Forum of Mathematics, Sigma (2016), Vol. 4, e28, 34 pages
doi:10.1017/fms.2016.23

2-ADIC INTEGRAL CANONICAL MODELS

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Received 15 December 2015; accepted 25 July 2016

Abstract

We use Lau's classification of 2-divisible groups using Dieudonné displays to construct integral canonical models for Shimura varieties of abelian type at 2-adic places where the level is hyperspecial.

2010 Mathematics Subject Classification: 14G35 (primary); 11G18 (secondary)

1. Introduction

In this note, we complete the construction of integral canonical models from [13] at places of hyperspecial level, so that it also works at 2-adic places, without any additional restrictions. Therefore, we obtain the following theorem.

THEOREM 1. *Let* $\mathrm{Sh}_K(G, X)$ *be a Shimura variety of abelian type associated with a Shimura datum* (G, X) *and a neat level* $K \subset G(\mathbb{A}_f)$*, defined over the reflex field* $E = E(G, X)$*. Suppose that for a prime* p *the* p*-primary part* K_p *is hyperspecial. Then* $\mathrm{Sh}_K(G, X)$ *admits an integral canonical model over* $\mathscr{O}_{E,(v)}$ *for any place* $v|p$ *of* E*.*

The nontrivial input is Lau's classification of p-divisible groups over a very large class of 2-adic rings in terms of Dieudonné displays [16], and its compatibility with a p-adic Hodge theoretic construction of Kisin [15].

These results will find use in a joint project of the second author with Andreatta, Goren and Howard on an averaged version of the Colmez conjecture on heights of abelian varieties with complex multiplication [2]. Moreover, they can be used to support Kisin's proof of the Langlands–Rapoport conjecture [13] for Shimura varieties of abelian type even at the 2-adic places.

As a more immediate application, an appendix to the paper contains a proof of the Tate conjecture for K3 surfaces in characteristic 2.

A remark on notation: Given a map $f : R \to S$ of commutative rings, and a module M over R, we will use the geometric notation f^*M for the change of scalars $S \otimes_{f,R} M$. Given a ring R and an object $\mathcal{M}$ in an R-linear rigid tensor category $\mathbf{C}$, we will write $\mathcal{M}^{\otimes}$ for the ind-object over $\mathbf{C}$ given by the direct sum of the tensor, exterior and symmetric powers of $\mathcal{M}$ and its dual. Given a scheme X over a ring R and a map $R \to R'$ of rings, we will write $X_{R'}$ or $R' \otimes_R X$ for the base change of X over R'. Given a finite free R-module M, we will write $M^{\otimes}$ for the direct sum of all R-modules that can be formed via the operations of taking duals, tensor products, and symmetric and exterior powers of M.

For any other possibly unfamiliar notions, we refer the reader to [12, Appendix] and [13, Section 1.1].

2. Lau's classification and integral p-adic Hodge theory

Fix a perfect field k in characteristic p. Set $W = W(k)$ and let $K_0 = \mathrm{Frac}(W)$ be its fraction field. Write $\sigma : W \to W$ for the canonical lift of the p-power Frobenius automorphism of k. Let K/K_0 be a finite totally ramified extension. Choose a uniformizer $\pi \in K$, and let $\mathcal{E}(u) \in W[u]$ be the associated Eisenstein polynomial with constant term $\mathcal{E}(0) = p$, so that $e = \deg \mathcal{E}(u)$ is the ramification index of K. Fix an algebraic closure K_0^{alg} of K_0, as well as an embedding $K \subset K_0^{\mathrm{alg}}$. Let $\Gamma_K = \mathrm{Gal}(K_0^{\mathrm{alg}}/K)$ be the absolute Galois group of K.

2.1. Let $\mathfrak{S} = W[\![u]\!]$ be the power series ring in one variable over W and equip it with the Frobenius lift $\varphi : \mathfrak{S} \to \mathfrak{S}$ satisfying $\varphi(u) = u^p$.

Let S be the p-adic completion of the divided power envelope of the surjection $W[u] \to \mathcal{O}_K$ carrying u to π. Explicitly, S is the p-adic completion of the subring

$$W\left[u, \frac{u^{ei}}{i!} : i \in \mathbb{Z}_{\geqslant 1}\right] \subset K_0[u].$$

The natural map $W[u] \to S$ extends to an embedding $\mathfrak{S} \hookrightarrow S$, and the Frobenius lift $\varphi : \mathfrak{S} \to \mathfrak{S}$ extends continuously to an endomorphism $\varphi : S \to S$.

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2.2. A *Breuil–Kisin module* over $\mathcal{O}_K$ (with respect to ϖ) is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M}[\mathcal{E}^{-1}] \xrightarrow{\simeq} \mathfrak{M}[\mathcal{E}^{-1}]$ is an isomorphism of $\mathfrak{S}$-modules.

Usually, the map $\varphi_{\mathfrak{M}}$ will be clear from context and we will denote the Breuil–Kisin module by its underlying $\mathfrak{S}$-module $\mathfrak{M}$.

For any integer i, we will write $\mathbf{1}(i)$ for the Breuil–Kisin module whose underlying $\mathfrak{S}$-module is just $\mathfrak{S}$ equipped with $\mathcal{E}(u)^{-i}$-times the canonical identification $\varphi^*\mathfrak{S}[\mathcal{E}(u)^{-1}] = \mathfrak{S}[\mathcal{E}(u)^{-1}]$. When $i = 0$, we will write $\mathbf{1}$ instead of $\mathbf{1}(0)$.

There are natural notions of tensor, exterior and symmetric products on the category of Breuil–Kisin modules. For any Breuil–Kisin module $\mathfrak{M}$, we will set $\mathfrak{M}(i) = \mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{1}(i)$.

The **dual** $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ of a Breuil–Kisin module $\mathfrak{M}$ is the dual $\mathfrak{S}$-module $\mathfrak{M}^\vee$ equipped with the isomorphism

$$\varphi_{\mathfrak{M}^\vee} = (\varphi_{\mathfrak{M}}^\vee)^{-1} : \varphi^*\mathfrak{M}^\vee[\mathcal{E}(u)^{-1}] \xrightarrow{\simeq} \mathfrak{M}^\vee[\mathcal{E}(u)^{-1}],$$

where $\varphi_{\mathfrak{M}}^\vee$ is the $\mathfrak{S}$-linear dual of $\varphi_{\mathfrak{M}}$.

2.3. By [13, Theorem (1.2.1)], there is a (covariant) fully faithful tensor functor $\mathfrak{M}$ from the category of $\mathbb{Z}_p$-lattices in crystalline Γ_K-representations to the category of Breuil–Kisin modules over $\mathcal{O}_K$. It has various useful properties. To describe them, fix a crystalline $\mathbb{Z}_p$-representation Λ. Then:

- If $\Lambda = \mathbb{Z}_p(i)$ is the rank-one representation of Γ_K attached to the ith-power of the p-adic cyclotomic character $\chi_p : \Gamma_K \to \mathbb{Z}_p^\times$, then there is a natural identification
$$\mathfrak{M}(\mathbb{Z}_p(i)) = \mathbf{1}(i). \tag{2.3.1}$$
- There is a canonical isomorphism of Breuil–Kisin modules:
$$\mathfrak{M}(\Lambda^\vee) \xrightarrow{\simeq} \mathfrak{M}(\Lambda)^\vee, \tag{2.3.2}$$
where $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p)$.
- For any $i \in \mathbb{Z}_{\geqslant 0}$, there are canonical isomorphisms:
$$\begin{aligned} \mathfrak{M}(\mathrm{Sym}^i \Lambda) &\xrightarrow{\simeq} \mathrm{Sym}^i \mathfrak{M}(\Lambda); \\ \mathfrak{M}(\wedge^i \Lambda) &\xrightarrow{\simeq} \wedge^i \mathfrak{M}(\Lambda) \end{aligned} \tag{2.3.3}$$
of Breuil–Kisin modules.

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- There is a canonical isomorphism of F-isocrystals over $\mathrm{Frac}(W)$:
$$\varphi^*\mathfrak{M}(\Lambda)/u\varphi^*\mathfrak{M}(\Lambda)[p^{-1}] \xrightarrow{\simeq} D_{\mathrm{cris}}(\Lambda) = (\Lambda \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}})^{\Gamma_K}. \tag{2.3.4}$$
- Equip $\varphi^*\mathfrak{M}(\Lambda)$ with the descending filtration $\mathrm{Fil}^\bullet \varphi^*\mathfrak{M}$ given by:
$$\mathrm{Fil}^i \varphi^*\mathfrak{M}(\Lambda) = \{x \in \varphi^*\mathfrak{M}(\Lambda) : \varphi_{\mathfrak{M}(\Lambda)}(x) \in \mathcal{E}(u)^i\mathfrak{M}(\Lambda)\}.$$
  Then there is a canonical isomorphism of filtered E-vector spaces
$$(\varphi^*\mathfrak{M}(\Lambda)/\mathcal{E}(u)\varphi^*\mathfrak{M}(\Lambda))[p^{-1}] \xrightarrow{\simeq} K \otimes_{\mathrm{Frac}(W)} D_{\mathrm{cris}}(\Lambda) = D_{\mathrm{dR}}(\Lambda). \tag{2.3.5}$$
  Here, the left-hand side is equipped with the filtration induced from $\mathrm{Fil}^\bullet \varphi^*\mathfrak{M}(\Lambda)$.
- The functor $\mathfrak{M}$ is compatible with unramified base change: If k'/k is a finite extension, set $K' = K \otimes_W W(k')$, and fix an embedding $K' \hookrightarrow K_0^{\mathrm{alg}}$. If $\mathfrak{S}_{k'} = W(k')[\![u]\!]$, we obtain a functor $\mathfrak{M}_{k'}$ from $\mathbb{Z}_p$-lattices in crystalline representations of $\Gamma_{K'} = \mathrm{Gal}(K_0^{\mathrm{alg}}/K')$ to Breuil–Kisin modules over $\mathcal{O}_{K'}$ consisting of pairs $(\mathfrak{M}', \varphi_{\mathfrak{M}'})$ with $\mathfrak{M}'$ finite free over $\mathfrak{S}_{k'}$. For any crystalline $\mathbb{Z}_p$-representation Λ of Γ_K, we now have a canonical isomorphism
$$\mathfrak{M}_{k'}(\Lambda|_{\Gamma_{K'}}) \xrightarrow{\simeq} \mathfrak{S}_{k'} \otimes_{\mathfrak{S}} \mathfrak{M}(\Lambda) \tag{2.3.6}$$
  of Breuil–Kisin modules.

2.4. A crucial property of the functor $\mathfrak{M}$ is what Kisin calls 'the Key Lemma'. To explain this, let Λ be as above. and let
$$\mathfrak{M} := \mathfrak{M}(\Lambda)$$
be its associated Breuil–Kisin module. Suppose that we are given a collection of Γ_K-invariant tensors $\{s_\alpha\} \subset \Lambda^\otimes$, which we can view as Γ_K-equivariant maps
$$s_\alpha : \mathbb{Z}_p \to \Lambda^\otimes.$$
By (2.3.1), (2.3.2) and (2.3.3), these give rise to maps:
$$s_{\alpha,\mathfrak{S}} : \mathbf{1} \to \mathfrak{M}^\otimes,$$
which we can view as a collection of φ-invariant elements $\{s_{\alpha,\mathfrak{M}}\} \subset \mathfrak{M}^\otimes$.

Set $M_{\mathrm{cris}} = \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M}$ and $M_{\mathrm{dR}} = \varphi^*\mathfrak{M}/\mathcal{E}(u)\varphi^*\mathfrak{M}$. Write $\mathrm{Fil}^\bullet M_{\mathrm{dR}}$ for the Hodge filtration on M_{dR}: this is the filtration of M_{dR} obtained by taking for each $i \in \mathbb{Z}$
$$\mathrm{Fil}^i M_{\mathrm{dR}} = M_{\mathrm{dR}} \cap \mathrm{Fil}^i M_{\mathrm{dR}}[p^{-1}],$$

where the filtration on $M_{\mathrm{dR}}[p^{-1}]$ is the quotient filtration induced from that on $\varphi^*\mathfrak{M}(\Lambda)[p^{-1}]$.

From $\{s_{\alpha,\mathfrak{S}}\}$, we obtain φ-invariant tensors $\{s_{\alpha,\mathrm{cris}}\} \subset M_{\mathrm{cris}}^{\otimes}$, as well as tensors $\{s_{\alpha,\mathrm{dR}}\} \subset \mathrm{Fil}^0 M_{\mathrm{dR}}^{\otimes}$.

THEOREM 2.5. *Suppose that the pointwise stabilizer $G \subset \mathrm{GL}(\Lambda)$ of $\{s_\alpha\}$ is a connected reductive group over $\mathbb{Z}_p$. Assume that k is either finite or algebraically closed. Then there is an isomorphism*

$$\mathfrak{S} \otimes_{\mathbb{Z}_p} \Lambda \xrightarrow{\simeq} \mathfrak{M}$$

carrying $1 \otimes s_\alpha$ to $s_{\alpha,\mathfrak{S}}$ for each α. Therefore, the stabilizer in $\mathrm{GL}(M_{\mathrm{cris}})$ of $\{s_{\alpha,\mathrm{cris}}\}$ (respectively in $\mathrm{GL}(M_{\mathrm{dR}})$ of $\{s_{\alpha,\mathrm{dR}}\}$) is isomorphic to G_W (respectively to $G_{\mathcal{O}_K}$). Moreover, the filtration $\mathrm{Fil}^\bullet M_{\mathrm{dR}}$ is split by a cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}_K} \to G_{\mathcal{O}_K}$.

Proof. This follows from Corollary 1.3.5 and the argument from [13, Corollary 1.4.3]. $\square$

REMARK 2.6. Although this is not explicitly stated in [13], if we assume that the pointwise stabilizer $G_{\mathfrak{S}} \subset \mathrm{GL}(\mathfrak{M})$ is connected reductive, but we do *not* assume that G is so, then the proof of [13, Proposition 1.3.4] shows that we still have an isomorphism

$$\mathfrak{S} \otimes_{\mathbb{Z}_p} \Lambda \xrightarrow{\simeq} \mathfrak{M}$$

carrying $1 \otimes s_\alpha$ to $s_{\alpha,\mathfrak{S}}$ for each α: Indeed, the reductivity of G is only used in Step 5 of the proof, and its use there can be replaced with that of the reductivity of $G_{\mathfrak{S}}$. In particular, since $\mathfrak{S}$ is faithfully flat over $\mathbb{Z}_p$, we conclude *a posteriori* that G is connected reductive over $\mathbb{Z}_p$.

2.7. Given an integer $a \in \mathbb{Z}_{\geqslant 0}$, a **Breuil window of level** a is a triple $(\mathfrak{P}_a, \mathrm{Fil}^1 \mathfrak{P}_a, \tilde{\varphi}_a)$, where:

- $\mathfrak{P}_a$ is a finite free module over $\mathfrak{S}_a := \mathfrak{S}/u^{a+1}\mathfrak{S}$;
- $\mathrm{Fil}^1 \mathfrak{P}_a \subset \mathfrak{P}_a$ is a free $\mathfrak{S}_a$-submodule containing $\mathcal{E}(u)\mathfrak{P}_a$ such that $\mathfrak{P}_a/\mathrm{Fil}^1 \mathfrak{P}_a$ is a finite free module over $\mathcal{O}_K/\pi^{a+1}$;
- $\tilde{\varphi}_a : \varphi^* \mathrm{Fil}^1 \mathfrak{P}_a \xrightarrow{\simeq} \mathfrak{P}_a$ is an isomorphism of $\mathfrak{S}_a$-modules.

Breuil windows of level a form an exact $\mathbb{Z}_p$-linear category for the obvious notion of morphism and short exact sequences.

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2.8. Breuil windows of level a are a special case of a definition from [16, Section 2.1]. Define a φ-semilinear map

$$\begin{aligned}\varphi_1 : \mathcal{E}(u)\mathfrak{S}_a &\to \mathfrak{S}_a\\ \mathcal{E}(u)x &\mapsto \varphi(x).\end{aligned}$$

In the notation of [16, Section 2.1], the tuple

$$\mathscr{B}_a = (\mathfrak{S}_a, \mathcal{E}(u)\mathfrak{S}_a, \mathcal{O}_K/\pi^{a+1}, \varphi, \varphi_1)$$

is a *lifting frame*. Lau considers the category of windows over $\mathscr{B}_a$. Windows are tuples (P, Q, F, F_1), where P is a free $\mathfrak{S}_a$-module, $Q \subset P$ is a $\mathfrak{S}_a$-submodule such that P/Q is finite free over $\mathcal{O}_K/\pi^{a+1}$, (note that, since $\mathcal{E}(u)$ is a non zero divisor, this implies that Q is necessarily free over $\mathfrak{S}_a$), $F : P \to P$ a φ-semilinear map, and $F_1 : Q \to P$ is another φ-semilinear map satisfying

$$F_1(\mathcal{E}(u) \cdot m) = F(m),$$

for all $m \in Q$, and whose image generates P as an $\mathfrak{S}_a$-module. Since $\mathcal{E}(u)$ is not a zero divisor, we see that F is uniquely determined by F_1, and so the category of windows over $\mathscr{B}_a$ is equivalent to the category of triples (P, Q, F_1), which is simply the category of Breuil windows of level a.

2.9. Given a p-divisible group $\mathcal{H}$ over a p-adically complete ring R, we will consider the contravariant Dieudonné F-crystal $\mathbb{D}(\mathcal{H})$ (see for instance [4]).

Given any nilpotent thickening $R' \to R$, whose kernel is equipped with divided powers, we can evaluate $\mathbb{D}(\mathcal{H})$ on R' to obtain a finite projective R'-module $\mathbb{D}(\mathcal{H})(R')$ (this construction depends on the choice of divided power structure, which will be specified or evident from context). If R' admits a Frobenius lift $\varphi : R' \to R'$, then we get a canonical map

$$\varphi : \varphi^*\mathbb{D}(\mathcal{H})(R') \to \mathbb{D}(\mathcal{H})(R')$$

obtained from the F-crystal structure on $\mathbb{D}(\mathcal{H})$.

An example of a (formal) divided power thickening is any surjection of the form $R' \to R'/pR'$, where we equip pR' with the canonical divided power structure induced from that on $p\mathbb{Z}_p$. Another example is the surjection $S \to \mathcal{O}_K$ from (2.1).

The evaluation on the trivial thickening $R \to R$ gives us a projective R-module $\mathbb{D}(\mathcal{H})(R)$ of finite rank equipped with a short exact sequence of projective R-modules:

$$0 \to (\operatorname{Lie}\mathcal{H})^\vee \to \mathbb{D}(\mathcal{H})(R) \to \operatorname{Lie}\mathcal{H}^\vee \to 0,$$

where $\mathcal{H}^\vee$ is the Cartier dual of $\mathcal{H}$.

We will set

$$\mathrm{Fil}^1 \mathbb{D}(\mathcal{H})(R) := (\mathrm{Lie}\,\mathcal{H})^\vee \subset \mathbb{D}(\mathcal{H})(R).$$

The associated descending 2-step filtration $\mathrm{Fil}^\bullet \mathbb{D}(\mathcal{H})(R)$ concentrated in degrees 0 and 1 will be called the **Hodge filtration**. Sometimes, we will abuse terminology and refer to the summand $\mathrm{Fil}^1 \mathbb{D}(\mathcal{H})(R)$ itself as the Hodge filtration.

2.10. We will say that a Breuil–Kisin module $\mathfrak{M}$ has *$\mathcal{E}$-height* 1 if the isomorphism $\varphi_{\mathfrak{M}}$ arises from a map $\varphi^*\mathfrak{M} \to \mathfrak{M}$ whose cokernel is killed by $\mathcal{E}(u)$. Write $\mathrm{BT}_{/\mathfrak{S}}$ for the category of Breuil–Kisin modules of $\mathcal{E}$-height 1.

Note that, for any p-divisible group $\mathcal{H}$ over $\mathcal{O}_K$, the Breuil–Kisin module $\mathfrak{M}(T_p(\mathcal{H})^\vee)$ has $\mathcal{E}$-height 1. Here, $\mathfrak{M}$ is the functor from (2.3).

With each Breuil–Kisin module $\mathfrak{M}$, we can functorially associate a triple $(\mathfrak{P}, \mathrm{Fil}^1 \mathfrak{P}, \tilde{\varphi}_{\mathfrak{P}})$, where $\mathfrak{P} = \varphi^*\mathfrak{M}$; $\mathrm{Fil}^1 \mathfrak{P} \subset \mathfrak{P}$ is $\mathfrak{M}$, viewed as a submodule of $\mathfrak{P}$ via the unique map $V_{\mathfrak{M}} : \mathfrak{M} \to \varphi^*\mathfrak{M}$ whose composition with $\varphi_{\mathfrak{M}}$ is multiplication by $\mathcal{E}(u)$; and $\tilde{\varphi}_{\mathfrak{P}} : \varphi^* \mathrm{Fil}^1 \mathfrak{P} \xrightarrow{\simeq} \mathfrak{P}$ is the obvious isomorphism.

Observe that $\mathrm{coker}\,\tilde{\varphi}_{\mathfrak{P}}$ is free over $\mathcal{O}_K = \mathfrak{S}/\mathcal{E}(u)\mathfrak{S}$, and that $\mathcal{E}(u)$ is a nonzero divisor in $\mathfrak{S}_a$. From this, it follows that the natural map

$$\mathrm{Fil}^1 \mathfrak{P} \otimes_{\mathfrak{S}} \mathfrak{S}_a \to \mathfrak{P} \otimes_{\mathfrak{S}} \mathfrak{S}_a$$

of finite free $\mathfrak{S}_a$-modules is injective, with cokernel finite free over $\mathcal{O}_K/\pi^{a+1}$.

Therefore, reducing the triple $(\mathfrak{P}, \mathrm{Fil}^1 \mathfrak{P}, \tilde{\varphi}_{\mathfrak{P}})$ mod u^{a+1} gives us a canonical functor

$$\mathcal{B}^{\mathcal{O}_K}_{\mathcal{O}_K/\pi^{a+1}} : \mathrm{BT}_{/\mathfrak{S}} \to \mathrm{BT}_{/\mathfrak{S}_a}.$$

Similarly, for any $a, b \in \mathbb{Z}_{\geqslant 0}$ with $b \geqslant a$, we also have a canonical reduction functor

$$\mathcal{B}^{\mathcal{O}_K/\pi^{b+1}}_{\mathcal{O}_K/\pi^{a+1}} : \mathrm{BT}_{/\mathfrak{S}_b} \to \mathrm{BT}_{/\mathfrak{S}_a}$$

obtained by reducing triples in $\mathrm{BT}_{/\mathfrak{S}_b}$ modulo u^{a+1}.

2.11. For any commutative ring R, write $(p\text{-div})_R$ for the category of p-divisible groups over R. Given a ring homomorphism $R \to R'$, write $\mathcal{B}^R_{R'}$ for the base change functor from $(p\text{-div})_R$ to $(p\text{-div})_{R'}$.

THEOREM 2.12. *There are exact anti-equivalences of categories*

$$\mathfrak{M} : (p\text{-div})_{\mathcal{O}_K} \xrightarrow{\simeq} \mathrm{BT}_{/\mathfrak{S}};$$
$$\mathfrak{P}_a : (p\text{-div})_{\mathcal{O}_K/\pi^{a+1}} \xrightarrow{\simeq} \mathrm{BT}_{/\mathfrak{S}_a}$$

with the following properties:

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(2.12.1) *For each $a \in \mathbb{Z}_{\geqslant 0}$, we have a canonical isomorphism of functors*

$$\mathfrak{P}_a \circ \mathcal{B}^{\mathcal{O}_K}_{\mathcal{O}_K/\pi^{a+1}} \xrightarrow{\simeq} \mathcal{B}^{\mathcal{O}_K}_{\mathcal{O}_K/\pi^{a+1}} \circ \mathfrak{M}$$

from $(p\text{-div})_{\mathcal{O}_K}$ *to* $\mathrm{BT}_{/\mathfrak{S}_a}$, *and if $b \geqslant a$, we have a canonical isomorphism of functors*

$$\mathfrak{P}_a \circ \mathcal{B}^{\mathcal{O}_K/\pi^{b+1}}_{\mathcal{O}_K/\pi^{a+1}} \xrightarrow{\simeq} \mathcal{B}^{\mathcal{O}_K/\pi^{b+1}}_{\mathcal{O}_K/\pi^{a+1}} \circ \mathfrak{P}_b$$

from $(p\text{-div})_{\mathcal{O}_K/\pi^{b+1}}$ *to* $\mathrm{BT}_{/\mathfrak{S}_a}$.

(2.12.2) *For each p-divisible group $\mathcal{H}$ over $\mathcal{O}_K$ there is a canonical isomorphism*

$$\mathfrak{M}(\mathcal{H}) \xrightarrow{\simeq} \mathfrak{M}(T_p(\mathcal{H})^\vee)$$

in $\mathrm{BT}_{/\mathfrak{S}}$.

(2.12.3) *The φ-equivariant composition*

$$\varphi^*\mathfrak{M}(\mathcal{H})/u\varphi^*\mathfrak{M}(\mathcal{H}) \xrightarrow[\simeq]{(2.3.4)} D_{\mathrm{cris}}(T_p(\mathcal{H})^\vee) \xrightarrow{\simeq} \mathbb{D}(\mathcal{H})(W)[p^{-1}]$$

maps $\varphi^*\mathfrak{M}(\mathcal{H})/u\varphi^*\mathfrak{M}(\mathcal{H})$ isomorphically onto $\mathbb{D}(\mathcal{H})(W)$.

(2.12.4) *The filtered isomorphism*

$$\begin{aligned}\varphi^*\mathfrak{M}(\mathcal{H})/\mathcal{E}(u)\varphi^*\mathfrak{M}(\mathcal{H})[p^{-1}] &\xrightarrow[\simeq]{(2.3.5)} K \otimes_{\mathrm{Frac}(W)} D_{\mathrm{cris}}(T_p(\mathcal{H})^\vee)\\ &\xrightarrow{\simeq} \mathbb{D}(\mathcal{H})(\mathcal{O}_K)[p^{-1}]\end{aligned}$$

maps $\varphi^*\mathfrak{M}(\mathcal{H})/\mathcal{E}(u)\varphi^*\mathfrak{M}(\mathcal{H})$ isomorphically onto $\mathbb{D}(\mathcal{H})(\mathcal{O}_K)$, and hence maps $\mathrm{Fil}^1\,\mathfrak{M}(\mathcal{H})$ *onto* $\mathrm{Fil}^1\,\mathbb{D}(\mathcal{H})(\mathcal{O}_K)$.

(2.12.5) *There is a canonical φ-equivariant isomorphism*

$$S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}(\mathcal{H}) \xrightarrow{\simeq} \mathbb{D}(\mathcal{H})(S)$$

whose reduction along the map $S \to \mathcal{O}_K$ gives the filtration preserving isomorphism in (2.12.4).

Proof. When $p > 2$ most of this, except for the existence of the equivalences $\mathfrak{P}_a$, follows from [13, (1.4.2) and (1.4.3)]. When $p = 2$, again, most of this follows [11]. However, the results of Lau from [16] lead to a uniform proof in all cases, and, more importantly, also at the same time give us the functors $\mathfrak{P}_a$ classifying p-divisible groups over the artinian rings $\mathcal{O}_K/\pi^{a+1}$.

To begin, the existence of the functors $\mathfrak{M}$ and $\mathfrak{P}_a$, as well as the compatibility between them asserted in (2.12.1) follows from [16, Theorem 6.6 and Corollary 5.4], and the definition of a crystalline homomorphism of frames; see the paragraph above [16, Theorem 2.2]. We note, however, that our functors are the Cartier duals of the exact equivalences of categories defined in [16]. Assertion (2.12.5) is [16, Proposition 7.1].

We now turn to assertion (2.12.2). When $p \neq 2$, this is [14, Theorem (1.1.6)]. (The original statement can be found in [13, Theorem (1.4.2)], but the assertion there is off by a Tate twist.) An independent proof of this assertion, which also works when $p = 2$, has been given by Lau in [15].

We will now explain how Lau's results imply what we need. First, since the functor $\mathfrak{M}$ of the theorem is an equivalence of categories, we can find a p-divisible group $\mathcal{H}'$ over $\mathcal{O}_K$ characterized by the property that there is an identification

$$\mathfrak{M}(\mathcal{H}') = \mathfrak{M}(T_p(\mathcal{H})^\vee)$$

in $\mathrm{BT}_{/\mathfrak{S}}$. We need to show that there is a canonical isomorphism $\mathcal{H} \xrightarrow{\simeq} \mathcal{H}'$ of p-divisible groups over $\mathcal{O}_K$. By Tate's full faithfulness theorem [27, Theorem 4], it is enough to show that there is a canonical Γ_K-equivariant isomorphism $T_p(\mathcal{H}) \xrightarrow{\simeq} T_p(\mathcal{H}')$ of p-adic Tate modules.

Let $K_\infty \subset K_0^{\mathrm{alg}}$ be as in [15, Section 7.2], so that it is the extension of K generated by a compatible family of p-power roots of the uniformizer π. Let $\Gamma_\infty \subset \Gamma_K$ be the absolute Galois group of K_∞. Then by a result of Kisin [12, Corollary 2.1.14], it is in fact sufficient to exhibit a canonical Γ_∞-equivariant isomorphism $T_p(\mathcal{H}) \xrightarrow{\simeq} T_p(\mathcal{H}')$: It will automatically be Γ_K-equivariant.

Let $\mathfrak{S}^{\mathrm{nr}}$ be the $\mathfrak{S}$-algebra considered in [15, Section 6]: it is a p-adically complete domain equipped with an action of Γ_∞, as well as a lift of Frobenius $\varphi : \mathfrak{S}^{\mathrm{nr}} \to \mathfrak{S}^{\mathrm{nr}}$ extending that on $\mathfrak{S}$.

Now we come to the key point: Lau shows in [15, Proposition 7.4] that there is a canonical Γ_∞-equivariant isomorphism

$$T_p(\mathcal{H}') \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}(\mathcal{H}'), \mathfrak{S}^{\mathrm{nr}}). \tag{2.12.1}$$

Here, the right-hand side is the space of φ-equivariant maps of $\mathfrak{S}$-modules equipped with the Γ_∞-action arising from the one on $\mathfrak{S}^{\mathrm{nr}}$.

On the other hand, by [12, Lemma 2.1.15], there is a canonical Γ_∞-equivariant isomorphism (note that our functor $\Lambda \rightsquigarrow \mathfrak{M}(\Lambda)$ differs from [12, Lemma 2.1.15] by duality)

$$T_p(\mathcal{H}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}(T_p(\mathcal{H})^\vee), \mathfrak{S}^{\mathrm{nr}}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}(\mathcal{H}'), \mathfrak{S}^{\mathrm{nr}}). \tag{2.12.2}$$

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Combining (2.12.1) and (2.12.2) gives us the desired Γ_∞-equivariant isomorphism $T_p(\mathcal{H}) \xrightarrow{\simeq} T_p(\mathcal{H}')$.

The remaining assertions now follow from the properties of the functor $\mathfrak{M}$. $\square$

3. Deformation theory

Suppose now that k is a finite field. Let $\mathcal{G}_0$ be a p-divisible group over k, and let $M_0 = \mathbb{D}(\mathcal{G}_0)(W)$ be the Dieudonné F-crystal associated with it. Suppose that we have φ-invariant tensors $\{s_{\alpha,0}\} \subset M_0^{\otimes}$, whose stabilizer is a connected reductive subgroup $G \subset \mathrm{GL}(M_0)$, and such that the Hodge filtration

$$\mathrm{Fil}^1(M_0 \otimes k) \subset M_0 \otimes k = \mathbb{D}(\mathcal{G}_0)(k)$$

is split by a cocharacter $\mu_0 : \mathbb{G}_m \to G_k$ (see [13, Section 1.1] for the terminology).

3.1. With M_0 we can associate a Breuil window of level 0 $(\mathfrak{P}_0, \mathrm{Fil}^1 \mathfrak{P}_0, \tilde{\varphi}_0)$ as follows: Let $V_{M_0} : M_0 \to \sigma^* M_0$ be the Verschiebung, so that $\varphi_{M_0} \circ V_{M_0}$ is the multiplication by p endomorphism of M_0. We take $\mathfrak{P}_0 = M_0$ and $\mathrm{Fil}^1 \mathfrak{P}_0 = (\sigma^{-1})^* M_0$, which we view as a submodule of $\mathfrak{P}_0$ via the map $(\sigma^{-1})^* V_{M_0}$. We have

$$\tilde{\varphi}_0 : \sigma^* \mathrm{Fil}^1 \mathfrak{P}_0 = M_0 \xrightarrow{\simeq} \mathfrak{P}_0.$$

The sequence

$$M_0 \otimes k \xrightarrow{V_{M_0} \otimes 1} \sigma^* M_0 \otimes k \xrightarrow{\varphi_{M_0} \otimes 1} M_0 \otimes k$$

is exact. Moreover, $\mathrm{Fil}^1(M_0 \otimes k) \subset M_0 \otimes k$ is characterized by the property that $\sigma^* \mathrm{Fil}^1(M_0 \otimes k)$ is the kernel of $\varphi_{M_0} \otimes 1$.

Therefore, we find that we could have also defined $\mathrm{Fil}^1 \mathfrak{P}_0$ to be the preimage of $\mathrm{Fil}^1(M_0 \otimes k)$ in $\mathfrak{P}_0 = M_0$.

3.2. Suppose that we are given a finite extension L/K_0 contained in K_0^{alg}, and a lift $\mathcal{G}$ of $\mathcal{G}_0$ to a p-divisible group over $\mathcal{O}_L$. Let $L_0 \subset L$ be the maximal unramified subextension with residue field k', and fix a uniformizer $\pi_L \in L$ with associated Eisenstein $\mathcal{E}_L(u) \in L_0[u]$, so that the theory of Section 2 applies with K, K_0, $\mathcal{E}(u)$ replaced with L, L_0, $\mathcal{E}_L(u)$. Therefore, we can associate with $\mathcal{G}$ the object $\mathfrak{M}(\mathcal{G}) \in \mathrm{BT}_{/\mathfrak{S}_{k'}}$, where

$$\mathfrak{S}_{k'} = W(k')[[u]].$$

Set $\mathfrak{P}(\mathcal{G}) = \varphi^* \mathfrak{M}(\mathcal{G})$. Then we have canonical isomorphisms

$$\mathfrak{P}(\mathcal{G})/u\mathfrak{P}(\mathcal{G}) \xrightarrow{\simeq} M_0; \quad \mathfrak{P}(\mathcal{G})/\mathcal{E}_L(u)\mathfrak{P}(\mathcal{G}) \xrightarrow{\simeq} \mathbb{D}(\mathcal{G})(\mathcal{O}_L).$$

By a standard argument (this is Dwork's trick: take any lift, and repeatedly apply φ to make it φ-invariant in the limit; see [13, (1.5.5)]), the φ-invariant tensors

$$\{1 \otimes s_{\alpha,0}\} \subset W(k') \otimes_W M_0^{\otimes}$$

lift uniquely to φ-invariant tensors

$$\{s_{\alpha,\mathfrak{S},\mathcal{G}}\} \subset L_0[\![u]\!] \otimes_{\mathfrak{S}_{k'}} \mathfrak{P}(\mathcal{G})^{\otimes}.$$

Here, we are viewing $\mathfrak{P}(\mathcal{G})$ as an object in the category of Breuil–Kisin modules over $\mathcal{O}_L$, and it is over this category that the ind-object $\mathfrak{P}(\mathcal{G})^{\otimes}$ is defined.

DEFINITION 3.3. We will say that $\mathcal{G}$ is **G-adapted** or **adapted to G** if the following conditions hold:

(3.3.1) The tensors $\{s_{\alpha,\mathfrak{S},\mathcal{G}}\}$ lie in $\mathfrak{P}(\mathcal{G})^{\otimes}$.

(3.3.2) There is an isomorphism

$$\mathfrak{S}_{k'} \otimes_W M_0 \xrightarrow{\simeq} \mathfrak{P}(\mathcal{G})$$

carrying, for each index α, $1 \otimes s_{\alpha,0}$ to $s_{\alpha,\mathfrak{S},\mathcal{G}}$, so that the stabilizer of $\{s_{\alpha,\mathfrak{S},\mathcal{G}}\}$ in $\mathrm{GL}(\mathfrak{P}(\mathcal{G}))$ can be identified with $G_{\mathfrak{S}_{k'}}$.

(3.3.3) The Hodge filtration on

$$\mathbb{D}(\mathcal{G})(\mathcal{O}_L) = \mathcal{O}_L \otimes_{\varphi,\mathfrak{S}_{k'}} \mathfrak{P}(\mathcal{G})$$

is split by a cocharacter $\mu_{\mathcal{G}} : \mathbb{G}_m \to G_{\mathcal{O}_L}$ lifting μ_0.

We can now state the main technical result of this note.

PROPOSITION 3.4. *Let R^{univ} be the universal deformation ring for $\mathcal{G}_0$ over W. Let $P_k \subset G_k$ be the stabilizer of the Hodge filtration*

$$\mathrm{Fil}^1(M_0 \otimes k) \subset M_0 \otimes k.$$

Then there is a quotient R^{univ}_G of R^{univ} that is formally smooth over W of dimension

$$d = \dim G_k - \dim P_k,$$

and is characterized by the following property: given a finite extension L/K_0, a map $x : R^{\mathrm{univ}} \to \mathcal{O}_L$ factors through R^{univ}_G if and only if the corresponding lift $\mathcal{G}_x$ of $\mathcal{G}_0$ over $\mathcal{O}_L$ is G-adapted.

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When $p>2$ or when $\mathcal{G}_0$ is connected, this is the content of [13, (1.5.8)]. For the general case, we will essentially repeat the same line of reasoning, except that we will replace the use of Grothendieck–Messing style crystalline deformation theory with the results of Lau summarized in Theorem 2.12.

3.5. For the proof of Proposition 3.4, we will need a notion of G-adaptedness for lifts $\mathcal{G}_a$ of $\mathcal{G}_0$ over $\mathcal{O}_L/\pi_L^{a+1}$. This is defined just as for lifts over $\mathcal{O}_L$. Set $\mathfrak{S}_{k',a}=\mathfrak{S}_{k'}/u^{a+1}\mathfrak{S}_{k'}$.

Associated with the lift is the object

$$\mathfrak{P}_a(\mathcal{G}_a)=(\mathfrak{P}_a(\mathcal{G}_a),\operatorname{Fil}^1\mathfrak{P}_a(\mathcal{G}_a),\tilde{\varphi}_a)$$

in $\mathrm{BT}_{/\mathfrak{S}_{k',a}}$.

Now, note that $\tilde{\varphi}_a$ induces an isomorphism

$$\varphi^*\mathfrak{P}_a(\mathcal{G}_a)[\mathcal{E}_L(u)^{-1}]\xrightarrow{\simeq}\mathfrak{P}_a(\mathcal{G}_a)[\mathcal{E}_L(u)^{-1}].$$

There are now unique lifts

$$\{s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}\}\subset L_0[\![u]\!]/(u^{a+1})\otimes_{\mathfrak{S}_{k',a}}\mathfrak{P}_a(\mathcal{G}_a)^{\otimes}$$

of $\{1\otimes s_{\alpha,0}\}\subset W(k')\otimes_W M_0^{\otimes}$, which satisfy $\tilde{\varphi}_a(\varphi^*s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a})=s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}$. In other words, the tensors $\{s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}\}$ are the unique φ-invariant lifts of $\{s_{\alpha,0}\}$.

Set $\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a)=\mathfrak{P}_a(\mathcal{G}_a)/\mathcal{E}_L(u)\mathfrak{P}_a(\mathcal{G}_a)$: this is a finite free module over $\mathcal{O}_L/\pi_L^{a+1}$. Equip it with the $\mathcal{O}_L/\pi_L^{a+1}$-linear direct summand

$$\operatorname{Fil}^1\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a)=\operatorname{Fil}^1\mathfrak{P}_a(\mathcal{G}_a)/\mathcal{E}_L(u)\mathfrak{P}_a(\mathcal{G}_a)\subset\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a).$$

This gives a 2-step descending filtration $\operatorname{Fil}^\bullet\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a)$ on $\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a)$ concentrated in degrees 0 and 1.

DEFINITION 3.6. We will say that $\mathcal{G}_a$ is ***G*-adapted** or **adapted to *G*** if the following conditions hold:

(3.6.1) The tensors $\{s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}\}$ lie in $\mathfrak{P}_a(\mathcal{G}_a)^{\otimes}$.

(3.6.2) There is an isomorphism

$$\mathfrak{S}_{k',a}\otimes_W M_0\xrightarrow{\simeq}\mathfrak{P}_a(\mathcal{G}_a)$$

carrying, for each index α, $1\otimes s_{\alpha,0}$ to $s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}$, so that the stabilizer of $\{s_{\alpha,\mathfrak{S}_a,\mathcal{G}_a}\}$ in $\mathrm{GL}(\mathfrak{P}_a(\mathcal{G}_a))$ can be identified with $G_{\mathfrak{S}_{k',a}}$.

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(3.6.3) The filtration $\mathrm{Fil}^\bullet \mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_a)$ is split by a cocharacter $\mu_{\mathcal{G}} : \mathbb{G}_m \to G_{\mathcal{O}_L/\pi_L^{a+1}}$ lifting μ_0.

LEMMA 3.7. *Set*

$$\mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1}) = \{x : R^{\mathrm{univ}} \to \mathcal{O}_L/\pi_L^{a+1} : \mathcal{G}_x \textit{ is } G\textit{-adapted}\}.$$

Then $\mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1})$ has at most q^{ad} elements, where $q = \#k'$ is the size of k' and $d = \dim G_k - \dim P_k$.

Proof. Fix any cocharacter $\mu : \mathbb{G}_m \to G$ lifting μ_0, and let $\mathrm{Fil}^1 M_0 \subset M_0$ be the corresponding lift of the Hodge filtration. We have a direct sum decomposition

$$M_0 = \mathrm{Fil}^1 M_0 \oplus M_0',$$

where $M_0' \subset M_0$ is the subspace on which $\mu(\mathbb{G}_m)$ acts trivially.

Fix an $x : R^{\mathrm{univ}} \to \mathcal{O}_L/\pi_L^{a+1}$ in $\mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1})$. This corresponds to a p-divisible group $\mathcal{G}_x$ over $\mathcal{O}_L/\pi_L^{a+1}$ lifting $\mathcal{G}_0$, and is such that the tensors $\{\boldsymbol{s}_{\alpha,\mathfrak{S}_a,\mathcal{G}_x}\}$ satisfy the conditions of (3.6).

Therefore, we have an isomorphism

$$\xi_a : \mathfrak{S}_{k',a} \otimes_W M_0 \xrightarrow{\simeq} \mathfrak{P}_a(\mathcal{G}_x)$$

carrying, for each α, $1 \otimes \boldsymbol{s}_{\alpha,0}$ to $\boldsymbol{s}_{\alpha,\mathfrak{S}_a,\mathcal{G}_x}$. We will use this isomorphism to identify $\mathfrak{P}_a(\mathcal{G}_x)$ with $\mathfrak{S}_{k',a} \otimes_W M_0$.

Moreover, the Hodge filtration $\mathrm{Fil}^\bullet \mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_x)$ on $\mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_x)$ is split by a cocharacter of $G_{\mathcal{O}_L/\pi_L^{a+1}}$. Lift $\mathrm{Fil}^\bullet \mathfrak{P}_{\mathrm{dR}}(\mathcal{G}_x)$ to a filtration $\mathrm{Fil}^\bullet(\mathfrak{S}_{k',a} \otimes_W M_0)$ split by a cocharacter of $G_{\mathfrak{S}_{k',a}}$. This is always possible by [13, Proposition 1.1.5]. In addition, using [10, Corollaire 3.3], we find that, by replacing ξ_a with $\xi_a \circ g$, for some $g \in G(\mathfrak{S}_{k',a})$ if necessary, we can assume that

$$\mathrm{Fil}^\bullet(\mathfrak{S}_{k',a} \otimes_W M_0) = \mathfrak{S}_{k',a} \otimes_W \mathrm{Fil}^\bullet M_0.$$

We now have

$$\xi_a^{-1}(\mathrm{Fil}^1 \mathfrak{P}_{a-1}(\mathcal{G}_x)) = (\mathcal{E}_L(u)\mathfrak{S}_{k',a} \otimes_W M_0') \oplus (\mathfrak{S}_{k',a} \otimes_W \mathrm{Fil}^1 M_0) \subset \mathfrak{S}_{k',a} \otimes_W M_0),$$

and the isomorphism $\tilde{\varphi}_a : \varphi^* \mathrm{Fil}^1 \mathfrak{P}_a(\mathcal{G}_x) \xrightarrow{\simeq} \mathfrak{P}_a(\mathcal{G}_x)$ translates to an isomorphism

$$\tilde{\varphi}_a : (\mathcal{E}_L(u)\mathfrak{S}_{k',a} \otimes_W \sigma^* M_0') \oplus (\mathfrak{S}_{k',a} \otimes_W \sigma^* \mathrm{Fil}^1 M_0) \xrightarrow{\simeq} \mathfrak{S}_{k',a} \otimes_W M_0,$$

which lifts the isomorphism

$$\tilde{\varphi}_0 : p(\sigma^* M_0') \oplus \sigma^* \mathrm{Fil}^1 M_0 \xrightarrow{\simeq} M_0$$

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obtained from the F-crystal structure on M_0 (see (3.1)) and is such that the induced isomorphism

$$\varphi_a : (\mathfrak{S}_{k',a} \otimes_W \sigma^* M_0)[\mathcal{E}_L(u)^{-1}] \xrightarrow{\simeq} (\mathfrak{S}_{k',a} \otimes_W M_0)[\mathcal{E}_L(u)^{-1}]$$

carries, for each α, $1 \otimes \sigma^* \boldsymbol{s}_{\alpha,0}$ to $1 \otimes \boldsymbol{s}_{\alpha,0}$.

Let $\widetilde{\mathcal{D}}_G(\mathcal{O}_L/\pi_L^{a+1})$ be the set of such isomorphisms $\tilde{\varphi}_a$. Using Theorem 2.12, we now find that we have constructed a surjection

$$\widetilde{\mathcal{D}}_G(\mathcal{O}_L/\pi_L^{a+1}) \to \mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1})$$

carrying an isomorphism $\tilde{\varphi}_a$ to the p-divisible group over $\mathcal{O}_L/\pi_L^{a+1}$ corresponding to the Breuil window

$$\Theta_a(\tilde{\varphi}_a) := (\mathfrak{S}_{k',a} \otimes_W M_0, (\mathcal{E}_L(u)\mathfrak{S}_{k',a} \otimes_W M_0') \oplus (\mathfrak{S}_{k',a} \otimes_W \operatorname{Fil}^1 M_0), \tilde{\varphi}_a).$$

Now, the action $\tilde{\varphi}_a \mapsto g \circ \tilde{\varphi}_a$, for $g \in G(\mathfrak{S}_{k',a})$ makes $\widetilde{\mathcal{D}}_G(\mathcal{O}_L/\pi_L^{a+1})$ a torsor under $G(\mathfrak{S}_{k',a})$. Fix $\tilde{\varphi}_a \in \mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1})$. Suppose now that we have $g_1, g_2 \in \mathfrak{S}_{k',a}$ such that

$$h := g_2 g_1^{-1} \in \mathcal{K}_a := \ker(G(\mathfrak{S}_{k',a}) \to G(\mathfrak{S}_{k',a-1})).$$

Then $\varphi(h) \in G(\mathfrak{S}_{k',a})$. Therefore, it is easy to see that h induces an isomorphism $\Theta_a(\tilde{\varphi}_a \circ g_1) \xrightarrow{\simeq} \Theta_a(\tilde{\varphi}_a \circ g_2)$ of Breuil windows if and only if it preserves the subspace

$$(\mathcal{E}_L(u)\mathfrak{S}_{k',a} \otimes_W M_0') \oplus (\mathfrak{S}_{k',a} \otimes_W \operatorname{Fil}^1 M_0) \subset \mathfrak{S}_{k',a} \otimes_W M_0.$$

Write $\mathcal{I}_a \subset \mathcal{K}_a$ for the subspace of elements that preserve this subspace.

Then we have shown that each nonempty fiber of the reduction map

$$\widetilde{\mathcal{D}}_G(\mathcal{O}_L/\pi_L^{a+1}) \to \widetilde{\mathcal{D}}_G(\mathcal{O}_L/\pi_L^{a})$$

is in bijection with the set $\mathcal{K}_a/\mathcal{I}_a$. Therefore, to finish the proof of the lemma, it is enough to show that $\mathcal{K}_a/\mathcal{I}_a$ has q^d-elements. For this, simply observe that, if $P \subset G$ is the parabolic subgroup preserving the subspace $\operatorname{Fil}^1 M_0 \subset M_0$, then the reduction map $G(\mathfrak{S}_{k',a}) \to G(\mathcal{O}_K/\pi_L^{a+1})$ identifies $\mathcal{K}_a/\mathcal{I}_a$ with

$$\frac{\ker(G(\mathcal{O}_L/\pi_L^{a+1}) \to G(\mathcal{O}_L/\pi_L^{a}))}{\ker(P(\mathcal{O}_L/\pi_L^{a+1}) \to P(\mathcal{O}_L/\pi_L^{a}))} \xrightarrow{\simeq} \frac{\pi^a \mathcal{O}_L}{\pi^{a+1}\mathcal{O}_L} \otimes_W \frac{\operatorname{Lie} G}{\operatorname{Lie} P} = \frac{\operatorname{Lie} G_k}{\operatorname{Lie} P_k}. \qquad \square$$

LEMMA 3.8. *Suppose that R_G is a quotient of R^{univ} that is formally smooth over W of relative dimension d, and is such that, for all $x : R^{\mathrm{univ}} \to \mathcal{O}_L$ factoring through R_G, $\mathcal{G}_x$ is G-adapted. Then every $x : R^{\mathrm{univ}} \to \mathcal{O}_L$ such that $\mathcal{G}_x$ is G-adapted factors through R_G.*

Proof. Since R_G is formally smooth, every map $x_a : R_G \to \mathcal{O}_L/\pi_L^{a+1}$ lifts to a map $x : R_G \to \mathcal{O}_L$. Therefore, the hypothesis implies that, for every $a \in \mathbb{Z}_{\geqslant 0}$, we have:

$$\{x_a : R^{\mathrm{univ}} \to \mathcal{O}_L/\pi_L^{a+1} : x \text{ factors through } R_G\} \subset \mathcal{D}_G(\mathcal{O}_L/\pi_L^{a+1}). \qquad (3.8.1)$$

The set on the left-hand side has size q^{ad}, since R_G is formally smooth of relative dimension d over W. The set on the right has at most q^{ad} elements by Lemma 3.7. Therefore, the inclusion (3.8.1) has to be an equality for every a. This easily implies the corollary. □

3.9. To finish the proof of Proposition 3.4, it is now enough to construct a formally smooth quotient R_G of R^{univ} as in Lemma 3.8. For this, we follow a construction of Faltings [9]. Fix any cocharacter $\mu : \mathbb{G}_m \to G$ lifting μ_0, and let $\mathrm{Fil}^1 M_0 \subset M_0$ be the corresponding lift of the Hodge filtration. We have a direct sum decomposition

$$M_0 = \mathrm{Fil}^1 M_0 \oplus M_0',$$

where $M_0' \subset M_0$ is the subspace on which $\mu(\mathbb{G}_m)$ acts trivially.

Let $P \subset G$ be the parabolic subgroup stabilizing $\mathrm{Fil}^1 M_0$. Let $U_G^{\mathrm{op}} \subset G$ and $U^{\mathrm{op}} \subset \mathrm{GL}(M_0)$ be the *opposite* unipotents determined by μ, so that

$$\mathrm{Lie}\, U^{\mathrm{op}} \subset \mathrm{End}(M_0), \quad \mathrm{Lie}\, U_G^{\mathrm{op}} \subset \mathrm{Lie}\, G_W$$

are the -1-eigenspaces for μ.

We have $\dim U_G^{\mathrm{op}} = d = \dim G - \dim P$.

3.10. Let R (respectively R_G) be the complete local ring of U^{op} (respectively U_G^{op}) at the identity.

Choose a basis for $\mathrm{Lie}\, U_G^{\mathrm{op}}$ and extend it to a basis for $\mathrm{Lie}\, U^{\mathrm{op}}$. This gives us compatible coordinates

$$R_G \xrightarrow{\simeq} W[\![u_1, \ldots, u_d]\!], \quad R \xrightarrow{\simeq} W[\![u_1, \ldots, u_d, u_{d+1}, \ldots, u_n]\!].$$

Let $\varphi : R \to R$ be the Frobenius lift satisfying $\varphi(u_i) = u_i^p$; it preserves R_G.

Now set $M^\diamond = R \otimes_W M_0$ (respectively $M^\diamond_{R_G} = R_G \otimes_W M_0$). Equip $M^\diamond$ and $M^\diamond_{R_G}$ with the constant Hodge filtrations

$$\mathrm{Fil}^\bullet M^\diamond = R \otimes_W \mathrm{Fil}^\bullet M_0, \quad \mathrm{Fil}^\bullet M^\diamond_{R_G} = R_G \otimes_W \mathrm{Fil}^\bullet M_0.$$

We will also equip $M^\diamond$ with the map

$$\varphi_{M^\diamond} = g(1 \otimes \varphi_0) : \varphi^* M^\diamond \to M^\diamond.$$

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Here, $g \in U^{\mathrm{op}}(R)$ is the tautological element, and $\varphi_0 : \varphi^* M_0 \to M_0$ is the map obtained from the F-crystal structure on M_0. Note that $\varphi_{M^\diamond}$ induces a map

$$\varphi_{M^\diamond_{R_G}} : \varphi^* M^\diamond_{R_G} \to M^\diamond_{R_G}.$$

PROPOSITION 3.11. *There exists a choice of the lift μ, for which there is a p-divisible group $\mathcal{G}^\diamond$ over R deforming $\mathcal{G}_0$ and equipped with an isomorphism:*

$$\mathbb{D}(\mathcal{G}^\diamond)(R) \xrightarrow{\simeq} M^\diamond$$

of filtered R-modules equipped with φ-semilinear endomorphisms. The associated map $R^{\mathrm{univ}} \to R$ of local W-algebras obtained from the universal property of R^{univ} is an isomorphism.

Proof. First, choose any G-adapted lift $\mathcal{G}_W$ of $\mathcal{G}_0$ over W. Such a lift always exists, as shown in the proof of [14, Proposition 1.1.13].

We then have a canonical isomorphism

$$\mathbb{D}(\mathcal{G}_W)(W) \xrightarrow{\simeq} M_0,$$

and the image of the Hodge filtration $\mathrm{Fil}^1 \mathbb{D}(\mathcal{G}_W)(W)$ in M_0 will give a filtration $\mathrm{Fil}^1 M_0$ that will be split by a cocharacter $\mu : \mathbb{G}_m \to G$ lifting μ_0.

The deformation $\mathcal{G}_W$ corresponds to a map $R^{\mathrm{univ}} \to W$. Let $\mathcal{G}^{\mathrm{univ}}$ be the universal deformation of $\mathcal{G}_0$ over R^{univ}. Then there is an isomorphism

$$W \otimes_{R^{\mathrm{univ}}} \mathbb{D}(\mathcal{G}^{\mathrm{univ}})(R^{\mathrm{univ}}) \xrightarrow{\simeq} M_0 \tag{3.11.1}$$

of filtered F-crystals over W.

Now, [9, Theorem 10], which is a purely linear algebraic result, implies that there is a unique map $f : R^{\mathrm{univ}} \to R$ such that if $e : R \to W$ corresponds to the identity section of U^{op}, then the composition

$$R^{\mathrm{univ}} \to R \to W$$

corresponds to lift $\mathcal{G}_W$ of $\mathcal{G}_0$, and such that there exists an isomorphism

$$R \otimes_{R^{\mathrm{univ}}} \mathbb{D}(\mathcal{G}^{\mathrm{univ}})(R^{\mathrm{univ}}) \xrightarrow{\simeq} M^\diamond$$

of filtered R-modules lifting (3.11.1) and compatible with the F-crystal structure in a sense that is made precise in [9].

That this map is an isomorphism follows from a versatility argument; see [20, Section 1.4.2] for details. □

By the above proposition, we can identify R_G with a quotient of R^{univ}. The proof of Proposition 3.4 is now completed by

PROPOSITION 3.12. *For every $x : R \to \mathcal{O}_L$ factoring through R_G, $\mathcal{G}_x$ is G-adapted.*

Proof. This is essentially shown in [14, Proposition 1.1.13]. We recall the reasoning here.

By construction, for each index α, the constant tensor

$$\boldsymbol{s}_{\alpha,R_G} = 1 \otimes \boldsymbol{s}_{\alpha,0} \in (M^{\diamond}_{R_G})^{\otimes} = \mathbb{D}(\mathcal{G}^{\diamond})(R_G)^{\otimes}$$

is φ-invariant, and lies in $\mathrm{Fil}^0(M^{\diamond}_{R_G})^{\otimes}$. Moreover, the Hodge filtration

$$\mathrm{Fil}^1\,\mathbb{D}(\mathcal{G}^{\diamond})(R_G) \subset \mathbb{D}(\mathcal{G}^{\diamond})(R_G)$$

is split by the cocharacter $1 \otimes \mu : \mathbb{G}_m \to R_G \otimes_W G = G_{R_G}$.

For every point $x : R_G \to \mathcal{O}_L$, the specialization of $\boldsymbol{s}_{\alpha,R_G}$ gives us a tensor $\boldsymbol{s}_{\alpha,\mathrm{dR},x} \in \mathbb{D}(\mathcal{G}_x)(\mathcal{O}_L)^{\otimes}$. The stabilizer of $\{\boldsymbol{s}_{\alpha,\mathrm{dR},x}\}$ is isomorphic to $G_{\mathcal{O}_L}$, and the Hodge filtration on $\mathbb{D}(\mathcal{G}_x)(\mathcal{O}_L)$ is split by a cocharacter $\mu_x : \mathbb{G}_m \to G_{\mathcal{O}_L}$ lifting μ_0.

Let $T_p(\mathcal{G}^{\diamond})$ be the lisse p-adic sheaf over $\mathrm{Spec}\,R_G[p^{-1}]$ obtained from the p-adic Tate module of $\mathcal{G}^{\diamond}$. By [14, Lemma 1.1.17], we can find a global section

$$\boldsymbol{s}_{\alpha,p,R_G} \in H^0(\mathrm{Spec}\,R_G[p^{-1}], T_p(\mathcal{G}^{\diamond})^{\otimes}[p^{-1}])$$

whose specialization at any point $x : R_G \to \mathcal{O}_L$ gives a Γ_L-invariant tensor $\boldsymbol{s}_{\alpha,p,x} \in T_p(\mathcal{G}_x)^{\otimes}[p^{-1}]$, characterized by the property that its de Rham realization is $\boldsymbol{s}_{\alpha,\mathrm{dR},x}$, and its crystalline realization is $\boldsymbol{s}_{\alpha,0}$.

We claim that $\boldsymbol{s}_{\alpha,p,R_G}$ is a section of $T_p(\mathcal{G}^{\diamond})^{\otimes}$. To see this, we can check at any point of $\mathrm{Spec}\,R_G[p^{-1}]$, which we choose to be the identity $x_0 \in \widehat{U}_G(W)$. Then, by the construction in Proposition 3.11, $\mathcal{G}_{x_0}$ is G-adapted. Therefore, for each α, we have a φ-invariant tensor $\boldsymbol{s}_{\alpha,\mathfrak{S},x_0} \in \mathfrak{P}(\mathcal{G}_{x_0})^{\otimes}$, which, via the functor of (2.3), corresponds to the tensor $\boldsymbol{s}_{\alpha,p,x_0}$. In particular, this implies that $\boldsymbol{s}_{\alpha,p,x_0}$ belongs to $T_p(\mathcal{G}_{x_0})^{\otimes}$, as desired. Moreover, by Remark 2.6 and Theorem 2.12, there exists an isomorphism

$$\mathfrak{S} \otimes_{\mathbb{Z}_p} T_p(\mathcal{G}_{x_0})^{\vee} \xrightarrow{\simeq} \mathfrak{P}(\mathcal{G}_{x_0})$$

carrying $1 \otimes \boldsymbol{s}_{\alpha,p,x_0}$ to $\boldsymbol{s}_{\alpha,\mathfrak{S},x_0}$, for all α.

Let $G_{\mathbb{Z}_p} \subset \mathrm{GL}(T_p(\mathcal{G}_{x_0}))$ be the stabilizer of $\{\boldsymbol{s}_{\alpha,p,x_0}\}$: this is a reductive group over $\mathbb{Z}_p$, isomorphic over W to G. It now follows that, for *every* $x : R_G \to \mathcal{O}_L$, the Γ_L-invariant tensor $\boldsymbol{s}_{\alpha,p,x}$ lies in $T_p(\mathcal{G}_x)^{\otimes}$. Moreover, the stabilizer of the collection $\{\boldsymbol{s}_{\alpha,p,x}\}$ in $\mathrm{GL}(T_p(\mathcal{G}_x))$ is isomorphic to $G_{\mathbb{Z}_p}$. Therefore, it now follows from Theorems 2.5 and 2.12 that $\mathcal{G}_x$ is adapted to G. □

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4. Integral canonical models

4.1. Let (G, X) be a Shimura datum, and let H be a faithful representation of G over $\mathbb{Q}$, equipped with a symplectic pairing $H \times H \to \mathbb{Q}$, affording an embedding of reductive $\mathbb{Q}$-groups

$$G \hookrightarrow \mathrm{GSp}(H, \psi)$$

into the groups of symplectic similitudes for (H, ψ). We assume that, for each $x \in X$, the associated homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$ induces a Hodge structure on $H_{\mathbb{C}}$ of weights $(-1, 0)$, $(0, -1)$, which is polarized by ψ.

Fix a $\mathbb{Z}$-lattice $H_{\mathbb{Z}} \subset H$ and let $K \subset G(\mathbb{A}_f)$ be a neat compact open such that K stabilizes $H_{\widehat{\mathbb{Z}}} = H_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}} \subset H_{\mathbb{A}_f}$. Associated with this is the Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ over the reflex field $E = E(G, X)$.

Let

$$H_{\mathbb{Z}}^{\vee} = \{h \in H_{\mathbb{Q}} : \ \psi(h, H_{\mathbb{Z}}) \subset \mathbb{Z}\}$$

be the dual lattice for $H_{\mathbb{Z}}$ with respect to ψ, and let $m \in \mathbb{Z}_{\geqslant 1}$ be such that $m^2 = [H_{\mathbb{Z}}^{\vee} : H_{\mathbb{Z}}]$ is the discriminant of $H_{\mathbb{Z}}$.

4.2. A classical construction (see for instance [18, (2.1.8)]) associates with (H, ψ) and the lattice $H_{\mathbb{Z}} \subset H$, a polarized variation of (pure) $\mathbb{Z}$-Hodge structures over $\mathrm{Sh}_K(\mathbb{C})$ of weights $(0, -1)$, $(-1, 0)$, and thus a family of polarized abelian varieties $A_{\mathrm{Sh}_K(\mathbb{C})} \to \mathrm{Sh}_K(\mathbb{C})$. The theory of canonical models for Shimura varieties now implies that this family arises from a canonical polarized abelian scheme $A \to \mathrm{Sh}_K$, which corresponds to a finite and unramified map

$$\mathrm{Sh}_K \to \mathcal{X}_{d,m,\mathbb{Q}}, \tag{4.2.1}$$

where, $\mathcal{X}_{d,m}$ is the moduli stack over $\mathbb{Z}$ of polarized abelian schemes of dimension $d = \frac{1}{2} \dim H$ and degree m^2.

4.3. More generally, the classical construction referenced above associates with a pair $(N, N_{\widehat{\mathbb{Z}}})$ consisting of an algebraic $\mathbb{Q}$-representation N of G and a K-stable lattice $N_{\widehat{\mathbb{Z}}} \subset N_{\mathbb{A}_f}$ a variation of pure $\mathbb{Z}$-Hodge structures

$$(\boldsymbol{N}_B, \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})}, \mathrm{Fil}^{\bullet} \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})}),$$

where $\boldsymbol{N}_B$ is the underlying $\mathbb{Z}$-local system over $\mathrm{Sh}_K(\mathbb{C})$, $\boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})} = \mathcal{O}_{\mathrm{Sh}_K(\mathbb{C})} \otimes \boldsymbol{N}_B$ is the associated vector bundle with integrable connection, and $\mathrm{Fil}^{\bullet} \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})}$ is the filtration on it by subvector bundles inducing a pure Hodge structure at every point of $\mathrm{Sh}_K(\mathbb{C})$.

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The theory of canonical models implies that, for every prime ℓ, the associated ℓ-adic local system $\mathbb{Z}_\ell \otimes \boldsymbol{N}_B$ descends to a lisse ℓ-adic sheaf $\boldsymbol{N}_\ell$ over Sh_K. When $(N, N_{\widehat{\mathbb{Z}}}) = (H, H_{\widehat{\mathbb{Z}}})$, this is simply the ℓ-adic Tate module associated with A.

More interestingly, Deligne's theory of absolute Hodge structures (see [13, Section 2.2]) shows that the filtered vector bundle $(\boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})}, \mathrm{Fil}^\bullet \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K(\mathbb{C})})$ with its integrable connection has a canonical descent

$$(\boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K}, \mathrm{Fil}^\bullet \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K})$$

to a filtered vector bundle over Sh_K with integrable connection.

4.4. Fix a prime p, and a place $v|p$ of E. Given an algebraic stack $\mathcal{X}$ over $\mathcal{O}_{E,(v)}$, and a normal algebraic stack Y over E equipped with a finite map $j_E : Y \to \mathcal{X}_E$, the *normalization* of $\mathcal{X}$ in Y is the finite $\mathcal{X}$-stack $j : \mathcal{Y} \to \mathcal{X}$, characterized by the property that $j_*\mathcal{O}_{\mathcal{Y}}$ is the integral closure of $\mathcal{O}_{\mathcal{X}}$ in $(j_E)_*\mathcal{O}_Y$. It is also characterized by the following universal property: given a finite morphism $\mathcal{Z} \to \mathcal{X}$ with $\mathcal{Z}$ a normal algebraic stack, flat over $\mathcal{O}_{E,(v)}$, any map of $\mathcal{X}_E$-stacks $\mathcal{Z}_E \to Y$ extends uniquely to a map of $\mathcal{X}$-stacks $\mathcal{Z} \to \mathcal{Y}$.

4.5. We now obtain an integral model $\mathcal{S}_K$ for Sh_K over $\mathcal{O}_{E,(v)}$ by taking the normalization of $\mathcal{O}_{E,(v)} \otimes_{\mathbb{Z}} \mathcal{X}_{d,m}$ in Sh_K. By construction A extends to a polarized abelian scheme over $\mathcal{S}_K$, which we denote once again by A.

Fix a finite extension k of $k(v)$, and set $W = W(k)$, $L_0 = \mathrm{Frac}(W)$. Fix an algebraic closure L_0^{alg} of K_0. Suppose that we have a point $t \in \mathcal{S}_K(k)$. Let $\mathcal{G}_t = A_t[p^\infty]$ be the p-divisible group over k associated with the fiber at t of A. Let $\mathcal{O}_t$ be the complete local ring of $\mathcal{S}_K$, and let R_t be the universal deformation ring of $\mathcal{G}_t$. Then R_t is noncanonically isomorphic to a power series ring over W in g^2-variables. By Serre–Tate deformation theory, $\mathcal{O}_t$ is the normalization of a quotient of R_t.

PROPOSITION 4.6. *Suppose that G admits a reductive model $G_{(p)}$ over $\mathbb{Z}_{(p)}$ such that $K_p = G_{(p)}(\mathbb{Z}_p)$. Then $\mathcal{O}_t$ is formally smooth over W.*

For the proof, we will need the following lemma.

LEMMA 4.7. *Set $H_{(p)} = H_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$. Then there exists a collection of tensors $\{s_\alpha\} \subset H_{(p)}^{\otimes}$ whose pointwise stabilizer is $G_{(p)} \subset \mathrm{GL}(H_{(p)})$.*

Proof. This essentially follows from [13, Lemma 2.3.1 and Proposition 1.3.2]. The only issue is that in the statement of Lemma 2.3.1, Kisin places the following restriction, which follows from one in a result that he uses of Prasad and Yu [26, (1.3)]: When $p = 2$, G must not have factors of type B. However, this restriction is unnecessary, since by Deligne's classification in [8, (1.3)], any factor of type B must have a simply connected derived group, and the required result of Prasad and Yu applies to *all* reductive groups with a simply connected derived group, without any restriction on residue characteristic or type. $\square$

Proof of Proposition 4.6. Set $H_{(p)} = H_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$, and fix any collection of tensors $\{s_\alpha\} \subset H_{(p)}^{\otimes}$ whose pointwise stabilizer is $G_{(p)} \subset \mathrm{GL}(H_{(p)})$. These tensors give rise to global sections $\{\boldsymbol{s}_{\alpha,p}\} \subset H^0(\mathrm{Sh}_K, \boldsymbol{H}_p^{\otimes})$.

Choose any lift $\tilde{t} \in \mathcal{S}_K(\mathcal{O}_L)$ of t to a point valued in the ring of integers $\mathcal{O}_L$ of a finite extension L/L_0 contained in L_0^{alg}. By enlarging k if necessary, we can assume that L is totally ramified over L_0. Let $\Gamma_L = \mathrm{Gal}(L^{\mathrm{alg}}/L)$ be the absolute Galois group of L. We obtain Γ_L-invariant tensors $\{\boldsymbol{s}_{\alpha,p,\tilde{t}}\} \subset \boldsymbol{H}_{p,\tilde{t}}^{\otimes}$, whose stabilizer is isomorphic to $G_{(p),\mathbb{Z}_p}$.

Set $\boldsymbol{H}_{\mathrm{cris},t} = \mathbb{D}(\mathcal{G}_t)(W)$ and $\boldsymbol{H}_{\mathrm{dR},\tilde{t}} = H^1_{\mathrm{dR}}(A_{\tilde{t}}/\mathcal{O}_L)^\vee$. Then Theorems 2.5 and 2.12 give us φ-invariant tensors

$$\{\boldsymbol{s}_{\alpha,\mathrm{cris},t}\} \subset \boldsymbol{H}_{\mathrm{cris},t}^{\otimes},$$

whose stabilizer is isomorphic to G_W, as well as tensors

$$\{\boldsymbol{s}_{\alpha,\mathrm{dR},\tilde{t}}\} \subset \mathrm{Fil}^0\, \boldsymbol{H}_{\mathrm{dR},\tilde{t}}^{\otimes},$$

whose stabilizer is isomorphic to $G_{\mathcal{O}_L}$. Moreover, the Hodge filtration on $\boldsymbol{H}_{\mathrm{dR},\tilde{t}}$ can be split via a cocharacter of $G_{\mathcal{O}_L}$.

By a theorem of Blasius–Wintenberger [5] on the compatibility of the de Rham comparison isomorphism with homological realizations of Hodge cycles on abelian varieties, the tensors $\{\boldsymbol{s}_{\alpha,\mathrm{dR},\tilde{t}}\}$ coincide with those obtained from $\{s_\alpha\}$ via the de Rham functor described in (4.3).

Also, by construction, under the canonical identification

$$\boldsymbol{H}_{\mathrm{cris},t} \otimes_W k = H^1_{\mathrm{dR}}(A_t/k)^\vee = \boldsymbol{H}_{\mathrm{dR},\tilde{t}} \otimes_{\mathcal{O}_L} k,$$

the tensors $\{\boldsymbol{s}_{\alpha,\mathrm{cris},t} \otimes 1\}$ are carried to $\{\boldsymbol{s}_{\alpha,\mathrm{dR},\tilde{t}} \otimes 1\}$. In particular, the Hodge filtration on

$$\boldsymbol{H}_{\mathrm{dR},t} := H^1_{\mathrm{dR}}(A_t/\mathbb{F}_p^{\mathrm{alg}})^\vee$$

is split by a cocharacter $\mu_0 : \mathbb{G}_m \to G_{\mathbb{F}_p^{\mathrm{alg}}}$.

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This implies that we can apply the theory of Section 3 with $\mathcal{G}_0 = \mathcal{G}_t$, and $\{s_{\alpha,0}\} = \{s_{\alpha,\mathrm{cris},t}\}$. Therefore, by Proposition 3.4, we have a canonical formally smooth quotient R_G^{univ} of R_t characterized by the property that any point $\tilde{t}' : R_t \to \mathcal{O}_L$ factors through R_G^{univ} if and only if the corresponding lift $\mathcal{G}_{\tilde{t}'}$ is G_W-adapted. We claim that R_G^{univ} is identified with $\mathcal{O}_t$.

This is done precisely as in the proof of [13, Proposition 2.3.5]. First, an easy dimension count, using the fact that μ_0 is conjugate to the inverse of the Hodge cocharacter associated with the Shimura datum (G, X), shows that it is enough to prove that $\mathcal{O}_t$ is a quotient of R_G^{univ}. In other words, we must show that every map $\tilde{t}' : \mathcal{O}_t \to \mathcal{O}_L$ is G-adapted. But, every such $\tilde{t}'$ gives us tensors $\{s_{\alpha,p,\tilde{t}'}\} \subset \boldsymbol{H}^{\otimes}_{p,\tilde{t}'}$, which give rise, as in Theorem 2.5, to φ-invariant tensors $\{s_{\alpha,\mathfrak{S},\tilde{t}'}\} \subset \mathfrak{M}(\mathcal{G}_{\tilde{t}'})^{\otimes}$, whose stabilizer is isomorphic to $G_{\mathfrak{S}}$.

These tensors in turn give rise to φ-invariant tensors $\{s_{\alpha,\mathrm{cris},\tilde{t}'}\} \subset \boldsymbol{H}^{\otimes}_{\mathrm{cris},t}$, and de Rham tensors $\{s_{\alpha,\mathrm{dR},\tilde{t}'}\} \subset \mathrm{Fil}^0\, \boldsymbol{H}^{\otimes}_{\mathrm{dR},\tilde{t}'}$.

To finish the proof, it only remains to check that the collection $\{s_{\alpha,\mathrm{cris},\tilde{t}}\}$ coincides with the collection $\{s_{\alpha,\mathrm{cris},t}\}$ constructed from the initial choice of lift $\tilde{t}$. This follows as in the proof of [13, Proposition 2.3.5], using a parallel transport argument. □

The explicit construction of R_G^{univ} in Section 3 now implies (see, for instance, [19, Corollary 4.13])

Proposition 4.8. *The functor from algebraic $\mathbb{Q}$-representations N of G to filtered vector bundles $(\boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K}, \mathrm{Fil}^{\bullet}\, \boldsymbol{N}_{\mathrm{dR},\mathrm{Sh}_K})$ extends canonically to an exact tensor functor $N \to \boldsymbol{N}_{\mathrm{dR}}$ from algebraic $\mathbb{Z}_{(p)}$-representations N of $G_{(p)}$ to filtered vector bundles on $\mathcal{S}_K$ equipped with an integrable connection. When $N = H_{(p)}$, the associated filtered vector bundle with integrable connection is simply $\boldsymbol{H}_{\mathrm{dR}}$, the relative first de Rham homology of $A \to \mathcal{S}_K$.*

4.9. Write $\widehat{\mathcal{S}}_K$ for the formal completion of $\mathcal{S}_K$ along $\mathcal{S}_{K,k(v)}$. The relative first crystalline cohomology of A over $\mathcal{S}_{K,(v)}$ gives a Dieudonné F-crystal $\boldsymbol{H}^{\vee}_{\mathrm{cris}}$ over $\mathcal{S}_{K,k(v)}$ whose evaluation on $\widehat{\mathcal{S}}_K$ is canonically isomorphic to the p-adic completion of $\boldsymbol{H}^{\vee}_{\mathrm{dR}}$ as a vector bundle with integrable connection.

We will now expand our definition of an F-crystal over $\mathcal{S}_{K,k(v)}$ to mean a crystal of vector bundles $\boldsymbol{N}$ over $\mathcal{S}_{K,k(v)}$ equipped with an isomorphism $\mathrm{Fr}^*\boldsymbol{N} \xrightarrow{\simeq} \boldsymbol{N}$ in the $\mathbb{Q}_p$-linear *isogeny* category associated with the category of crystals over $\mathcal{S}_{K,k(v)}$.

Given an algebraic $\mathbb{Z}_{(p)}$-representation $N_{(p)}$ of $G_{(p)}$, the restriction of the vector bundle $\boldsymbol{N}_{\mathrm{dR}}$ to $\widehat{\mathcal{S}}_K$ is the evaluation of a canonical crystal $\boldsymbol{N}_{\mathrm{cris}}$ of vector bundles over $\mathcal{S}_{K,k(v)}$. Since $H_{(p)}$ is a faithful representation of $G_{(p)}$, we now have

PROPOSITION 4.10. *There is a unique structure of an F-crystal on $\boldsymbol{N}_{\mathrm{cris}}$, which when $N_{(p)} = H^{\vee}_{(p)}$ agrees with the canonical F-crystal structure on the Dieudonné F-crystal $\boldsymbol{H}^{\vee}_{\mathrm{cris}}$. This gives an exact tensor functor $N_{(p)} \mapsto \boldsymbol{N}_{\mathrm{cris}}$ from $\mathbb{Z}_{(p)}$-representations of $G_{(p)}$ to F-crystals over $\mathcal{S}_{K,k(v)}$.*

THEOREM 4.11. *Let (G, X) be a Shimura datum of abelian type, and let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial compact open subgroup. Then, for any place $v|p$ of the reflex field $E = E(G, X)$, the pro-Shimura variety*

$$\mathrm{Sh}_{K_p} = \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathrm{Sh}_{K_p K^p}$$

over E admits an integral canonical model $\mathcal{S}_{K_p}$ over $\mathcal{O}_{E,(v)}$. That is, $\mathcal{S}_{K_p}$ is regular and formally smooth over $\mathcal{O}_{E,(v)}$ with generic fiber Sh_{K_p}, and, given any other regular and formally smooth scheme S over $\mathcal{O}_{E,(v)}$, any map $S_E \to \mathrm{Sh}_{K_p}$ extends to a map $S \to \mathcal{S}_{K_p}$.

When (G, X) is of Hodge type, so that G admits a polarizable faithful representation H of weights $(0, -1), (-1, 0)$, we have

$$\mathcal{S}_{K_p} = \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathcal{S}_{K_p K^p}$$

where $\mathcal{S}_{K_p K^p}$ is constructed using the representation H as in (4.5).

Proof. Note that the stated extension property completely characterizes the integral canonical model $\mathcal{S}_{K_p}$.

When (G, X) is of Hodge type, the required extension property of the pro-scheme $\mathcal{S}_{K_p}$ follows as in the proof of [13, Theorem 2.3.8], using Proposition 4.6. The only additional point to note is that Kisin uses a purity result of Faltings for abelian schemes over smooth $\mathbb{Z}_p$-schemes [22, 3.6], which has some restrictions when $p = 2$. However, these restrictions have since been removed by Vasiu and Zink; see [28, Corollary 5].

For a general Shimura datum of abelian type, the theorem is deduced from the case of Hodge type using Kisin's twisting construction [13, Section 3]; see especially [13, Corollary 3.4.14]. □

Acknowledgements

W. Kim is supported by the Herchel Smith Postdoctoral Fellowship, and the EPSRC (Engineering and Physical Sciences Research Council) in the form of EP/L025302/1. K. Madapusi Pera is supported by NSF grant DMS-1502142.

https://doi.org/10.1017/fms.2016.23 Published online by Cambridge University Press

Appendix A. The Tate conjecture in characteristic 2

This appendix is due to the second named author. We prove:

THEOREM A.1. *The Tate conjecture holds for K3 surfaces over finitely generated fields.*

Of course, the case when the field has characteristic $\neq 2$, this is already known by the results of the second author in [17], and also by earlier work by Maulik [21] and Charles [7] (among many others). The new ingredient here is the characteristic 2 case.

We will assume that the reader is familiar with the methods of [17], and so our treatment will be a bit terse.

A.2. We will need a particular class of Shimura varieties of abelian type associated with a quadratic lattice L of signature $(n, 2)$ with $n \geqslant 1$. This theory is summarized in [17, Section 4]; see also the work of André [1].

From L we obtain a Shimura datum (G_L, X_L), where $G_L = \mathrm{SO}(L_{\mathbb{Q}})$ and X_L is the symmetric domain of oriented negative definite planes in $L_{\mathbb{R}}$. Associated with L is also the discriminant kernel $K_L \subset G_L(\mathbb{A}_f)$, which is the largest subgroup that stabilizes $L_{\widehat{\mathbb{Z}}}$ and acts trivially on the discriminant module $L^{\vee}/L$.

Associated with any compact open subgroup $K \subset G(\mathbb{A}_f)$ is the Shimura variety (or rather algebraic stack)

$$\mathrm{Sh}_K(L) := \mathrm{Sh}_K(G_L, X_L),$$

which is defined over $\mathbb{Q}$. When $K = K_L$, we will set $\mathrm{Sh}(L) = \mathrm{Sh}_{K_L}(L)$.

As in [17], we will assume that L contains a hyperbolic plane, so that we have

$$\mathrm{Sh}(L)(\mathbb{C}) = \Gamma_L \backslash X_L,$$

where $\Gamma_L \subset \mathrm{SO}(L)(\mathbb{Z})$ is the discriminant kernel.

A.3. The Shimura variety $\mathrm{Sh}(L)$ carries a canonical family of $\mathbb{Z}$-motives (here, this means absolute Hodge motives; see [17, Section 2]) $\boldsymbol{L}$ associated with the standard representation $L_{\mathbb{Q}}$ of G_L, and the lattice $L_{\widehat{\mathbb{Z}}}$. More precisely, for every point $s \to \mathrm{Sh}(L)$, the cohomological realizations of the $\mathbb{Z}$-motive $\boldsymbol{L}_s$ will be the fibers at s of the sheaves $\boldsymbol{L}_B$, $\boldsymbol{L}_\ell$, and $\boldsymbol{L}_{\mathrm{dR}}$ associated with $(L_{\mathbb{Q}}, L_{\widehat{\mathbb{Z}}})$.

Moreover, as in [17, Section 4.4], we find that there exists a finite étale cover $\widetilde{\mathrm{Sh}}(L) \to \mathrm{Sh}(L)$ associated with the central extension $\mathrm{GSpin}(L_{\mathbb{Q}}) \to G_L$, and an abelian scheme $A^{\mathrm{KS}} \to \widetilde{\mathrm{Sh}}(L)$, called the **Kuga–Satake family**, whose realizations are associated with the left regular representation H of $\mathrm{GSpin}(L_{\mathbb{Q}})$ on the Clifford algebra $C(L_{\mathbb{Q}})$, and thus give us the family of motives $\boldsymbol{H}$ over $\widetilde{\mathrm{Sh}}(L)$. The family of motives $\underline{\mathrm{End}}(\boldsymbol{H})$ associated with $\mathrm{End}(H)$ descends over $\mathrm{Sh}(L)$, as does the sheaf $\boldsymbol{E}$ of endomorphisms of the abelian scheme A^{KS}, along with the homological realization map

$$\boldsymbol{E} \to \underline{\mathrm{End}}(\boldsymbol{H}).$$

The action of L on the Clifford algebra $C(L)$ by left multiplication induces an embedding of families of motives

$$\boldsymbol{L} \hookrightarrow \underline{\mathrm{End}}(\boldsymbol{H})$$

over $\mathrm{Sh}(L)$.

Given any scheme $T \to \mathrm{Sh}(L)$, a **special endomorphism** over T will be a section of $\boldsymbol{E}$ over T, whose homological realizations land in the image of $\boldsymbol{L}$ at all geometric points of T. We will write $L(T)$ for the group of special endomorphisms over T. Composition in $\underline{\mathrm{End}}(\boldsymbol{H})$ induces a canonical positive definite quadratic form on $L(T)$ with values in $\mathbb{Z}$.

A.4. Let $L^\diamond$ be another quadratic lattice of signature $(n^\diamond, 2)$ such that we have an isometric embedding $L \hookrightarrow L^\diamond$ onto a direct summand of $L^\diamond$. Let $\Lambda \subset L^\diamond$ be the orthogonal complement of L, so that we can view G_L as the subgroup of $G_{L^\diamond}$ that acts trivially on Λ. We then have an embedding of Shimura data

$$(G_L, X_L) \hookrightarrow (G_{L^\diamond}, X_{L^\diamond})$$

and thus a map $\mathrm{Sh}(L) \to \mathrm{Sh}(L^\diamond)$ of Shimura varieties.

For any $T \to \mathrm{Sh}(L^\diamond)$, write $L^\diamond(T)$ for the associated space of special endomorphisms over T. Then there is a canonical isometric embedding $\Lambda \hookrightarrow L^\diamond(\mathrm{Sh}(L))$ such that, for any $T \to \mathrm{Sh}(L)$, we have a canonical identification

$$L(T) = \Lambda^\perp \subset L^\diamond(T).$$

A.5. Suppose now that $L_{(p)} := L_{\mathbb{Z}_{(p)}}$ is self-dual. Then the associated special orthogonal group $G_{(p)} := \mathrm{SO}(L_{(p)})$ is a reductive model for G_L over $\mathbb{Z}_{(p)}$, and $K_{L,p} = G_{(p)}(\mathbb{Z}_p)$ is a hyperspecial compact open subgroup of $G_L(\mathbb{Q}_p)$. Therefore, by Theorem 4.11, we have a smooth integral canonical model $\mathcal{S}(L)_{(p)}$ over $\mathbb{Z}_{(p)}$ for $\mathrm{Sh}(L)$. More precisely, the theorem gives us an integral canonical model $\mathcal{S}_{K^pK_{L,p}}(L)_{(p)}$ over $\mathbb{Z}_{(p)}$ for $\mathrm{Sh}_{K^pK_{L,p}}(L)$, when $K^p \subset G_L(\mathbb{A}_f^p)$ is a sufficiently small neat compact open subgroup. We obtain $\mathcal{S}(L)_{(p)}$ as the quotient stack of this integral model by the natural action of the finite group K_L^p/K^p.

By Proposition 4.10, we now have crystalline realizations $\boldsymbol{L}_{\mathrm{cris}}$ and $\underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{cris}})$ of $\boldsymbol{L}$ and $\underline{\mathrm{End}}(\boldsymbol{H})$, respectively: These are F-crystals over the special fiber $\mathcal{S}(L)_{\mathbb{F}_p}$ of $\mathcal{S}(L)_{(p)}$, and we have a canonical inclusion $\boldsymbol{L}_{\mathrm{cris}} \hookrightarrow \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{cris}})$ of F-crystals, whose quotient is once again an F-crystal over $\mathcal{S}(L)_{\mathbb{F}_p}$.

Moreover, the Kuga–Satake abelian scheme A^{KS} extends over a finite étale cover of $\mathcal{S}(L)_{(p)}$ – namely, the integral canonical model for $\widetilde{\mathrm{Sh}}(L)$ – and its endomorphism scheme $\boldsymbol{E}$ descends over $\mathcal{S}(L)_{(p)}$, and is equipped with a crystalline realization map $\boldsymbol{E} \to \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{cris}})$. (That the integral canonical model for the GSpin Shimura variety is finite étale over that for the SO Shimura variety follows from Kisin's construction of models of abelian type from those of Hodge type, using his twisting construction; see in particular (3.4.6) and (3.4.10) of [13].)

Given a point $s \to \mathcal{S}(L)_{(p)}$ in characteristic p, a **special endomorphism** over s will be a section of $\boldsymbol{E}$ whose crystalline realization lands in the subspace

$$\boldsymbol{L}_{\mathrm{cris},s} \subset \mathrm{End}(\boldsymbol{H}_{\mathrm{cris},s}).$$

Given $T \to \mathcal{S}(L)_{(p)}$, a **special endomorphism** over T will be a section of $\boldsymbol{E}$, which induces a special endomorphism at every geometric point of T. Write $L(T)$ for the space of special endomorphisms over T. Composition in $\boldsymbol{E}$ induces a positive definite quadratic form $Q : L(T) \to \mathbb{Z}$; see [19, Lemma 6.12].

A.6. In general, for each lattice L as above, we have a unique *normal* integral model $\mathcal{S}(L)$ for $\mathrm{Sh}(L)$ over $\mathbb{Z}$, characterized by the following properties:

- If $L_{(p)}$ is self-dual for a prime p, then

$$\mathcal{S}(L)_{\mathbb{Z}_{(p)}} = \mathcal{S}(L)_{(p)}$$

  is the integral canonical model for $\mathrm{Sh}(L)$ over $\mathbb{Z}_{(p)}$.

- If $L \hookrightarrow L^\diamond$ is as in (Appendix A.4), then the map $\mathrm{Sh}(L) \to \mathrm{Sh}(L^\diamond)$ extends to a finite map $\mathcal{S}(L) \to \mathcal{S}(L^\diamond)$ of $\mathbb{Z}$-stacks.

Moreover, for every $T \to \mathcal{S}(L)$, we have a functorially associated space of special endomorphisms $L(T)$ characterized by the following properties:

- If T factors through $\mathrm{Sh}(L)$, then $L(T)$ agrees with the space defined in (Appendix A.3).

- If $L_{(p)}$ is self-dual and T factors through $\mathcal{S}(L)_{(p)}$, then $L(T)$ agrees with the space defined in (Appendix A.5).

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- Suppose that $L \hookrightarrow L^\diamond$ and Λ are as in (Appendix A.4), and that $L^\diamond(T)$ is the space of special endomorphisms over T viewed as a scheme over $\mathcal{S}(L^\diamond)$. Then there is an isometric embedding

$$\iota : \Lambda \hookrightarrow L^\diamond(\mathcal{S}(L)),$$

and a canonical identification

$$L(T) = \Lambda^\perp = \{x \in L^\diamond(T) :\subset L^\diamond(T).$$

For more details on all this, see [2, Section 4], especially 4.4.6, 4.4.7 and 4.5.5 of [2, Section 4].

A.7. For every prime ℓ, the ℓ-adic realization $\boldsymbol{L}_\ell$ over $\mathrm{Sh}(L)$ extends to an ℓ-adic sheaf over $\mathcal{S}(L)[\ell^{-1}]$, which we will denote by the same symbol. The ℓ-adic realizations of any special endomorphism are sections of $\boldsymbol{L}_\ell$.

Let $s \to \mathcal{S}(L)$ be a point defined over a field $k(s)$. Fix a separable closure $k(s)^{\mathrm{sep}}$ for $k(s)$, and let $\Gamma_s = \mathrm{Gal}(k(s)^{\mathrm{sep}}/k(s))$ be the associated absolute Galois group. Let $s^{\mathrm{sep}} \to \mathcal{S}(L)$ be the induced $k(s)^{\mathrm{sep}}$-valued point. Then the fiber $\boldsymbol{L}_{\ell,s^{\mathrm{sep}}}$ of $\boldsymbol{L}_\ell$ is a Γ_s-representation.

THEOREM A.8. *Let k be the prime field of $k(s)$. Suppose that U is a smooth connected variety over k with the following properties:*

- *The field of rational functions on U is isomorphic to $k(s)$, and the map $s \to \mathcal{S}(L)$, viewed as map from the generic point of U, extends to a map $U \to \mathcal{S}(L)$.*

- *If $k = \mathbb{F}_p$ for a prime p, there exists a point $s_0 \in U(\mathbb{F}_{p^m})$ such that the integer*

$$r_\ell = \dim\left(\bigcup_n \boldsymbol{L}_{\ell,s_0^{\mathrm{sep}}}^{\Gamma_{s_0,n}}\right)$$

is independent of the prime $\ell \neq p$. Here, $\Gamma_{s_0,m} \subset \Gamma_{s_0}$ is the unique subgroup of index n.

Then, for every prime ℓ distinct from the characteristic of $k(s)$, the ℓ-adic realization map

$$L(s) \otimes \mathbb{Q}_\ell \to \boldsymbol{L}_{\ell,s^{\mathrm{sep}}}^{\Gamma_s} \tag{A.1}$$

is an isomorphism.

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Proof. As in the proof of [17, Corollary 6.11], the hypotheses, along with Zarhin's proof of the Tate conjecture for abelian varieties over function fields [29], allow us to reduce to the case where $k(s)$ is finite over its prime field. When $k(s)$ is in characteristic 0, then the result is an immediate consequence of Faltings's isogeny theorem for abelian varieties over number fields. When $k(s)$ is in characteristic $p > 0$, then as in [17, Lemma 6.6], we can reduce to the case where $L(s) \neq 0$ and such that $L_{(p)}$ is self-dual. Here, the argument used to prove [17, Theorem 6.4] works also without the hypothesis $p > 2$ to give us the theorem.

For the convenience of the reader, we now sketch the argument: By the ℓ-independence hypothesis, it is enough to prove that (A.1) is an isomorphism for *some* prime $\ell \neq p$. After enlarging the field of definition of s if necessary, we can assume that we have

$$\boldsymbol{L}_{\ell,s^{\mathrm{sep}}}^{\Gamma_s} = \left(\bigcup_n \boldsymbol{L}_{\ell,s^{\mathrm{sep}}}^{\Gamma_{s,n}}\right).$$

The main point now is that the map (A.1) is equivariant for the action of a reductive $\mathbb{Q}$-group I_s, which preserves the rational subspace $L(s)$ on the left-hand side and acts irreducibly on the right-hand side. Since $L(s) \neq 0$, this will prove the theorem.

It remains to exhibit the group I_s, whose construction and properties are due to Kisin. Let $H_{(p)} \subset H$ be the $\mathbb{Z}_{(p)}$-lattice provided by the Clifford algebra of the self-dual lattice $L_{(p)} \subset L_{\mathbb{Q}}$. As in the proof of Proposition 4.6, one can find tensors $\{s_\alpha\} \subset H_{(p)}^{\otimes}$ whose pointwise stabilizer is $\mathrm{GSpin}(L_{(p)})$.

Choose a lift s' of s to a point of the integral canonical model $\widetilde{\mathcal{S}}(L)_{(p)}$ of the GSpin Shimura variety. In the proof of Proposition 4.6, we showed that from the collection $\{s_\alpha\}$, we obtain a *canonical* collection of φ-invariant tensors $\{\boldsymbol{s}_{\alpha,\mathrm{cris},s'}\} \subset \boldsymbol{H}_{\mathrm{cris},s'}^{\otimes}$, whose pointwise stabilizer is isomorphic to $\mathrm{GSpin}(L_{(p)})_W$; here, $W = W(k(s))$. At the same time, for every prime $\ell' \neq p$, we obtain Γ_s-invariant ℓ'-adic realizations $\{\boldsymbol{s}_{\alpha,\ell,s'}\} \subset \boldsymbol{H}_{\ell',s',\mathrm{sep}}^{\otimes}$.

Associated with the point s' is the Kuga–Satake abelian variety $A_{s'}^{\mathrm{KS}}$. We now define I_s to be the subgroup of $\underline{\mathrm{Aut}}^{\circ}(A_s^{\mathrm{KS}})$ (see [17, Section 6.9] for the notation) consisting of those automorphisms, which fix the crystalline realizations $\{\boldsymbol{s}_{\alpha,\mathrm{cris},s'}\}$, as well as all the ℓ'-adic realizations $\{\boldsymbol{s}_{\alpha,\ell',s'}\}$. It is not hard to show that this group acts on both sides of (A.1) and that it preserves $L(s)$. In [17, Section 6.9], this is done by exhibiting an explicit $\mathrm{GSpin}(L_{(p)})$-invariant projector on $C(L_{(p)})$, whose image is the natural inclusion $L_{(p)} \hookrightarrow C(L_{(p)})$.

It remains then to show that I_s acts irreducibly on the right hand side for a certain choice of ℓ. We will choose an ℓ, so that $\mathrm{GSpin}(L)_{\mathbb{Q}_\ell}$ is split, and such that the characteristic polynomial of Frobenius acting on the ℓ-adic cohomology of A splits completely over $\mathbb{Q}_\ell$. We define a subgroup $I_{\ell,s} \subset \mathrm{GL}(\boldsymbol{H}_{\ell,s',\mathrm{sep}})$ consisting of automorphisms that are $\Gamma_{s,n}$-equivariant, for some divisible enough n, and which

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stabilize the ℓ-adic realizations $\{\boldsymbol{s}_{\alpha,\ell,s'}\}$. As shown in [17, Lemma 6.8], $I_{\ell,s}$ acts irreducibly on the right- hand side of (A.1), so it suffices to show that the natural map

$$I_s \otimes \mathbb{Q}_\ell \to I_{\ell,s}$$

is an isomorphism.

This will be accomplished via a translation by Kisin of Tate's proof of his conjecture for endomorphisms of abelian varieties over finite fields. It will only use the (rather simple, in the scheme of things) property that the point s' canonically determines the crystalline tensors $\{\boldsymbol{s}_{\alpha,\mathrm{cris},s'}\}$, coupled with the finiteness of the set of $\mathbb{F}_{p^n}$-valued points of finite covers of $\widetilde{\mathcal{S}}(L)_{(p)}$.

Since $I_{\ell,s}$ is split reductive by our hypotheses, a group theoretic argument (see [17, Proposition 6.10]) reduces to showing that the ℓ-adic manifold $I_s(\mathbb{Q}_\ell)\backslash I_{\ell,s}(\mathbb{Q}_\ell)$ is compact.

For this, we choose a lift of s' to a point in the pro-scheme $\widetilde{\mathcal{S}}_p(L)$ over $\mathbb{Z}_{(p)}$ built as in Theorem 4.11 as the limit of the integral models of finite level covering $\widetilde{\mathcal{S}}(L)(p)$. In other words, we have chosen a compatible family of prime-to-p level structures on $A^{\mathrm{KS}}_{s'}$, which respect $\mathrm{GSpin}(L)$-structures. The tautological action on $I_{\ell,s}$ on the cohomology of $A^{\mathrm{KS}}_{s'}$ induces an action on level structures at ℓ, and therefore a map

$$I_{\ell,s}(\mathbb{Q}_\ell) \to \widetilde{\mathcal{S}}_p(L)(k(s)^{\mathrm{sep}})$$

carrying the identity to our choice of lift of s'. This is explained more precisely in the proof of [17, Proposition 6.10].

We now claim that if two elements $g_1, g_2 \in I_{\ell,s}(\mathbb{Q}_\ell)$ have the same image under this map, then we have $g_2 = hg_1$, for some $h \in I_s(\mathbb{Q}_\ell)$. Indeed, the assumption implies that g_1 and g_2 also have the same image in the moduli of polarized abelian varieties with full prime-to-p level structure. This implies that there is an isogeny $h \in \mathrm{Aut}^\circ(A^{\mathrm{KS}}_{s^{\mathrm{sep}}})$ such that $g_2 = hg_1$.

We claim that h lies in $I_s(\mathbb{Q})$. As seen in the proof of [17, Proposition 6.10], from its very definition, h preserves all the ℓ'-adic realizations $\boldsymbol{s}_{\alpha,\ell,s'}$, for $\ell' \neq p$. To finish, we need to know that h also preserves the crystalline realization $\boldsymbol{s}_{\alpha,\mathrm{cris},s'}$. But this is clear: Since g_1 and g_2 map to the same point, the associated crystalline realizations of the s_α must also be the same at the points associated with both g_1 and g_2!

The claim in the previous paragraph implies that $I_s(\mathbb{Q}_\ell)\backslash I_{\ell,s}(\mathbb{Q}_\ell)$ is a quotient of a space mapping injectively into $\widetilde{\mathcal{S}}_p(L)(k(s)^{\mathrm{sep}})$. From this, it is not hard to deduce (see the proof of [17, Proposition 6.10] again) that, for some compact open subgroup $U_\ell \subset I_{\ell,s}(\mathbb{Q}_\ell)$, the double coset space

$$I_s(\mathbb{Q}_\ell)\backslash I_{\ell,s}(\mathbb{Q}_\ell)/U_\ell$$

is finite. This finishes the proof of the theorem. □

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REMARK A.9. The ℓ-independence condition imposed above has been shown to always hold by Kisin [14, Corollary 2.3.2] when $p > 2$. The argument there is valid without this hypothesis: The only reason the restriction intervenes is for the lack of an explicit description of the complete local rings of $\widetilde{\mathcal{S}}(L)$ over $\mathbb{Z}_2$, but this is now available, by Proposition 4.6. In particular, the above theorem is true with only the simple (and necessary) condition that $k(s)$ be finitely generated over its prime field.

A.10. Fix an integer $d > 0$, and let M_{2d} be the moduli stack over $\mathbb{Z}$ of primitively quasipolarized K3 surfaces of degree $2d$; see [17, Section 3].

Let $(\boldsymbol{\mathcal{X}}, \boldsymbol{\xi}) \to \mathrm{M}_{2d}$ be the universal quasipolarized K3 surface. For each prime ℓ, we have its ℓ-adic primitive cohomology sheaf $\boldsymbol{P}^2_{\ell}$ over $\mathrm{M}_{2d}[\ell^{-1}]$. For each prime p, we also have the associated F-crystal $\boldsymbol{P}^2_{\mathrm{cris}}$ over $\mathrm{M}_{2d,\mathbb{F}_p}$. Finally, we have the filtered vector bundle $\boldsymbol{P}^2_{\mathrm{dR}}$ over M_{2d} obtained from the primitive relative de Rham cohomology of $\boldsymbol{\mathcal{X}}$.

We also have the full de Rham cohomology $\boldsymbol{H}^2_{\mathrm{dR}}$ of $\boldsymbol{\mathcal{X}} \to \mathrm{M}_{2d}$. The canonical Poincaré pairing on it is induced from a quadratic form

$$\boldsymbol{Q} : \boldsymbol{H}^2_{\mathrm{dR}} \to \mathscr{O}_{\mathrm{M}_{2d}}.$$

This is obvious when 2 is invertible, and follows from [24, Theorem 4.7] in general: Indeed, Ogus shows there that the Poincaré pairing on de Rham cohomology is *even* and therefore arises from an integral valued quadratic form. This form in turn induces one on $\boldsymbol{P}^2_{\mathrm{dR}} \to \mathscr{O}_{\mathrm{M}_{2d}}$.

A.11. As in [17, Section 3.10], let N be the self-dual lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic plane. Choose a hyperbolic basis e, f for the first copy of U. We will now take our quadratic lattice to be

$$L_d = \langle e - df \rangle^{\perp} \subset N.$$

Associated with this is the Shimura variety $\mathrm{Sh}(L_d)$, with its integral model $\mathcal{S}(L_d)$ over $\mathbb{Z}$, and the family of motives $\boldsymbol{L}$ associated with L_d, along with its various cohomological realizations $\boldsymbol{L}_?$ over $\mathrm{Sh}(L_d)$; see (Appendix A.3). Moreover, for each prime ℓ, the ℓ-adic realization $\boldsymbol{L}_{\ell}$ extends to a lisse ℓ-adic sheaf over $\mathcal{S}(L_d)[\ell^{-1}]$, which we once again denote by $\boldsymbol{L}_{\ell}$.

There exists a canonical 2-fold étale cover $\tilde{\mathrm{M}}_{2d} \to \mathrm{M}_{2d}$, whose restriction to $\mathrm{M}_{2d}[\ell^{-1}]$ parameterizes isometric trivializations $\det(L_d) \otimes \mathbb{Z}_{\ell} \xrightarrow{\simeq} \underline{\det}(\boldsymbol{P}^2_{\ell})$ of the rank-one sheaf $\underline{\det}(\boldsymbol{P}^2_{\ell})$.

There is now a canonical period map (see [17, Corollary 5.4])

$$\iota^{\mathrm{KS}}_{\mathbb{Q}} : \tilde{\mathrm{M}}_{2d,\mathbb{Q}} \to \mathrm{Sh}(L_d).$$

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Moreover, for $? = \mathrm{dR}, \ell$, we have a canonical isometry (see [17, Proposition 4.6]):

$$\alpha_? : \boldsymbol{L}_?(-1)|_{\tilde{\mathrm{M}}_{2d,\mathbb{Q}}} \xrightarrow{\simeq} \boldsymbol{P}^2_?|_{\tilde{\mathrm{M}}_{2d,\mathbb{Q}}}.$$

PROPOSITION A.12. *Let $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d} \subset \tilde{\mathrm{M}}_{2d}$ be the complement of the nonsmooth loci in all its special fibers. Then $\iota^{\mathrm{KS}}_{\mathbb{Q}}$ extends to an étale morphism*

$$\iota^{\mathrm{KS}} : \tilde{\mathrm{M}}^{\mathrm{sm}}_{2d} \to \mathcal{S}(L_d).$$

Proof. It is enough to prove the proposition over $\mathbb{Z}_{(p)}$ for a fixed prime p. Let $L_d \hookrightarrow L^\diamond$ be an isometric embedding as in (Appendix A.4) with $L^\diamond_{(p)}$ a self-dual lattice.

Now, $\mathcal{S}(L_d) \to \mathcal{S}(L^\diamond)$ is the normalization of $\mathcal{S}(L^\diamond)$ in $\mathrm{Sh}(L_d)$, and so it is enough to show that the composition

$$\tilde{\mathrm{M}}_{2d,\mathbb{Q}} \xrightarrow{\iota^{\mathrm{KS}}_{\mathbb{Q}}} \mathrm{Sh}(L_d) \to \mathrm{Sh}(L^\diamond)$$

extends to a map $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to \mathcal{S}(L^\diamond)_{(p)}$. This is shown just as in the proof of [17, Proposition 4.7], using the fact that $\mathcal{S}(L^\diamond)_{(p)}$ is an integral canonical model for its generic fiber.

It remains to show that the map is étale. Since $L_d^\vee/L_d$ is a cyclic abelian group of order $2d$, we can assume that we have chosen $L^\diamond$, so that $\Lambda = L_d^\perp \subset L^\diamond$ has rank one, and is spanned by an element v with $v^2 = 2d$. Consider the stack

$$\mathcal{Z}(2d)_{(p)} \to \mathcal{S}(L^\diamond)_{(p)}$$

parameterizing $f \in L^\diamond(T)$ with $f \circ f = 2d$. This is in fact a Deligne–Mumford stack, finite over $\mathcal{S}(L^\diamond)_{(p)}$; see [19, Proposition 6.13].

The canonical embedding $\Lambda \hookrightarrow L^\diamond(\mathcal{S}(L))$ determines a finite morphism

$$\mathcal{S}(L)_{\mathbb{Z}_{(p)}} \to \mathcal{Z}(2d)_{(p)},$$

which is an open and closed immersion in the generic fiber; see [19, Lemma 7.1].

Now, it is enough to show that the induced map

$$\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to \mathcal{Z}(2d)_{(p)} \tag{A.1}$$

is étale.

Arguing as in the proof of [19, Lemma 6.16(4)], we first find that (A.1) factors through the open substack $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$ of $\mathcal{Z}(2d)_{(p)}$, where the de Rham realization $\boldsymbol{f}_{\mathrm{dR}}$ of the tautological element $f \in L^\diamond(\mathcal{Z}(2d)_{(p)})$ spans a local direct summand of $\boldsymbol{L}^\diamond_{\mathrm{dR}}|_{\mathcal{Z}^{\mathrm{pr}}(m)_{(p)}}$. Its orthogonal complement gives us a vector subbundle

$$\boldsymbol{L}_{\mathrm{dR}} := \langle \boldsymbol{f}_{\mathrm{dR}} \rangle^\perp \subset \boldsymbol{L}^\diamond_{\mathrm{dR}}|_{\mathcal{Z}^{\mathrm{pr}}(m)_{(p)}}$$

over $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$, whose restriction over $\mathrm{Sh}(L)$ is canonically identified with $\boldsymbol{L}_{\mathrm{dR},\mathbb{Q}}$. Moreover, the isotropic line $\mathrm{Fil}^1 \boldsymbol{L}^{\diamond}_{\mathrm{dR}} \subset \boldsymbol{L}^{\diamond}_{\mathrm{dR}}$ is orthogonal to $\boldsymbol{f}_{\mathrm{dR}}$ over $\mathcal{Z}(2d)_{(p)}$, and so gives us a rank-one local direct summand

$$\mathrm{Fil}^1 \boldsymbol{L}_{\mathrm{dR}} \subset \boldsymbol{L}_{\mathrm{dR}}$$

over $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$.

Now, the deformation theory of $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$ is governed by $\mathrm{Fil}^1 \boldsymbol{L}_{\mathrm{dR}}$. More precisely, given a k-valued point s of the stack, its deformations over the ring of dual numbers $k[\epsilon]$ are in canonical bijection with lifts of $\mathrm{Fil}^1 \boldsymbol{L}_{\mathrm{dR},s}$ to isotropic direct summands of $\boldsymbol{L}_{\mathrm{dR},s} \otimes k[\epsilon]$. This is shown exactly as in [19, Proposition 5.16], using the explicit description of the deformation rings of $\mathcal{S}(L^{\diamond})_{(p)}$ provided here by Proposition 4.6.

It is also known that the deformation theory of M_{2d} is governed by the rank-one local direct summand $\mathrm{Fil}^2 \boldsymbol{P}^2_{\mathrm{dR}}$ of $\boldsymbol{P}^2_{\mathrm{dR}}$ corresponding to the degree 2 part of the Hodge filtration in precisely the same manner.

Therefore, to show that (A.1) is étale, it suffices to show that the isometry α_{dR} in the generic fiber extends to an isometry

$$\boldsymbol{L}_{\mathrm{dR}}(-1)|_{\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}} \xrightarrow{\simeq} \boldsymbol{P}^2_{\mathrm{dR}}$$

of filtered vector bundles over $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$. See the proof of [17, Theorem 5.8] for an explanation of this.

To show that α_{dR} extends, it is enough to do so over the ordinary locus of $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$; see [17, Proposition 5.11]. The extension over the ordinary locus is accomplished exactly as in Lemma 5.10 of the same article. The main ingredient is an integral comparison isomorphism of Bloch–Kato [6] for ordinary varieties, which holds without any hypothesis on p. $\square$

LEMMA A.13. *Suppose that we have a point $s \to \tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}$ defined over a perfect field k of characteristic $p > 0$, and a nonzero element $f \in L(\iota^{\mathrm{KS}}(s))$. Then the deformation space of the triple $(\boldsymbol{\mathcal{X}}_s, \boldsymbol{\xi}_s, f)$ over $W(k)$ admits a component that is flat over $W(k)$.*

In particular, there exists a finite extension E of the field of fractions of $W(k)$, and a lift $\tilde{s} : \mathcal{O}_E \to \tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}$ such that f lifts to an element $\tilde{f} \in L(\iota^{\mathrm{KS}}(\tilde{s}))$.

Proof. Set $t = \iota^{\mathrm{KS}}(s)$. In the notation of the proof of Proposition A.12, giving the point t is equivalent to giving the corresponding point of $\mathcal{Z}^{\mathrm{pr}}(2d)$, which in turn is equivalent to giving a pair $(t^{\diamond}, x)$, where $t^{\diamond}$ is the point of $\mathcal{S}(L^{\diamond})$ obtained from t, and $x \in L(t^{\diamond})$ satisfies $x \circ x = 2d$. We have a canonical identification $L(t) = \langle x \rangle^{\perp} \subset L(t^{\diamond})$, so that we can view f as an element of $L(t^{\diamond})$ that is perpendicular to x.

The proof of the proposition also shows that in fact the map $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to \mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$ is étale. This means that, to prove the lemma, it is enough to show that the deformation space of the triple $(t^\diamond, x, f)$ admits a flat component. In fact, we will see that this space is already flat over $\mathbb{Z}_{(p)}$.

For this, let $f \circ f = m \in \mathbb{Z}_{>0}$, let $\boldsymbol{x} \in L(\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)})$ be the tautological element, and consider the finite morphism

$$\mathcal{Z}^{\mathrm{pr}}(2d, m)_{(p)} \to \mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$$

parameterizing, for schemes $T \to \mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$, elements $y \in \langle \boldsymbol{x} \rangle^{\perp} \subset L(T)$ such that $y \circ y = m$. It is now enough to show that the restriction of the source of this morphism over the smooth locus of the target is flat over $\mathbb{Z}_{(p)}$. This is shown exactly as in [19, Corollary 7.18]. □

A.14. Since $\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}$ is normal, the isometry α_ℓ over the generic fiber extends:

$$\alpha_\ell : \boldsymbol{L}_\ell(-1)|_{\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}[\ell^{-1}]} \xrightarrow{\simeq} \boldsymbol{P}^2_\ell|_{\tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}[\ell^{-1}]}.$$

PROPOSITION A.15. *For every point $s \to \tilde{\mathrm{M}}^{\mathrm{sm}}_{2d}$, there is an isometric inclusion*

$$L(\iota^{\mathrm{KS}}(s)) \hookrightarrow \langle \boldsymbol{\xi}_s \rangle^{\perp} \subset \mathrm{Pic}(\boldsymbol{\mathcal{X}}_s)$$

compatible with ℓ-adic realizations on both sides via the isometry α_ℓ, for any prime ℓ distinct from the characteristic of $k(s)$.

Proof. If $k(s)$ has characteristic 0, this is a consequence of the Lefschetz $(1,1)$ theorem. If it has characteristic $p > 0$, choose $f \in L(\iota^{\mathrm{KS}}(s))$. Then, as in the proof of [17, Theorem 5.17], combining Lemma A.13 with the Lefschetz $(1,1)$ theorem will give us the (necessarily unique) divisor class in $\langle \boldsymbol{\xi}_s \rangle^{\perp}$, whose ℓ-adic realizations, for $\ell \neq p$, agree with those of f. □

Proof of Theorem A.1. Suppose that X is a K3 surface over a finitely generated field k. Fix a separable closure k^{sep} of k, and let $\Gamma_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be the associated absolute Galois group. We have to show that, for all ℓ prime to the characteristic of k, the realization map

$$\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell \to H^2_{\text{ét}}(X_{k^{\mathrm{sep}}}, \mathbb{Q}_\ell)(1)^{\Gamma_s}$$

is an isomorphism.

We can assume that X admits a quasipolarization ξ of degree $2d$ for some integer d, and thus is associated with a k-valued point $s \to \mathrm{M}_{2d}$. We can assume that s admits a lift to $\tilde{\mathrm{M}}_{2d}$, which we will once again denote by s.

Suppose first that s factors through $\tilde{\mathrm{M}}_{2d}^{\mathrm{sm}}$. We can find a smooth scheme U of finite type over the prime field of $k(s)$, whose field of rational functions is $k(s)$, and which is equipped with a map $U \to \tilde{\mathrm{M}}_{2d}^{\mathrm{sm}}$ extending s. Then we can use the map ι^{KS} to view U as a scheme over $\mathcal{S}(L)$. The ℓ-independence hypothesis of Theorem A.8 holds for every point of U valued in a finite field; see [17, Remark 6.3].

The Tate conjecture now follows for X from Proposition A.15 and Theorem A.8. Indeed, their combination proves the inequality

$$\mathrm{rank}\ \mathrm{Pic}(X) \geqslant \dim H^2_{\text{ét}}(X_{k^{\mathrm{sep}}}, \mathbb{Q}_\ell)(1)^{\Gamma_s}.$$

If s is not a smooth point of M_{2d}, X is a superspecial K3 surface. In other words, the de Rham Chern class $\mathrm{ch}_{\mathrm{dR}}(\xi)$ lies in $\mathrm{Fil}^2 H^2_{\mathrm{dR}}(X/k)$; see [23, 2.2]. Now, since there are no nonconstant families of quasipolarized superspecial K3 surfaces (see [23, Remark 2.7]), and since every irreducible component of the supersingular locus of $\mathrm{M}_{2d,\mathbb{F}_p}$ has dimension 9 (see [25, Theorem 15]), the theorem in general follows from the validity of the Tate conjecture for points in $\tilde{\mathrm{M}}_{2d,\mathbb{F}_p}^{\mathrm{sm}}$, and Artin's result on the constancy of Picard rank in families of supersingular K3 surfaces [3, Corollary (1.3)]. □

REMARK A.16. Given the Tate conjecture, one should be able to show that every superspecial K3 surface is actually the Kummer surface associated with a superspecial abelian surface, using which one should be able to extend the period map ι^{KS} over all of $\tilde{\mathrm{M}}_{2d}$.

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