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2:1:1 Resonance in the Quasi-Periodic Mathieu Equation

RICHARD RAND* and TINA MORRISON
Department of Theoretical & Applied Mechanics, Cornell University, Ithaca, NY 14853, U.S.A.;
**Author for correspondence (e-mail: rhr2@cornell.edu; fax: +1-607-255-2011)*

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Abstract. We present a small ϵ perturbation analysis of the quasi-periodic Mathieu equation

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)x = 0$$

in the neighborhood of the point $\delta = 0.25$ and $\omega = 0.5$. We use multiple scales including terms of $O(\epsilon^2)$ with three time scales. We obtain an asymptotic expansion for an associated instability region. Comparison with numerical integration shows good agreement for $\epsilon = 0.1$. Then we use the algebraic form of the perturbation solution to approximate scaling factors which are conjectured to determine the size of instability regions as we go from one resonance to another in the $\delta - \omega$ parameter plane.

Key words: parametric excitation, resonance, quasi-periodic Mathieu equation

1. Introduction

The following quasi-periodic Mathieu equation,

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)x = 0 \tag{1}$$

has been the topic of a number of recent research papers [1–7]. In particular, the stability of Equation (1) has been investigated. (For given parameters $(\epsilon, \delta, \omega)$, Equation (1) is said to be stable if all solutions are bounded, and unstable if an unbounded solution exists.) A stability chart for $\epsilon = 0.1$ is shown in Figure 1. This chart was obtained in [1] by numerical integration of Equation (1). A striking feature of this complicated figure is that various details appear to be repeated at different length scales. For example, the shape of the instability regions around the point $\delta = 0.25$ and $\omega = 1$ appear to be similar to those near the point $\delta = 0.25$ and $\omega = 0.5$, except that the latter regions are smaller in scale. The purpose of this work is to investigate this similarity, and to obtain a scaling law which relates these two regions. Our approach will be based on approximate solutions obtained by perturbation methods. In the case of the instability regions near the point $\delta = 0.25$ and $\omega = 1$, a perturbation analysis has been performed in [5], giving approximate closed form expressions for the transition curves separating regions of stability from regions of instability. For example, the following approximate expression was derived for the largest of the local instability regions near $\delta = 0.25$ and $\omega = 1$, see Figure 2:

$$\omega = 1 + \epsilon\left(S \pm \left(\frac{1}{2} + \frac{\delta_1}{S}\right)\right) + O(\epsilon^2) \tag{2}$$

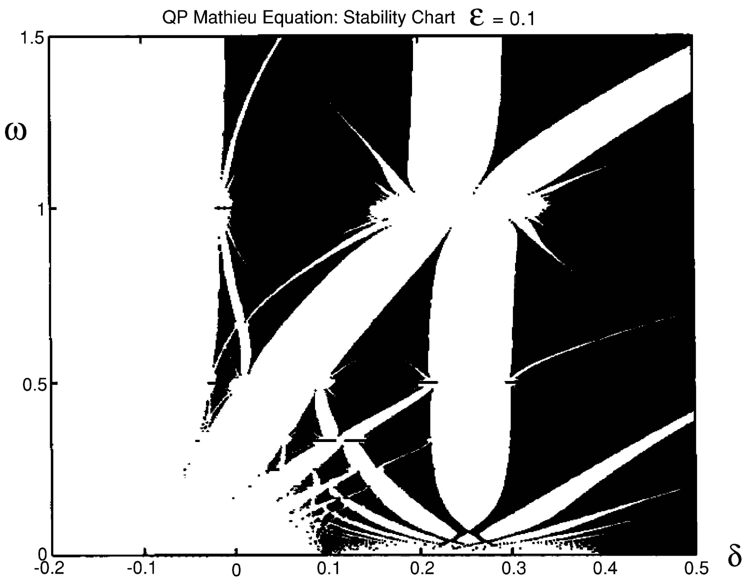

Figure 1. Stability diagram of Equation (1) from [1]. White regions are unstable, black regions are stable.

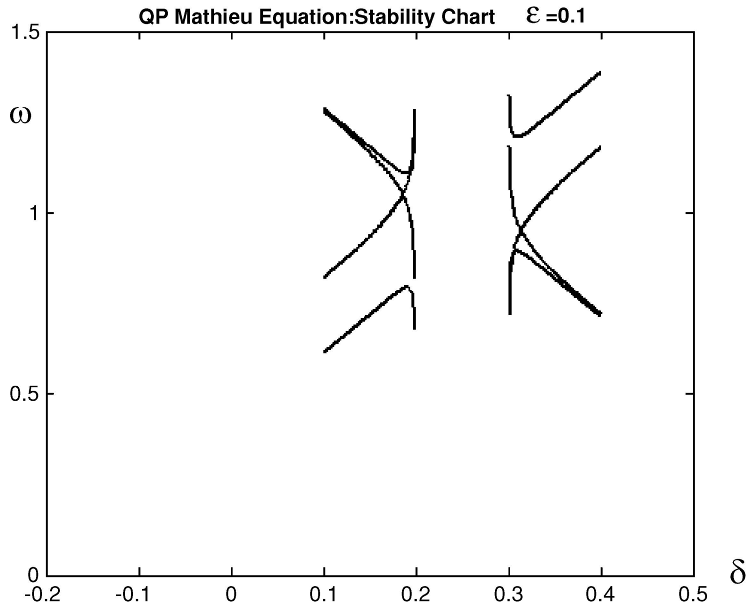

Figure 2. Transition curves near $\delta = 0.25$ and $\omega = 1$ based on Equation (2) from [5]. Compare with Figure 1.

where

$$\delta = \frac{1}{4} + \delta_1 \epsilon \quad \text{and} \quad S = \sqrt{4\delta_1^2 - 1} \tag{3}$$

In the present work we use a different perturbation method to obtain a comparable approximate expression for the largest of the local instability regions near $\delta = 0.25$ and $\omega = 0.5$. By comparing the two expressions we are able to determine scaling factors which relate their relative sizes.

The title of this paper is explained by noting that Equation (1) may be viewed as an oscillator with natural frequency $\sqrt{\delta}$ which is parametrically forced with frequencies 1 and ω. Near the point $\delta = 0.25$ and $\omega = 0.5$, the three frequencies are in the ratio of 2:1:1.

2. Perturbation Method

We use the method of multiple time scales on Equation (1), with the goal of obtaining an approximate expression for transition curves in the neighborhood of $\delta = 0.25$ and $\omega = 0.5$. To accomplish this, we found it necessary to go to $O(\epsilon^2)$ and to use three time scales:

$$\xi = t, \ \eta = \epsilon t, \quad \text{and} \quad \zeta = \epsilon^2 t \tag{4}$$

As usual in this method [8] the dependent variable x becomes a function of ξ, η, and ζ, giving a new expression for the second derivative of x in Equation (1):

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial \xi^2} + 2\epsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \epsilon^2 \frac{\partial^2 x}{\partial \eta^2} + 2\epsilon^2 \frac{\partial^2 x}{\partial \xi \partial \zeta} + O(\epsilon^3) \tag{5}$$

We expand x, ω, and δ as a power series of ϵ:

$$\begin{aligned} x &= x_0 + \epsilon\, x_1 + \epsilon^2 x_2 + \cdots \\ \omega &= \tfrac{1}{2} + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \\ \delta &= \tfrac{1}{4} + \epsilon\delta_1 + \epsilon^2 \delta_2 + \cdots \end{aligned} \tag{6}$$

and substitute these series into Equations (1), (5) and collect terms in ϵ. Using subscripts to represent partial differentiation, we obtain:

$$\epsilon^0 : x_{0,\xi\xi} + \frac{1}{4} x_0 = 0 \tag{7}$$

$$\begin{aligned} \epsilon^1 : x_{1,\xi\xi} + \frac{1}{4} x_1 = {} & -2x_{0,\xi\eta} - \delta_1 x_0 - x_0, \cos\xi \\ & - x_0 \cos\left(\frac{\xi}{2} + \omega_1\eta + \omega_2\zeta\right) \end{aligned} \tag{8}$$

$$\begin{aligned} \epsilon^2 : x_{2,\xi\xi} + \frac{1}{4} x_2 = {} & -2x_{0,\xi\zeta} - x_{0,\eta\eta} - 2x_{1,\xi\eta} - \delta_2 x_0 - \delta_1 x_1 \\ & - x_1 \cos\xi - x_1 \ \cos\left(\frac{\xi}{2} + \omega_1\eta + \omega_2\zeta\right) \end{aligned} \tag{9}$$

We write the solution to Equation (7) in the form

$$x_0(\xi, \eta, \zeta) = A(\eta, \zeta) \cos\frac{\xi}{2} + B(\eta, \zeta) \sin\frac{\xi}{2} \tag{10}$$

where A and B are as yet undetermined slowly varying coefficients. Substituting Equation (10) into Equation (8) and removing secular terms gives the following equations on A and B:

$$\begin{aligned} A_\eta &= \left(\delta_1 - \frac{1}{2}\right) B \\ B_\eta &= -\left(\delta_1 + \frac{1}{2}\right) A \end{aligned} \tag{11}$$

Equations (11) have the solution:

$$A(\eta, \zeta) = A_1(\zeta)\cos\frac{S}{2}\eta + A_2(\zeta)\sin\frac{S}{2}\eta \tag{12}$$

and a similar expression for $B(\eta, \zeta)$. Here $S = \sqrt{4\delta_1^2 - 1}$ and A_1 and A_2 are as yet undetermined slowly varying coefficients.

Having removed secular terms from Equation (8), we may solve for x_1, which may be written in the abbreviated form:

$$x_1(\xi, \eta, \zeta) = C(\eta, \zeta)\cos\frac{\xi}{2} + D(\eta, \zeta)\sin\frac{\xi}{2} + \text{periodic terms} \tag{13}$$

where the first two terms on the RHS of Equation (13) are the complementary solution of Equation (8) involving as yet undetermined slowly varying coefficients $C(\eta, \zeta)$ and $D(\eta, \zeta)$. The "periodic terms" represent a particular solution of Equation (8). Although the expressions for these are too long give here, we note that they consist of sinusoidal terms with arguments

$$\frac{3}{2}\xi \pm \frac{S}{2}\eta, \quad \omega_1\eta + \omega_2\zeta \pm \frac{S}{2}\eta, \quad \text{and} \quad \xi + \omega_1\eta + \omega_2\zeta \pm \frac{S}{2}\eta \tag{14}$$

Next, we substitute the expressions for x_0 and x_1, Equations (10) and (13), and the expressions for A and B, Equation (12), into the x_2 equation, Equation (9), and eliminate secular terms. This gives equations on C and D which may be written in the following form:

$$\begin{aligned} C_\eta &= \left(\delta_1 - \frac{1}{2}\right)D + \text{periodic terms} \\ D_\eta &= -\left(\delta_1 + \frac{1}{2}\right)C + \text{periodic terms} \end{aligned} \tag{15}$$

Although the expressions for the 'periodic terms' are too long to give here, we note that they consist of sinusoidal terms with arguments

$$\frac{S}{2}\eta, \quad \left(2\omega_1 \pm \frac{S}{2}\right)\eta + 2\omega_2\zeta, \quad \text{and} \quad \left(2\omega_1 \pm \frac{S}{2}\right)\eta - 2\omega_2\zeta \tag{16}$$

Note that Equations (15) on the arbitrary coefficients C, D of x_1 are similar in form to the Equations (11) on the arbitrary coefficients A,B of x_0, except that the C-D eqs. are nonhomogeneous. Thus our next step is to remove secular terms from Equations (15). Since $S = \sqrt{4\delta_1^2 - 1}$, we see that the periodic terms in Equations (15) which have argument $\frac{S}{2}\eta$ are resonant. For general values of ω_1, these are the only resonant terms. However, if $\omega_1 = S/2$, then some of the other periodic terms in Equations (15) will be resonant as well, because in that case $(2\omega_1 - \frac{S}{2})\eta = \frac{S}{2}\eta$.

We first consider the case in which ω_1 does not equal $S/2$. Eliminating secular terms in Equations (15) turns out to give the following equations on the slow flow coefficients A_1 and A_2 which appeared in the expressions for A and B, Equation (12):

$$\begin{aligned} A_{1\zeta} &= -p\,A_2 \\ A_{2\zeta} &= p\,A_1 \end{aligned} \tag{17}$$

where

$$p = \delta_1 \frac{\left(24\delta_2 - 24\delta_1^2 - 7\right)}{12\,S} \tag{18}$$

Since all solutions to Equations (17) are periodic and bounded, no instability is possible.

We next consider the resonant case in which $\omega_1 = S/2$. Elimination of secular terms in Equations (15) gives:

$$\begin{aligned} A_{1\zeta} &= -p\,A_2 + q\,(-A_1 \sin 2\omega_2\zeta + A_2 \cos 2\omega_2\zeta) \\ A_{2\zeta} &= p\,A_1 + q\,(A_2 \sin 2\omega_2\zeta + A_1\, \cos 2\omega_2\zeta) \end{aligned} \tag{19}$$

where p is given by Equation (18), and where q is given by the following equation:

$$q = \frac{1}{2} + \frac{\delta_1}{S} \tag{20}$$

As we show in the next section, Equations (19) exhibit unbounded solutions. Parameter values for which unbounded solutions occur in (19) correspond to regions of instability in the stability diagram.

3. Analysis of the Slow Flow

In order to investigate the stability of the slow flow system (19), we write it in the form:

$$\begin{aligned} \dot{u} &= -pv + q(-u \sin \Omega t + v \cos \Omega t) \\ \dot{v} &= pu + q(v \sin \Omega t + u \cos \Omega t) \end{aligned} \tag{21}$$

where A_1, A_2, $2\omega_2$ and ζ have been replaced respectively by u, v, Ω and t for convenience.

We begin by transforming (21) to polar coordinates, $u = r\cos\theta$, $v = r\sin\theta$:

$$\begin{aligned} \dot{r} &= -qr\,\sin(\Omega t - 2\theta) \\ \dot{\theta} &= p + q\cos(\Omega t - 2\theta) \end{aligned} \tag{22}$$

Next we replace θ by $\phi = \Omega t - 2\theta$:

$$\begin{aligned} \dot{r} &= -qr\,\sin\phi \\ \dot{\phi} &= \Omega - 2p - 2q\cos\phi \end{aligned} \tag{23}$$

Writing Equations (23) in first order form,

$$\frac{dr}{d\phi} = \frac{-qr\,\sin\phi}{\Omega - 2p - 2q\cos\phi} \tag{24}$$

we are easily able to obtain the integral:

$$r = \frac{\text{const}}{\sqrt{\Omega - 2p - 2q\cos\phi}} \tag{25}$$

Equation (25) represents a curve in the u–v plane. If the denominator of the RHS vanishes for some value of ϕ, then the curve extends to infinity and the corresponding motion is unstable. This case corresponds to the existence of an equilibrium point in the ϕ equation (the second of Equations (23)). In the contrary case in which the denominator of the RHS of Equation (25) does not vanish, the motion remains bounded (stable) and the ϕ equation has no equilibria. Thus the condition for instability is that the following equation has a real solution:

$$\cos\phi = \frac{\Omega - 2p}{2q} \tag{26}$$

The transition case between stable and unstable is given by the condition:

$$\frac{\Omega - 2p}{2q} = \pm 1 \quad \Rightarrow \quad \Omega = 2p \pm 2q \tag{27}$$

Substituting Equations (18),(20) and using $\Omega = 2\omega_2$, Equation (27) becomes:

$$\omega_2 = \delta_1 \frac{\left(24\delta_2 - 24\delta_1^2 - 7\right)}{12\,S} \pm \left(\frac{1}{2} + \frac{\delta_1}{S}\right) \tag{28}$$

where $S = \sqrt{4\delta_1^2 - 1}$. In addition, for resonance we required that

$$\omega_1 = \frac{S}{2} \tag{29}$$

Thus we obtain the following expression for transition curves near $\delta = 0.25$ and $\omega = 0.5$:

$$\omega = \frac{1}{2} + \epsilon\frac{S}{2} + \epsilon^2\left(\delta_1 \frac{\left(24\delta_2 - 24\delta_1^2 - 7\right)}{12\,S} \pm \left(\frac{1}{2} + \frac{\delta_1}{S}\right)\right) + \cdots \tag{30}$$

where

$$\delta = \frac{1}{4} + \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots \tag{31}$$

Equation (30) gives a value of ω for a given value of δ, the latter defined by δ_1, δ_2 and ϵ. However, for a given value of ϵ, any value of δ close to 0.25 can be achieved in the form $1/4 + \epsilon\delta_1$, that is by choosing $\delta_2 = 0$. So in our numerical evaluation of Equations (30) and (31), we take $\delta_2 = 0$. See Figure 3 where the transition curves (30) are displayed for $\epsilon = 0.1$ along with the results of previous work, Equation (2), already displayed in Figure 2 cf. Figure 1.

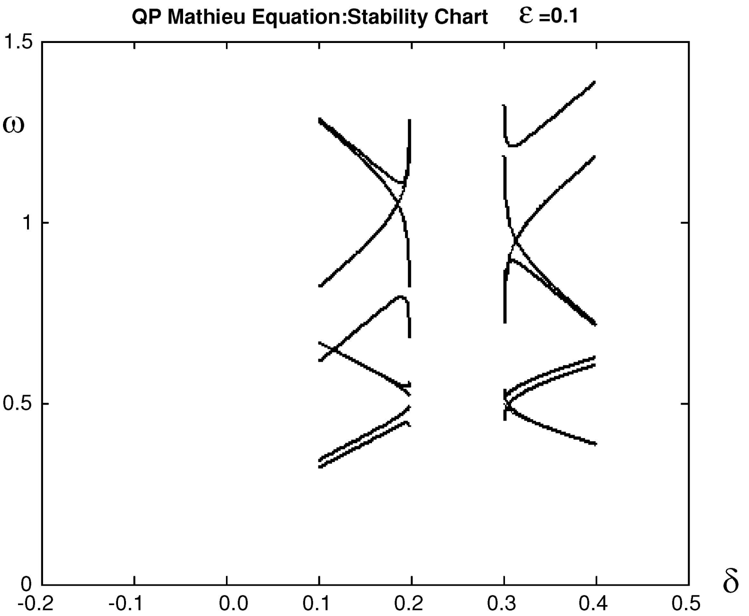

Figure 3. Transition curves near $\delta = 0.25$ and $\omega = 0.5$ based on Equation (30). Also shown for comparison are the transition curves near $\delta = 0.25$ and $\omega = 1$ based on Equation (2) from [5]. Compare with Figures 1 and 2.

4. Discussion

We now have asymptotic approximations for corresponding instability regions at two points in the $\delta - \omega$ parameter plane, and we wish to compare them. Based on an expansion about the point $\delta = 0.25$ and $\omega = 1$, we have Equation (2) from [5]. And from the work presented in this paper, we have Equation (30), valid in the neighborhood of $\delta = 0.25$ and $\omega = 0.5$. We repeat these expansions here for the convenience of the reader, using the subscript A for Equation (2), and the subscript B for Equation (30):

$$\omega_A = 1 + \epsilon\left(S \pm \left(\frac{1}{2} + \frac{\delta_1}{S}\right)\right) + O(\epsilon^2) \tag{32}$$

$$\omega_B = \frac{1}{2} + \epsilon\frac{S}{2} + \epsilon^2\left(\delta_1\frac{\left(-24\delta_1^2 - 7\right)}{12\,S} \pm \left(\frac{1}{2} + \frac{\delta_1}{S}\right)\right) + O(\epsilon^3) \tag{33}$$

Note that although these expansions are truncated at different orders of ϵ, both are the lowest order approximations which respectively yield two transition curves (as represented by the $\pm$ sign), and which therefore allow the thickness of the associated instability region to be computed.

We compare these two expressions in two ways: 1) the centerline of the region, and 2) the thickness of the region.

The respective centerlines are given by the following approximations:

$$\omega_A = 1 + \epsilon S + O(\epsilon^2) \tag{34}$$

$$\omega_B = \frac{1}{2} + \epsilon\frac{S}{2} + O(\epsilon^2) \tag{35}$$

From these we may conjecture that the scaling of the centerline in the ω direction goes like ω_0, being the ω value of the point of expansion. That is, in the case of Equation (34), $\omega_0 = 1$, while for Equation (35), $\omega_0 = 1/2$, and we observe that the two curves have comparable shape, but that they are stretched in the ω direction in proportion to their values of ω_0.

Moving on to the question of the thickness of the two instability regions, these are obtained by subtracting the expressions for the upper and lower transition curves, and are given by the following approximations:

$$\text{thickness}_A = 2\epsilon\left(\frac{1}{2} + \frac{\delta_1}{S}\right) + O(\epsilon^2) \tag{36}$$

$$\text{thickness}_B = 2\epsilon^2\left(\frac{1}{2} + \frac{\delta_1}{S}\right) + O(\epsilon^3) \tag{37}$$

We see that once again the two expressions have the same general form, but that the smaller region is a factor of ϵ thinner than the larger region. This leads us to the conjecture that this is due to the difference in the order of the resonance. For example, this would lead to the guess that the comparable instability region associated with the point $\delta = 0.25$ and $\omega = 1/3$ (which would correspond to a $2 : \frac{2}{3} : 1$ resonance) would have a similar equation for its thickness, but with an ϵ^3 in the leading term.

5. Conclusions

We have presented a small ϵ perturbation analysis of the quasi-periodic Mathieu equation (1) in the neighborhood of the point $\delta = 0.25$ and $\omega = 0.5$. We used multiple scales and found that we needed to go to $O(\epsilon^2)$ and use three time scales in order to obtain a minimal representation of an instability region. Comparison with numerical integration of Equation (1) showed good agreement for $\epsilon = 0.1$.

We used the perturbation approximation to estimate the scaling of instability regions as we go from one resonance to another in the $\delta - \omega$ parameter plane. This is an interesting use of perturbation approximations. A comparable result could not be easily achieved by purely numerical methods. For example, inspection of Figure 1 would lead us to conclude that the instability region near $\delta = 0.25$ and $\omega = 1$ is much thicker than the comparable region near $\delta = 0.25$ and $\omega = 0.5$. However, it would be difficult to conclude using only numerical results that the ratio of thicknesses was approximately ϵ, as we showed in this work.

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