See also N. Belnap's foreword to Bressan [11], especially point 11, where this "nonextensional predication" (nonextensional in the modal sense, not in the stronger sense used in this chapter) is called Bressan's cardinal innovation.

intensional models, as they will be called here, derive from the structures considered in the proofs of cut elimination in Takahashi [49] and Prawitz [45]. The latter also play an important role in Andrews' [1] proof that his (non-extensional) resolution calculus corresponds to the first six axioms of Church [14]. The structures considered by these authors are proof-generated and are defined on the basis of a purely syntactic notion (Schütte's [47] semivaluations), but recently purely semantic, stand-alone, generalizations of such models have been offered in Fitting [19] ('generalized Henkin models') and in Benzmüller et al. [9] ('Σ-models'). Fitting's models involve a non-standard interpretation of abstraction, while the models of Benzmüller et al. have a non-standard form of application, but these complications seem unnecessary, as our intensional models will do without them.

Intensional models will serve two purposes. The first is that they deal with problems of logical omniscience. A second use is technical: the notion of entailment one gets from intensional models is easily axiomatized with the help of a cut free tableau calculus. This second point will be dwelled upon below; for the first point consider the following example. While it is reasonable to assume that sentences (15a) and (15b) determine the same set of possible worlds, it is not reasonable to assume that applying the function "Mary knows that p" to (15a) necessarily results in the same value as applying that function to (15b): (15c) might be true while (15d) is false. Intensional models provide a way to make the necessary distinction. The idea will be that co-entailment, or, more generally, having the same extension in all models, will not imply identity, i.e. the axiom of Extensionality will not hold.

(15) a. The cat is out if the dog is in
  b. The dog is out if the cat is in
  c. Mary knows that the cat is out if the dog is in
  d. Mary knows that the dog is out if the cat is in

4.1 Types and Terms

Unlike **IL**, which is based on hierarchies of *functions*, the logic **MTT** will be based on hierarchies of *relations* (Orey [42], Schütte [47]), as relational models are pleasant to work with. Some definitions therefore must be changed and we shall start with the definition of *types*. Assume that some set $\mathcal{B}$ of *basic types*, among which must be the type s of possible worlds, is given.

Definition 7. *The set $\mathcal{T}$ of* types *is the smallest set of strings over the alphabet $\mathcal{B} \cup \{\rangle, \langle\}$ such that (i) $\mathcal{B} \subseteq \mathcal{T}$ and (ii) if $\alpha_1, \ldots, \alpha_n \in \mathcal{T}$ $(n \geq 0)$ then $\langle \alpha_1 \ldots \alpha_n \rangle \in \mathcal{T}$.*

Types formed with clause (ii) of this definition will be called *complex*. The complex type $\langle\rangle$, obtained by letting $n = 0$ in (ii), will be the type of *propositions*; this will also be the type of formulas, which will have sets of possible worlds as their extensions. In general, extensions for terms of type $\langle \alpha_1 \ldots \alpha_n \rangle$ will be

$n + 1$-ary relations, with one argument place for a possible world (the world where the relation is evaluated) and one for each of the α_i. Note that we have defined types to be certain strings, so that there is a difference between (say) the type s and the type $\langle s \rangle$. The latter is associated with a set of possible worlds in each world, or, equivalently, with the type of binary relations between worlds. Any of these relations can be viewed as an *accessibility* relation.

A *language* will be a countable set non-logical constants such that each constant has a unique type. If $\mathcal{L}$ is a language, the set of constants from $\mathcal{L}$ having type α is denoted $\mathcal{L}_\alpha$. For each $\alpha \in \mathcal{T}$ we assume the existence of a denumerably infinite set $\mathcal{V}_\alpha$ of variables of type α, such that $\mathcal{V}_\alpha \cap \mathcal{V}_\beta = \varnothing$ if $\alpha \neq \beta$. We let $\mathcal{V} = \bigcup_{\alpha \in \mathcal{T}} \mathcal{V}_\alpha$. In proofs it will occasionally be useful to be able to refer to fixed well-orderings $<_\mathcal{L}$ and $<_\mathcal{V}$ on languages $\mathcal{L}$ and on the set $\mathcal{V}$ respectively, so we will assume that these are in place as well.

The following definition gives terms in all types. Apart from variables and non-logical vocabulary, there will be application and abstraction, and a basis for defining the usual connectives and quantifiers. Moreover, for any term R of type $\langle s \rangle$ there will be a modal operator $\langle R \rangle$ and a term $R^\smile$ intended to denote the converse of R.

Definition 8. *Let $\mathcal{L}$ be a language. Define sets $T^\mathcal{L}_\alpha$ of* terms *of $\mathcal{L}$ of type α, for each $\alpha \in \mathcal{T}$, as follows.*

(i) $\mathcal{L}_\alpha \subseteq T^\mathcal{L}_\alpha$ *and* $\mathcal{V}_\alpha \subseteq T^\mathcal{L}_\alpha$ *for each* $\alpha \in \mathcal{T}$

(ii) *If* $A \in T^\mathcal{L}_{\langle \beta\alpha_1 \ldots \alpha_n \rangle}$ *and* $B \in T^\mathcal{L}_\beta$, *then* $(AB) \in T^\mathcal{L}_{\langle \alpha_1 \ldots \alpha_n \rangle}$

(iii) *If* $A \in T^\mathcal{L}_{\langle \alpha_1 \ldots \alpha_n \rangle}$ *and* $x \in \mathcal{V}_\beta$, *then* $(\lambda x.A) \in T^\mathcal{L}_{\langle \beta\alpha_1 \ldots \alpha_n \rangle}$

(iv) $\bot \in T^\mathcal{L}_{\langle\rangle}$

(v) *If* $\varphi \in T^\mathcal{L}_{\langle\rangle}$ *and* $\psi \in T^\mathcal{L}_{\langle\rangle}$ *then* $\varphi \to \psi \in T^\mathcal{L}_{\langle\rangle}$

(vi) *If* $\varphi \in T^\mathcal{L}_{\langle\rangle}$ *and* $x \in \mathcal{V}_\alpha$ *then* $\forall x \varphi \in T^\mathcal{L}_{\langle\rangle}$

(vii) *If* $R \in T^\mathcal{L}_{\langle s \rangle}$ *and* $\varphi \in T^\mathcal{L}_{\langle\rangle}$ *then* $\langle R \rangle \varphi \in T^\mathcal{L}_{\langle\rangle}$

(viii) *If* $R \in T^\mathcal{L}_{\langle s \rangle}$ *then* $R^\smile \in T^\mathcal{L}_{\langle s \rangle}$

The operation of taking converses will be useful in applications where the notion arises naturally, such as in temporal logic where, if $<$ is used to denote the relation of temporal precedence, $\langle < \rangle$ will be Prior's future operator F and $\langle <^\smile \rangle$ (or $\langle > \rangle$ after an obvious abbreviation) his past operator P.

We will write $T^\mathcal{L}$ for the set of all terms of the language $\mathcal{L}$, i.e. for the union $\bigcup_{\alpha \in \mathcal{T}} T^\mathcal{L}_\alpha$. If A is a term of type α, we may indicate this by writing A_α and we will use φ, ψ, χ for terms of type $\langle\rangle$, i.e. formulas. The notions *free* and *bound* occurrence of a variable and the notion B *is free for* x *in* A are defined as usual, as are *closed* terms and *sentences*. *Substitutions* are functions σ from variables to terms such that $\sigma(x)$ has the same type as x. If σ is a substitution then the substitution σ' such that $\sigma'(x) = A$ and $\sigma'(y) = \sigma(y)$ for all $y \not\equiv x$ is denoted as $\sigma[x := A]$. If A is a term and σ is a substitution, $A\sigma$, the extension of σ to A, is defined in the usual way. The substitution σ such that $\sigma(x_i) = A_i$ and $\sigma(y) = y$ if $y \notin \{x_1, \ldots, x_n\}$ is written as $\{x_1 := A_1, \ldots, x_n := A_n\}$.

Parentheses in terms will often be dropped on the understanding that association is to the left, i.e. ABC is $((AB)C)$. The operators $\top$, $\neg$, $\wedge$, $\vee$, $\leftrightarrow$ and $\exists$ are obtained as usual. The following definition gives some other useful operators.

Definition 9. *We will write*

$$
\begin{aligned}
&=_{\langle\alpha\alpha\rangle} \textit{ for } \lambda x_\alpha \lambda y_\alpha \forall z_{\langle\alpha\rangle}(zx \to zy),\\
&[R]\varphi \textit{ for } \neg\langle R\rangle\neg\varphi,\\
&\Diamond\varphi \textit{ for } \langle\lambda x_s.\top\rangle\varphi,\\
&\Box\varphi \textit{ for } [\lambda x_s.\top]\varphi, \textit{ and}\\
&\dot{A} \textit{ for } \forall x_{\langle s\rangle}[x]x^{\smile}A_s.
\end{aligned}
$$

The first of these abbreviations gives equalities of type $\langle\alpha\alpha\rangle$ for each α. Of course we will usually write $A = B$ instead of $=AB$. The second abbreviation introduces the usual dual to $\langle\cdot\rangle$ and, for example, allows us to write $[<]$ for Prior's G and $[>]$ for his H. The second and third conventions let us write $\Diamond$ and $\Box$ for the *global* possibility and necessity operators, which have the universal relation on worlds as their underlying accessibility relation. The abbreviation $\dot{A}$, lastly, introduces what are called *nominals* (see Blackburn et al. [10] or Areces and ten Cate, chapter 14 of this handbook, for much more on these). As will become apparent below, $\dot{A}$ will be true in a world w if and only if w is denoted by A.

4.2 Standard Models

Before we introduce the intensional models that will interpret **MTT** terms, let us have a brief look at a class of models that, in order to conform to general usage, will be called *standard* (even though for many practical purposes the intensional models defined below will be preferred).

Definition 10. *A* standard collection of domains *is a set $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ such that $D_\alpha \neq \varnothing$ if α is basic, $D_\alpha \cap D_\beta = \varnothing$ if $\alpha \neq \beta$ and α and β are basic, while $D_{\langle\alpha_1\ldots\alpha_n\rangle} = \mathcal{P}(D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n})$ for each $D_{\langle\alpha_1\ldots\alpha_n\rangle}$. A* standard model *is a pair $\langle D, J\rangle$ such that $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ is a standard collection of domains and J is a function with the set of all constants as its domain, such that $J(c) \in D_\alpha$ for each constant c of type α. J is called the* interpretation function *of $\langle D, J\rangle$.*

Letting the interpretation function J send constants of type α directly to D_α diverges from the set-up in **IL**, but is in conformity with Church's original logic. It will bring the behaviour of constants in line with that of free variables.

An *assignment* a for a standard collection of domains $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ is a function which has the set of variables $\mathcal{V}$ as domain and has the property that $a(x) \in D_\alpha$ if $x \in \mathcal{V}_\alpha$. The usual notational conventions for assignments obtain: If a is an assignment, $x_1, \ldots, x_n$ are pairwise distinct variables, and $d_1, \ldots, d_n$ are objects such that $d_i \in D_\alpha$ if x_i is of type α, then $a[d_1/x_1, \ldots, d_n/x_n]$ is the assignment a' defined by letting $a'(x_i) = d_i$ and $a'(y) = a(y)$, if $y \notin \{x_1, \ldots, x_n\}$.

When working with hierarchies of relations it is often expedient to have a way of interpreting relations as certain functions. The following definition provides one (compare Muskens [39]).

Definition 11. *Let R be an n-ary relation ($n > 0$) and let $0 < k \leq n$. Define the* k-th slice function *$F^k_R(d)$ of R by:*

$$F^k_R(d) = \{\langle d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n\rangle \mid \langle d_1, \ldots, d_{k-1}, d, d_{k+1}, \ldots, d_n\rangle \in R\}$$

So $F^k_R(d)$ is the $n-1$-ary relation that is obtained from R by fixing its k-th argument place by d. Note that if R is a relation in $\mathcal{P}(D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n})$ its first slice function is a function from possible worlds to relations in $\mathcal{P}(D_{\alpha_1} \times \cdots \times D_{\alpha_n})$ and can therefore be identified with what Montague would call a *relation-in-intension*. This motivated the choice of letting $D_{\langle\alpha_1\ldots\alpha_n\rangle}$ equal $\mathcal{P}(D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n})$ in definition 10.

The next definition provides terms with values in standard models. Clauses (i) and (iv)–(viii) will probably not surprise the reader, as they are essentially standard; for clauses (ii) and (iii) *second* slice functions provide motivation. For (ii) lets $V(a, AB)$ be equal to the result of applying the second slice function of $V(a, A)$ to $V(a, B)$, while (iii) defines $V(a, \lambda x_\beta.A)$ as the relation whose second slice function is the function F such that, for all $d \in D_\beta$, $F(d) = V(a[d/x], A)$.

Definition 12. *The* value $V_M(a, A)$ *of a term A on a standard model $M = \langle D, J\rangle$ under an assignment a to M is defined as follows (we drop subscripts M):*

(i) $V(a, c) = J(c)$ *if c is a constant;* $V(a, x) = a(x)$ *if x is a variable;*

(ii) $V(a, AB) = \{\langle w, \vec{d}\rangle \mid \langle w, V(a, B), \vec{d}\rangle \in V(a, A)\}$;

(iii) $V(a, \lambda x_\beta.A) = \{\langle w, d, \vec{d}\rangle \mid d \in D_\beta$ *and* $\langle w, \vec{d}\rangle \in V(a[d/x], A)\}$;

(iv) $V(a, \bot) = \varnothing$;

(v) $V(a, \varphi \to \psi) = D_s - (V(a, \varphi) - V(a, \psi))$;

(vi) $V(a, \forall x_\alpha \varphi) = \bigcap_{d \in D_\alpha} V(a[d/x], \varphi)$;

(vii) $V(a, \langle R\rangle\varphi) = \{w \mid \exists w' \in V(a, \varphi)$ *such that* $\langle w, w'\rangle \in V(a, R)\}$;

(viii) $V(a, R^\smile) = \{\langle w, w'\rangle \mid \langle w', w\rangle \in V(a, R)\}$.

4.3 Intensional Models

Intensional models generalize the standard models just given in two ways. The first generalization follows Henkin [28] in not necessarily associating domains $D_{\langle\alpha_1\ldots\alpha_n\rangle}$ with the full powerset $\mathcal{P}(D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n})$, but to be contented with some subset of this relational space. When this generalization is made it becomes possible to prove (generalized) completeness for the logic. However, if a tableau system is used it will contain a Cut rule. In order to avoid invoking the latter it seems to be necessary to adopt a second generalization and to move to a class of structures that do not necessarily validate the axiom of Extensionality, which says that two predicates are identical when they can be predicated of the same objects. The strategy of taking out Extensionality, pioneered by Takahashi [49] and Prawitz [45], allows one to prove the completeness of a cut-free system, after which Extensionality can be added to the logic again if that should be desired.

In the present set-up, which is inspired by Fitting [19], we will get rid of Extensionality by distinguishing between the *intension* and the *extension* of a term of complex type. The basic idea will be that any object in a domain D_α can be the intension of some term. Intensions of complex type will not be constructed set-theoretically out of those of a less complex type. Extensions, on the other hand, will be relations over the relevant domains of intensions, with their identity criteria therefore given by set membership. One and the same extension may be determined by two or more different intensions.

Let us see how this can be done. A *collection of domains* will be a set of non-empty sets $\{D_\alpha \mid \alpha \in \mathcal{T}\}$, such that $D_\alpha \cap D_\beta = \varnothing$ if $\alpha \neq \beta$. There are no further constraints on collections of domains. *Assignments* and the notational conventions pertaining to assignments are defined as before. The set of all assignments for a collection of domains D is denoted $\mathcal{A}_D$. The *intension functions* defined below send terms to almost arbitrary domain elements. There are a few restrictions on these functions but they are rather liberal.

Definition 13. *An* intension function *for a collection of domains $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ and a language $\mathcal{L}$ is a function I with domain $\mathcal{A}_D \times T^{\mathcal{L}}$ such that*

(i) $I(a, A) \in D_\alpha$*, if A is of type α*

(ii) $I(a, x) = a(x)$*, if x is a variable*

(iii) $I(a, A) = I(a', A)$*, if a and a' agree on all variables free in A*

(iv) $I(a, A\{x := B\}) = I(a[I(a, B)/x], A)$*, if B is free for x in A*

Before we continue with also defining *extension* functions, let us pay some attention to the nitty-gritty and observe that the intension functions just defined behave well when the language is restricted or extended. The following property will be used a couple of times below.

Proposition 1. *(i) Let I be an intension function for D and $\mathcal{L}$ and let $\mathcal{L}' \subseteq \mathcal{L}$. Then the restriction I' of I to $\mathcal{A}_D \times T^{\mathcal{L}'}$ is an intension function for D and $\mathcal{L}'$. (ii) Let I be an intension function for D and $\mathcal{L}$, let $\mathcal{L} \subseteq \mathcal{L}'$ and let f be a function with domain $\mathcal{L}' \backslash \mathcal{L}$ such that $f(c) \in D_\alpha$ if $c \in \mathcal{L}'_\alpha \backslash \mathcal{L}_\alpha$. Then there is an intension function I' for D and $\mathcal{L}'$ such that I and I' agree on $\mathcal{A}_D \times T^{\mathcal{L}}$ and $I'(a, c) = f(c)$ for every $c \in \mathcal{L}' \backslash \mathcal{L}$.*

Proof. (i) is trivial, so let us verify (ii). Let A be an arbitrary term in $T^{\mathcal{L}'}$ and let $c_1, \ldots, c_n$ be the constants occurring in A that are in $\mathcal{L}'$ but not in $\mathcal{L}$ such that $c_i <_{\mathcal{L}'} c_j$ if $i < j$. Let $A^\dagger$ be the result of replacing each c_i in A with the first variable x_i in $<_{\mathcal{V}}$ such that x_i is not free in A, has the type of c_i and is distinct from each of the x_j $(j < i)$. Clearly $A = A^\dagger\{x_1 := c_1, \ldots, x_n := c_n\}$. Let $I'(a, A) = I(a[f(c_1)/x_1, \ldots, f(c_n)/x_n], A^\dagger)$ and check that I' meets the requirements. □

The next definition provides the promised extension functions, which send objects of complex type to certain relations. We first give very general constraints; more requirements will follow in definition 16.

Definition 14. *An* extension function *for* $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ *is a function* E *with domain* $\cup\{D_\alpha \mid \alpha$ *is complex*$\}$ *such that* $E(d) \subseteq D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n}$ *if* $d \in D_{\langle\alpha_1\ldots\alpha_n\rangle}$.

Note that there is no requirement that the restriction of an extension function to any $D_{\langle\alpha_1\ldots\alpha_n\rangle}$ should be *onto* $\mathcal{P}(D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n})$ or that extension functions should be injective. This reflects the two generalizations discussed above. The possible lack of surjectivity is Henkin's generalization and the possible lack of injectivity reflects the move that Prawitz and Takahashi made.

Definition 15. *A* generalized frame *for the language* $\mathcal{L}$ *is a triple* $\langle D, I, E\rangle$ *such that* D *is a collection of domains,* I *is an intension function for* D *and* $\mathcal{L}$*, and* E *is an extension function for* D*.*

We are interested in the extensions $E(I(a, A))$ of terms A. For the sake of readability we will often write $V(a, A)$ for these, letting V denote the composition of E and I. The following definition puts a series of constraints on extension functions that make things start to behave in a desired way.

Definition 16. *A generalized frame* $\langle D, I, E\rangle$ *for* $\mathcal{L}$ *is a* intensional model *for* $\mathcal{L}$ *if*

(i) $V(a, AB) = \{\langle w, \vec{d}\rangle \mid \langle w, I(a, B), \vec{d}\rangle \in V(a, A)\}$;

(ii) $V(a, \lambda x_\beta.A) = \{\langle w, d, \vec{d}\rangle \mid d \in D_\beta$ *and* $\langle w, \vec{d}\rangle \in V(a[d/x], A)\}$;

(iii) $V(a, \bot) = \varnothing$;

(iv) $V(a, \varphi \to \psi) = D_s - (V(a, \varphi) - V(a, \psi))$;

(v) $V(a, \forall x_\alpha \varphi) = \bigcap_{d \in D_\alpha} V(a[d/x], \varphi)$;

(vi) $V(a, \langle R\rangle\varphi) = \{w \mid \exists w' \in V(a, \varphi)$ *such that* $\langle w, w'\rangle \in V(a, R)\}$;

(vii) $V(a, R^\smile) = \{\langle w, w'\rangle \mid \langle w', w\rangle \in V(a, R)\}$.

The clauses here are identical to the relevant ones in definition 12, with one important exception: in the clause for AB the (second slice function of) the value of A is no longer applied to the *extension* of B, but to its *intension.* The idea is that the extension of a predicate A determines and is determined by all the things that A can truthfully be predicated of while the intension of A determines and is determined by all the predicates that hold of A.

Do intensional models exist? One answer is that the standard models defined in the previous section obviously correspond to a subclass of the class of intensional models in which E is the identity function, but one would like to see intensional models that are not standard. For the latter we refer to the construction in the section on elementary model theory below.

Having the notion of intensional model in place we can define what it means for a sentence to be made true by an intensional model in a given world or to be valid in an intensional model.

Definition 17. *Let $M = \langle D, I, E \rangle$ be an intensional model for $\mathcal{L}$, let $w \in D_s$ and let φ be a sentence of $\mathcal{L}$. M and w* satisfy *φ (or* make *φ* true*), $M, w \models \varphi$, if $w \in V(a, \varphi)$ for any a. We also say that M satisfies φ if there is some $w \in D_s$ such that $M, w \models \varphi$ and that φ is* satisfiable *if some M satisfies φ. If $M, w \models \varphi$ for all $w \in D_s$ then φ is said to be* valid *in M and we write $M \models \varphi$.*

The corresponding notion of entailment is defined as follows.

Definition 18. *Let Π and Σ be sets of sentences in $\mathcal{L}$. Π is said to* intensionally entail *or* i-entail *Σ, $\Pi \models_i \Sigma$, if, for every intensional model $M = \langle D, I, E \rangle$ for $\mathcal{L}$ and every $w \in D_s$, if $M, w \models \varphi$ for all $\varphi \in \Pi$ then $M, w \models \varphi$ for some $\varphi \in \Sigma$.*

This gives a rather weak logic in comparison with other type logics. In applications it will usually be necessary to strengthen the logic with sets of sentences $\mathcal{S}$ which may typically contain modal axioms, but may also contain classical axioms, such as instantiations of the Extensionality scheme, the Axiom of Descriptions, or axioms regulating λ-conversion. About the latter notion the following proposition lists some useful facts.

Proposition 2. *Let $M = \langle D, I, E \rangle$ be an intensional model, and let a be an assignment for D. Then, for all A and B of appropriate types,*

(i) $V(a, \lambda x.A) = V(a, \lambda y.A\{x := y\})$*, if y is free for x in A;*

(ii) $V(a, (\lambda x.A)B) = V(a, A\{x := B\})$*, if B is free for x in A;*

(iii) $V(a, \lambda x.Ax) = V(a, A)$*, if x is not free in A.*

Proof. Left to the reader. □

These statements show that λ-conversion preserves identity of extension, but that does not imply that intensional identity is also preserved and that V can be replaced uniformly with I in the proposition above. If such intensional identities are wanted, and in most applications one will certainly want to have at least the possibility of α and β conversion in any context, an axiomatic extension of the logic may provide them. See 5.2 below.

5 Tableaus for Modal Type Theory

5.1 Tableaus

In this section the proof theory of **MTT** will be given in the form of a tableau system. The calculus will be set up as a form of labeled deduction, with labels storing information about worlds and truth values. Formally, a *labeled sentence* of $\mathcal{L}$ will be a triple $\langle \mathsf{S}, u, \varphi \rangle$ consisting of a *sign* S, which can either be T or F, a constant $u \in \mathcal{L}_s$, and a sentence φ of $\mathcal{L}$. Labeled sentences $\langle \mathsf{S}, u, \varphi \rangle$ will typically be written as $\mathsf{S}u{:}\,\varphi$, where $\mathsf{T}u{:}\,\varphi$ can be read as 'φ is true in world u' and $\mathsf{F}u{:}\,\varphi$ as expressing that φ is false in u.

Tableaus will be defined as certain sets of branches. A *branch* in its turn will be a set of labeled sentences. The notion of *satisfaction* can easily be

$$\frac{\Gamma}{\Gamma'}\,\mathrm{W} \quad \text{if } \Gamma' \subseteq \Gamma$$

$$\frac{\Gamma, \mathsf{T}u\colon \bot}{\quad}\,\mathsf{T}\bot \qquad \frac{\Gamma, \mathsf{T}u\colon \varphi,\ \mathsf{F}u\colon \varphi}{\quad}\,\mathrm{Ax}$$

$$\frac{\Gamma, \mathsf{S}u\colon (\lambda x.A)B\vec{C}}{\Gamma, \mathsf{S}u\colon A\{x := B\}\vec{C}}\,\beta\text{-ext} \qquad \frac{\Gamma, \mathsf{S}u\colon R^{\smile} u'}{\Gamma, \mathsf{S}u'\colon Ru}\,\smile$$

$$\frac{\Gamma, \mathsf{T}u\colon \varphi \to \psi}{\Gamma, \mathsf{F}u\colon \varphi \;\big|\; \Gamma, \mathsf{T}u\colon \psi}\,\mathsf{T}{\to} \qquad \frac{\Gamma, \mathsf{F}u\colon \varphi \to \psi}{\Gamma, \mathsf{T}u\colon \varphi,\ \mathsf{F}u\colon \psi}\,\mathsf{F}{\to}$$

$$\frac{\Gamma, \mathsf{T}u\colon \forall x \varphi}{\Gamma, \mathsf{T}u\colon \varphi\{x := A\}}\,\mathsf{T}\forall \qquad \frac{\Gamma, \mathsf{F}u\colon \forall x \varphi}{\Gamma, \mathsf{F}u\colon \varphi\{x := c\}}\,\mathsf{F}\forall \ (c \text{ not in the premise})$$

$$\frac{\Gamma, \mathsf{T}u\colon \langle R\rangle \varphi}{\Gamma, \mathsf{T}u\colon Ru', \mathsf{T}u'\colon \varphi,}\,\mathsf{T}\langle\cdot\rangle \ (u' \text{ not in the premise}) \qquad \frac{\Gamma, \mathsf{F}u\colon \langle R\rangle \varphi}{\Gamma, \mathsf{F}u\colon Ru' \;\big|\; \Gamma, \mathsf{F}u'\colon \varphi}\,\mathsf{F}\langle\cdot\rangle$$

Table 1: Tableau rules for **MTT**.

extended from sentences to labeled sentences and branches, for we can define an intensional model $M = \langle D, I, E\rangle$ to satisfy $\mathsf{T}u\colon \varphi$ if $I(a, u) \in V(a, \varphi)$ for some (and hence every) a, while letting M satisfy $\mathsf{F}u\colon \varphi$ if $I(a, u) \notin V(a, \varphi)$ for any a. M is said to *satisfy* a branch Γ if it satisfies all $\vartheta \in \Gamma$. If no model M satisfies Γ, Γ is said to be *unsatisfiable*; otherwise Γ is *satisfiable*.

We will use the usual sequent notation for branches, writing Γ, θ for $\Gamma \cup \{\theta\}$, etc. Diverging slightly from the usual set-up of tableaus, tableau rules will be defined as certain relations between branches, not as relations between labeled sentences. The interpretation of these rules (that are given in Table 1) is one of replacement of branches, for example the interpretation of $\mathsf{T}{\to}$ in Table 1 is that the branch $\Gamma, \mathsf{T}u\colon \varphi \to \psi$ can be replaced by the two branches $\Gamma, \mathsf{F}u\colon \varphi$ and $\Gamma, \mathsf{T}u\colon \psi$ in any tableau. The format also allows the formulation of a *weakening* rule W that allows the removal of signed formulas from a branch.

Compare $\mathsf{T}{\to}$ with a more usual approach where one would have a rule

$$\frac{\mathsf{T}u\colon \varphi \to \psi}{\mathsf{F}u\colon \varphi \;\big|\; \mathsf{T}u\colon \psi},$$

meaning that whenever a branch is found to contain $\mathsf{T}u\colon \varphi \to \psi$ it may be split, $\mathsf{F}u\colon \varphi$ may be added to one side and $\mathsf{T}u\colon \psi$ to another. Of course the two approaches very much boil down to the same thing. The present set-up is close to that of a Gentzen calculus for the logic: read T as 'left' and F as 'right' and turn the rules in Table 1 upside down.

A convention that is adopted in Table 1 (and that we shall continue to use) is that wherever the notation $A\{x := B\}$ is used B must be free for x in A.

$$\frac{\mathsf{T}u\colon \neg\varphi}{\mathsf{F}u\colon \varphi}\,\mathsf{T}\neg \qquad \frac{\mathsf{F}u\colon \neg\varphi}{\mathsf{T}u\colon \varphi}\,\mathsf{F}\neg \qquad \frac{\mathsf{T}u\colon \varphi \wedge \psi}{\mathsf{T}u\colon \varphi,\ \mathsf{T}u\colon \psi}\,\mathsf{T}\wedge \qquad \frac{\mathsf{F}u\colon \varphi \wedge \psi}{\mathsf{F}u\colon \varphi \mid \mathsf{F}u\colon \psi}\,\mathsf{F}\wedge$$

$$\frac{\mathsf{T}u\colon \varphi \vee \psi}{\mathsf{T}u\colon \varphi \mid \mathsf{T}u\colon \psi}\,\mathsf{T}\vee \qquad \frac{\mathsf{F}u\colon \varphi \vee \psi}{\mathsf{F}u\colon \varphi,\ \mathsf{F}u\colon \psi}\,\mathsf{F}\vee \qquad \frac{\mathsf{T}u\colon \exists x\varphi}{\mathsf{T}u\colon \varphi\{x := c\}}\,\mathsf{T}\exists\ (c \text{ fresh}) \qquad \frac{\mathsf{F}u\colon \exists x\varphi}{\mathsf{F}u\colon \varphi\{x := A\}}\,\mathsf{F}\exists$$

$$\frac{\mathsf{T}u\colon \varphi \leftrightarrow \psi}{\mathsf{T}u\colon \varphi,\ \mathsf{T}u\colon \psi \mid \mathsf{F}u\colon \varphi,\ \mathsf{F}u\colon \psi}\,\mathsf{T}\leftrightarrow \qquad \frac{\mathsf{F}u\colon \varphi \leftrightarrow \psi}{\mathsf{T}u\colon \varphi,\ \mathsf{F}u\colon \psi \mid \mathsf{F}u\colon \varphi,\ \mathsf{T}u\colon \psi}\,\mathsf{F}\leftrightarrow$$

$$\frac{\mathsf{T}u\colon [R]\varphi}{\mathsf{F}u\colon Ru' \mid \mathsf{T}u'\colon \varphi}\,\mathsf{T}[\cdot] \qquad \frac{\mathsf{F}u\colon [R]\varphi}{\mathsf{T}u\colon Ru', \mathsf{F}u'\colon \varphi}\,\mathsf{F}[\cdot]\ (u' \text{ fresh}) \qquad \frac{\mathsf{T}u'\colon \dot{u}}{\mathsf{T}u'\colon u = u'}\,\mathsf{T}^{\cdot} \qquad \frac{\mathsf{F}u\colon \dot{u}}{\ }\,\mathsf{F}^{\cdot}$$

$$\frac{\mathsf{T}u\colon \Box\varphi}{\mathsf{T}u'\colon \varphi}\,\mathsf{T}\Box \qquad \frac{\mathsf{F}u\colon \Box\varphi}{\mathsf{F}u'\colon \varphi}\,\mathsf{F}\Box\ (u' \text{ fresh}) \qquad \frac{\mathsf{T}u\colon \Diamond\varphi}{\mathsf{T}u'\colon \varphi}\,\mathsf{T}\Diamond\ (u' \text{ fresh}) \qquad \frac{\mathsf{F}u\colon \Diamond\varphi}{\mathsf{F}u'\colon \varphi}\,\mathsf{F}\Diamond$$

$$\frac{\mathsf{T}u\colon A = B,\ \mathsf{S}u\colon \varphi\{x := A\}}{\mathsf{S}u\colon \varphi\{x := B\}}\,\mathrm{LL} \qquad \frac{\ }{\mathsf{T}u\colon A = A}\,\mathrm{id} \qquad \frac{\mathsf{T}u\colon A = B}{\mathsf{T}u'\colon A = B}\,\mathrm{U}$$

$$\frac{\mathsf{S}u\colon \varphi,\ \mathsf{T}u\colon u = u'}{\mathsf{S}u'\colon \varphi}\,\mathrm{LL}' \qquad \frac{\ }{\mathsf{T}u\colon \varphi \mid \mathsf{F}u\colon \varphi}\,\mathrm{Cut}$$

Table 2: Derived tableau rules and Cut (abbreviated forms).

An alternative notation for tableau rules, better suited for inline environments, is $\Gamma/\Gamma_1;\ldots;\Gamma_n$, where / replaces the horizontal line and ; the vertical lines in any rule. The following definition tells how we can expand sets of branches and obtain tableaus.

Definition 19. *A set of branches T' is a* one step expansion *of a set of branches T if $T' = (T\backslash\Gamma) \cup \{\Gamma_1,\ldots,\Gamma_n\}$ for some tableau rule $\Gamma/\Gamma_1;\ldots;\Gamma_n$. T' is an* expansion *of T if there is a sequence $T_1,\ldots,T_n$ such that $T_1 = T$, $T_n = T'$ and each T_{k+1} is a one step expansion of T_k. A set of branches T is a* tableau *if it is an expansion of $\{\Gamma\}$ for some finite branch Γ.*

Thus while no finiteness condition was imposed on branches per se, tableaus are stipulated to originate from finite branches. Note that the $\mathsf{T}\bot$ and Ax rules can cause branches to disappear from a tableau while it is being expanded. This can lead to the closure of tableaus as defined in the following definition.

Definition 20. *A finite branch* Γ has a closed tableau *if $\varnothing$ is an expansion of $\{\Gamma\}$. If Π and Σ are sets of sentences then $\Pi \vdash \Sigma$ holds if, for some finite $\Pi_0 \subseteq \Pi$, some finite $\Sigma_0 \subseteq \Sigma$ and some $u \in \mathcal{L}_s$ that does not occur in any sentence in $\Pi_0 \cup \Sigma_0$, $\{\mathsf{T}u\colon \varphi \mid \varphi \in \Pi_0\} \cup \{\mathsf{F}u\colon \varphi \mid \varphi \in \Sigma_0\}$ has a closed tableau.*

Name	Modal axiom	Corresponding R rule
$\mathbf{T}_\forall$	$\Box\forall p([R]p \to p)$	$/\mathsf{T}u\colon Ru$
$\mathbf{D}_\forall$	$\Box\forall p([R]p \to \langle R\rangle p)$	$/\mathsf{T}u_1\colon Ru_2$ (u_2 fresh)
$\mathbf{4}_\forall$	$\Box\forall p([R]p \to [R][R]p)$	$\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_2\colon Ru_3/\mathsf{T}u_1\colon Ru_3$
$\mathbf{5}_\forall$	$\Box\forall p(\langle R\rangle p \to [R]\langle R\rangle p)$	$\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_1\colon Ru_3/\mathsf{T}u_2\colon Ru_3$

Table 3: Correspondences between modal axioms and certain rules.

We employ the usual notational conventions with respect to $\vdash$. *A formula* φ *is called* tableau provable *if* $\vdash \varphi$.

For ease of reference Table 2 lists some rules that are derivable from those already given in Table 1. We leave it to the reader to show that these rules are indeed derivable (most cases are entirely trivial, some easy but amusing). Another exercise is to show that $\vdash \exists p(p \land \forall q(q \to [R](p \to q)))$. Table 2 also displays the Cut rule, which we will see is admissable. Here we have not bothered to write all the Γs of our official rule presentation and have reverted to the more usual way of presenting tableau rules.

Clearly the rules were chosen in a way that makes it possible to show Soundness to hold.

Theorem 3 (Soundness). *If* Γ *has a closed tableau then* Γ *is unsatisfiable. Hence* $\Pi \vdash \Sigma$ *implies* $\Pi \models_i \Sigma$.

Proof. For each tableau rule $\Gamma/\Gamma_1; \ldots; \Gamma_n$, if Γ is satisfiable, one of the Γ_i is satisfiable. Verifying this will involve proposition 1 for some cases. By induction, if T is an expansion of $\{\Gamma\}$ then, if Γ is satisfiable, some $\Gamma' \in T$ must be satisfiable. Hence if Γ has a closed tableau, Γ can not be satisfiable. This proves the first statement of the theorem. Suppose $\Pi \vdash \Sigma$. Then for some finite $\Pi_0 \subseteq \Pi$ and $\Sigma_0 \subseteq \Sigma$ and some $u \in \mathcal{L}_s$ that does not occur in $\Pi_0 \cup \Sigma_0$, $\{\mathsf{T}u\colon \varphi \mid \varphi \in \Pi_0\} \cup \{\mathsf{F}u\colon \varphi \mid \varphi \in \Sigma_0)\}$ has a closed tableau and hence is unsatisfiable. It follows that $\Pi \models_i \Sigma$ □

5.2 Axiomatic Extensions

If, in some setting, one wants to restrict attention to a class of models that validate some set of sentences $\mathcal{S}$ then it becomes natural to define $\Pi \models_{\mathcal{S}} \Sigma$ as $\mathcal{S} \cup \Pi \models_i \Sigma$. Similarly, $\Pi \vdash_{\mathcal{S}} \Sigma$ can be defined as $\mathcal{S} \cup \Pi \vdash \Sigma$ and the soundness theorem gives that $\Pi \vdash_{\mathcal{S}} \Sigma$ implies $\Pi \models_{\mathcal{S}} \Sigma$ (while completeness, yet to be shown to hold, gives the converse). Prime candidates for inclusion in such a theory $\mathcal{S}$ are the usual rules for lambda conversion. These are the universal closures of any instantiation of one of the following schemes.

(α) $\lambda x.A = \lambda y.A\{x := y\}$, if y is free for x in A;

(β) $(\lambda x.A)B = A\{x := B\}$, if B is free for x in A;

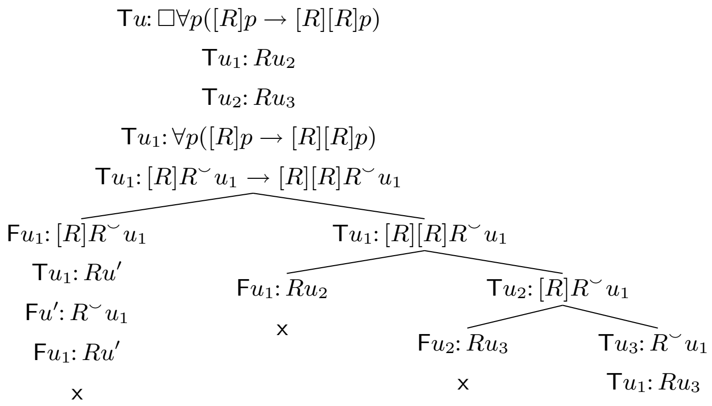

Figure 1: Derivation of $\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_2\colon Ru_3/\mathsf{T}u_1\colon Ru_3$ from $\mathbf{4}_\forall$

(η) $\lambda x.Ax = A$, if x is not free in A.

It is clearly consistent to add these rules to **MTT**, as they are valid in standard models. Once they are added, the derived rules U and LL ensure that

$$\frac{\mathsf{S}u\colon \varphi\{x := A\}}{\mathsf{S}u\colon \varphi\{x := B\}}\,\lambda \qquad \text{if } A =_{\beta\eta} B$$

also becomes a derived rule. Since one can work with the standard notion of reduction $\twoheadrightarrow_{\beta\eta}$ here, it is clear that the resulting logic is Church-Rosser. This will also hold if, for some reason, it is decided that $\mathcal{S}$ should contain (α), but only one of the rules (β) and (η). Note that the (β) rule scheme discussed here should well be distinguished from the rule we have called β-ext, which is much weaker, as it only allows β-conversions in *head* position.

Other obvious candidates for inclusion in a theory $\mathcal{S}$ are the usual modal axioms for modalities $\langle R\rangle$. For instance, one could ensure validity of **T** by including the scheme $\Box([R]\varphi \to \varphi)$ (the leading $\Box$ ensures that one gets validity, not just truth of **T** in the initial world). Another way to express the same idea, more natural perhaps in the present context, is by quantification over propositions, as in $\Box\forall p([R]p \to p)$, which is called $\mathbf{T}_\forall$ in Table 3, where also quantified analogues of **D**, **4** and **5** are found. If such axioms are adopted it is often possible to use derived rules in one's tableaus that closely mirror the usual frame correspondences in modal logic (for the latter see e.g. chapter 1 of this handbook, by Blackburn and Van Benthem). In fig. 1, for example, is a tableau verifying that $\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_2\colon Ru_3/\mathsf{T}u_1\colon Ru_3$ is a derived rule in the presence of the $\mathbf{4}_\forall$ axiom. On the other hand, if $\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_2\colon Ru_3/\mathsf{T}u_1\colon Ru_3$ should be adopted as an additional rule, $\Box\forall p[R]p \to [R][R]p$ becomes tableau provable, as fig. 2 shows. Table 3 lists some more of these correspondences (use a nominal $\dot{u}_3$ when showing the correctness of the last one) and the reader will have no difficulty providing many more. Note that, with the help of nominals, it is

$$\mathsf{F}u\colon \Box\forall p([R]p \to [R][R]p)$$
$$\mathsf{F}u_1\colon \forall p([R]p \to [R][R]p)$$
$$\mathsf{F}u_1\colon [R]c_{\langle\rangle} \to [R][R]c$$
$$\mathsf{T}u_1\colon [R]c$$
$$\mathsf{F}u_1\colon [R][R]c$$
$$\mathsf{T}u_1\colon Ru_2$$
$$\mathsf{F}u_2\colon [R]c$$
$$\mathsf{T}u_2\colon Ru_3$$
$$\mathsf{F}u_3\colon c$$
$$\mathsf{T}u_1\colon Ru_3$$
$$\mathsf{F}u_1\colon Ru_3 \qquad\qquad \mathsf{T}u_3\colon c$$
$$\times \qquad\qquad\qquad \times$$

Figure 2: Derivation of $\mathbf{4}_\forall$ in the presence of $\mathsf{T}u_1\colon Ru_2, \mathsf{T}u_2\colon Ru_3/\mathsf{T}u_1\colon Ru_3$

also possible to directly express properties of accessibility relations, even those that are not modally definable in the usual set-up. For example, irreflexivity of R can be expressed as $\Box\forall x_s(\dot{x} \to \neg\langle R\rangle\dot{x})$. See Blackburn et al. [10] for more information on expressing first order relational properties with the help of nominals.

6 Elementary Model Theory

In this section we will prove some basic modeltheoretic properties of **MTT**: Generalized Completeness, the Generalized Löwenheim-Skolem property, and the admissability of the Cut rule, all via a Model Existence theorem in the way Smullyan [48] did it for first order logic (see also Fitting [17, 19]). None of the techniques employed here is new, but we include full proofs for two reasons. The first of these being that, since our definition of an intensional model deviates from existing notions in the literature and since the devil is always in the details, it is good to have an explicit sanity check on those definitions. The second reason is that readers not already familiar with these kind of proofs may find examples here in a relatively streamlined setting.

Before we tackle the main modeltheoretic properties of **MTT**, some attention must be paid to the notion of *identity* in intensional models, as this relation may not be the identity of the metalanguage.

6.1 Identity and Indiscernability

The decision to let the relations $=_{\langle\alpha\alpha\rangle}$ be abbreviations of $\lambda x_\alpha \lambda y_\alpha \forall z_{\langle\alpha\rangle}(zx \to zy)$, as it was done in definition 9, derives directly from Russell, and via Russell from Leibniz, as the abbreviation equates identity with *indistinguishability*. It is clear that in standard models identity and indistinguishability coincide, but, as was noted by Andrews [2] for the non-modal case, in nonstandard models it may happen that two objects d_1 and d_2 that are in fact not identical may fail to be distinguished because there simply is no set to keep them apart. This may be thought of as an anomaly and one may be tempted to restrict attention to intensional models that are *normal* in the following sense.

Definition 21. *An intensional model $M = \langle D, I, E\rangle$ is* normal *if, for any α, any $d, d' \in D_\alpha$ and arbitrary a, $d = d'$ if $\langle w, d, d'\rangle \in V(a, =)$ for some $w \in D_s$.*

In fact restriction to normal models will not buy us any new truths as will be shown shortly. First some facts that will come in handy.

Proposition 4. *Let $M = \langle D, I, E\rangle$ be an intensional model, and let a be an assignment for D. Then, for all A, B and B' of appropriate types,*

(i) *$V(a, A = B) = \varnothing$ or $V(a, A = B) = D_s$;*

(ii) *$V(a, A = A) = D_s$;*

(iii) *$V(a, A\{x := B\} = A\{x := B'\}) = D_s$ if $V(a, B = B') = D_s$, provided B and B' are free for x in A.*

Proof. (i) Suppose $w \in V(a, A_\alpha = B_\alpha)$, i.e. $w \in V(a, \forall x(xA \to xB))$. Choosing $\lambda y_\alpha.\Box\forall z_{\langle\alpha\rangle}(zA \to zy)$ for x, it is easily shown that $w \in V(a, \Box\forall z(zA \to zB))$. Hence $w' \in V(a, \forall x(xA \to xB))$ for all $w' \in D_s$ and we are done. (ii) Trivial. (iii) Assume that $w \in V(a, B = B'$, i.e. $w \in V(a, \forall y(yB \to yB'))$. Choose $\lambda v.A\{x := B\} = A\{x := v\}$ (with fresh v) for y and derive that $w \in V(a, A\{x := B\} = A\{x := B'\})$. □

The following proposition shows that, if desired, one can always 'normalize' models by 'dividing out' the indistinguishability relation. The proof implicitly uses the axiom of choice.

Proposition 5. *Let $M = \langle D, I, E\rangle$ be an intensional model and let $w \in D_s$. There are a normal intensional model $\overline{M} = \langle \overline{D}, \overline{I}, \overline{E}\rangle$ and a $\overline{w} \in \overline{D}_s$ such that, for each sentence φ, $\overline{w}$ satisfies φ in $\overline{M}$ iff w satisfies φ in M.*

Proof. Suppose $M = \langle D, I, E\rangle$. We define the relation $\sim$ between objects of identical type in M's domains as follows. For any α, any $d, d' \in D_\alpha$ and arbitrary a let $d \sim d'$ iff, for some (and therefore every) $w \in D_s$, $\langle w, d, d'\rangle \in V(a, =_{\langle\alpha\alpha\rangle})$. Clearly, $\sim$ is an equivalence relation. Using proposition 4 and definition 13 it is straightforward to show that, for any term A,

$$(16)\ \ d \sim d' \Longrightarrow I(a[d/x], A) \sim I(a[d'/x], A)\ .$$

It is also worth noting that, for any w, w' and any φ and a

$$(17)\ \ w \sim w' \Longrightarrow (w \in V(a, \varphi) \Longrightarrow w' \in V(a, \varphi))\ .$$

The way to show this is to observe that, if neither x_s nor y_s is free in φ,

(18) $V(a[w/x], (\lambda y.\varphi)^\smile x) = D_s \Longleftrightarrow w \in V(a, \varphi)$,

and to then use the definition of $w \sim w'$.

Define $\overline{d} = \{d' \mid d \sim d'\}$, and let $\overline{D}_\alpha = \{\overline{d} \mid d \in D_\alpha\}$, while $\overline{D} = \{\overline{D}_\alpha \mid \alpha \in \mathcal{T}\}$. Let f be a function such that $f(\overline{d}) \in \overline{d}$, if $\overline{d} \in \overline{D}_\alpha$. For any assignment a for $\overline{D}$, let a° be the assignment for D defined by $a^\circ(x) = f(a(x))$, for all x. Let $\overline{I}(a, A) = \overline{I(a^\circ, A)}$, for each assignment a for $\overline{D}$ and each term A. Then $\overline{I}$ is an intension function for $\overline{D}$. The first three requirements of definition 13 are easily checked, so let us check the last requirement. Note that

$$\begin{array}{rcl} I(a^\circ, A\{x := B\}) & = & \text{(by definition 13)} \\ I(a^\circ[I(a^\circ, B)/x], A) & \sim & \text{(by (16))} \\ I(a^\circ[f(\overline{I(a^\circ, B)})/x]), A) & = & \text{(by the definition of } \overline{I}\text{)} \\ I(a^\circ[f(\overline{I}(a, B))/x]), A) & = & \text{(by the definition of } \circ\text{)} \\ I((a[\overline{I}(a, B)/x])^\circ, A) \ . & & \end{array}$$

From this conclude that $\overline{I}(a, A\{x := B\}) = \overline{I}(a[\overline{I}(a, B)/x], A)$.

Define $\overline{E}$ by letting $\overline{E}(\overline{d}_\alpha) = \{\langle \overline{w}, \overline{d_1}, \ldots, \overline{d_n}\rangle \mid \langle w, d_1, \ldots, d_n\rangle \in E(d)\}$, if α is complex. In order to show that this is well-defined assume that $w \sim w'$, $d \sim d'$, and $d_i \sim d_i'$. Then, if $R, x_1, \ldots, x_n$ are distinct variables of appropriate types

$$\begin{array}{rcl} \langle w, d_1, \ldots, d_n\rangle \in E(d) & \Longleftrightarrow & \text{(by def. (16))} \\ w \in V(a[d/R, d_1/x_1, \ldots, d_n/x_n], Rx_1 \ldots x_n) & \Longleftrightarrow & \text{(by (17))} \\ w' \in V(a[d/R, d_1/x_1, \ldots, d_n/x_n], Rx_1 \ldots x_n) & \Longleftrightarrow & \text{(equational reas.)} \\ w' \in V(a[d'/R, d_1'/x_1, \ldots, d_n'/x_n], Rx_1 \ldots x_n) & \Longleftrightarrow & \text{(by def. (16))} \\ \langle w', d_1', \ldots, d_n'\rangle \in E(d') & & \end{array}$$

so that the definition was legitimate.

Write $\overline{V}(a, A)$ for $\overline{E}(\overline{I}(a, A))$ and observe that

(19) $\langle \overline{w}, \overline{d_1}, \ldots, \overline{d_n}\rangle \in \overline{V}(a, A)$ iff $\langle w, d_1, \ldots, d_n\rangle \in V(a^\circ, A)$

From this it follows by a straightforward induction on term complexity that $\overline{M} = \langle \overline{D}, \overline{I}, \overline{E}\rangle$ is a intensional model. It also follows that $\overline{w}$ satisfies φ in $\overline{M}$ iff w satisfies φ in M and that $\overline{M}$ is normal. □

So the notion of identity of the logic may diverge from the notion of identity employed in the metalanguage, but for notions like satisfiability and entailment this does not matter.

6.2 Model Existence

We now come to the Model Existence theorem and its proof. This theorem (Theorem 7 below) roughly says that if a branch Γ does not have a certain kind of property, here called a 'sound unsatisfiability property', it is satisfiable by an intensional model. The proof proceeds in two steps. One step shows that such a branch Γ can be extended to a branch Γ^* that is *downward saturated* in a sense to be defined shortly. The other step shows that downward saturated branches are satisfiable. We will prove the last of these two statements first. Let us start with the definition of a downward saturated branch.

Definition 22. *A branch Γ of $\mathcal{L}$ is called* downward saturated *if the following hold:*

(a) $\{\mathsf{T}u{:}\,\varphi,\ \mathsf{F}u{:}\,\varphi\} \not\subseteq \Gamma$ *for any sentence φ and constant u;*

(b) $\mathsf{T}u{:}\,\bot \notin \Gamma$*;*

(c) $\mathsf{S}u{:}\,(\lambda x.A)B\vec{C} \in \Gamma \Longrightarrow \mathsf{S}u{:}\,A\{x := B\}\vec{C} \in \Gamma$*, if $\lambda x.A$, B, and the sequence of terms $\vec{C}$ are closed and of appropriate types;*

(d) $\mathsf{T}u{:}\,\varphi \to \psi \in \Gamma \Longrightarrow \mathsf{F}u{:}\,\varphi \in \Gamma$ *or* $\mathsf{T}u{:}\,\psi \in \Gamma$*;*

(e) $\mathsf{F}u{:}\,\varphi \to \psi \in \Gamma \Longrightarrow \{\mathsf{T}u{:}\,\varphi,\ \mathsf{F}u{:}\,\psi\} \subseteq \Gamma$*;*

(f) $\mathsf{T}u{:}\,\forall x_\alpha \varphi \in \Gamma \Longrightarrow \mathsf{T}u{:}\,\varphi\{x := A\} \in \Gamma$ *for all closed A of type α;*

(g) $\mathsf{F}u{:}\,\forall x_\alpha \varphi \in \Gamma \Longrightarrow \mathsf{F}u{:}\,\varphi\{x := c\} \in \Gamma$ *for some $c \in \mathcal{L}_\alpha$*

(h) $\mathsf{T}u{:}\,\langle R\rangle\varphi \in \Gamma \Longrightarrow \{\mathsf{T}u{:}\,Ru', \mathsf{T}u'{:}\,\varphi\} \subseteq \Gamma$ *for some $u' \in \mathcal{L}_s$;*

(i) $\mathsf{F}u{:}\,\langle R\rangle\varphi \in \Gamma \Longrightarrow \mathsf{F}u{:}\,Ru' \in \Gamma$ *or* $\mathsf{F}u'{:}\,\varphi \in \Gamma$ *for all $u' \in \mathcal{L}_s$;*

(j) $\mathsf{S}u{:}\,R^{\smile}u' \in \Gamma \Longrightarrow \mathsf{S}u'{:}\,Ru \in \Gamma$*;*

A downward saturated branch Γ of $\mathcal{L}$ is said to be complete *if $\mathsf{T}u{:}\,\varphi \in \Gamma$ or $\mathsf{F}u{:}\,\varphi \in \Gamma$ for each sentence φ of $\mathcal{L}$ and each $u \in \mathcal{L}_s$.*

That downward saturated branches are satisfiable is the content of the next lemma. Here an intensional model is constructed using the method employed by Takahashi and Prawitz.

Lemma 6 (Hintikka Lemma). *If Γ is a downward saturated branch in a language $\mathcal{L}$ such that $\mathcal{L}_\alpha \neq \varnothing$ for each basic type α then Γ is satisfied by a intensional model. If Γ is complete, then Γ is satisfied by a normal countable intensional model.*

Proof. Let Γ be a downward saturated branch in a language $\mathcal{L}$ as indicated. We will find an intensional model satisfying Γ using the Takahashi-Prawitz construction. The following induction on type complexity defines domains D_α as sets of pairs $\langle A, e\rangle$, where A is a closed term of type α and e is called a *possible extension* of A.

(i) If α is basic let $D_\alpha = \{\langle c, c\rangle \mid c \in \mathcal{L}_\alpha\}$;

(ii) If $\alpha = \langle \alpha_1 \dots \alpha_n\rangle$ let $\langle A_\alpha, e\rangle \in D_\alpha$ iff A is closed, $e \subseteq D_s \times D_{\alpha_1} \times \dots \times D_{\alpha_n}$ and, whenever $\langle B_1, e_1\rangle \in D_{\alpha_1}, \dots, \langle B_n, e_n\rangle \in D_{\alpha_n}$

  (a) If $\mathsf{T}u{:}\,AB_1 \dots B_n \in \Gamma$ then $\langle\langle u, u\rangle, \langle B_1, e_1\rangle, \dots, \langle B_n, e_n\rangle\rangle \in e$;

  (b) If $\mathsf{F}u{:}\,AB_1 \dots B_n \in \Gamma$ then $\langle\langle u, u\rangle, \langle B_1, e_1\rangle, \dots, \langle B_n, e_n\rangle\rangle \notin e$.

Each D_α is non-empty. For basic types α this follows from the requirement that $\mathcal{L}_\alpha \neq \varnothing$; for complex types $\langle \alpha_1 \dots \alpha_n\rangle$ consider $\langle \lambda x_{\alpha_1} \dots \lambda x_{\alpha_n}.\bot, \varnothing\rangle$. Since induction on term complexity easily shows that terms have unique types, we also have that $D_\alpha \cap D_\beta = \varnothing$ if $\alpha \neq \beta$. Hence $\{D_\alpha \mid \alpha \in \mathcal{T}\}$ is a collection of domains. It is worth observing that each D_α is a function if Γ is complete. In that case each D_α will be countable.

The set $D = \{D_\alpha \mid \alpha \in \mathcal{T}\}$ will be the collection of domains of the intensional model we are after. We will define a function I which will turn out to be an intension function for D. First some handy notation. If π is an ordered pair, write π^1 and π^2 for the first and second elements of π respectively, so that $\pi = \langle \pi^1, \pi^2 \rangle$. If f is a function whose values are ordered pairs, write f^1 and f^2 for the functions with the same domain as f, such that $f^1(z) = (f(z))^1$ and $f^2(z) = (f(z))^2$ for any argument z. Let a be an assignment for D. The substitution $\overleftarrow{a}$ is defined by $\overleftarrow{a}(x) = a^1(x)$ and we let $I^1(a, A) = A\overleftarrow{a}$ for any term A. The second component of I is defined using an induction on term complexity.

(a) $I^2(a, x) = a^2(x)$, if x is a variable;
$I^2(a, c_\alpha) = c$, if α is basic;
$I^2(a, c_\alpha) = \{\langle \langle u, u \rangle, \langle A_1, e_1 \rangle, \ldots, \langle A_n, e_n \rangle \rangle \mid \langle A_i, e_i \rangle \in D_{\alpha_i}$
$\& \top u{:}\, cA_1 \ldots A_n \in \Gamma\}$, if α is complex;

(b) $I^2(a, AB) = \{\langle w, \vec{d} \rangle \mid \langle w, I(a, B), \vec{d} \rangle \in I^2(a, A)\}$;

(c) $I^2(a, \lambda x_\beta.A) = \{\langle w, d, \vec{d} \rangle \mid d \in D_\beta$ and $\langle w, \vec{d} \rangle \in I^2(a[d/x], A)\}$;

(d) $I^2(a, \bot) = \varnothing$;

(e) $I^2(a, \varphi \to \psi) = D_s - (I^2(a, \varphi) - I^2(a, \psi))$;

(f) $I^2(a, \forall x_\alpha \varphi) = \bigcap_{d \in D_\alpha} I^2(a[d/x], \varphi)$;

(g) $I^2(a, \langle R \rangle \varphi) = \{w \mid \exists w'$ with $\langle w, w' \rangle \in I^2(a, R)$ and $w' \in I^2(a, \varphi)\}$;

(h) $I^2(a, R^\smile) = \{\langle w, w' \rangle \mid \langle w', w \rangle \in I^2(a, R)\}$.

The definition obviously follows definition 16 save in its first clause. Note that well-definedness does not depend on the question whether I is an intension function for D and $\mathcal{L}$, and indeed the latter is not immediately obvious. We need to check the requirements in definition 13. That $I(a, x) = a(x)$ for any variable x is immediate and that $I(a, A) = I(a', A)$ if a and a' agree on the variables free in A follows by a standard property of substitutions and an easy induction. Suppose that B is free for x in A. Then

$$I^1(a, A\{x := B\}) = A\{x := B\}\overleftarrow{a} = A\overleftarrow{a}[x := B\overleftarrow{a}] =$$
$$A\overleftarrow{a}[x := I^1(a, B)] = A\overleftarrow{a[I(a, B)/x]} = I^1(a[I(a, B)/x], A) \ .$$

That $I^2(a, A\{x := B\}) = I^2(a[I(a, B)/x], A)$ follows by a straightforward induction on the complexity of A which we leave to the reader. It follows that $I(a, A\{x := B\}) = I(a[I(a, B)/x], A)$ if B is free for x in A.

It remains to be shown that $I(a, A) \in D_\alpha$ for any assignment a and term A of type α. This is done by induction on the complexity of A. That $I(a, x_\alpha) \in D_\alpha$ if x is a variable follows from the stipulation that $I(a, x) = a(x)$ and that $I(a, c_\alpha) \in D_\alpha$ if α is basic is immediate. In the remaining cases the type of A is complex. Since it is easily seen by a separate induction that $I^2(a, A_\alpha) \subseteq$

$D_s \times D_{\alpha_1} \times \cdots \times D_{\alpha_n}$ if $\alpha = \langle \alpha_1 \ldots \alpha_n \rangle$, it suffices to prove that, whenever $\alpha = \langle \alpha_1 \ldots \alpha_n \rangle$, and $\langle B_1, e_1 \rangle \in D_{\alpha_1}, \ldots, \langle B_n, e_n \rangle \in D_{\alpha_n}$

(a) If $\mathsf{T}u{:}\, A\overleftarrow{a}\, B_1 \ldots B_n \in \Gamma$ then $\langle \langle u, u \rangle, \langle B_1, e_1 \rangle, \ldots, \langle B_n, e_n \rangle \rangle \in I^2(a, A)$;

(b) If $\mathsf{F}u{:}\, A\overleftarrow{a}\, B_1 \ldots B_n \in \Gamma$ then $\langle \langle u, u \rangle, \langle B_1, e_1 \rangle, \ldots, \langle B_n, e_n \rangle \rangle \notin I^2(a, A)$.

We will consider some remaining cases of the induction, leaving others to the reader. IH will be short for 'induction hypothesis'.

- $A_\alpha \equiv c$ and $\alpha = \langle \alpha_1 \ldots \alpha_n \rangle$. The requirement follows from the definition of $I^2(a, c)$ and clause (i) of definition 22.

- $A \equiv B_{\langle \beta \alpha_1 \ldots \alpha_n \rangle} C_\beta$. Suppose $\langle B_1, e_1 \rangle \in D_{\alpha_1}, \ldots, \langle B_n, e_n \rangle \in D_{\alpha_n}$. From the induction hypothesis it follows that $I(a, C) = \langle C\overleftarrow{a}, I^2(a, C) \rangle \in D_\beta$. Hence

$$\begin{aligned} \mathsf{T}u{:}\, (BC)\overleftarrow{a}\, B_1 \ldots B_n \in \Gamma &\iff \\ \mathsf{T}u{:}\, B\overleftarrow{a}\, C\overleftarrow{a}\, B_1 \ldots B_n \in \Gamma &\implies && \text{(IH)} \\ \langle \langle u, u \rangle, I(a, C), \langle B_1, e_1 \rangle, \ldots, \langle B_n, e_n \rangle \rangle \in I^2(a, B) &\iff && \text{(def. of } I) \\ \langle \langle u, u \rangle, \langle B_1, e_1 \rangle, \ldots, \langle B_n, e_n \rangle \rangle \in I^2(a, BC) & \end{aligned}$$

  This proves the (a) part of the case; the (b) part is similar.

- $A \equiv (\lambda x_{\alpha_1} C_{\langle \alpha_2 \ldots \alpha_n \rangle})$. Again suppose $d_i = \langle B_i, e_i \rangle \in D_{\alpha_i}$, and reason as follows.

$$\begin{aligned} \mathsf{F}u{:}\, (\lambda x.C)\overleftarrow{a}\, B_1 \ldots B_n \in \Gamma &\iff \\ \mathsf{F}u{:}\, (\lambda x.C\overleftarrow{a}\,[x := x]) B_1 \ldots B_n \in \Gamma &\implies && \text{def. 22, } B_1 \text{ is closed} \\ \mathsf{F}u{:}\, C\overleftarrow{a}\,[x := B_1] B_2 \ldots B_n \in \Gamma &\iff \\ \mathsf{F}u{:}\, C\overleftarrow{a[d_1/x]} B_2 \ldots B_n \in \Gamma &\implies && \text{(IH)} \\ \langle \langle u, u \rangle, d_2, \ldots, d_n \rangle \notin I^2(a[d_1/x], C) &\iff && \text{(def. of } I^2) \\ \langle \langle u, u \rangle, d_1, d_2, \ldots, d_n \rangle \notin I^2(a, \lambda x.C) & \end{aligned}$$

  This proves the (b) part, which is similar to the (a) part.

- $A_{\langle\rangle} \equiv \forall x_\beta \varphi$. Let $d \in D_\beta$ be arbitrary. Then $d = \langle B, e \rangle$ for some closed term B. In order to prove the (a) part of the statement we reason as follows.

$$\begin{aligned} \mathsf{T}u{:}\, (\forall x \varphi)\overleftarrow{a} \in \Gamma &\iff \\ \mathsf{T}u{:}\, \forall x \varphi\overleftarrow{a}\,[x := x] \in \Gamma &\implies && \text{def. 22} \\ \mathsf{T}u{:}\, \varphi\overleftarrow{a}\,[x := x]\{x := B\} \in \Gamma &\iff \\ \mathsf{T}u{:}\, \varphi\overleftarrow{a[d/x]} \in \Gamma &\implies && \text{(IH)} \\ \langle u, u \rangle \in I^2(a[d/x], \varphi) & \end{aligned}$$

  Since d was arbitrary, we conclude that $\langle u, u \rangle \in I^2(a, \forall x \varphi)$. The (b) part is similar.

- The cases $A_{\langle\rangle} \equiv \bot$, $A_{\langle\rangle} \equiv \varphi \to \psi$, $A_{\langle\rangle} \equiv \langle R\rangle\varphi$ and $A_{\langle s\rangle} \equiv R^{\smile}$ are straightforward.

This concludes the proof that I is an intension function for D and $\mathcal{L}$. Now define the function E by letting $E(\langle A, e\rangle) = e$ if $\langle A, e\rangle \in D_\alpha$ for any complex α. Clearly, $E(I(a, A)) = I^2(a, A)$ for any A, E is an extension function for D, and $M = \langle D, I, E\rangle$ is an intensional model for the language $\mathcal{L}$. It is easy to see that M satisfies Γ.

In order to prove the second part of the lemma, assume that Γ is complete. We have already established that M is countable in that case, and proposition 5 gives a normal countable intensional model satisfying Γ. □

We now come to the first step sketched in the introduction to this section. The notion of an *unsatisfiability property* (related to the *abstract consistency properties* of Smullyan [48] and Fitting [17]) is defined and it is shown that branches that do not have a 'sound' unsatisfiability property can in fact be extended to a downward saturated branch and hence are satisfiable. The interest in the theorem comes from the fact that many interesting notions can in fact be related to sound unsatisfiability properties as we shall see below.

Definition 23. *Let $\mathcal{U}$ be a set of branches in the language $\mathcal{L}$. $\mathcal{U}$ is an* unsatisfiability property *in $\mathcal{L}$ if, for each tableau rule $\Gamma/\Gamma_1; \ldots; \Gamma_n$, $\{\Gamma_1, \ldots, \Gamma_n\} \subseteq \mathcal{U}$ implies $\Gamma \in \mathcal{U}$.*

An unsatisfiability property $\mathcal{U}$ in $\mathcal{L}$ is sound *if no $\Gamma \in \mathcal{U}$ is satisfied by an intensional model for $\mathcal{L}$.*

Theorem 7 (Model Existence). *Let $\mathcal{L}$ and $\mathcal{C}$ be languages such that each $\mathcal{C}_\alpha$ is denumerably infinite and $\mathcal{L} \cap \mathcal{C} = \varnothing$. Assume that $\mathcal{U}$ is a sound unsatisfiability property in $\mathcal{L} \cup \mathcal{C}$ and that Γ is a branch in the language $\mathcal{L}$. If $\Gamma \notin \mathcal{U}$ then Γ is satisfied by a countable normal intensional model.*

Proof. Let $\mathcal{U}$ and Γ be as described. We construct a downward saturated branch Γ^* such that $\Gamma \subseteq \Gamma^*$. Let $\vartheta_1, \ldots, \vartheta_n, \ldots$ be an enumeration of all labeled sentences in $\mathcal{L} \cup \mathcal{C}$. Write $\#(\vartheta)$ for the index that the labeled sentence ϑ obtains in this enumeration. Let $\Gamma_0 = \Gamma$ and define each Γ_{n+1} by distinguishing the following four cases.

- $\Gamma_{n+1} = \Gamma_n$, if $\Gamma_n \cup \{\vartheta_n\} \in \mathcal{U}$;
- $\Gamma_{n+1} = \Gamma_n \cup \{\vartheta_n\}$, if $\Gamma_n \cup \{\vartheta_n\} \notin \mathcal{U}$ and ϑ_n is not of the form $\mathsf{F}u{:}\, \forall x\varphi$ or of the form $\mathsf{T}u{:}\, \langle R\rangle\varphi$;
- $\Gamma_{n+1} = \Gamma_n \cup \{\vartheta_n, \mathsf{F}u{:}\, \varphi\{x := c\}\}$, where c is the first constant in $\mathcal{C}_\alpha$ which does not occur in $\Gamma_n \cup \{\vartheta_n\}$, if $\Gamma_n \cup \{\vartheta_n\} \notin \mathcal{U}$ and $\vartheta_n \equiv \mathsf{F}u{:}\, \forall x_\alpha\varphi$;
- $\Gamma_{n+1} = \Gamma_n \cup \{\vartheta_n, \mathsf{T}u{:}\, Ru', \mathsf{T}u'{:}\, \varphi\}$, where u' is the first constant in $\mathcal{C}_s$ which does not occur in $\Gamma_n \cup \{\vartheta_n\}$, if $\Gamma_n \cup \{\vartheta_n\} \notin \mathcal{U}$ and $\vartheta_n \equiv \mathsf{T}u{:}\, \langle R\rangle\varphi$.

This is well-defined since each Γ_n contains only a finite number of constants from $\mathcal{C}$. That $\Gamma_n \notin \mathcal{U}$ for each n follows by a simple induction which uses the definition of an unsatisfiability property and the fact that $\mathsf{F}\forall$ and $\mathsf{T}\langle\cdot\rangle$ are tableau rules. Define $\Gamma^* = \bigcup_n \Gamma_n$. For all finite sets $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\}$ and all $k \geq \max\{k_1, \ldots, k_n\}$

$$(20)\ \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^* \Leftrightarrow \Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{U}$$

In order to show this, let $k \geq \max\{k_1, \ldots, k_n\}$ and let $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^*$. Then there is some ℓ such that $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma_\ell$. Let $m = \max\{k, \ell\}$. We have that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma_m$. Since $\Gamma_m \notin \mathcal{U}$ and $\mathcal{U}$ is closed under supersets (because of the weakening rule W), it follows that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{U}$. For the reverse direction, suppose that $\Gamma_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathcal{U}$. Then, since $\mathcal{U}$ is closed under supersets, $\Gamma_{k_i} \cup \{\vartheta_{k_i}\} \notin \mathcal{U}$, for each of the k_i. By the construction of Γ^* each $\vartheta_{k_i} \in \Gamma^*$ and $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Gamma^*$.

Γ^* is a downward saturated branch. The conditions (g) and (h) of definition 22 immediately follow from the construction of Γ^*. For checking the other conditions of definition 22 the equivalence in (20) can be used. Here we check condition (i), which may serve as an example for the other cases. Assume $\mathsf{F}u\colon \langle R\rangle\varphi \in \Gamma^*$ and let u' be a constant of type s. Let k be the maximum of $\#(\mathsf{F}u\colon \langle R\rangle\varphi)$, $\#(\mathsf{F}u\colon Ru')$, and $\#(\mathsf{F}u'\colon \varphi)$. Since, by (20), $\Gamma_k \cup \{\mathsf{F}u\colon \langle R\rangle\varphi\} \notin \mathcal{U}$, it must be the case by definition 23 and the fact that $\mathsf{F}\langle\cdot\rangle$ is a tableau rule that either $\Gamma_k \cup \{\mathsf{F}u\colon Ru'\} \notin \mathcal{U}$ or $\Gamma_k \cup \{\mathsf{F}u'\colon \varphi\} \notin \mathcal{U}$. Using (20) again, we find that either $\mathsf{F}u\colon Ru' \in \Gamma^*$ or $\mathsf{F}u'\colon \varphi \in \Gamma^*$.

We conclude that Γ^* is satisfied by an intensional model M. In order to prove that there is a normal countable intensional model satisfying Γ^* and hence Γ it suffices to show that Γ^* is complete. Let φ be any sentence of $\mathcal{L} \cup \mathcal{C}$ and assume that $\mathsf{T}u\colon \varphi \notin \Gamma^*$ and $\mathsf{F}u\colon \varphi \notin \Gamma^*$. Then, by (20), $\Gamma_k \cup \{\mathsf{T}u\colon \varphi\} \in \mathcal{U}$ and $\Gamma_k \cup \{\mathsf{F}u\colon \varphi\} \in \mathcal{U}$, for sufficiently large k. But M satisfies Γ_k and therefore must either satisfy $\Gamma_k \cup \{\mathsf{T}u\colon \varphi\}$ or $\Gamma_k \cup \{\mathsf{F}u\colon \varphi\}$, contradicting the soundness of $\mathcal{U}$. Thus Γ^* is complete and some normal countable intensional model satisfies Γ^* and Γ. □

We can now reap a harvest of corollaries by relating various notions to unsatisfiability properties. In the following Γ will always be a branch in some language $\mathcal{L}$ while Δ ranges over branches in $\mathcal{L} \cup \mathcal{C}$, where $\mathcal{L}$ and $\mathcal{C}$ are as in the formulation of Theorem 7.

Corollary 8 (Generalized Compactness). *If each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable then Γ is satisfiable.*

Proof. $\{\Delta \mid$ some finite $\Delta_0 \subseteq \Delta$ is unsatisfiable$\}$ is a sound unsatisfiability property. □

Corollary 9 (Generalized Löwenheim–Skolem). *If Γ is satisfiable then Γ is satisfiable by a countable intensional model.*

Proof. $\{\Delta \mid \Delta$ is unsatisfiable$\}$ is a sound unsatisfiability property. □

Corollary 10 (Generalized Completeness). *If* Γ *is unsatisfiable then* Γ *has a closed tableau.*

Proof. $\{\Delta \mid \Delta$ has a closed tableau$\}$ is a sound unsatisfiability property. □

Corollary 11 (Admissability of Cut). *If* $\Gamma \cup \{\mathsf{T}u\colon \varphi\}$ *and* $\Gamma \cup \{\mathsf{F}u\colon \varphi\}$ *have closed tableaus then* Γ *has a closed tableau.*

Proof. Use soundness and generalized completeness. □

7 Conclusion

This chapter has looked at some of the motivations for combining modality with quantification and abstraction over objects of higher order. Montague's logic **IL** was reviewed and was found to have some shortcomings: it is not Church-Rosser and it is not intensional in Whitehead and Russell's original sense. An alternative higher order modal logic **MTT** was then introduced. **MTT** imports many ideas from the higher order logics in Fitting [19], but is based on a simpler notion of model. We have dubbed the generalized models on which **MTT** is based *intensional* models. As was shown above, the usual rules of α, β and η conversion can consistently be added to the logic in which case the logic sports the Church-Rosser property.

The logic is also fully intensional (or "hyperintensional") in the sense that co-entailing expressions need not be identical and we shall use the rest of this conclusion to discuss some points that arise in relation with this. Consider (21–24), where in each case a natural language sentence is accompanied by its **MTT** rendering. (Here *fido*, *fritz* and *mary* are constants of individual type e, *in* is a predicate of type $\langle e \rangle$, and *know* is a relation of type $\langle\langle\rangle e\rangle$.)

(21) a. Fritz is out if Fido is in
b. $\mathit{in\ fido} \rightarrow \neg(\mathit{in\ fritz})$

(22) a. Fido is out if Fritz is in
b. $\mathit{in\ fritz} \rightarrow \neg(\mathit{in\ fido})$

(23) a. Mary knows Fritz is out if Fido is in
b. $\mathit{know}\ (\mathit{in\ fido} \rightarrow \neg(\mathit{in\ fritz}))\ \mathit{mary}$

(24) a. Mary knows Fido is out if Fritz is in
b. $\mathit{know}\ (\mathit{in\ fritz} \rightarrow \neg(\mathit{in\ fido}))\ \mathit{mary}$

Simple tableaus will verify that (21b) and (22b) co-entail, as they should. But (23b) and (24b) do *not* co-entail: Note that $\{\mathsf{T}u\colon (23b), \mathsf{F}u\colon (24b)\}$ is downward saturated and thus will have an intensional model refuting one direction of the entailment.

It may be protested that there is at least one sense in which Mary knows that Fido is out if Fritz is in if she knows that Fritz is out if Fido is in: While she may have failed to derive the contraposed statement *explicitly*, there is still a sense in which she is *implicitly* committed to it. Such a notion of *implicit knowledge* is also available in **MTT**. Let K be a constant of type $\langle es \rangle$. K can be given the role of an *epistemic alternative relation* by adopting the following meaning postulate.

(25) $\Box\forall x_e \forall w_s (Kxw \leftrightarrow \forall p_{\langle\rangle}(\mathit{know}\, px \rightarrow \Diamond(\dot{w} \wedge p)))$

This says effectively that a world w is an epistemic alternative for a person x if w is in the intersection of the extensions of all propositions that x explicitly knows to hold.[8] A tableau will show that (25) entails (26).

(26) $\forall x_e \forall p_{\langle\rangle}(\mathit{know}\, px \rightarrow [Kx]p)$

Thus it can be deduced that (27), where the modal operator $[K\ \mathit{mary}]$ was used to model Mary's implicit beliefs, follows from (23). In fact *implicit* beliefs are closed under consequence and hence co-entailment.

(27) a. Mary implicitly knows Fido is out if Fritz is in
b. $[K\ \mathit{mary}](\mathit{in\, fritz} \rightarrow \neg(\mathit{in\, fido}))$

The non-equivalence of (23b) and (24b) discussed above illustrates that **MTT** is intensional in Whitehead and Russell's sense of the term. Relations, including zero-place relations, can be co-extensional without being identical. This means that linguistic expressions that are assumed to denote relations are no longer predicted to be intersubstitutable if they have the same extension, not even if they have the same extension in all possible worlds.

This is not unimportant since many expressions in natural language are undoubtedly relational and a nasty foundational problem will no longer be associated with them, but there seems to be a rest category of problems with expressions of *basic* type. Above we have treated *proper names* as having a basic type e, and this leads to the well-known Hesperus–Phosphorus, or Cicero–Tully, kind of problem. If *Hesperus* is translated as $\mathit{hesperus}_e$, *Phosphorus* as $\mathit{phosphorus}_e$, and the identity statement *Hesperus is Phosphorus* as $\mathit{hesperus} = \mathit{phosphorus}$, the consequence will be the false prediction that the two names can be substituted for one another in any context *salva veritate*.

There are two reactions to this. One possible reaction is an adaptation of the logic. One could introduce some domain of individual concepts and allow many-one correspondences between individual concepts and individuals. Such a strategy is followed by Fox and Lappin [20] in a different set-up, but in our case it would lead to a complication of the logic, be it probably a mild one.

The second reaction leaves the logic as is, but adapts the rendering of natural language expressions. If names can be treated as predicates in some way, the intension–extension distinction will come for free for them as well. In fact, the existing literature contains several proposals for treating names as based

[8] Note that the present set-up distinguishes between propositions (the elements of $D_{\langle\rangle}$) and sets of possible worlds. The extension of a proposition will be a set of worlds. Different propositions may determine the same extension.

on predicates and not on individual constants. Russell's description theory of names is an early example and Montague [38] offers another example by essentially treating names as being of the "raised" type $\langle\langle e\rangle\rangle$, not simply of type e. In combination with a treatment of identity as co-extensionality (in all possible worlds) this would avoid the problems if our logic is used. A third proposal that in effect treats names as relations comes from the literature on plurality. Many authors on this subject, starting with Bartsch [5] and Bennett [6] (see Lønning [34] for an overview), have argued that both singular and plural individuals should in fact be treated as sets, with the semantic property of being a singleton corresponding to the grammatical notion of singularity. In the present set-up this effect can be obtained by redefining type e as a complex type $\langle 0\rangle$, where 0 is a new basic type for abstract individuals. Type 0 objects will now correspond one-to-one with the *extensions* of those type e objects that have singleton extensions, i.e. to singular individuals, but there are many intensional models in which $hesperus_e$ and $phosphorus_e$ are co-extensional (with a singleton extension) in all worlds but are not identical. Let $A_e \approx B_e$ be an abbreviation of $\Box\forall x_0(Ax \leftrightarrow Bx)$, i.e. let $A \approx B$ express necessary co-extensionality, and assume that natural language *is* (the "*is* of identity") in fact expresses $\approx$. Then the argument in (28) will be rendered as (29) and will therefore be predicted to be invalid.

(28) $$\frac{\text{Hesperus is Phosphorus} \qquad \text{Mary knows that Hesperus is Hesperus}}{\text{Mary knows that Hesperus is Phosphorus}}$$

(29) $$\frac{hesperus \approx phosphorus \qquad know\ (hesperus \approx hesperus)\ mary}{know\ (hesperus \approx phosphorus)\ mary}$$

Again, the invalidity of the argument depends on the fact that Mary's knowledge was taken to be Mary's *explicit* knowledge. If implicit knowledge is taken, the argument will turn out to be valid, as the reader will have no difficulty to verify.

We conclude that the logic **MTT** is truly intensional, as it will distinguish between the meaning of one relation and another necessarily co-extensive with it. This can be used to avoid many substitution problems in natural language semantics and other areas. If it is moreover accepted that proper names should in fact be treated as constants of complex type, they will also be treated hyperintensionally. For example, letting them be of type $\langle 0\rangle$, a move which may be argued for on independent grounds having to do with the treatment of plurality, will make them start to act as naming individual concepts and substitution problems with them will be avoided.

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