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BORROWING CONSTRAINTS AND INTERNATIONAL COMOVEMENTS*

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I. *Introduction*

It is now well understood how the presence of borrowing constraints can affect the time series properties of aggregate economic data. In particular the results in Scheinkman and Weiss [9] show that borrowing constraints may cause the appearance of economic fluctuations in an economy where, if the perfect risk sharing implied by a full set of contingent claims markets was available, no aggregate fluctuations would be observed.

Departures from perfect risk-sharing across countries would also have several implications for the behavior of the international comovements of economic time series. Scheinkman [8] suggested that correlation of consumption series across countries could be used to test for the presence of a full set of contingent claims markets. Also, as it is shown formally below, if the output of different countries are Pareto substitutes in consumption, in a complete markets setting, the correlation of output series should be smaller (algebraically) than that of the corresponding productivity series. In this paper we construct a formal model of a two country economy that allows us to derive implications of the presence of borrowing constraints on the behavior of economic time series. Simulations of the model reveal that it is capable of generating significant positive correlation across output series even in the presence of uncorrelated productivity shocks. This result suggests that borrowing constraints can be used to explain the substantial positive correlation of output growth across countries in the presence of almost no correlation of productivity growth series (cf. Costello [5]). Further the model can generate a much lower consumption correlation than that implied by a complete set of contingent claims market what again seems to be in accordance with empirical observations.

The model we use is a version of the one developed in Scheinkman and Weiss [9]. Agents in each country are engaged in the production of a single consumption good using labor as its sole input but labor productivity is random. Their utility depends on the consumption of the two goods and on leisure. They can trade their output for the other country's production or for a single durable "asset." This asset is assumed to have a fixed

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nominal return of zero and is thus held solely to permit higher consumption level in "lean" times. The absence of complete contingent claims markets gives rise to a precautionary demand for wealth and, in particular to a non-zero price for the asset.

The qualitative features of the equilibrium can be described simply, especially in the case where the utility function of each agent is separable and the marginal utility of leisure is constant. Suppose each country has a high and a low level of productivity. Consider an increase in the productivity of labor in country one. If complete insurance was present, the price of good one would drop enough so that the consumption of good two would not be altered. In our case the only form of insurance available is the holding of the asset. On the average, when the productivity of country one goes from low to high, country one individuals would have a small share of asset holdings and will now try to increase their holding of the asset and this results is an increase in the price of the asset in term of good one. Hence country two's individuals will have a capital gain if we measure their wealth with good one as the numeraire, and thus their holdings of money does serve as partial insurance. However, individuals in country two still face the same trade-off between leisure and consumption of good two and hence the shock has no effect on their demand for good two. Individuals in country one on the other hand will, at fixed prices, consume more of both goods. The net result is an increase in output in country two as well as an increase in the relative price of good two in terms of good one, although weaker than the relative price change in a complete market economy. Thus the productivity shock in country one causes, through its effect on the equilibrium price of good two an increase in the output of good two generating a positive correlation of output across countries.

There are of course other ways in which one could generate these co-movements. If intermediate goods were introduced, then an increase in the productivity in one country could cheapen inputs in the other country sufficiently to generate an increase in output. This would of course imply that these intermediate goods would have countercyclical prices what seems to be countrary to the available evidence (cf. Murphy, Schleifer and Vishny [6]).

The model has other implications for economic time series. Since increases in productivity lead to a cheapening of the output in a country net exports are, in the model, procyclical. This seems to be at odds with the data (see e.g. Bachus, Kehoe and Kydland [1]). Also the model generates a negative correlation between the value of exports and the relative price of exportables in terms of importables. Though the data on aggregate export and import prices is by nature unreliable this doesn't seem to be rejected for the U.S.[1] In any case, the mechanism proposed here is, at best, responsible for a fraction of the observed patterns and deviations are to be expected.

The simulations also show that at least for certain parameter values the utility losses incurred are small relative to the changes in output correlation observed. Individual's optimization seem to lead to large effects on quantities while avoiding big utility losses. Since this model is too unrealistic to be matched quantitatively to actual data, this should not be taken to mean that actual markets display in fact allocations that are almost optima,

[1] The correlation between quaterly changes in GNP between 1948 and 1987 and changes in the logarithm of the price of exports devided by the price of imports is essentially zero but equals- 24 if we omit the years of 1974 and 1979/80 when the two big "oil shocks" hapenned.

but rather that relative large changes in some observed statistics relative to what would prevail in complete markets do not necessarily imply that large improvements are feasible.

The proof of existence of an equilibrium is entirely constructive and allows us to simulate paths as well as to compute numerical statistics for sample economies. The particular algorithm we devised satisfy monotonicity properties that are used to compare equilibria when parameter values are changed.

The paper is organized as follows: in section 2 we present the formal model and discuss the competitive equilibrium with complete markets as well as the equilibrium in the presence of borrowing constraints. As in Scheinkman and Weiss [9] and in Conze, Lasry and Scheinkman [4] the equilibrium under borrowing constraints is shown to be characterized by a martingale property. In section 2 we also state the main propositions that are used to show the existence of an equilibrium. Section 3 discuss the stationary distribution of asset holdings, while section 4 presents simulations of the model. Section 5 discusses some conclusions and the appendix contains the formal proofs.

II. *The Model*

There are two countries of equal size and one good produced in each country. In each country production displays constant returns and involves only the use of labor input. The amount produced in country a ($a=1, 2$) by one unit of labor at time t is given by a random variable θ_t^a that may assume any of a finite number of values α_j^a, $j=1, \dots, J_a$. More precisely we postulate the existence of a probability space $(\Omega, \mathscr{F}, P)$ and of two stochastic processes $\{\theta_t^1\}$, $\{\theta_t^2\}$ defined on this space.

To simplify notation, we set $I=J_1J_2$ amd $s_t=(\theta_t^1, \theta_t^2)$. The process $\{s_t\}$ takes values in

$$\{s_t,\ i=1,\dots,I\} \equiv \{(\alpha_j^1, \alpha_k^2),\ j=1,\dots,\ J_1,\ k=1,\dots,J_2\}.$$

We assume that the transition probability of $\{s_t\}$ is given by [2]

$$P(s_{t+\tau}=s_j | s_t=s_i)=\lambda_{i,j}\tau+o(\tau),\ j\neq i. \tag{1}$$

The consumer's utility function for stochastic streams of consumption and labor is given by

$$U^a=E\left[\int_0^{+\infty} e^{-rt}u^a(c_{1,t}^a,\ c_{2,t}^a,\ l_t^a)dt\right] \tag{2}$$

where $c_{1,t}^a$ (resp. $c_{2,t}^a$) is the consumption at time t by the a-th agent of the good produced in country 1 (resp country 2), and l_t^a is the amount of labor at time t of the a-th agent. The function u^a, $a=1,2$ is assumed to be twice continuously differentiable on $R_{+,*}^3$, strictly increasing in its first two arguments and strictly decreasing in its third argument.

Agents observe the history of the process $\{s_t\} = \{(\theta_t^1, \theta_t^2)\}$ and make their choices con-

[2] In all numerical simulations, we will consider the particular case where $J_1=J_2=2$ and $\{\theta_t^1\}$ and $\{\theta_t^2\}$ are independent, with transition probabilities given by

$$P(\theta_{t+\tau}^a=\alpha_2^a | \theta_t^a=\alpha_1^a)=P(\theta_{t+\tau}^a=\alpha_1^a | \theta_t^a=\alpha_2^a)=\lambda^a\tau+o(\tau),\ a=1,2$$

In other words, in this particular case, productivities in the two countries are independent, can take two values, and their switchwes are governed by independent Poisson counting processes.

ditional on these observations. We write $\mathscr{F}_t$ for the information available at time t.[3]

2.1 The Competitive Equilibrium with Complete Markets

Before proceeding further with the competitive equilibrium under borrowing constraints, we will briefly discuss some properties of the market allocation if there was a complete set of Arrow-Debreu contingent markets. Such markets would allow agents to purchase at time 0 at a price $\pi_{1,t}(\omega)$ (resp. $\pi_{2,t}(\omega)$) the right to delivery of one unit of consumption of the good produced in country 1 (resp. country 2) at time t in state $\omega \in \Omega$.

The problem faced by agents of type a is to maximise (2) subject to the budget constraint

$$E\left[\int_0^{+\infty} (\pi_{1,t} c_{1,t}^a + \pi_{2,t} c_{2,t}^a) dt\right] \leq E\left[\int_0^{+\infty} \pi_{a,t} \theta_t^a l_t^a dt\right].$$

Since the competitive equilibrium is Pareto optimal, we know that its allocation solves

$$\max \{E[U^1] + \gamma E[U^2]\}$$

subject to[4]

$$c_{1,t}^1 + c_{1,t}^2 = \theta_t^1 l_t^1,$$

$$c_{2,t}^1 + c_{2,t}^2 = \theta_t^2 l_t^2.$$

with $\gamma > 0$. Since the U^a's have not been specified, we can assume without loss of generality that $\gamma = 1$. Notice that in (2) the discount rate r is the same for both types of individuals. Therefore the competitive equilibrium allocation $\{(c_{1,t}^1, c_{2,t}^1, c_{1,t}^2, c_{2,t}^2)\}$ actually solves at each instant t

$$\max_{(c_1^1, c_2^1, c_1^2, c_2^2)} \left\{u^1\left(c_1^1, c_2^1, \frac{c_1^1 + c_1^2}{\theta_t^1}\right) + u^2\left(c_1^2, c_2^2, \frac{c_2^1 + c_2^2}{\theta_t^2}\right)\right\}.$$

We wish to show that if labor and consumption are separable in the utility functions, and if we assume that an increase in consumption of one good lowers the one period marginal utility of the other good, an increase in one country's productivity leads to a fall in the other country's output. This in particular shows, that under these assumptions, if complete markets prevails any positive output correlation must be explained by a positive correlation of productivities. We assume that

$$u^a(c_1, c_2, l) = v^a(c_1, c_2) - w^a(l).$$

Let K_1 (resp. K_2) denote output of country 1 (resp. country 2). (K_1, K_2) solves

$$\max_{K_1, K_2} V(\theta^1, \theta^2, K_1, K_2) \tag{3}$$

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[3] i.e. consumer's choices are $\mathscr{F}_t$-measurable where $\mathscr{F} = \{\mathscr{F}_t\}$ is the minimal filtration generated by $\{s_t\}$, $\sigma(s_u, 0 \leq u \leq t)$.

[4] Here and in what follows equalities and inequalities are assumed to hold with probability one.

with

$$V(\theta^1,\theta^2, K_1, K_2)=\max_{c_1^1,c_2^1}\left\{v^1(c_1^1, c_2^1)-w^1\left(\frac{K_1}{\theta^1}\right)+v^2(K_1-c_1^1, K_2-c_2^1)-w^2\left(\frac{K_2}{\theta^2}\right)\right\} \quad (4)$$
$$=-w^1\left(\frac{K_1}{\theta^1}\right)-w^2\left(\frac{K_2}{\theta_2}\right)+\max_{c_1^1,c_2^1}\{v^1(c_1^1, c_2^1)+v^2(K_1-c_1^1, K_2-c_2^1)\}$$
$$=-w^1\left(\frac{K_1}{\theta^1}\right)-w^2\left(\frac{K_2}{\theta^2}\right)+U(K_1, K_2).$$

We also assume that the two goods are substitutable for both types of individuals, that is

$$v_{12}^1=\frac{\partial^2 v^1}{\partial c_1 \partial c_2}\leq 0,\ v=\frac{\partial^2 v^2}{\partial c_1 \partial c_2}\leq 0,$$

and that v^1 and v^2 are strongly concave, w^1 and w^2 are convex. Then in (3) and (4) the maximum is strictly interior, so that the optimum (K_1,K_2) satisfies $\partial V/\partial K_1=0$ and $\partial V/\partial K_2=0$. Differentiating with respect to θ^1, we get

$$\begin{bmatrix} -U_{11}+w^{1\prime\prime}/(\theta^1)^2 & -U_{12} \\ -U_{12} & -U_{22}+w^{2\prime\prime}/(\theta^2)^2 \end{bmatrix}\begin{bmatrix} \partial K_1/\partial\theta^1 \\ \partial K_2/\partial\theta^1 \end{bmatrix}=\begin{bmatrix} w^{1\prime}/(\theta^1)^2 \\ 0 \end{bmatrix} \quad (5)$$

where $U_{ij}=\partial^2 U/(\partial K_i \partial K_j)$.

We want to show that $\partial K_1/\partial\theta^1\geq 0$ and $\partial K_2/\partial\theta^1\leq 0$. From (5) a sufficient condition is that $U_{11}\leq 0$, $U_{22}\leq 0$ and $U_{12}\leq 0$. It is clear that U is strictly concave, so we have the two first inequalities. Now let $(\bar{c}_1^1, \bar{c}_2^1)$ be the pair such that

$$U(K_1,K_2)=v^1(\bar{c}_1^1,\bar{c}_2^1)+v^2(K_1-\bar{c}_1^1 K_2-\bar{c}_2^1).$$

Then

$$\frac{\partial U}{\partial K_1}=\frac{\partial v^1}{\partial c_1}(\bar{c}_1^1, \bar{c}_2^1),$$
$$U_{12}=v_{11}^1\frac{\partial \bar{c}_1^1}{\partial K_2}+v_{12}^1\frac{\partial \bar{c}_2^1}{\partial K_2}.$$

Also for $a=1,2$,

$$\frac{\partial v^1}{\partial c_a}(c_1^1, c_2^1)-\frac{\partial v^2}{\partial c_a}(K_1-\bar{c}_1^1, K_2-\bar{c}_2^1)=0$$

which implies

$$\begin{bmatrix} v_{11}^1+v_{11}^2 & v_{11}^1+v_{11}^2 \\ v_{12}^1+v_{12}^2 & v_{22}^1+v_{22}^2 \end{bmatrix}\begin{bmatrix} \partial\bar{c}_1^1/\partial K_2 \\ \partial\bar{c}_2^1/\partial K_2 \end{bmatrix}=\begin{bmatrix} v_{12}^2 \\ v_{22}^2 \end{bmatrix}$$

It is then easy to show that

$$U_{12}=\frac{v_{12}^1[v_{11}^2 v_{22}^2-(v_{12}^2)^2]+v_{12}^2[v_{11}^1 v_{22}^1-(v_{12}^1)^2]}{\Delta}$$

where $\Delta=(v_{11}^1+v_{11}^2)(v_{22}^1+v_{22}^2)-(v_{12}^1+v_{12}^2)^2$ is strictly positive from the strong concavity of v^1 and v^2. Therefore $U_{12}\leq 0$.

Notice that when $U_{12}=0$, $\partial K_2/\partial\theta^1=0$ and $\partial K_1/\partial\theta^2=0$. In this case the correlation between K_1 and K_2 is zero if the shocks θ^1 and θ^2 are uncorrelated. This limit case corresponds to the case where both v^1 and v^2 are separable in (c_1,c_2).

2.2 The Competitive Equilibrium and Borrowing Constraints

We assume the existence of a fixed stock of "money" whose units are chosen such that the average per-capita holdings equals one half. Agents are assumed to know the initial distribution of "money" holdings.

The typical individuals in country a takes as given the stochastic processes of prices $\{p_{1,t}\}$ and $\{p_{2,t}\}$ for the good produced in country 1 and 2 respectively, and solve problem (P^a) (equations (6) to (9) below) in order to choose, among other things, the amount y_t^a of "money" that he will hold:

$$\max E\left[\int_0^{+\infty} e^{-rt}u^a(c_{1,t}^a,\, c_{2,t}^a,\, l_t^a)dt\right] \tag{6}$$

subject to

$$y_0^a \text{ given} \tag{7}$$

$$\dot{y}_t^a=\theta_t^a p_{a,t}l_t^a-p_{1,t}c_{1,t}^a-p_{2,t}c_{2,t}^a \tag{8}$$

$$y_t^a\geq 0,\ l_t^a\geq 0,\ c_{1,t}^a\geq 0,\ c_{2,t}^a\geq 0. \tag{9}$$

Notice that (9) implies that no borrowing is allowed. An equilibrium is a pair of stochastic processes $\{(p_{1,t},\, p_{2,t})\}$ such that if $\{(y_t^a,\, c_{1,t}^a,\, c_{2,t}^a,\, l_t^a)\}$ solves (P^a), then for all $t\geq 0$,

$$y_t^1+y_t^2=1, \tag{10}$$

$$c_{1,t}^1+c_{1,t}^2=\theta_t^1 l_t^1, \tag{11}$$

$$c_{2,t}^1+c_{2,t}^2=\theta_t^2 l_t^2. \tag{12}$$

We will also assume that the marginal utilities of consumption at zero consumption are infinite. This guarantees that $c_{1,t}^a>0$, $c_{2,t}^a>0$ and $l_{1,t}^a>0$ for all t and $a=1,2$.

Let $\{z_t\}$ be the stochastic process representing the average amount of "money" held by agents in the first country.

As in Scheinkman and Weiss [9] and Conze, Lasry and Scheinkman [4] we will first motivate heuristically a candidate equilibrium. We should then prove that our candidate is in fact an equilibrium. The proof here would be completely similar to the proof of this result for the models in Scheinkman and Weiss [9] and Conze, Lasry and Shceinkman [4], to which the reader is refered.

Any competitive equilibrium at each instant t, conditional on the amount of "money" received by the typical individual of country one from the typical individual of country two, is Pareto optimal. Hence the competitive allocation $\{(c_{1,t}^1,\, c_{2,t}^1,\, c_{1,t}^2,\, c_{2,t}^2)\}$ must solve at each instant t

$$\max_{(c_1^1, c_2^1, c_1^2, c_2^2)} \left\{ u^1\left(c_1^1, c_2^1, \frac{c_1^1+c_1^2}{\theta_t^1}\right) + \gamma_t u^2\left(c_1^2, c_2^2, \frac{c_2^1+c_2^2}{\theta_t^2}\right) \right\}.$$

Further if

$$q_t^1 = \frac{1}{p_{1,t}} \frac{\partial u^1}{\partial c_1}\left(c_{1,t}^1, c_{2,t}^1, \frac{c_{1,t}^1+c_{1,t}^2}{\theta_t^1}\right) \tag{13}$$

$$q_t^2 = \frac{1}{p_{2,t}} \frac{\partial u^2}{\partial c_2}\left(c_{1,t}^2, c_{2,t}^2, \frac{c_{2,t}^1+c_{2,t}^2}{\theta_t^2}\right) \tag{14}$$

that is $\{q_t^a\}$ is the stochastic process describing the marginal utilities of money of individual a, then

$$\gamma_t = \frac{q_t^1}{q_t^2} \tag{15}$$

We will assume that u^1 and u^2 are such that the function

$$V(c_1^1, c_2^1, c_1^2, c_2^2) = u^1\left(c_1^1, c_2^1, \frac{c_1^1+c_1^2}{\theta_t^1}\right) + \gamma_t u^2\left(c_1^2, c_2^2, \frac{c_2^1+c_2^2}{\theta_t^2}\right)$$

is strongly concave, i.e. D^2V is negative definite. Since by assumption its maximum is interior it follows from the implicit functions theorem that

$$c_{1,t}^1 = c_1^1(\theta_t^1, \theta_t^2, \gamma_t), \tag{16}$$

$$c_{2,t}^1 = c_2^1(\theta_t^1, \theta_t^2, \gamma_t), \tag{17}$$

$$c_{1,t}^2 = c_1^2(\theta_t^1, \theta_t^2, \gamma_t), \tag{18}$$

$$c_{2,t}^2 = c_2^2(\theta_t^1, \theta_t^2, \gamma_t), \tag{19}$$

and that all these functions are, at least, continuously differentiable.

We will look for an equilibrium in which (almost all) sample paths of $\{z_t\}$ are absolutely continuous functions of t, that is $z_t = \int_0^t \dot{z}_u du$ for some process $\{\dot{z}_t\}$. Further, we only consider equilibria in which each individual of a given type starts with the same amount of "money." With strict concavity, this implies that y_t^1 equals z_t and y_t^2 equals $1-z_t$. Since

$$\dot{z}_t = p_{1,t} c_{1,t}^2 - p_{2,t} c_{2,t}^1,$$

we may write, using equations (13) to (15) as well as equations (16) to (19),

$$\begin{aligned} \dot{z}_t &= f(s_t, q_t^1, q_t^2) \\ &= \frac{1}{q_t^1 \theta_t^1} g^1\left(\theta_t^1, \theta_t^2, \frac{q_t^2}{q_t^1}\right) - \frac{1}{q_t^2 \theta_t^2} g^2\left(\theta_t^1, \theta_t^2, \frac{q_t^1}{q_t^2}\right) \end{aligned} \tag{20}$$

where

$$g^1(\theta^1, \theta^2, x)=\theta^1 \frac{\partial u^1}{\partial c_1}\left(c_1^1\left(\theta^1, \theta^2, \frac{1}{x}\right), c_2^1\left(\theta^1, \theta^2, \frac{1}{x}\right), \frac{c_1^1(\theta^1, \theta^2, 1/x)+c_1^2(\theta^1, \theta^2, 1/x)}{\theta^1}\right)$$
$$\times c_1^2\left(\theta^1, \theta^2, \frac{1}{x}\right),$$
$$g^2(\theta^1, \theta^2, x)=\theta^2 \frac{\partial u^2}{\partial c_2}\left(c_1^2(\theta^1, \theta^2, x), c_2^2(\theta^1, \theta^2, x), \frac{c_2^1(\theta^1, \theta^2, x)+c_2^2(\theta^1, \theta^2, x)}{\theta^2}\right)$$
$$\times c_2^1(\theta^1, \theta^2, x).$$

Notice that from (20), f is homogeneous of degree-1 in (q^1, q^2).

We will look for an equilibrium in which $q_t^1=q^1(s_t, z_t)$ and $q_t^2=q^2(s_t, z_t)$. In this case, from (20) we can infer that the process $\{(s_t, z_t)\}$ is Markovian. We will now proceed to further characterize this equilibrium.

As money yields no direct utility it is natural to guess that, in equilibrium, the expected discounted marginal utility of money at $t+dt$, conditional on the information available at time t, equals the marginal utility of money at time t, i.e. the processes $\{e^{-rt}q_t^a\}$, $a=1,2$ are $\mathscr{F}$-martingales. From the Markov property of $\{(s_t, z_t)\}$ and (1) we get

$$\begin{aligned} & E[e^{-r(t+\tau)}q_{t+\tau}^a|\mathscr{F}_t] \\ & =E[e^{-r(t+\tau)}q_{t+\tau}^a|s_t, z_t] \\ & =e^{-rt}q^a(s_i, z_t)+e^{-rt}\left\{\frac{dq^a}{dz}(s_i, z_t)f(s_i, q^1(s_i, z_t), q^2(s_i, z_t))\right\}\tau \\ & \quad +e^{-rt}\left\{\sum_{j\neq i} \lambda_{i,j}[q^a(s_j, z_t)-q^a(s_i, z_t)]-rq^a(s_i, z_t)\right\}\tau+o(\tau) \end{aligned}$$

with $s_i=s_t$. When $\tau \to 0$, the martingale condition implies that

$$\begin{aligned} & \frac{dq^a}{dz}(s_i, z_t)f(s_t, q^1(s_i, z_t), q^2(s_i, z_t)) \\ & \quad +\sum_{j\neq i} \lambda_{i,j}[q^a(s_j, z_t)-q^a(s_i, z_t)]-rq^a(s_i, z_t)=0. \end{aligned}$$

Also the no-borrowing condition implies that $\dot{z}_t|_{z_t=0}\geq 0$ and $\dot{z}_t|_{z_t=1}\leq 0$, that is for all $i\in\{1,\ldots,I\}$,

$$f(s_i, q^1(s_i, 0), q^2(s_i, 0))\geq 0$$
$$f(s_i, q^1(s_i, 1), q^2(s_i, 1))\leq 0.$$

For simplicity of notations, we set $v_i=r+\sum_{j\neq i}\lambda_{i,j}$, $q_i^1(z)=q^1(s_i, z)$ and $q_i^2(z)=q^2(s_i, z)$. Writing the previous results together, we obtain system (S) (equations (21) to (24)) on [0,1]: for all $i\{\in 1,\ldots,I\}$,

$$\frac{dq_i^1}{dz}(z)f(s_i, q_i^1(z), q_i^2(z))+\sum_{j\neq i} \lambda_{i,j}q_j^1(z)-v_i q_i^1(z)=0 \tag{21}$$

$$\frac{dq_i^2}{dz}(z)f(s_i, q_i^1(z), q_i^2(z))+\sum_{j\neq i} \lambda_{i,j}q_j^2(z)-v_i q_i^2(z)=0 \tag{22}$$

$$f(s,q_i^1(0),q_i^2(0)) \geq 0 \tag{23}$$

$$f(s,q_i^1(1),q_i^2(1)) \leq 0 \tag{24}$$

where $f(s,q^1,q^2)$ is C^1 and homogeneous of degree-1 in $(q^1,q^2) \in R^2_{+,*}$. For simplicity of notations, we denote by q the vector of the $2I$ functions $(q_i^1,q_i^2)_{i \in \{1,\dots,I\}}$.

Notice first that system (S) always has the trivial solution $q_i^a=0$, $i=1,\dots,I$, $a=1,2$. This corresponds to the case where money has no value. In order to guarantee the existence of a non-trivial solution we need assumptions 1 to 3 below. As we explain below assumptions 1 and 2 are satisfied in the separable case with constant marginal utility of leisure whenever relative risk aversion is not too big.

Theorem 1 that follows states the existence of at least one non-trivial solution to system (S) under certain conditions. Before going further, we make the following assumptions:

Assumption 1 |

- $\forall s \in \{s_i,\ i=1,\dots,I\}$, $\forall q^2 \in \mathbb{R}_{+,*}$, *the function* $q^1 \in \mathbb{R}_{+,*} \mapsto f(s,\ q^1,\ q^2)$ *is strictly increasing.*
- $\forall s \in \{s_i,\ i=1,\dots,I\}$, $\forall q^1 \in \mathbb{R}_{+,*}$, *the function* $q^2 \in \mathbb{R}_{+,*} \mapsto f(s,\ q^1,\ q^2)$ *is strictly decreasing.*

Since we will look to solutions q such that q_i^1 is decreasing and q_i^2 is increasing, assumption 1 implies that $\dot{z}$ will be decreasing as a function of z.

Assumption 2 |

- $\forall i \in \{1,\dots,I\}$, $f(s_i,\ x,\ 1) > 0$ for $x \in \mathbb{R}_{+,*}$, *big enough.*
- $\forall i \in \{1,\dots,I\}$, $f(s_i,\ x,\ 1) < 0$ for $x \in \mathbb{R}_{+,*}$ *small enough.*

Assumption 2 is necessary in order for equations (23) and (24) to be satisfied.

We can get an idea of the restrictions imposed by assumptions 1 and 2 by examining the separable case, with constant marginal utility of leisure and same utility function for both groups. Without loss of generality, we can take the marginal utility of leisure equal to 1, and the utility functions is

$$u(c_1,c_2,l)=v(c_1)+w(c_2)-l.$$

In this case, we get

$$f(\theta^1,\ \theta^2,\ q^1,\ q^2)=\frac{1}{q^1\theta^1}g^1\left(\frac{q^2}{q^1\theta^1}\right)-\frac{1}{q^2\theta^2}g^2\left(\frac{q^1}{q^2\theta^2}\right)$$

with $g^1=(v')^{-1}$ and $g^2=(w')^{-1}$. Since marginal utilities of consumption at zero consumption are infinite, it is clear that assumption 2 is satisfied. Now since g^1 and g^2 are strictly decreasing, assumption 1 is satisfied if and only if $x \mapsto xg^1(x)$ and $x \mapsto xg^2(x)$ are strictly decreasing functions, that is if and only if

$$\forall c>0, \quad c\frac{v''(c)}{v'(c)}>-1 \quad \text{and} \quad c\frac{w''(c)}{w'(c)}>-1. \tag{25}$$

Equation (25) says that the relative risk aversion is less than one.

From assumptions 1 and 2 and the fact that $f(s,q^1,q^2)$ is homogeneous in (q^1,q^2), there exist for every $s\in\{s_i,i=1,\ldots,I\}$ a unique $h(s)>0$ such that $f(s,q^1,q^2)=0$ is equivalent to $q^2=q^1h(s)$. Moreover, $f(s,q^1,q^2)>0$ if and only if $q^2<q^1h(s)$ and $f(s,q^1,q^2)<0$ if and only if $q^2>q^1h(s)$.

To every application $K:\{1,\ldots,I\}\to\{1,2\}$ we associate the real matrix $I\times I$ defined by $M(K)=(m_{i,j})$ where $m_{i,i}=0$, and if $j\neq i$,

$$m_{i,j}=\frac{\lambda_{i,j}}{\nu_i}\mathbf{1}_{K(i)=K(j)}+\frac{\lambda_{i,j}}{\nu_i}\frac{1}{h(s_j)}\mathbf{1}_{K(i)=1,K(j)=2}+\frac{\lambda_{i,j}}{\nu_i}h(s_j)\mathbf{1}_{K(i)=2,K(j)=1}$$

For every K, we denote by $\rho(K)$ the greatest positive eigenvalue of $M(K)$, which is well defined by the Perron-Frobenius theorem (see for instance Nikaido [7]). Also we set $\rho=\sup_K \rho(K)$.

THEOREM 1 *System* (S) *has at least one solution q satisfying*

$$\forall i\in\{1,\ldots,I\},\ q_i^1\in C^0([0,1])\cap C^1([0,1])\setminus\{z_i^*\}), \tag{26}$$

$$\forall i\in\{1,\ldots,I\},\ q_i^2\in C^0([0,1])\cap C^1([0,1])\setminus\{z_i^*\}), \tag{27}$$

$$\forall i\in\{1,\ldots,I\},\ q_i^1 \textit{ is strictly positive and strictly decreasing,} \tag{28}$$

$$\forall i\in\{1,\ldots,I\},\ q_i^2 \textit{ is strictly positive and strictly increasing,} \tag{29}$$

if and only if $\rho>1$, *where* z_i^* *is the only point in* [0,1] *such that*

$$z_i^*=0 \quad \textit{if} \quad f(s_i,q_i^1(0),q_i^2(0))<0,$$

$$z_i^*=1 \quad \textit{if} \quad f(s_i,q_i^1(1),q_i^2(1))>0,$$

$$f(s_i,q_i^1(z_i^*),q_i^2(z_i^*))=0 \textit{ otherwise.}$$

Notice that in theorem 1, uniqueness of z_i^* follows from the fact that $z\mapsto f(s_i,q_i^1(z),q_i^2(z))$ is strictly decreasing. The proof of the theorem is in appendix. It is very similar to the proof of the main result in Conze, Lasry and Scheinkman [4]. We will refer to this paper as much as possible.

In section 2.3 below we state the main propositions needed to establish theorem 1. The proofs are in appendix.

The condition $\rho>1$ does not hold in all cases even under assumptions 1 and 2. Intuitively if agents discount the future heavily or if their productivities does not vary enough across states then money commands a zero price. The fact that productivities vary enough can be stated as:

ASSUMPTION 3 *There exist i and j in* $\{1,\ldots,I\}$ *such that* $h(s_i)>1$ *and* $h(s_j)<1$.

In the separable case mentioned above, when $w=v$, it is easy to check that $h(\theta,\theta)=1$ and that $\partial h/\partial\theta^1>0$. Hence assumption 3 is verified provided there are two states with relative productivities respectively greater and less than 1.

THEOREM 2 *Assume* 1 *to* 3. *Then there exist* $\bar{r}>0$ *such that* $\rho>1$ *if and only if* $r<\bar{r}$.

The proof of theorem 2 is given in appendix.

2.3 Existence of an Equilibrium

The proof of existence of a solution to system (S) when $\rho>1$ is entirely constructive. This allows us to establish results concerning the comparison of solutions as parameters change as well as to obtain numerical simulations of the model that can be used to compute the correlations among the different equilibrium prices and quantities.

The argument consists in transforming the problem of solving system (S) into a fixed point problem. In order to do this let E (resp. F) be the space of strictly positive and strictly decreasing (resp. increasing) functions which are defined in [0,1] and continuous. Let $s\in]0,1[\times]0,1[$ and $\nu>0$ be constants. Let $(\tilde{u},\tilde{v})\in E\times F$. We define the swtich point associated to $(s,\tilde{u},\tilde{v})$ by

$$z^*=0 \quad \text{if} \quad f(s,\tilde{u}(0),\tilde{v}(0))\leq 0,$$

$$z^*=1 \quad \text{if} \quad f(s,\tilde{u}(1),\tilde{v}(1))\geq 0,$$

$$f(s,\tilde{u}(z^*),\tilde{v}(z^*))=0 \text{ otherwise.}$$

We introduce the following system, which we call reduced system (equation (30) to (34) below):

$$\frac{du}{dz}(z)f(s,u(z),v(z))-\nu u(z)+\tilde{u}(z)=0, \tag{30}$$

$$\frac{dv}{dz}(z)f(s,u(z),v(z))-\nu v(z)+\tilde{v}(z)=0, \tag{31}$$

$$u(z^*)=\frac{\tilde{u}(z^*)}{\nu} \quad \text{if} \quad z^*>0, \tag{32}$$

$$v(z^*)=\frac{\tilde{v}(z^*)}{\nu} \quad \text{if} \quad z^*<1 \tag{33}$$

$$f(s,u(z^*),v(z^*))=0. \tag{34}$$

Notice that if $0<z^*<1$, then $v(z^*)/u(z^*)=\tilde{v}(z^*)/\tilde{u}(z^*)=h(s)$ and (34) actually follows from (32) and (33).

Given functions $(q_i^1,q_i^2)\in E\times F$ for each $i=1,\dots,I$ proposition 1 below shows existence and unicity of $(q_i^1,q_i^2)\in E\times F$ the solution to system (30) to (34) with $s=s_i$, $\nu=\nu_i$ and

$$\tilde{u}(z)=\sum_{j\neq i}\lambda_{i,j}\bar{q}_j^1(z),$$

$$\tilde{v}(z)=\sum_{j\neq i}\lambda_{i,j}\bar{q}_j^2(z).$$

We may think of this as defining a map Φ on $(E\times F)^I$. Obviously a fixed point of Φ is a solution to (S).

The next four propositions establish the existence of at least one fixed point for Φ and a constructive method to compute the fixed point. Proposition 1 shows that in fact Φ maps $(E\times F)^I$ into $(E\times F)^I$ and further that Φ is increasing i.e. if $\tilde{q}\geq\tilde{p}$ then $\Phi(\tilde{q})\geq\Phi(\tilde{p})$.[5] Proposition 2 shows that there exists a $\bar{q}$ such that $\Phi(\bar{q})\leq\bar{q}$, i.e. $\bar{q}$ is a supersolution. Proposition 3 establishes the existence of a subsolution, i.e. of $\underline{q}$ such that $\Phi(\underline{q})\geq\underline{q}$. Finally proposition 4 shows that if we let $\bar{q}_n=\Phi(\bar{q}_{n-1})$ and $\bar{q}_0=\bar{q}$ then $\bar{q}_n$ is a decreasing sequence that converges to a $\bar{q}_*$ that satisfies $\Phi(\bar{q}_*)=\bar{q}_*$. Also, if we let $\underline{q}_n=\Phi(\underline{q}_{n-1})$ and $\underline{q}_0=\underline{q}$ then $\underline{q}_n$ is an increasing sequence that converges to a $\underline{q}_*$ that is also a fixed point of Φ. Further $\bar{q}_*$ and $\underline{q}_*$ are solutions of (S) satisfying conditions (26) to (29).

Notice that we have thus obtained two solutions to (S). In all simulations we found that $\bar{q}_*=\underline{q}_*$ but we have no proof that this equality always holds.

The precise results are as follows.

PROPOSITION 1 *System* (30) *to* (34) *has a unique solution* $(u,v)\in E\times F$ *with* u *and* v *in* $C^1([0,1]\backslash\{z^*\})$. *Moreover, let* $(\tilde{u}_1,\tilde{v}_1)\in E\times F$ *and* $(\tilde{u}_2,\tilde{v}_2)\in E\times F$ *with* $\tilde{u}_1\geq\tilde{u}_2$ *and* $\tilde{v}_1\geq\tilde{v}_2$. *Let* (u_1,v_1) *and* (u_2,v_2) *be the corresponding solutions of system* (30) *to* (34). *Then* $u_1\geq u_2$ *and* $v_1\geq v_2$.

PROPOSITION 2 *For all* $(q^1,q^2)\in\mathbb{R}^2_{+,*}$, *let*

$$\tilde{f}(q^1,q^2)=\min\left\{\min_i f(s_i,q^1,q^2),\ -\frac{2}{q^1}+\frac{1}{q^2}\right\}.$$

There exist a function $y\in E\cap C^1([0,1])$ *such that*

$$\frac{dy}{dz}(z)\tilde{f}(y(z),y(0))-ry(z)=0,$$

$$\frac{y(1)}{y(0)}<\min_{i\in\{1,\ldots,I\}}\left\{h(s_i),\ \frac{1}{h(s)}\right\}.$$

Define $\bar{q}=(\bar{q}_i^1,\bar{q}_i^2)_{i\in\{1,\ldots,I\}}$ *by* $\bar{q}_i^1(z)=y(z)$ *and* $\bar{q}_i^2=y(1-z)$ *for all* $i=1,\ldots,I$. *Then* $\bar{q}$ *is a supersolution, that is with* $q=\Phi(\bar{q}),q_i^1(z)\leq q_i^1(z)$ *and* $q_i^2(z)\leq\bar{q}_i^2(z)$ *for all* $z\in[0,1]$ *and* $i\in\{1,\ldots,I\}$.

Proposition 3 *Assume* $\rho>1$. *Let* $K:\{1,\ldots,I\}\to\{1,2\}$ *such that* $\rho(K)=\rho$. *There exist a strictly positive eigenvector* $[a_i]\in\mathbb{R}^I$ *(i.e.* $a_i>0\forall i$*) associated to* $(M(K),\rho)$. *Define* $(a_i^1,a_i^2)_{i\in\{1,\ldots,I\}}\in\mathbb{R}^{2I}_{+,*}$ *by* $a_i^1=a_i$ *if* $K(i)=1$, $a_i^2=a_i$ *if* $K(i)=2$ *and* $a_i^2=a_i^1h(s_i)$. *Then for* ε *and* η *small enough,* $\underline{q}=(\underline{q}_i^1,\underline{q}_i^2)_{i\in\{1,\ldots,I\}}$ *defined by* $\underline{q}_i^1(z)=\varepsilon(a_i^1-\eta z)$ *and* $\underline{q}_i^2(z)=\varepsilon(a_i^1-\eta(1-z))$ *is a subsolution, that is with* $q=\Phi(\underline{q})$, $q_i^1(z)\geq\underline{q}_i^1(z)$ *and* $q_i^2(z)\geq\underline{q}_i^2(z)$ *for all* $z\in[0,1]$ *and* $i\in\{1,\ldots,I\}$.

PROPOSITION 4 *Let the sequence* $\bar{q}_n=(\bar{q}_{i,n}^1,\bar{q}_{i,n}^2)_{i\in\{1,\ldots,I\}}\in(E\times F)^I$ *be defined by* $\bar{q}^0=\bar{q}$ *and* $\bar{q}_n=$

[5] Here, $q\geq p$ means that for each $a\in\{1,2\}$, for each $i\in\{1,\ldots,I\}$, for each $z\in[0,1]$, $q_i^a(z)\geq p_i^a(z)$.

$\Phi(\bar{q}_{n-1})$. *Let the sequence* $\underline{q}_n=(\underline{q}^1_{i,n},\underline{q}^2_{i,n})_{i\in\{1,\ldots,I\}}\in(E\times F)^I$ *be defined by* $\underline{q}_0=\underline{q}$ *and* $\underline{q}_n=\Phi(\underline{q}_{n-1})$. *Then the first sequence is decreasing and converges to* $\bar{q}_*=(\bar{q}^1_{i,*},\bar{q}^2_{*,i})_{i\in\{1,\ldots,I\}}\in(E\times F)^I$, *the second sequence is increasing and converges to* $\underline{q}_*=(\underline{q}^1_{i,*},\underline{q}^2_{i,*})_{i\in\{1,\ldots,I\}}\in(E\times F)^I$. *Moreover the convergence of* $(\bar{q}^1_{i,n},\bar{q}^2_{i,n})$ (resp. $(\underline{q}^1_{i,n},\underline{q}^1_{i,n})$) *to* $(\bar{q}^1_{i,*},\bar{q}^2_{i,*})$ (resp. $(\underline{q}^1_{i,*},\underline{q}^2_{i,*})$) is uniform *on* $[0.1]\backslash\{\bar{z}^*_i\}$ (resp. $[0,1]\backslash\{\underline{z}^*_i\}$) *where* $\bar{z}^*_i$ (resp. $\underline{z}^*_i$) *is the switch point associated to* $(s_i, \bar{q}^1_{i,*}, \bar{q}^2_{i,*})$ (resp. $(s_i,\underline{q}^1_{i,*},\underline{q}^2_{i,*})$), *and* $(\bar{q}^1_{i,*},\bar{q}^2_{i,*})$ (resp. $(\underline{q}^1_{i,*},\underline{q}^2_{i,*})$) *is* C^1 *on* $[0,1]\backslash\{\bar{z}^*_i\}$ *(resp.* $[0,1]\backslash\{\underline{z}^*_i\}$). $\bar{q}$ *and* $\underline{q}$ *are solutions to system* (S) *satisfying conditions* (26) *to* (29).

The method used to obtain the existence of a solution to system (S) yields several results concerning comparison of solutions. These are illustrated by the following result:

PROPOSITION 5 *If* $\bar{r}>\hat{r}\geq r$ *and if* q *is a solution to* (S) *satisfying* (26) *to* (29) *when the discount rate is* r, *then there exists a solution* $\hat{q}$ *to* (S) *satisfying* (26) *to* (29) *when the discount rate is* $\hat{r}$ *such that* $\hat{q}^a_i(z)\leq q^a_i(z)$, $a\in\{1,2\}$, $i\in\{1,\ldots,I\}$, $z\in[0,1]$.

Proof: *Let* Φ_r *denote the application* Φ *defined above when the dependence to* r *is made explicit. Since* q *solves system* (S), *it is striaghtforward to check that* (q^1_i,q^2_i) *solves the reduced system* (30) *to* (34) *with* $\nu=\hat{q}+\sum_j \lambda_{i,j}$, *and*

$$\tilde{u}(z)=(\hat{r}-r)q^1_i(z)+\sum_j \lambda_{i,j}q^1_j(z),\quad \tilde{v}(z)=(\hat{r}-r)q^2_i(z)+\sum_j \lambda_{i,j}q^2_j(z)$$

Furthermore, $\tilde{u}\geq\sum_j \lambda_{i,j}q^1_j$ and $\tilde{v}\geq\sum_j \lambda_{i,j}q^2_j$, so that by proposition 1 $\Phi_{\hat{q}}(q)\leq q$, i.e. q is a supersolution to the fixed point problem associate with $\Phi_{\hat{q}}$. Since $\bar{r}>\hat{q}$, theorem 2, proposition 3 and proposition 4 imply the existence of a solution to (S) when the discount rate is $\hat{r}$. From proposition 4 this solution $\hat{q}$ satisfies $\hat{q}\leq q$. Hence the result. □

III. *The Stationary Distribution of Asset Holding*

Starting from an initial distribution of asset holdings and state of labor productivity at time t_0, the distribution of asset holdings and labor productivity at time t_1 is random, as it depends upon the realization of the random path of labor productivity.

A characteristic of the model is that the process $\{(z_t,s_t)\}$ has a strong ergodic property, that is there exist a unique probability measure π on $[0,1]\times\{s_1,\ldots,s_I\}$ (together with the Borel sigma-algebra) such that for all bounded function f on $[0,1]\times\{s_1,\ldots,s_I\}$,

$$\lim_{t\to+\infty} E[f(z_t, s_t)|z_0, s_0]=\int f(z, s)d\pi(z, s) \tag{35}$$

independently of the initial distribution of (z_0,s_0). This property is proved in Conze [3], theorem 1. In particular, (35) implies the mean-ergodic property

$$\lim_{T\to+\infty} E\left[\frac{1}{T}\int_0^T f(z_t, s_t)dt\,|\,z_0, s_0\right]=\int f(z, s)d\pi(z, s),$$

which enables us to use space averages to compute time averages. The ergodic distrunition π is of course also invariant for the process $\{(z_t,s_t)\}$, that is for all $t\geq 0$,

$$\int E[f(z_t, s_t)|z_0=z, s_0=s]d\pi(z, s)=\int f(z, s)d\pi(z, s).$$

Further, π can be characterized by a set of equations. Let $p_i=\pi([0,1]\times\{s_i\})$ and $F_i(z)=\pi([0,z]\times\{s_i\})$. Conditional on $s_{t+dt}=s_i$, the state s_t takes value s_i with probability $1-\lambda_i dt$ and value $s_j, j\neq i$, with probability $\lambda_{j,i}dt$. Hence,

$$p_i=p_i(1-\lambda_i dt)+\sum_{j\neq i} p_j\lambda_{j,i}dt,$$

that is

$$\lambda_i p_i-\sum_{j\neq i}\lambda_{j,i}p_j=0. \tag{36}$$

Now if $s_t=j$, then $z_t=z_{t+dt}-f(s_j,q_j^1(z_t),q_j^2(z_t))$, and a first order expansion leads to

$$F_i(z)=\left[F_i(z)-\frac{dF_i}{dz}(z)f(s_i,q_j^1(z),q_i^2(z))dt\right](1-\lambda_i dt)+\sum_{j\neq i}F_j(z)\lambda_{j,i}dt.$$

Hence

$$\frac{dF_i}{dz}(z)f(s_i, q_j^1(z), q_i^2(z))+\lambda_i F_i(z)-\sum_{j\neq i}\lambda_{j,i}F_j(z)=0 \tag{37}$$

Moreover

$$F_i(0)=0 \text{ if } z_i^*>0 \tag{38}$$

$$F_i(1)=p_i \tag{39}$$

Here the derivation of system (36) to (39) was heuristic. Nevertheless, it is shown in Conze [3] theorem 2 that there is a unique solution to this system satisfying $p_i>0$ for all $i\in\{1,\ldots,I\}$, $\sum_i p_i=1$, F_i positive, increasing and continuous on $[0,1]\backslash\{z_i^*\}$, and that $(p_i,F_i)_{i\in\{1,\ldots,I\}}$ corresponds to the invariant probability measure π. The discontinuity of F_i at z_i^* means that the invariant distribution associates a strictly positive mass with the event $s_t=s_i$ and $z_t=z_i^*$. The explanation is that conditional on the productivity state being s_i, the point z_i^* plays the role of an attractor for the dynamics of $\{z_t\}$.

IV. *Simulations of the Model*

In this section we present the results of the numerical simulations of the model that are used to compute the correlation among the different equilibrium prices and quantities.

All the results exhibited here were computed in the following manner: first we apply the fixed point algorithm that is used to prove the existence of a solution to system (S) to calculate numerically the equilibrium marginal utility functions of the asset $q_i^a(z)$. Once we have obtained the functions $q_i^a(z)$, we are able to compute all the relevant equilibrium quantities or prices as a function of the productivity vector s and the average amount of the asset held by agents in the first country z. In order to compute the relevant correlations, it now suffices to compute the ergodic distribution of the pair (s_t, z_t). This is accomplished by using a fixed point algorithm, as described in the proof of theorem 2 in Conze [3].

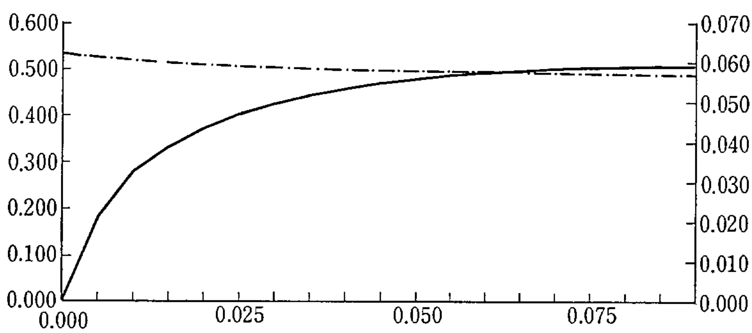
The simulations are for the utility function $u(c_1,c_2,l)=(c_1^\delta+c_2^\delta)/2+l$, two states of productivity for each "country" and independence of productivity changes across countries.

In the complete markets case, the output correlation is, as we proved above, zero. At zero discount rates we have essentially complete markets (cf. Bewley [2]) and hence output correlation is also zero. As the discount rate increases output correlation also increases but is a concave function of the discount rate (cf. Figure 1). Though we do not have a way of changing the stringency of the borrowing constraint it seems intuitive that the effect of increasing the severity of the borrowing constraint should be similar to that of increasing the discount rate. Hence it is reasonable to conjecture that a relatively mild borrowing constraint would lead to a large level of output correlation. In Figure 1 we also plot the effect of a change in the discount rate on average utility. A higher discount rate leads individuals to be less willing to work today in exchange for money to be spent in lean times and this leads to a fall in the average utility.

Figure 2 shows that the higher the relative risk-aversion coefficient $(1-\delta)$ the lower the output correlation obtained. The tendency towards risk-neutrality lowers, in equilibrium, the value of money and hence increases the correlation of output.

Figure 3 plots the effect of risk-aversion on the correlation of (the value of) consumption across countries. It is clear that in a complete markets equilibrium the correlation of consumption across countries is unity. Here lower risk aversion may lead to a higher

FIGURE 1. OUTPUT CORRELATIONS AND AVERAGE UTILITY VERSUS DISCOUNT RATE



—correlation between productions—average utility

$J_1=J_2=2$

$\alpha_1^1=\alpha_1^2=0.1$

$\alpha_2^1=\alpha_2^2=0.9$

$P(\theta_{t+\tau}^1=2|\theta_t^1=1)=0.2\tau+o(\tau)$

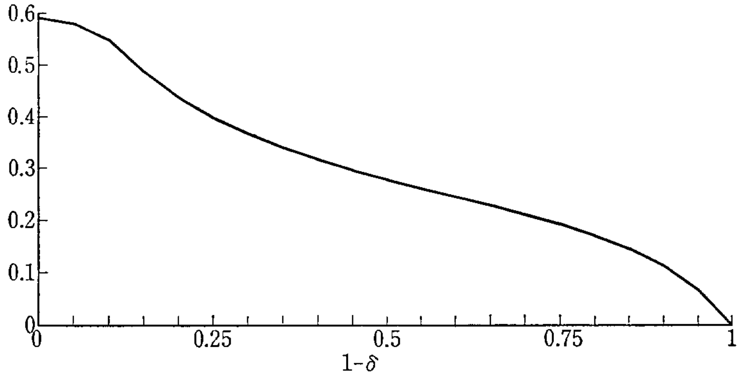
$P(\theta_{t+\tau}^1=1|\theta_t^1=2)=0.2\tau+o(\tau)$

$P(\theta_{t+\tau}^2=2|\theta_t^2=1)=0.2\tau+o(\tau)$

$P(\theta_{t+\tau}^2=1|\theta_t^2=2)=0.2\tau+o(\tau)$

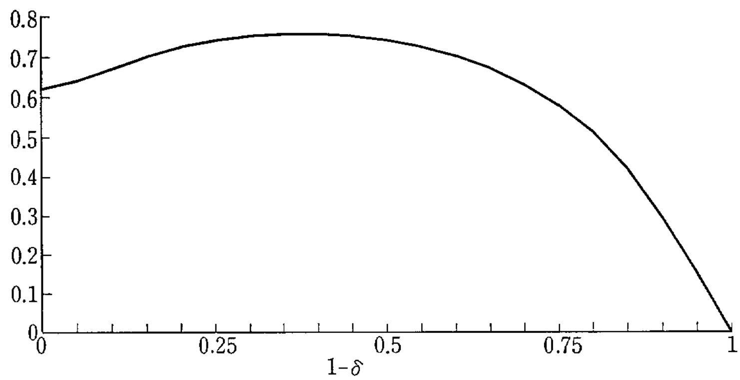
$\delta=0.5$

FIGURE 2. OUTPUT CORRELATIONS VERSUS RELATIVE RISK-AVERSION



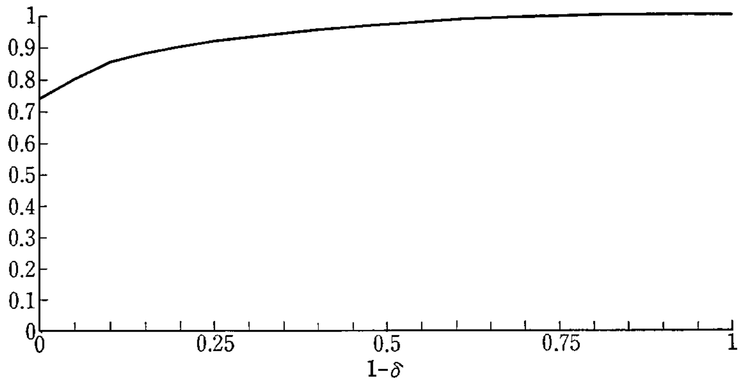
$J_1=J_2=2$
$\alpha_1^1=\alpha_1^2=0.1$
$\alpha_2^1=\alpha_2^2=0.9$
$P(\theta_{t+\tau}^1=2|\theta_t^1=1)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^1=1|\theta_t^1=2)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^2=2|\theta_t^2=1)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^2=1|\theta_t^2=2)=0.2\tau+o(\tau)$
$r=0.01$

FIGURE 3. CORRELATION OF CONSUMPTION ACROSS COUNTRIES VERSUS RELATIVE RISK-AVERSION



$J_1=J_2=2$
$\alpha_1^1=\alpha_1^2=0.1$
$\alpha_2^1=\alpha_2^2=0.9$
$P(\theta_{t+\tau}^1=2|\theta_t^1=1)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^1=1|\theta_t^1=2)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^2=2|\theta_t^2=1)=0.2\tau+o(\tau)$
$P(\theta_{t+\tau}^2=1|\theta_t^2=2)=0.2\tau+o(\tau)$
$r=0.01$

FIGURE 4. RATIO OF UTILITY UNDER BORROWING CONSTRAINTS TO THE UTILITY IN A COMPLETE MARKET VERSUS RELATIVE-RISK AVERSION



$$J_1 = J_2 = 2$$
$$\alpha_1^1 = \alpha_1^2 = 0.1$$
$$\alpha_2^1 = \alpha_2^2 = 0.9$$
$$P(\theta_{t+\tau}^1 = 2 \mid \theta_t^1 = 1) = 0.2\tau + o(\tau)$$
$$P(\theta_{t+\tau}^1 = 1 \mid \theta_t^1 = 2) = 0.2\tau + o(\tau)$$
$$P(\theta_{t+\tau}^2 = 2 \mid \theta_t^2 = 1) = 0.2\tau + o(\tau)$$
$$P(\theta_{t+\tau}^2 = 1 \mid \theta_t^2 = 2) = 0.2\tau + o(\tau)$$
$$r = 0.01$$

or lower correlation of consumption across countries. This results from a higher correlation in output combined with less insurance across countries in equilibrium, the lower the risk aversion. This last point is illustrated in Figure 4 where we plot the ratio of utility of type 1 in equilibrium to the utility of the same type in a complete markets equilibrium, as a function of the relative risk-aversion parameter. The results of the simulations used to derive Figure 4 also point out to the fact that, at least for a range of parameters values, large output correlations can be associated with very small losses in utility. Individuals' optimization as well as market mechanisms seem to avoid much of the utility losses while at the same time causing big changes in certain quantities.

Simulations also show that in the model "exports" are positively correlated with output and negatively correlated with the relative price of the exported good.

V. *Conclusion*

This paper has presented a model where failure of perfect risk-sharing across countries can be used to explain a positive correlation of output series in the presence of independent productivity shocks.[6] In the model, the distribution of financial assets across countries

[6] The fact that no migration is allowed also plays an important role in generating the comovements. In fact, in our model, the only risks that need to be shared are the shocks to the productivity of labor in the different countries and if migration was costless wages in each country would equalize. Murphy Shleifer and Vishny [6] used, in a static model, immobile labor to generate an increase in output in one sector in response to an increase in productivity in another sector.

evolves endogenously and in turn affects the distribution of outputs even after controlling for the productivity shocks.

In order to focus on the effect of the borrowing constraints we considered the case where the productivity shocks were independent and the utility functions in both countries were identical and separable. In this case we showed that the model was capable of generating substantial positive cross-country correlation of output and a lower consumption correlation than that implied under complete markets. Simulations also revealed that the model generates a negative correlation between the value of output and the relative price of exportables in terms of importables. This should not be surprising since most of the output changes in one country would be the result of a change in productivity in that same country. As mentioned in footnote 1 in the introduction, the correlation between quarterly changes in US GNP between 1948 and 1987 and changes in the logarithm of the price of exports divided by the price of imports[7] is not significantly distinct from zero but if we omit the years 1975 and 1979/1980 when the two "oil shocks" occurred it equals -.24.

Our model cannot accommodate the presence of intermediate goods as oil or any of the monetary aspects that are surely important in determining the transmission of output shocks. Nonetheless we believe it is useful in illustrating how incomplete markets can help explain some of the aspects of this international transmission. Further the mathematical techniques discussed here should be useful in dealing with the missing ingredients.

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VI. *Appendix*

6.1 Proof of proposition 1

Here we prove proposition 1. The proof will first be done in the case $0<z^*<1$ and then extended to the general case by using an approximation procedure.

When $0<z^*<1$, the reduced system (30) to (34) is equivalent to the two following differential systems: a forward problem on $[z^*,1]$

$$\begin{cases} \dfrac{du}{dz}(z)f(s, u(z), v(z))=\nu u(z)-\tilde{u}(z),\ z\in]z^*, l] \\ \dfrac{dv}{dz}(z)f(s, u(z), v(z))=\nu v(z)-\tilde{v}(z),\ z\in]z^*,1] \\ u(z^*)=\dfrac{\tilde{u}(z^*)}{\nu} \\ v(z^*)=\dfrac{\tilde{v}(z^*)}{\nu} \end{cases} \tag{40}$$

[7] The data is from the International Financial Statistics.

and a backward problem on $[0,z^*]$

$$\left\{\begin{array}{l} \frac{du}{dz}(z)f(s, u(z), v(z))=\nu u(z)-\tilde{u}(z),\ z\in[0, z^*[\\ \frac{dv}{dz}(z)f(s, u(z), v(z))=\nu v(z)-\tilde{v}(z),\ z\in[0, z^*[\\ u(z^*)=\frac{\tilde{u}(z^*)}{\nu} \\ v(z^*)=\frac{\tilde{v}(z^*)}{\nu} \end{array}\right. \tag{41}$$

We will only deal with system (40). The resolution of system (41) follows by changing z in $1-z$ in (40).

Since $f(s,u(z^*),v(z^*))=0$, system (40) is degenerate. Let $\varepsilon>0$. We consider the following system on $[z^*,1]$:

$$\left\{\begin{array}{l} \frac{du_\varepsilon}{dz}(z)f_\varepsilon(s, u_\varepsilon(z), v_\varepsilon(z))=\nu u_\varepsilon(z)-\tilde{u}(z) \\ \frac{dv_\varepsilon}{dz}(z)f_\varepsilon(s, u_\varepsilon(z), v_\varepsilon(z))=\nu v_\varepsilon(z)-\tilde{v}(z) \\ u_\varepsilon(z^*)=\frac{\tilde{u}(z^*)}{\nu} \\ v_\varepsilon(z^*)=\frac{\tilde{v}(z^*)}{\nu} \end{array}\right. \tag{42}$$

with $f_\varepsilon(s,u,v)=f(s,u,v)-\varepsilon$. Notice that $f_\varepsilon(s,u_\varepsilon(z^*),v_\varepsilon(z^*))=-\varepsilon<0$.

LEMMA 1 *System* (42) *has a unique solution* $(u_\varepsilon,v_\varepsilon)$ *with* u_ε *(resp.* u_ε*) in* $C^1([z^*,1])$, *strictly positive and strictly decreasing (resp. increasing). Moreover, for all* $z\in[z^*,1]$,

$$\frac{\tilde{u}(z^*)}{\nu}>u_\varepsilon(z)>\frac{\tilde{u}(z)}{\nu}$$

$$\frac{\tilde{v}(z^*)}{\nu}<v_\varepsilon(z)<\frac{\tilde{v}(z)}{\nu}$$

PROOF: system (42) is a standard Cauchy problem. The proof follows the proof of proposition 5 in Conze, Lasry and Scheinkman [4], and is left to the reader. □

LEMMA 2 *The family* (u_ε) *(resp.* (v_ε)*) is increasing (resp. decreasing) with* ε.

PROOF: let $\varepsilon'>\varepsilon>0$. For all real pair $(u,v)\in[\tilde{u}(1)/\nu,\tilde{u}(z^*)/\nu]\times[\tilde{v}(z^*)/\nu,\tilde{v}(1)/\nu]$,

$$f_{\varepsilon'}(s, u, v)<f_\varepsilon(s, u, v)<0.$$

Hence,

$$\frac{du_\varepsilon}{dz}(z)\leq\frac{\nu u_\varepsilon(z)-\tilde{u}(z)}{f_{\varepsilon'}(s,\,u_\varepsilon(z),\,v_\varepsilon(z))},$$

$$\frac{dv_\varepsilon}{dz}(z)\geq\frac{\nu v_\varepsilon(z)-\tilde{v}(z)}{f_{\varepsilon'}(s,\,u_\varepsilon(z),\,v_\varepsilon(z))}.$$

Let $x(z)=(u_\varepsilon(z)-u_{\varepsilon'}(z))_+^2+(v_{\varepsilon'}(z)-v_\varepsilon(z))_+^2$ where $(.)_+=\max(.,0)$. Then

$$\frac{dx}{dz}=2(u_\varepsilon-u_{\varepsilon'})+\left(\frac{du_\varepsilon}{dz}-\frac{du_{\varepsilon'}}{dz}\right)+2(v_{\varepsilon'}-v_\varepsilon)+\left(\frac{dv_{\varepsilon'}}{dz}-\frac{dv_\varepsilon}{dz}\right).$$

Since $(u,v)\mapsto(\nu u-\tilde{u}(z))/f_{\varepsilon'}(s,u,v)$ and $(u,v)\mapsto(\nu v-\tilde{u}(z))/f_{\varepsilon'}(s,u,v)$ are Lipschitz on $[\tilde{u}(1)/\nu,\tilde{u}(z^*)/\nu]\times[\tilde{v}(z^*)/\nu,\tilde{v}(1)/\nu]$ uniformely in $z\in[z^*,1]$, there exist $R>0$ such that

$$\frac{du_\varepsilon}{dz}-\frac{du_{\varepsilon'}}{dz}\leq R(|u_\varepsilon-u_{\varepsilon'}|+|v_{\varepsilon'}-v_\varepsilon|)$$

$$\frac{du_\varepsilon}{dz}-\frac{du_{\varepsilon'}}{dz}\geq R(|u_\varepsilon-u_\varepsilon|_{'}+|v_{\varepsilon'}-v_\varepsilon|)$$

and we obtain $dx/dz\leq 3Rx$. From $x(z^*)=0$ and Gronwall's lemma $x(z)=0$ for all z in $[z^*,1]$. □

Since $u_\varepsilon(z)\geq\tilde{u}(z)/\nu$ and $v_\varepsilon(z)\leq\tilde{v}(z)/\nu$ for all z in $[z^*,1]$, the families (u_ε) and (v_ε) converge pointwise to u and v respectively, satisfying for $z\in[z^*,1]$

$$\frac{\tilde{u}(z^*)}{\nu}>u(z)\geq\frac{\tilde{u}(z)}{\nu},$$

$$\frac{\tilde{v}(z^*)}{\nu}<v(z)\leq\frac{\tilde{v}(z)}{\nu}.$$

LEMMA 3 *Let $\bar{z}>z^*$. Then (u_ε) and (v_ε) are Lipschitz on $[\bar{z},1]$ uniformly in ε.*

PROOF: the proof is quite simple, and is left to the reader. □

Using Ascoli's theorem, lemma 3 and the convergence of (u_ε) and (v_ε) imply their uniform convergence to u and v on $[\bar{z},1]$. It is then straightforward to check that (u,v) is a solution of (40) with u decreasing, v increasing, and both in $C^0([z^*,1])\cap C^1([z^*,1])$. Also the inequalities $u(z)\geq\tilde{u}(z)/\nu$ and $v(z)\leq\tilde{v}(z)/\nu$ are in fact strict for $z>z^*$. Therefore,

$$\frac{\tilde{u}(z^*)}{\nu}>u(z)>\frac{\tilde{u}(z)}{\nu}, \tag{43}$$

$$\frac{\tilde{v}(z^*)}{\nu}<v(z)<\frac{\tilde{v}(z)}{\nu}. \tag{44}$$

The strict monotonicity of u and v then follows. Similarly, we obtain a solution of (41) on $[0,z^*]$ with u strictly decreasing, v strictly increasing, and both in $C^0([0,z^*])\cap C^1([0,z^*])$, and

$$\frac{\tilde{u}(z)}{\nu} > u(z) > \frac{\tilde{u}(z^*)}{\nu}, \tag{45}$$

$$\frac{\tilde{v}(z)}{\nu} < v(z) < \frac{\tilde{v}(z^*)}{\nu}. \tag{46}$$

Hence a solution to the reduced system (30) to (34) with $u \in E \cap C^1([0,1] \setminus \{z^*\})$ and $v \in F \cap C^1([0,1] \setminus \{z^*\})$.

LEMMA 4 *(u,v) is unique.*

PROOF: let $(\bar{u},\bar{v})$ be an other solution. We will prove that $\bar{u}=u$ and $\bar{v}=v$ on $[z^*,1]$. The proof is similar on $[0,z^*]$. We start by proving $\bar{u} \leq u$ and $\bar{v} \geq v$. Let $\varepsilon > 0$. For all $(a,b) \in R^2_{+,*}$, $f_\varepsilon(s,a,b) < f(s,a,b)$. As in lemma 2's proof, we get $\bar{u} \leq u_\varepsilon$ and $\bar{v} \geq v_\varepsilon$. Taking the limit in $\varepsilon, \bar{u} \leq u$ and $\bar{v} \geq v$. This implies $f(s,\bar{u},\bar{v}) \leq f(s,u,v) \leq 0$, and finally

$$\frac{d\bar{u}}{dz} - \frac{du}{dz} \geq \frac{\nu\bar{u} - \nu u}{f(s,u,v)} \geq 0, \quad \forall z \in [z^*,1].$$

Hence, since $\bar{u}(z^*)=u(z^*)$, $\bar{u} \geq u$. Also $\bar{v} \leq v$. □

Hence the first part of proposition 1 in the case $0 < z^* < 1$. We now turn to the second part of proposition 1. Let $(\tilde{u}_1,\tilde{v}_1) \in E \times F$ and $(\tilde{u}_2,\tilde{v}_2) \in E \times F$ with $\tilde{u}_1 \geq \tilde{u}_2$ and $\tilde{v}_1 \geq \tilde{v}_2$. Let z_1^* (resp. z_2^*) be the switch point associated to $(s,\tilde{u}_1,\tilde{v}_1)$ (resp. $(s,\tilde{u}_2,\tilde{v}_2)$, and assume that $0 < z_1^* < 1$ and $0 < z_2^* < 1$. Let (u_1,v_1) and (u_2,v_2) be the solutions of (30) to (34) corresponding to respectively $(\tilde{u}_1,\tilde{v}_1)$ and $(\tilde{u}_2,\tilde{v}_2)$.

LEMMA 5 *For all z in $[0,1]$, $u_1(z) \geq u_2(z)$ and $v_1(z) \geq v_2(z)$.*

PROOF: as in Conze, Lasry and Scheinkman [4], we only have to consider the case $\tilde{v}_1 = \tilde{v}_2 = \tilde{v}$. It is easy to obtain

$$z_1^* \geq z_2^*$$

$$\tilde{u}_1(z_1^*) \geq \tilde{u}_2(z_2^*)$$

$$u_1(z_2^*) \geq u_1(z_1^*) \geq u_2(z_2^*) \geq u_2(z_1^*)$$

$$u_1(z) \geq u_2(z), \ \forall z \in [z_2^*, z_1^*]$$

$$v_1(z) \geq v_2(z), \ \forall z \in [z_2^*, z_1^*].$$

We want to prove $u_1 \geq u_2$ and $v_1 \geq v_2$ on $[0,z_2^*]$ and on $[z_1^*,1]$. The proof will be done for $z \in [z_1^*,1]$. Consider $(u_{1,\varepsilon},v_{1,\varepsilon})$ (resp. $(u_{2,\varepsilon},v_{2,\varepsilon})$) solutions of (42) with $z^* = z_1^*$ and $(\tilde{u},\tilde{v}) = (\tilde{u}_1,\tilde{v}_1)$ (resp. $z^* = z_2^*$ and $(\tilde{u},\tilde{v}) = (\tilde{u}_2,\tilde{v}_2)$). As in Conze, Lasry and Scheinkman [4], one can prove $u_{1,\varepsilon} \geq u_{2,\varepsilon}$ and $v_{1,\varepsilon} \geq v_{2,\varepsilon}$ on $[z_1^*,1]$, for all $\varepsilon > 0$. Taking limit in ε leads to the desired result. □

To obtain proposition 1 in the general case, we use the following approximation procedure. Let

$$e > \max\left\{\frac{\tilde{v}(0)}{h(s)}, \ \tilde{u}(1)h(s)\right\}$$

and set

$$\tilde{u}_{e,n}(z)=\tilde{u}(z)+\max\{e-enz, 0\}$$

$$\tilde{v}_{e,n}(z)=\tilde{v}(z)+\max\{e-en(1-z), 0\}$$

for all $n \in N_*$. It is easy to check that the switch point z_n^* associated to $(\tilde{u}_{e,n},\tilde{v}_{e,n})$ satisfies $0<z_n^*<1$. Let (u_n,v_n) be the solution of (30) to (34) for $\tilde{u}=\tilde{u}_{e,n}$ and $v=\tilde{v}_{e,n}$. Then as in Conze, Lasry and Scheinkman [4] section 5.2, (u_n,v_n) converges to (u,v) satisfying proposition 2.

6.2 Proofs of proposition 2, 3 and 4

To prove proposition 2, one may first check that $\tilde{f}$ is still homogeneous of degree -1, locally Lipchitz and that $\tilde{f}(1,1)<0$, and then proceed as in Conze, Lasry and Scheinkman [4], proposition 2.

We now prove proposition 3. The existence of the vector $[a_i]$ follows from the Peron-Frobenius theorem (see for instance Nikaido [7]). From $[a_i]<M(K)[a_i]$ we get

$$\forall i \in \{1, \ldots, I\},\ a_i^{K(i)} < \frac{1}{\nu_i} \sum_{j\neq i} \lambda_{i,j} a_j^{K(i)}. \tag{47}$$

Let

$$a_{i,\eta}^1(z)=a_i^1-\eta z,$$

$$a_{i,\eta}^2(z)=a_i^2-\eta(1-z),$$

$$\tilde{u}_{i,\eta}(z)=\sum_{j\neq i} \lambda_{i,j} a_{i,\eta}^1(z),$$

$$\tilde{v}_{i,\eta}(z)=\sum_{j\neq i} \lambda_{i,j} a_{i,\eta}^2(z)$$

$$q_{\varepsilon,\eta}=(q_{i,\varepsilon,\eta}^1, q_{i,\varepsilon,\eta}^2)_{i\in\{1,\ldots,I\}}=\Phi(\underline{q}),$$

and $z_{i,\eta}^*$ be the switch point associated to $(s_i,\tilde{u}_{i,\eta},\tilde{v}_{i,\eta})$. Assume for instance $K(i)=1$. We start with the case $z_{i,\eta}^*=1$. From (47), we get for η small enough

$$\frac{1}{\varepsilon} q_{i,\varepsilon,\eta}^1(1)=\frac{1}{\nu_i} \sum_{j\neq i} \lambda_{i,j} a_{j,\eta}^1(1) \geq a_{i,\eta}^1(z)+\delta,\ \ \forall z\in[0,1]$$

with $\delta>0$ independent of ε and η. Since $q_{i,\varepsilon,\eta}^1$ is decreasing,

$$\frac{1}{\varepsilon} q_{i,\varepsilon,\eta}^1(z) \geq a_{i,\eta}^1(z)+\delta,\ \ \forall z\in[0,1]. \tag{48}$$

In particular, $q_{i,\varepsilon,\eta}^1(z)\geq \underline{q}_i^1(z)$. From (48), we get

$$\begin{aligned}\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(1)&=\frac{1}{\varepsilon}q^1_{i,\varepsilon,\eta}(1)h(s_i)\\&\geq a^1_{i,\eta}(1)h(s_i)+\delta h(s_i)\\&\geq a^2_{i,\eta}(1)+\delta'\end{aligned}$$

with $\delta'=\delta h(s_i)>0$. Let

$$\bar{z}_*=\inf\left\{z\in[0,1]/\forall y\in[z,1],\ \frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(y)\geq a^2_{i,\eta}(y)+\frac{\delta'}{2}\right\}.$$

If $\bar{z}_*=0$, then $q^2_{i,\varepsilon,\eta}(z)\geq\varepsilon a^2_{i,\eta}(z)$ for all z in [0,1]. If not, then for $z\in[0,\bar{z}_*]$,

$$\begin{aligned}\frac{1}{\varepsilon}\frac{dq^2_{i,\varepsilon,\eta}}{dz}(z)&=\frac{1}{\varepsilon}\frac{\nu_i q^2_{i,\varepsilon,\eta}(z)-\tilde{v}_{i,\eta}(z)}{f(s_i,q^1_{i,\varepsilon,\eta}(z),q^2_{i,\varepsilon,\eta}(z))}\\&=\varepsilon\frac{\nu_i q^2_{i,\varepsilon,\eta}(z)/\varepsilon-\tilde{v}_{i,\eta}(z)/\varepsilon}{f(s_i,q^1_{i,\varepsilon,\eta}(z)/\varepsilon,q^2_{i,\varepsilon,\eta}(z)/\varepsilon)}.\end{aligned}$$

Also for $z\in[0,\bar{z}_*]$,

$$\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(z)\leq\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(\bar{z}_*)=a^2_{i,\eta}(\bar{z}_*)+\frac{\delta'}{2}\leq a^2_{i,\eta}(1)+\frac{\delta'}{2},$$

hence

$$\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(z)\leq\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(1)+\frac{\delta'}{2}-\delta'\leq\tilde{v}_{i,\eta}(1)-\frac{\delta'}{2}.$$

This implies for $z\in[0,\bar{z}_*]$

$$f(s_i,q^1_{i,\varepsilon,\eta}(z),q^2_{i,\varepsilon,\eta}(z))\geq f\left(s_i,\tilde{u}_{i,\eta}(1),\tilde{v}_{i,\eta}(1)-\frac{\delta'}{2}\right)>Q$$

with $Q>0$ independent of ε and η, and we finally obtain for $z\in[0,\bar{z}_*]$,

$$\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(z)\geq\frac{1}{\varepsilon}q^2_{i,\varepsilon,\eta}(\bar{z}_*)-R\varepsilon=a^2_{i,\eta}(\bar{z}_*)+\frac{\delta'}{2}-R\varepsilon\geq a^2_{i,\eta}(z)+\frac{\delta'}{2}-R\varepsilon$$

with R independent of ε and η. Hence, for ε and η small enough, $q^2_{i,\varepsilon,\eta}(z)\geq \underline{q}^2_i(z)$. If $z^*_{i,\eta}=0$, then

$$\tilde{v}_{i,\eta}(z)=\sum_{j\neq i}\lambda_{i,j}a^2_{j,\eta}(z)\geq\tilde{u}_{i,\eta}(z)h(s_i)=\sum_{j\neq i}\lambda_{i,j}a^1_{j,\eta}(z)h(s_i)$$

and we get from (47) that

$$a^2_{i,\eta}(z)\leq\sum_{j\neq i}\lambda_{i,j}a^2_{j,\eta}(z)+\delta,\quad\forall z\in[0,1]$$

with $\delta>0$ independent of ε and η. We then proceed as in the first case $z^*_{i,\eta}=1$. Finally, if $0<z^*_{i,\eta}<1$, then we proceed as in the first case for $z\leq z^*_{i,\eta}$ and as the second case when $z\geq z^*_{i,\eta}$. □

From the monotonicity of Φ and the existence of a subsolution and a supersolution, proposition 4 is a well known result when Φ maps a compact into itself. But here $(E\times F)^I$ is not compact. Nevertheless, the proof doesn't present any particular difficulty, and is completely similar to the proof of proposition 4 in Conze, Lasry and Scheinkman [4], to which the reader should refer.

6.3 The Case $\rho\leq 1$

PROPOSITION 6 *If* $\rho\leq 1$, *there is no solution to system* (S) *satisfying* (26) *to* (29).

PROOF: assume that $\rho\leq 1$ and that there is a solution q to (S) satisfying (26) to (29). For all i in $\{1,\ldots,I\}$, let $a_i=q^1_i(1/2)$ if $z^*_i\geq 1/2$ and $a_i=q^2_i(1/2)$ otherwise. If $z^*_i\geq 1/2$, then $f(s_i,q^1_i(1/2),q^2_i(1/2)\geq 0$ and from $(dq^1_i/dz)(1/2)\leq 0$ and equation (21), we get

$$\nu_i q^1_i\left(\frac{1}{2}\right)\leq \sum_{j\neq i}\lambda_{i,j}q^1_j\left(\frac{1}{2}\right). \tag{49}$$

Also

$$q^2_i\left(\frac{1}{2}\right)\leq q^1_i\left(\frac{1}{2}\right)h(s_i). \tag{50}$$

If $z^*_i<1/2$, then $f(s_i,q^1_i(1/2),\ q^2_i(1/2)\leq 0$ and from $(dq^2_i/dz)(1/2)\geq 0$ and equation (22) we get

$$\nu_i q^2_i\left(\frac{1}{2}\right)\leq \sum_{j\neq i}\lambda_{i,j}q^2_j\left(\frac{1}{2}\right). \tag{51}$$

Also

$$q^2_i\left(\frac{1}{2}\right)\geq q^1_i\left(\frac{1}{2}\right)h(s_i). \tag{52}$$

Let K: $\{1,\ldots,I\}\rightarrow\{1,2\}$ be defined by $K(i)=1$ if $z^*_i\geq 1/2$ and $K(i)=2$ otherwise. From (49) to (52)

$$[a_i]\leq M(K)[a_i] \tag{53}$$

where inequality between two vectors means inequality between their coordinates. If $\rho<1$, then $M(K)^n$ is a contraction for n big enough (see Nikaido [7]) and (53) implies that for all i, $a_i=0$, i.e. $q^1_i(1/2)=q^2_i(1/2)=0$, which contradicts (28) and (29). If $\rho=1$ and $[a_i]\neq M(K)[a_i]$, there exist $\varepsilon>0$ and $m\in N_*$ such that $(1+\varepsilon)[a_i]\leq M(K)^m[a_i]$. Also $(M(K)^m/(1+\varepsilon))^n$ is a contraction for n big enough (again see Nikaido [7]) and $q^1_i(1/2)=q^2_i(1/2)=0$ for all i. At least, if $\rho=1$ and $[a_i]=M(K)[a_i]$, then from $\nu_i>\sum_{j\neq i}\lambda_{i,j}$ we have that $q^1_i(1/2)=q^2_i(1/2)=0$ for all i. □

6.4 Proof of theorem 2

Here we prove theorem 2. From the definition of the matrices $M(K)$, it is obvious that the $\rho(K)$, and therefore ρ itself, are continuous and decreasing functions of r. By continuity, it is sufficient for theorem 2 to prove that $\rho>1$ when $r=0$.

Let K: $\{1,....I\}\rightarrow\{1,2\}$ and $M(K)=(m_{i,j})$. From the Perron-Frobenius theorem, the existence of $a\in R^I_{+,*}$ such that $M(K)a\geq a$ and $M(K)a\neq a$ is a sufficient condition to have $\rho(K)>1$.

Let a be the vector with all coordinates equal to 1. To get $\rho>1$, it suffices to prove that there exist $K:\{1,...,I\}\rightarrow\{1,2\}$ such that

$$\forall i\in\{1,...,I\},\ \sum_j m_{i,j}\geq 1 \tag{54}$$

with strict equality for at least one $i\in\{1,...,I\}$. Let K be defined by $K(i)=1$ if $h(s_i)>1$ and $K(i)=2$ otherwise. It is easy to check that $m_{i,j}\geq\lambda_{i,j}/\nu_i$ for all $(i,j)\in\{1,...,I\}^2, j\neq i$. If none of these inequalities is strict, then for all $(i,j)\in\{1,...,I\}\times\{1,...,I\}$ either $K_j=K_i$ or $h(s_j)=1$, which implies that $h(s_i)\geq 1$ for all $i\in\{1,...,I\}$ or $h(s_i)\leq 1$ for all $i\in\{1,...,I\}$. This contradicts assumption 3. □

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