2-Factors in Hamiltonian Graphs

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Abstract

We show that every hamiltonian claw-free graph with a vertex x of degree $d(x) \geq 7$ has a 2-factor consisting of exactly two cycles.

1 Introduction

All graphs considered in this paper are simple and undirected. The vertex set of a graph is V, and E is the edge set. For notation not defined here we refer the reader to [1]. The neighborhood of a vertex v is denoted by $N(v)$, the degree of a vertex v is $d(v) = |N(x)|$. If $X \subseteq V$ is a set of vertices, $G[X]$ stands for the subgraph on X induced by G. The complete bipartite graph $K_{1,3}$ is also called the claw, and a graph is said to be claw-free if it does not contain any induced copies of $K_{1,3}$.

In the paper, C will always be a hamiltonian cycle with some orientation. For a vertex $v \in V$, let v^+, v^{++}, v^{3+}, etc. denote the successors of v on C, and let v^-, v^{--}, v^{3-}, etc. denote the predecessors of v. The notation uCv stands for the $u-v$ path given by C and its orientation, uC^-v will be the $u-v$ path following C in reversed direction. Let $U := \{v \in V \mid v^-v^+ \notin E\}$. We will call a 2-factor consisting of exactly two cycles a $2C$-factor.

Hamiltonicity of graphs has been studied widely, and lately a lot of the conditions that imply a graph to be hamiltonian were shown to be sufficient to also guarantee the existence of a wide range of 2-factors. But what can we say when we assume hamiltonicity as one of the properties of the graph? What kind of conditions will yield what kind of 2-factors?

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Consider the following family $\mathcal{G}$ of graphs: Let $G(V, E)$ be a graph. Then G belongs to $\mathcal{G}$ if

1. For some $k \geq 5$, V is the disjoint union of vertex sets $V_1, V_2, V_3, \ldots V_k$ with (let $V_{k+1} = V_1$):
   (a) $|V_i| \geq 1$ for all $1 \leq i \leq k$,
   (b) $|V_i| = 1$ for at least five different indices,
   (c) $|V_i| + |V_{i+1}| \leq 4$ for all $1 \leq i \leq k$.
2. $E = \{uv \mid u, v \in V_i \cup V_{i+1}$ for some $1 \leq i \leq k\}$.

It is easy to observe that every graph in $\mathcal{G}$ is hamiltonian, but no graph in $\mathcal{G}$ contains a $2C$-factor. Further note that $\mathcal{G}$ contains graphs with minimum degree $\delta(G) = 4$, maximum degree $\Delta(G) = 6$ and average degree $\bar{d}(G) > 5 - \epsilon$ for every $\epsilon > 0$. Consider for instance the graph $G \in \mathcal{G}$ with $|V_1| = |V_3| = |V_5| = |V_7| = |V_9| = 1$, $|V_2| = |V_4| = |V_6| = |V_8| = 3$ and $|V_{10}| = |V_{11}| = \ldots = |V_k| = 2$.

No hamiltonian graphs with average degree $\bar{d}(G) \geq 5$ which do not contain a $2C$-factor are known. On the other hand, the best known bound for the minimum degree forcing the existence of a $2C$-factor is the following theorem by Gould and Jacobson.

Theorem 1. *[3] Let G be a hamiltonian graph on $n \geq 8$ vertices with minimum degree $\delta(G) \geq 5n/12$. Then G contains a $2C$-factor.*

There are no nontrivial bounds for the maximum degree in this setting of general graphs, as the graph obtained from joining an $(n-1)$-cycle with a single vertex is hamiltonian with maximum degree $n-1$, but has no $2C$-factor.

But, for the special class of claw-free graphs, we get the following sharp result.

Theorem 2. *Let G be a hamiltonian claw-free graph containing a vertex x with degree $d(x) \geq 7$. Then G has a 2-factor consisting of exactly two cycles.*

2 Proof

We will start with the following lemma.

Lemma 3. *Suppose G is a hamiltonian graph on at least 8 vertices that has no $2C$-factor. If $u, v \in U$ and $uv \in E$, then $|uCv| \leq 4$ or $|vCu| \leq 4$.*

Proof: Let us first suppose that $|uCv| \geq 6$ and $|vCu| \geq 6$ (see Figure 1). Since G is claw-free and $v \in U$, either $uv^+ \in E$ or $uv^- \in E$. Say, $uv^+ \in E$ (2). Now $vu^+ \notin E$ (3), otherwise a $2C$-factor can easily be constructed. By claw-freeness, $vu^- \in E$ (4). Next, $u^-v^+ \notin E$ (5) to prevent a $2C$-factor, thus $v^+u^+, v^-u^- \in E$ (6,7) to prevent claws in v, u, respectively. Now, $v^{++}u^+ \notin E$ (8), otherwise $C_1 = vuv^+v, C_2 =$

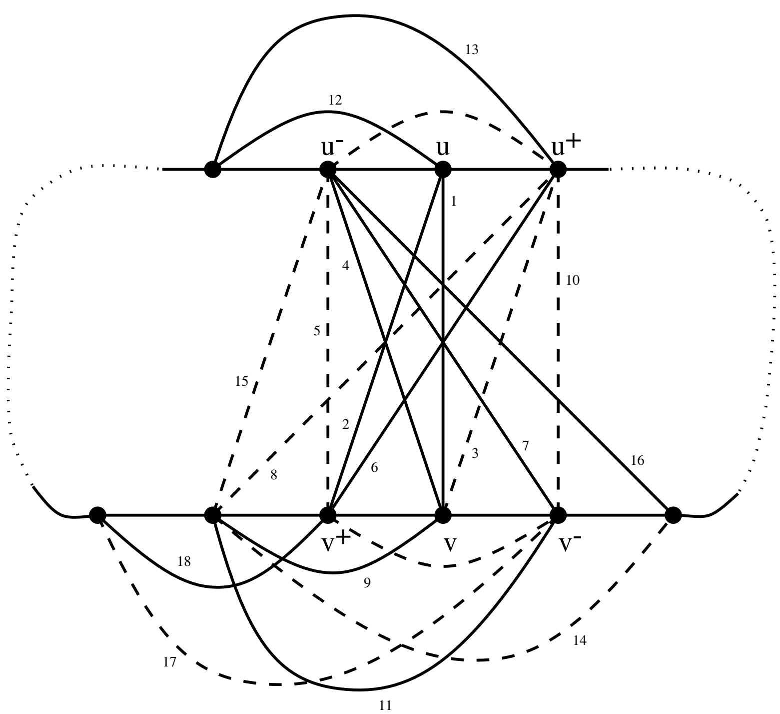

Figure 1: $|vCu| \geq 6$

$u^+Cv^-u^-\bar{C}v^{++}u^+$ is a $2C$-factor. By claw-freeness, $vv^{++} \in E$ (9). Again, $v^-u^+ \notin E$ (10), thus $v^{++}v^- \in E$ (11). By a symmetric argument, $u^{--}u, u^{--}u^+ \in E$ (12,13). Now, $v^{++}v^{--} \notin E$ (14), otherwise $C_1 = v^+vv^-u^-uv^+, C_2 = u^+Cv^{--}v^{++}Cu^{--}u^+$ is a $2C$-factor. Claw-freeness at v^- forces $v^{--}u^- \in E$ (16) as $v^{++}u^-$ (15) would yield a $2C$-factor. Now, $v^{3+}v^- \notin E$ (17), otherwise $C_1 = vv^+v^{++}v, C_2 = v^-v^{3+}Cv^-$ is a $2C$-factor. To avoid a claw at v^{++} ($v^+v^- \notin E$), $v^{3+}v^+ \in$

E (18). But now, $C_1 = vv^-v^{++}v, C_2 = v^+uCv^{--}u^-\bar{C}v^{3+}v^+$ is a $2C$-factor, a contradiction. Note that the above argument only requires $|vCu| \geq 6$ as it works even if $v^{3+} = u^{--}$.

To prove the lemma suppose that either $|uCv| = 5$ or $|vCu| = 5$, we may assume by symmetry $|uCv| = 5$ (see Figure 2). Note, that here $u^{++} = v^{--}$. If $uv^+ \in E$ (1), the argument from above will give the contradiction, as $|vCu| > 5$. Hence, $uv^-, vu^+ \in E$ (2,3), and, following an argument symmetric to the one used above, $v^-u^-, v^+u^+ \in E$ (4,5). Now $uu^{++}, uv^+ \notin E$ (6,7), so $u^{++}v^+ \in E$ (8) to avoid a claw at u^+. But now, $C_1 = uvu^+u, C_2 = u^-v^-u^{++}v^+Cu^-$ is a $2C$-factor, a contradiction. □

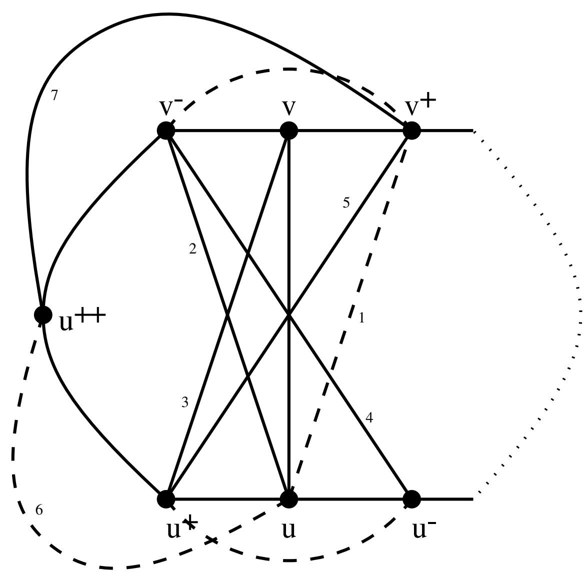

Figure 2: $|vCu| = 5$

Lemma 4. *Suppose G is a hamiltonian graph on at least 8 vertices that has no $2C$-factor. If $u, v \in U$, $uv \in E$, and $|uCv| \leq |vCu|$, then $G[uCv]$ is complete.*

Proof: By Lemma 3, we know that $|uCv| \leq 4$. If $|uCv| \leq 3$, there is nothing to prove, so assume that $|uCv| = 4$. If $G[uCv]$ is not complete, then $uv^+, vu^- \in E$ to avoid claws and a $2C$-factor. As $u^-v^+ \in E$ would yield a $2C$-factor, $u^-v^-, u^+v^+ \in E$ to avoid claws. If one of the edges uv^- and uu^{--} exists, a $2C$-factor is apparent. To avoid a claw centered at u^-, $u^{--}v^- \in E$ is forced. But now, $C_1 = uu^-vu, C_2 = u^{--}v^-u^+v^+Cu^{--}$ is a $2C$-factor, a contradiction. □

Proof of Theorem 2: Suppose again, for the sake of contradiction, that G contains no $2C$-factor. Faudree *et al.* [2] showed that the 2-color Ramsey number for a triangle and a $K_4 - e$ (the graph on 4 vertices with 5 edges) is

$$r(K_3, K_4 - e) = 7.$$

As $d(x) \geq 7$, we know that $G[N(x)]$ contains either an independent set of size 3 or a $K_4 - e$. The independent set would yield a claw, therefore $G[N(x)]$ contains a $K_4 - e$, say $x_1, x_2, x_3, x_4 \in N(x)$ and x_1x_2,x_1x_3,x_1x_4,x_2x_3, $x_2x_4 \in E$.

Depending on the location of the five vertices x, x_1, x_2, x_3, x_4 on C, we will consider seven cases. Note that $G[x, x_1, x_2, x_3, x_4]$ is either a $K_5 - e$ or a K_5.

Case 1. *Suppose that the five vertices are consecutive on C, i.e. there is a $v \in V$, such that $\{x, x_1, x_2, x_3, x_4\} = \{v^{--}, v^-, v, v^+, v^{++}\}$.*

If $v^{--}v^{++}, v^-v^+ \in E$, then $C_1 = vv^+v^-v, C_2 = v^{++}Cv^{--}v^{++}$ is a $2C$-factor. Thus, one of the two edges is missing.

Suppose first that $v^-v^+ \notin E$. If $v^{3-}v^- \in E$, then $C_1 = vv^{--}v^+v, C_2 = v^{++}Cv^{3-}v^-v^{++}$ is a $2C$-factor. Thus, $v^{3-}v^- \notin E$, and similarly $v^{3+}v^+ \notin E$. But this implies that $v^{--}, v^{++} \in U$, a contradiction with Lemma 3.

Thus, we may assume that $v^{--}v^{++} \notin E$, in fact we may assume that $x_3 = v^{++}, x_4 = v^{--}$. Note that $xx_4^- \notin E$, otherwise $C_1 = x_4x_1x_2x_4, C_2 = xx_3Cx_4^-x$ is a $2C$-factor. Similarly, $x_1x_4^-$,$x_2x_4^-$,xx_3^+, $x_1x_3^+$,$x_2x_3^+ \notin E$, and therefore $x_3, x_4 \in U$. As $d(x) \geq 7$, x has at least 3 neighbors other than x_1, x_2, x_3, x_4, say $y_1, y_2, y_3 \in N(x)$ appear in this order on C. To avoid the claw $G[x, x_3, x_4, y_2]$, at least one of the edges x_3y_2, x_4y_2 has to exist, we may assume that $x_3y_2 \in E$.

Suppose that $y_2 \in U$. As $G[y_2Cx_3]$ is not complete, $G[x_3Cy_2]$ is complete by Lemma 4 (and $|x_3Cy_2| = 4$). This yields the $2C$-factor $C_1 = x_1x_2x_3x_1, C_2 = xy_1x_3^+y_2Cx_4x$, a contradiction. Thus, $y_2^-y_2^+ \in E$. If $x_2y_2 \in E$, then $C_1 = xx_2y_2x, C_2 = x_1x_3Cy_2^-y_2^+Cx_4x_1$ is a $2C$-factor, thus $x_2y_2 \notin E$. To avoid the claw $G[x_3, x_3^+, x_2, y_2]$, we have $x_3^+y_2 \in E$. This yields the $2C$-factor $C_1 = x_1x_2x_3x_1, C_2 = xy_2x_3^+y_2^-y_2^+Cx_4x$, the contradiction finishing the case.

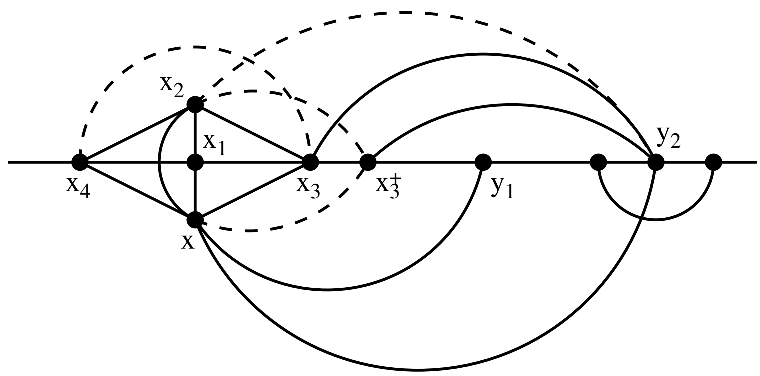

Figure 3: Case 1

Case 2. *Suppose four of the vertices* x, x_1, x_2, x_3, x_4 *appear consecutively on* C.

Let v be the vertex out of $\{x, x_1, x_2, x_3, x_4\}$ which is not a predecessor or a successor of one of the other four vertices in the $K_5 - e$. If $v \notin U$, then consider the cycle $C' = v^+Cv^-v^+$, and extend it through v by inserting v between two consecutive vertices in $\{x, x_1, x_2, x_3, x_4\}$. We can apply Case 1 to this situation to get a contradiction. Thus, $v \in U$.

Let $u \in V$ such that $\{u^{--}, u^-, u, u^+\} \cup \{v\} = \{x, x_1, x_2, x_3, x_4\}$. As $G[x, x_1, x_2, x_3, x_4]$ is a K_5 or a $K_5 - e$, at least one of u^-v and uv is an edge, by symmetry we may assume $uv \in E$. To avoid the claw $G[v, u, v^-, v^+]$, one of uv^- and uv^+ is an edge.

If $uv^+ \in E$, then $u^+v \notin E$ to avoid a $2C$-factor. Then $u^-v \in E$ and one of u^-v^- and u^-v^+ is an edge. Either one of these two edges produces a $2C$-factor, a contradiction.

On the other hand, if $uv^- \in E$, then $u^-v \notin E$ to avoid a $2C$-factor. But this implies $u^{--}v, u^-u^+ \in E$, and $C_1 = uu^-u^+Cv^-u, C_2 = vu^{--}Cv$ is a $2C$-factor, the contradiction finishing the case.

Case 3. *Suppose there are two vertices* $u, v \in V$ *such that* $\{x, x_1, x_2, x_3, x_4\} = \{u^-, u, u^+, v, v^+\}$.

In this case, a $2C$-factor is easy to find. Depending on which of the 10 edges is missing, either $C_1 = v^+Cu^-v^+, C_2 = uCvu$ or $C_1 = v^+Cuv^+, C_2 = u^+Cvu^+$ will do.

Case 4. *Suppose there are three vertices* $u, v, w \in V$ *such that* $\{x, x_1, x_2, x_3, x_4\} = \{u^-, u, u^+, v, w\}$.

By symmetry we may assume that $u^-v, uv, u^+v \in E$. If $v^-v^+ \in E$, we can find a different hamiltonian cycle and apply Case 2. Thus, $v \in U$. To avoid the claw $G[v, u, v^-, v^+]$, one of the edges uv^-, uv^+ has to exist. But either one produces a $2C$-factor, a contradiction.

Case 5. *Suppose there are three vertices $u, v, w \in V$ such that $\{x, x_1, x_2, x_3, x_4\} = \{u, u^+, v, v^+, w\}$.*

By symmetry we may assume that u, v, w appear on C in this order. If both $uv^+, u^+v \in E$, a $2C$-factor is immediate, so one of these two edges is missing. This implies that all other 8 possible edges within $\{u, u^+, v, v^+, w\}$ exist. Further, $w \in U$, otherwise we can find a different hamiltonian cycle and apply Case 3. If $vw^+ \in E$, a $2C$-factor is immediate, thus $vw^- \in E$ to avoid a claw centered at w. This yields the $2C$-factor $C_1 = wCuw, C_2 = v^+Cw^-vC^-u^+w^+$, a contradiction.

Case 6. *Suppose there are four vertices $u, v, w, y \in V$ such that $\{x, x_1, x_2, x_3, x_4\} = \{u, u^+, v, w, y\}$.*

By symmetry we may assume that u, v, w, y appear on C in this order. Suppose that $vy \in E$. By Lemma 3, at most one of v, y is in U, say $y \notin U$. If $v \in U$, then $v^-y \in E$ or $v^+y \in E$ to avoid a claw. But now we can reduce the case to Case 5. On the other hand, if $v \notin U$ we can find a different hamiltonian cycle by inserting v or y between u and u^+, depending on which of the edges is missing. Applying Case 4 to this situation gives a contradiction. Therefore, $vy \notin E$ and all other 9 possible edges inside $\{u, u^+, v, w, y\}$ exist.

If any of v, w, y is not in U, then we can reduce this case to Case 4 by inserting this vertex between u and u^+. Thus, we may assume that $v, w, y \in U$. Again by Lemma 3, $u^-u^+, uu^{++} \in E$, as $|wCu|, |u^+Cw| \geq 5$. To avoid a claw at v, one of uv^-, uv^+ is an edge. If $uv^+ \in E$, then $C_1 = u^+Cvu^+, C_2 = uv^+Cu$ is a $2C$-factor. If $uv^- \in E$, then $C_1 = uu^{++}Cv^-u, C_2 = u^+vCu^-u^+$ is a $2C$-factor, the contradiction finishing this case.

Case 7. *Suppose none of the vertices $\{u_1, u_2, u_3, u_4, u_5\} = \{x, x_1, x_2, x_3, x_4\}$ are consecutive on C.*

We may assume that u_1, u_2, u_3, u_4, u_5 appear on C in this order. If none of the five vertices are in U, a $2C$-factor is easy to find. By symmetry, we may assume that $u_3 \in U$. At least one of the edges u_3u_5, u_1u_3 exists, we may assume $u_3u_5 \in E$. By Lemma 3, $u_5 \notin U$. To avoid a claw, one of the edges $u_3^- u_5, u_3^+ u_5$ has to exist. In either case we can pick a different hamiltonian cycle and reduce the argument to Case 6. This finishes the proof of the theorem. □

References

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