

## Monotonic ratios of functions

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*Problem:* Show that if  $p > 1$ , then  $\sinh px / \sinh x$  increases with  $x$  for  $x > 0$ .

The most obvious approach is to try to show that the derivative is non-negative. This can, in fact, be achieved without too much difficulty, using the special properties of the functions  $\sinh$  and  $\cosh$ . However, one is left with the feeling that this might be a special case of something much more general. Does a similar statement apply to  $f(px)/f(x)$  for a wide range of functions  $f$ ? We will show that this is indeed the case whenever  $f$  is a polynomial, or a power series, with non-negative coefficients. In fact, we will establish a more general result applying to suitable ratios  $g(x)/f(x)$ .

We use the term “increasing” in the wide sense: if  $x_1 < x_2$ , then  $f(x_1) \leq f(x_2)$  (not excluding the case where  $f(x)$  is constant). Also, to avoid tedious repetition, we will say, for example, that  $f(x)$  “decreases with  $x$ ” to mean that it is a decreasing function of  $x$  (and similarly with  $n$  instead of  $x$ ). Our result is as follows.

*Theorem:* (i) Suppose that  $f(x) = \sum_{r=0}^n a_r x^r$  and  $g(x) = \sum_{r=0}^n c_r a_r x^r$ , where  $a_r \geq 0$  and  $c_r > 0$  for each  $r$ , with some  $a_{r_0} > 0$ . If  $c_r$  decreases with  $r$ , then  $g(x)/f(x)$  decreases with  $x$  for  $x > 0$ . If  $c_r$  increases, then  $g(x)/f(x)$  increases.

(ii) Now suppose that  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} c_n a_n x^n$  for  $|x| < R$ , where  $a_n \geq 0$  and  $c_n > 0$  for each  $n$ , with some  $a_{n_0} > 0$ . If  $c_n$  decreases with  $n$ , then  $g(x)/f(x)$  decreases with  $x$  on  $(0, R)$ . If  $c_n$  increases, then  $g(x)/f(x)$  increases.

Of course, (ii) follows from (i) simply by considering limits.

What happens if one tries to prove the theorem by showing that the derivative is non-negative? This is equivalent to showing that  $f(x)g'(x) - f'(x)g(x) \geq 0$ , perhaps by showing that all the coefficients in this expression are non-negative. It turns out that this approach just leads to unpleasantly complicated expressions, with no transparent route to the conclusion. To test this assertion, the reader could try writing out the case  $n = 3$ .

Our method will not use differentiation at all. Instead, we will use *Abel summation*, which is the following way to rewrite a sum of products. Given  $a_r, b_r$  for  $0 \leq r \leq n$ , write  $A_r = a_0 + a_1 + \dots + a_r$ . Then  $a_0 = A_0$  and  $a_r = A_r - A_{r-1}$  for  $r \geq 1$ , so

$$\sum_{r=0}^n a_r b_r = A_0 b_0 + (A_1 - A_0) b_1 + \dots + (A_n - A_{n-1}) b_n$$

$$= A_0(b_0 - b_1) + A_1(b_1 - b_2) + \cdots + A_{n-1}(b_{n-1} - b_n) + A_n b_n.$$

The other ingredient of our proof is the following obvious fact: if  $f(x)$  and  $g(x)$  are positive, then  $f(x)/g(x)$  is increasing if and only if  $g(x)/f(x)$  is decreasing. We will apply this repeatedly, in a switchback ride of successive inversions.

*Proof of the Theorem:* As already mentioned, we only need to prove (i). Also, it is enough to prove the statement for decreasing  $c_r$ . The statement for increasing  $c_r$  then follows, by considering  $f(x)/g(x)$  and noting that  $a_r = c_r^{-1}(c_r a_r)$ . Further, if  $r_0 > 0$ , then division top and bottom by  $x^{r_0}$  replaces  $f(x)$  by a polynomial with non-zero constant term, so it is enough to consider the case where  $a_0 > 0$ .

Write  $f_k(x) = \sum_{r=0}^k a_r x^r$  (so  $f_n(x) = f(x)$ ). By Abel summation,

$$g(x) = \sum_{k=0}^{n-1} (c_k - c_{k+1}) f_k(x) + c_n f(x).$$

Since  $c_k - c_{k+1} \geq 0$ , the required statement follows if we can show that for each  $k < n$ , the ratio  $f_k(x)/f(x)$  decreases with  $x$ . Consider the reciprocal:

$$\frac{f(x)}{f_k(x)} = 1 + \sum_{s=k+1}^n \frac{a_s x^s}{f_k(x)}.$$

Inverting again, we have for  $s > k$ ,

$$\frac{f_k(x)}{x^s} = \sum_{r=0}^k a_r x^{r-s}.$$

Here  $r - s < 0$ , so  $x^{r-s}$  decreases with  $x$ . Hence  $f_k(x)/x^s$  decreases, so  $x^s/f_k(x)$  increases. Therefore  $f(x)/f_k(x)$  increases, so  $f_k(x)/f(x)$  decreases, as required.

*Note.* For a minor generalisation, replace the terms  $x^r$  by positive functions  $u_r(x)$  satisfying the condition that  $u_r(x)/u_{r+1}(x)$  decreases with  $x$  on  $(0, \infty)$ . The proof is the same, with  $x^{r-s}$  replaced by  $u_r(x)/u_s(x)$ .

An immediate deduction is the result we stated first, slightly enhanced:

*Corollary:* Let  $f(x) = \sum_{r=0}^n a_r x^r$ , where  $a_r \geq 0$  for all  $r$ , with some  $a_{r_0} > 0$ . If  $p > q > 0$ , then  $f(px)/f(qx)$  increases with  $x$  on  $(0, \infty)$ . Similarly for infinite series  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  within the interval where  $f(px)$  converges.

*Proof:* Apply the Theorem with  $a_n$  replaced by  $a_n q^n$  and  $c_n = p^n/q^n$ .

In particular, we recover our original example  $\sinh px/\sinh x$ , together with (for example)  $\cosh px/\cosh x$ . We record a number of other particular cases.

*Example 1:* Applied to the infinite geometric series  $\sum_{n=0}^{\infty} x^n = 1/(1-x)$  (for  $|x| < 1$ ), the Corollary says that if  $p > q > 0$ , then  $(1-qx)/(1-px)$  is increasing for  $0 < x < \frac{1}{p}$ . However, this is obvious: the expression equates to  $q/p + (p-q)/(1-px)$ . But for the polynomial  $f_n(x) = 1 + x + \dots + x^{n-1}$ , the statement is that  $f_n(px)/f_n(qx)$  increases for all  $x > 0$ , and this is not at all trivial. Again, direct differentiation does not provide an easy proof, and the identity  $f_n(x) = (1-x^n)/(1-x)$  gives (for  $x$  not equal to  $\frac{1}{p}$  or  $\frac{1}{q}$ )

$$\frac{f_n(px)}{f_n(qx)} = \frac{1-qx}{1-px} \frac{1-p^n x^n}{1-q^n x^n} :$$

for  $x < \frac{1}{p}$ , the first factor is increasing (as just seen), while the second factor is decreasing.

*Example 2:* To simplify notation, write  $L(x) = -\log(1-x)$ , so that  $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$  for  $|x| < 1$ . So if  $0 < p < 1$ , then  $L(px)/L(x)$  decreases on  $(0, 1)$ . Of course, it follows with no further work that the derivative is non-positive: we can reason this way round just as well as conversely! Written out, this equates to the following inequality:

$$p(1-x)L(x) \leq (1-px)L(px).$$

A direct proof of this is possible, but it entails careful comparison of the series expressions for both sides. Also, substituting  $x = 1/y$ , we deduce that

$$\frac{\log y - \log(y-p)}{\log y - \log(y-1)}$$

increases with  $y$  for  $y > 1$ .

*Example 3:* The “dilogarithm function”  $\text{Li}_2(x)$  is defined for  $|x| \leq 1$  by  $\text{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$ . So our Theorem, with  $c_n = \frac{1}{n}$ , shows that  $\text{Li}_2(x)/L(x)$  decreases on  $[0, 1)$ . Similarly, for example,  $x \cosh x / \sinh x$  increases with  $x$ .

Some other expressions can be reduced to our type by substitutions. We give two examples.

*Example 4:* Let  $f(x) = (x^p - x^{-p})/(x - x^{-1})$  for  $x > 1$ . The substitution  $x = e^t$  transforms  $f(x)$  to  $\sinh pt / \sinh t$ , and  $x$  increases when  $t$  increases, so if  $p > 1$ , then  $f(x)$  increases for  $x > 1$ .

*Example 5:* Let

$$f(x) = \frac{(x-1)(x^p+1)}{x^{p+1}-1}$$

for  $x > 1$ . (Note: if  $p = 1$ , then  $f(x)$  has the constant value 1.) Substitute  $x = e^{2t}$ : then  $f(x) = g(t)$ , where

$$g(t) = \frac{(e^{2t}-1)(e^{2pt+1})}{e^{2(p+1)t}-1} = \frac{2 \sinh t \cosh pt}{\sinh(p+1)t} = 1 - \frac{\sinh(p-1)t}{\sinh(p+1)t},$$

since  $2 \sinh t \cosh pt = \sinh(p+1)t - \sinh(p-1)t$ . So if  $p \geq 1$ , then  $g(t)$  increases with  $t$ , hence  $f(x)$  increases with  $x$ .

*Further thoughts about  $f(px)/f(x)$ .* Let us say that a function  $f$  has *property (A)* if it is positive and for all  $p > 1$ , the ratio  $f(px)/f(x)$  increases with  $x$  on the positive interval within its domain of definition. The Corollary says that polynomials and power series with non-negative coefficients have property (A). Are there lots more functions with the property?

A rather trivial answer is that  $f(x) = x^r$  has the property for any  $r$  (positive or negative), since then  $f(px)/f(x)$  has the constant value  $p^r$ . Numerous further examples are now generated by the following observations. First, if  $f(x)$  and  $g(x)$  have property (A), then so does  $f(x)g(x)$ . Second, if  $f(x)$  has property (A), then so do the functions  $f(x)^r$  and  $f(x^r)$  for all  $r > 0$ . So the following functions all have property (A):

$$x + \frac{1}{x}, \quad \left(x + \frac{1}{x}\right) \sinh x, \quad (\sinh x)^{1/2}, \quad \sinh x^{1/2}.$$

Finally, let us compare property (A) with the class of *convex* functions. Recall that a differentiable function  $f$  is convex if its derivative  $f'$  is increasing. Hence  $x^r$  is convex for  $r \geq 1$  and  $r \leq 0$ , and concave for  $0 \leq r \leq 1$ . So the functions described in the Corollary are certainly convex. But there is no close match. The non-convex functions  $x^r$ , for  $0 < r < 1$ , have property (A). Meanwhile, the convex function  $e^{-x}$  does not have property (A), since  $e^{-px}/e^{-x} = e^{(1-p)x}$ , which is decreasing. Another such example, easily verified, is  $1/(x+1)$ .

*Acknowledgment:* The author is grateful to the referee for a number of useful suggestions.

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