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A new elementary proof of Euler's continued fractions

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Introduction

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}.$$

The value of a continued fraction is defined in a natural way. We construct the sequence of convergents $\{c_n\}$ as follows:

$$c_0 = a_0,$$

$$c_1 = a_0 + \frac{b_0}{a_1} = \frac{a_0 a_1 + b_0}{a_1},$$

$$c_2 = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2}} = \frac{a_0 a_1 a_2 + a_0 b_1 + a_2 b_0}{a_1 a_2 + b_1}, \ldots$$

and if this sequence $\{c_n\}$ converges then we say that the above infinite continued fraction *converges* and we write

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}} = \lim_{n \to \infty} c_n.$$

Continued fractions often reveal beautiful number patterns. The interested reader is referred to [1] for a collection of many interesting continued fractions of famous mathematical constants. Continued fractions also have applications in cryptography − the study of secret codes and data encryption [2, 3].

Euler was the first person who studied continued fractions systematically. In his foundational publication on the theory of continued fractions, *De Fractionibus Continuis Dissertatio* [4], Euler derived many interesting continued fraction identities. In this paper, we will present a new proof of the following two continued fractions of Euler's constant e:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}} = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{\ddots}}}}. \qquad (1)$$

Our proof is very elementary and based on simple manipulation of sequences.

For the first identity, suppose that the continued fraction converges to x. Then we can determine a closed form formula for its subfraction as follows:

$$u_n = n + \cfrac{n}{n+1+\cfrac{n+1}{n+2+\cfrac{n+2}{n+3+\cfrac{n+3}{\ddots}}}} = -\frac{n}{n+1} \times \frac{x\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!} - 1}{x\displaystyle\sum_{k=0}^{n+1}\frac{(-1)^k}{k!} - 1}.$$

This subfraction formula provides a surprising twist for a proof by contradiction. Indeed, if $x \neq e$ then

$$\lim_{n\to\infty} u_n = -\frac{xe^{-1} - 1}{xe^{-1} - 1} = -1.$$

which is utterly untrue as $u_n > n$.

Similarly, for the second identity, suppose that the continued fraction converges to x. Then

$$v_n = n + \cfrac{n+1}{n+1+\cfrac{n+2}{n+2+\cfrac{n+3}{\ddots}}} = -\frac{x\displaystyle\sum_{k=0}^{n-1}\frac{(-1)^k}{k!} - 1}{x\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!} - 1}$$

and if $x \neq e$ then

$$\lim_{n\to\infty} v_n = -\frac{xe^{-1} - 1}{xe^{-1} - 1} = -1 \qquad \text{which is false, as above.}$$

Euler-Wallis recurrence formulas

To prove that a continued fraction converges, we often need to determine its convergent sequence. The following theorem due to William Brouncker [1620-1684], the first President of The Royal Society, gives us recursive formulas to calculate the convergents. John Wallis [1616-1703] and Leonhard Euler [1707-1783] made extensive use of these formulas which are now called the Euler-Wallis formulas.

Theorem 1 [5]: For any $n \geqslant 0$, the nth convergent can be determined as $c_n = \dfrac{p_n}{q_n}$ where the numerator and the denominator sequences $\{p_n\}_{n \geqslant -2}$ and $\{q_n\}_{n \geqslant -2}$ are specified as follows (with the convention that $b_{-1} = 1$):

$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad p_n = a_n p_{n-1} + b_{n-1} p_{n-2}, \qquad \text{for all } n \geqslant 0,$$

$$q_{-2} = 1, \qquad q_{-1} = 0, \qquad q_n = a_n q_{n-1} + b_{n-1} q_{n-2}, \qquad \text{for all } n \geqslant 0.$$

The theorem can be proved easily by induction. By modifying the coefficient a_n to be $a_n + \frac{b_n}{a_{n+1}}$, the $(n + 1)$th convergent is equal to the modified nth convergent, and thus,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{\left(a_n + \frac{b_n}{a_{n+1}}\right)p_{n-1} + b_{n-1}p_{n-2}}{\left(a_n + \frac{b_n}{a_{n+1}}\right)q_{n-1} + b_{n-1}q_{n-2}}$$

$$= \frac{a_{n+1}(a_n p_{n-1} + b_{n-1}p_{n-2}) + b_n p_{n-1}}{a_{n+1}(a_n q_{n-1} + b_{n-1}q_{n-2}) + b_n q_{n-1}}$$

$$= \frac{a_{n+1}p_n + b_n p_{n-1}}{a_{n+1}q_n + b_n q_{n-1}}.$$

Using the Euler-Wallis recurrence formulas, it is easy to prove by induction that

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} b_0 b_1 \dots b_{n-1}, \text{ for all } n \geqslant 1,$$

$$p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n b_0 b_1 \dots b_{n-2}, \text{ for all } n \geqslant 2,$$

and so

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1} b_0 b_1 \dots b_{n-1}}{q_{n-1}q_n},$$

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n b_0 b_1 \dots b_{n-2}}{q_{n-2}q_n}.$$

It follows that, for positive coefficients a_i and b_i,

$$\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \dots > \frac{p_{2n+1}}{q_{2n+1}} > \dots > \frac{p_{2n}}{q_{2n}} > \dots > \frac{p_4}{q_4} > \frac{p_2}{q_2} > \frac{p_0}{q_0}.$$

Thus, in this case $\{c_{2n}\}$ and $\{c_{2n+1}\}$ always converge, and the continued fraction converges if, and only if,

$$\lim_{n \to \infty} \left(\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}\right) = 0.$$

Since $q_0 = 1$ and $q_n \geqslant a_n q_{n-1}$, we have $q_n \geqslant a_1 \dots a_n$ for all $n \geqslant 1$, and thus,

$$\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}} = \frac{b_0 b_1 \dots b_{2n}}{q_{2n}q_{2n+1}} \leqslant \frac{b_0 b_1 \dots b_{2n}}{a_1^2 \dots a_{2n}^2 a_{2n+1}}.$$

To prove the convergence of a continued fraction with positive coefficients, it suffices to show that

$$\lim_{n \to \infty} \frac{b_0 b_1 \dots b_{2n}}{a_1^2 \dots a_{2n}^2 a_{2n+1}} = 0.$$

For the two continued fractions considered here, we have $a_i \geqslant i$ and $b_i \leqslant i + 2$, so

$$\frac{b_0 b_1 \dots b_{2n}}{a_1^2 \dots a_{2n}^2 a_{2n+1}} \leqslant \frac{(2n + 2)!}{((2n)!)^2 (2n + 1)} = \frac{2n + 2}{(2n)!} \to 0$$

and the convergence is clearly guaranteed.

The first continued fraction for e

Theorem 2: For any integer $n \geqslant 1$,

$$n + \cfrac{n}{n+1+\cfrac{n+1}{n+2+\cfrac{n+2}{n+3+\cfrac{n+3}{\ddots}}}} = -\frac{n}{n+1} \times \frac{e\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!} - 1}{e\displaystyle\sum_{k=0}^{n+1}\frac{(-1)^k}{k!} - 1},$$

and

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}} = e.$$

Proof: We know that the first continued fraction of (1) converges. We let x be the convergence value and construct the sequences $\{u_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ as follows:

$$u_1 = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}} = \frac{1}{x - 2} = \frac{xa_1 + b_1}{xc_1 + d_1},$$

$$\begin{aligned} u_n &= n + \cfrac{n}{n + 1 + \cfrac{n + 1}{n + 2 + \cfrac{n + 2}{\ddots}}} \\ &= \frac{n - 1}{u_{n-1} - (n - 1)} \\ &= \frac{n - 1}{\dfrac{xa_{n-1} + b_{n-1}}{xc_{n-1} + d_{n-1}} - (n - 1)} \\ &= \frac{x(n - 1)c_{n-1} + (n - 1)d_{n-1}}{x\left(a_{n-1} - (n - 1)c_{n-1}\right) + \left(b_{n-1} - (n - 1)d_{n-1}\right)} \\ &= \frac{xa_n + b_n}{xc_n + d_n}. \end{aligned}$$

Then we have the following recursive formulas:

$a_1 = 0,\ b_1 = 1,\ c_1 = 1,\ d_1 = -2$

$a_n = (n-1)c_{n-1},\ b_n = (n-1)d_{n-1},$

$c_n = a_{n-1} - (n-1)c_{n-1},\ d_n = b_{n-1} - (n-1)d_{n-1},$ for all $n \geqslant 2$.

For any $n \geqslant 3$, we have

$$\frac{a_n}{n-1} = c_{n-1} = a_{n-2} - (n-2)c_{n-2} = a_{n-2} - a_{n-1}$$

$$\Rightarrow a_n + (n-1)a_{n-1} - (n-1)a_{n-2} = 0$$

$$\Rightarrow a_n - a_{n-2} + (n-1)a_{n-1} - (n-2)a_{n-2} = 0.$$

So

$$\sum_{k=3}^{n} \left(a_k - a_{k-2} + (k-1)a_{k-1} - (k-2)a_{k-2}\right) = 0$$

and this gives

$$a_n + a_{n-1} - a_2 - a_1 + (n-1)a_{n-1} - a_1 = 0.$$

It follows that for any $n \geqslant 2$,

$$a_n + na_{n-1} = a_2 + 2a_1.$$

Rewriting the above equation as

$$\frac{(-1)^n a_n}{n!} - \frac{(-1)^{n-1} a_{n-1}}{(n-1)!} = \left(a_2 + 2a_1\right)\frac{(-1)^n}{n!}$$

and taking the summation

$$\sum_{k=2}^{n}\left(\frac{(-1)^k a_k}{k!} - \frac{(-1)^{k-1} a_{k-1}}{(k-1)!}\right) = \left(a_2 + 2a_1\right)\sum_{k=2}^{n}\frac{(-1)^k}{k!}$$

we get

$$\frac{(-1)^n a_n}{n!} - \frac{(-1)a_1}{1!} = \left(a_2 + 2a_1\right)\sum_{k=2}^{n}\frac{(-1)^k}{k!}, \text{ for all } n \geqslant 1.$$

We derive the following closed form for a_n:

$$a_n = (-1)^n n!\left(-a_1 + (a_2 + 2a_1)\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right), \text{ for all } n \geqslant 1.$$

Similarly, we get the following closed form for b_n:

$$b_n = (-1)^n n!\left(-b_1 + (b_2 + 2b_1)\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right), \text{ for all } n \geqslant 1.$$

Substituting the values

$$a_1 = 0, \qquad b_1 = 1, \qquad a_2 = 1, \qquad b_2 = -2$$

we obtain

$$a_n = (-1)^n n! \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

$$b_n = (-1)^{n+1} n!\,.$$

It follows that

$$c_n = \frac{a_{n+1}}{n} = (-1)^{n+1} \frac{(n+1)!}{n} \sum_{k=0}^{n+1} \frac{(-1)^k}{k!},$$

$$d_n = \frac{b_{n+1}}{n} = (-1)^n \frac{(n+1)!}{n}.$$

Finally, we have

$$u_n = \frac{xa_n + b_n}{xc_n + d_n} = \frac{x(-1)^n n! \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!} + (-1)^{n+1} n!}{x(-1)^{n+1} \dfrac{(n+1)!}{n} \displaystyle\sum_{k=0}^{n+1} \frac{(-1)^k}{k!} + (-1)^n \dfrac{(n+1)!}{n}}$$

$$= -\frac{n}{n+1} \frac{x \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!} - 1}{x \displaystyle\sum_{k=0}^{n+1} \frac{(-1)^k}{k!} - 1}, \text{ for all } n \geqslant 1.$$

From here, it implies that $x = e$. This is because if $x \neq e$ then

$$\lim_{n \to \infty} u_n = -\frac{xe^{-1} - 1}{xe^{-1} - 1} = -1$$

which contradicts the obvious fact that $u_n > n$.

The second continued fraction for e

By using the same method as above, the second identity can be easily proved and we leave that to the interested reader.

Theorem 3: For any integer $n \geqslant 1$,

$$n + \cfrac{n+1}{n+1+\cfrac{n+2}{n+2+\cfrac{n+3}{\ddots}}} = -\frac{e \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} - 1}{e \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!} - 1},$$

and

$$2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{\ddots}}}} = e.$$

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