# $3+1$ formalism in general relativity 

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(1) The 3+1 foliation of spacetime
(2) 3+1 decomposition of Einstein equation
(3) The Cauchy problem

4 Conformal decomposition

## Outline

(1) The 3+1 foliation of spacetime
(2) 3+1 decomposition of Einstein equation
(3) The Cauchy problem

4 Conformal decomposition

## Framework: globally hyperbolic spacetimes

4-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$ :

- $\mathscr{M}$ : 4-dimensional smooth manifold
- $g$ : Lorentzian metric on $\mathscr{M}$ : $\operatorname{sign}(g)=(-,+,+,+)$
$(\mathscr{M}, \boldsymbol{g})$ is assumed to be time orientable: the light cones of $\boldsymbol{g}$ can be divided continuously over $\mathscr{M}$ in two sets (past and future)


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The spacetime $(\mathscr{M}, \boldsymbol{g})$ is assumed to be globally hyperbolic: $\exists$ a foliation (or slicing) of the spacetime manifold $\mathscr{M}$ by a family of spacelike hypersurfaces $\Sigma_{t}$ :
$\mathscr{M}=\bigcup_{t \in \mathbb{R}} \Sigma_{t}$
hypersurface $=$ submanifold of $\mathscr{M}$ of dimension 3

## Unit normal vector and lapse function


$n$ : unit normal vector to $\Sigma_{t}$
$\Sigma_{t}$ spacelike $\Longleftrightarrow \boldsymbol{n}$ timelike
$\boldsymbol{n} \cdot \boldsymbol{n}:=\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n})=-1$
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The 1-form $\underline{n}$ associated with $\boldsymbol{n}$ is proportional to the gradient of $t$ :
$\underline{\boldsymbol{n}}=-N \mathbf{d} t \quad\left(n_{\alpha}=-N \nabla_{\alpha} t\right)$
$N$ : lapse function ; $N>0$
Elapse proper time between $p$ and $p^{\prime}: \delta \tau=N \delta t$

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Elapse proper time between $p$ and $p^{\prime}: \delta \tau=N \delta t$
Normal evolution vector : $m:=N n$ $\langle\mathbf{d} t, \boldsymbol{m}\rangle=1 \Rightarrow \boldsymbol{m}$ Lie drags the hypersurfaces $\Sigma_{t}$

## Induced metric (first fundamental form)

The induced metric or first fundamental form on $\Sigma_{t}$ is the bilinear form $\gamma$ defined by

$$
\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_{p}\left(\Sigma_{t}\right) \times \mathcal{T}_{p}\left(\Sigma_{t}\right), \quad \gamma(\boldsymbol{u}, \boldsymbol{v}):=\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})
$$

$$
\Sigma_{t} \text { spacelike } \Longleftrightarrow \gamma \text { positive definite (Riemannian metric) }
$$

$D$ : Levi-Civita connection associated with $\gamma: D \gamma=0$
$\mathcal{R}$ : Riemann tensor of $\boldsymbol{D}$ :

$$
\forall \boldsymbol{v} \in \mathcal{T}\left(\Sigma_{t}\right),\left(D_{i} D_{j}-D_{j} D_{i}\right) v^{k}=\mathcal{R}^{k}{ }_{l i j} v^{l}
$$

$\boldsymbol{R}:$ Ricci tensor of $\boldsymbol{D}: R_{i j}:=R^{k}{ }_{i k j}$
$R$ : scalar curvature (or Gaussian curvature) of $(\Sigma, \gamma): R:=\gamma^{i j} R_{i j}$

## Orthogonal projector

Since $\gamma$ is not degenerate we have the orthogonal decomposition:

$$
\mathcal{T}_{p}(\mathscr{M})=\mathcal{T}_{p}\left(\Sigma_{t}\right) \oplus \operatorname{Vect}(\boldsymbol{n})
$$

The associated orthogonal projector onto $\Sigma_{t}$ is

$$
\begin{array}{rlc}
\vec{\gamma}: \mathcal{T}_{p}(\mathscr{M}) & \longrightarrow & \mathcal{T}_{p}(\Sigma) \\
\boldsymbol{v} & \longmapsto v+(\boldsymbol{n} \cdot \boldsymbol{v}) \boldsymbol{n}
\end{array}
$$

In particular, $\vec{\gamma}(n)=0$ and $\forall v \in \mathcal{T}_{p}\left(\Sigma_{t}\right), \vec{\gamma}(v)=v$
Components: $\gamma^{\alpha}{ }_{\beta}=\delta^{\alpha}{ }_{\beta}+n^{\alpha} n_{\beta}$

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"Extended" induced metric:

$$
\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_{p}(\mathscr{M}) \times \mathcal{T}_{p}(\mathscr{M}), \quad \gamma(\boldsymbol{u}, \boldsymbol{v}):=\gamma(\vec{\gamma}(\boldsymbol{u}), \vec{\gamma}(\boldsymbol{v}))
$$

$\gamma=\boldsymbol{g}+\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} \quad\left(\gamma_{\alpha \beta}=g_{\alpha \beta}+n_{\alpha} n_{\beta}\right)$
(hence the notation $\vec{\gamma}$ for the orthogonal projector)

## Extrinsic curvature (second fundamental form)

The extrinsic curvature (or second fundamental form) of $\Sigma_{t}$ is the bilinear form defined by

$$
\begin{array}{rllc}
\boldsymbol{K}: \mathcal{T}_{p}\left(\Sigma_{t}\right) \times \mathcal{T}_{p}\left(\Sigma_{t}\right) & \longrightarrow & \mathbb{R} \\
(\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & -\boldsymbol{u} \cdot \nabla_{\boldsymbol{v}} \boldsymbol{n}
\end{array}
$$

It measures the "bending" of $\Sigma_{t}$ in $(\mathscr{M}, \boldsymbol{g})$ by evaluating the change of direction of the normal vector $\boldsymbol{n}$ as one moves on $\Sigma_{t}$

Weingarten property: $\boldsymbol{K}$ is symmetric: $\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v})=\boldsymbol{K}(\boldsymbol{v}, \boldsymbol{u})$
Trace: $K:=\operatorname{tr}_{\gamma} \boldsymbol{K}=\gamma^{i j} K_{i j}=(3$ times $)$ the mean curvature of $\Sigma_{t}$

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$$
\begin{gathered}
\text { "Extended" } \boldsymbol{K}: \forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_{p}(\mathscr{M}) \times \mathcal{T}_{p}(\mathscr{M}), \quad \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}):=\boldsymbol{K}(\vec{\gamma}(\boldsymbol{u}), \vec{\gamma}(\boldsymbol{v})) \\
\Longrightarrow \nabla \underline{\boldsymbol{n}}=-\boldsymbol{K}-\boldsymbol{D} \ln N \otimes \underline{\boldsymbol{n}} \quad\left(\nabla_{\beta} n_{\alpha}=-K_{\alpha \beta}-D_{\alpha} \ln N n_{\beta}\right) \\
K=-\boldsymbol{\nabla} \cdot \boldsymbol{n}
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"Extended" $\boldsymbol{K}: \forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_{p}(\mathscr{M}) \times \mathcal{T}_{p}(\mathscr{M}), \quad \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}):=\boldsymbol{K}(\vec{\gamma}(\boldsymbol{u}), \vec{\gamma}(\boldsymbol{v}))$

$$
\Longrightarrow \begin{gathered}
\boldsymbol{\nabla} \underline{\boldsymbol{n}}=-\boldsymbol{K}-\boldsymbol{D} \ln N \otimes \underline{\boldsymbol{n}} \quad\left(\nabla_{\beta} n_{\alpha}=-K_{\alpha \beta}-D_{\alpha} \ln N n_{\beta}\right) \\
K=-\boldsymbol{\nabla} \cdot \boldsymbol{n}
\end{gathered}
$$

$\Sigma_{t}$ being part of a foliation, an alternative expression of $\boldsymbol{K}$ is available:

$$
\boldsymbol{K}=-\frac{1}{2} \mathcal{L}_{n} \gamma
$$

## Intrinsic and extrinsic curvatures

Examples in the Euclidean space

- intrinsic curvature: Riemann tensor $\mathcal{R}$
- extrinsic curvature: second fundamental form $\boldsymbol{K}$


$$
\begin{aligned}
& \boldsymbol{\mathcal { R }}=0 \\
& \boldsymbol{K}=0
\end{aligned}
$$

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$\mathcal{R}=0$
$\boldsymbol{K}=0$
cylinder

$\mathcal{R}=0$
$\boldsymbol{K} \neq 0$


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$\mathcal{R}=0$
$\boldsymbol{K}=0$
cylinder

$\boldsymbol{\mathcal { R }}=0$
$\boldsymbol{K} \neq 0$
sphere

$\mathcal{R} \neq 0$
$\boldsymbol{K} \neq 0$


## Link between the $\boldsymbol{\nabla}$ and $D$ connections

For any tensor field $T$ tangent to $\Sigma_{t}$ :

$$
D_{\rho} T^{\alpha_{1} \ldots \alpha_{p}}{ }_{\beta_{1} \ldots \beta_{q}}=\gamma^{\alpha_{1}}{ }_{\mu_{1}} \cdots \gamma^{\alpha_{p}}{ }_{\mu_{p}} \gamma^{\nu_{1}}{ }_{\beta_{1}} \cdots \gamma^{\nu_{q}}{ }_{\beta_{q}} \gamma^{\sigma}{ }_{\rho} \nabla_{\sigma} T^{\mu_{1} \ldots \mu_{p}}{ }_{\nu_{1} \ldots \nu_{q}}
$$

For two vector fields $u$ and $v$ tangent to $\Sigma_{t}, \boldsymbol{D}_{u} \boldsymbol{v}=\nabla_{u} \boldsymbol{v}+\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) \boldsymbol{n}$

## $3+1$ decomposition of the Riemann tensor

- Gauss equation: $\gamma^{\mu}{ }_{\alpha} \gamma^{\nu}{ }_{\beta} \gamma^{\gamma}{ }_{\rho} \gamma^{\sigma}{ }_{\delta}{ }^{4} \mathcal{R}^{\rho}{ }_{\sigma \mu \nu}=\mathcal{R}^{\gamma}{ }_{\delta \alpha \beta}+K^{\gamma}{ }_{\alpha} K_{\delta \beta}-K^{\gamma}{ }_{\beta} K_{\alpha \delta}$ contracted version :
$\gamma^{\mu} \gamma^{\nu}{ }_{\beta}{ }^{4} R_{\mu \nu}+\gamma_{\alpha \mu} n^{\nu} \gamma^{\rho}{ }_{\beta} n^{\sigma}{ }^{4} \mathcal{R}^{\mu}{ }_{\nu \rho \sigma}=R_{\alpha \beta}+K K_{\alpha \beta}-K_{\alpha \mu} K^{\mu}{ }_{\beta}$ trace: ${ }^{4} R+2^{4} R_{\mu \nu} n^{\mu} n^{\nu}=R+K^{2}-K_{i j} K^{i j} \quad$ (Theorema Egregium)


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- Codazzi equation: $\gamma^{\gamma}{ }_{\rho} n^{\sigma} \gamma^{\mu}{ }_{\alpha} \gamma^{\nu}{ }_{\beta}{ }^{4} \mathcal{R}^{\rho}{ }_{\sigma \mu \nu}=D_{\beta} K^{\gamma}{ }_{\alpha}-D_{\alpha} K^{\gamma}{ }_{\beta}$ contracted version : $\gamma^{\mu}{ }_{\alpha} n^{\nu}{ }^{4} R_{\mu \nu}=D_{\alpha} K-D_{\mu} K^{\mu}{ }_{\alpha}$


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- Ricci equation: $\gamma_{\alpha \mu} n^{\rho} \gamma^{\nu}{ }_{\beta} n^{\sigma 4} \mathcal{R}_{\rho \nu \sigma}^{\mu}=\frac{1}{N} \mathcal{L}_{m} K_{\alpha \beta}+\frac{1}{N} D_{\alpha} D_{\beta} N+K_{\alpha \mu} K_{\beta}^{\mu}$ combined with the contracted Gauss equation :
$\gamma_{\alpha}^{\mu} \gamma^{\nu}{ }_{\beta}{ }^{4} R_{\mu \nu}=-\frac{1}{N} \mathcal{L}_{m} K_{\alpha \beta}-\frac{1}{N} D_{\alpha} D_{\beta} N+R_{\alpha \beta}+K K_{\alpha \beta}-2 K_{\alpha \mu} K^{\mu}{ }_{\beta}$


## Outline

(1) The $3+1$ foliation of spacetime
(2) 3+1 decomposition of Einstein equation
(3) The Cauchy problem

4 Conformal decomposition

## Einstein equation

The spacetime $(\mathscr{M}, \boldsymbol{g})$ obeys Einstein equation

$$
{ }^{4} \boldsymbol{R}-\frac{1_{4}^{4}}{2} R \boldsymbol{g}=8 \pi \boldsymbol{T}
$$

where $\boldsymbol{T}$ is the matter stress-energy tensor

## $3+1$ decomposition of the stress-energy tensor

$\mathcal{E}:$ Eulerian observer $=$ observer of 4-velocity $\boldsymbol{n}$

- $E:=T(n, n)$ : matter energy density as measured by $\mathcal{E}$
- $p:=-\boldsymbol{T}(\boldsymbol{n}, \vec{\gamma}()$.$) : matter momentum density as measured by \mathcal{E}$
- $S:=T(\vec{\gamma}(),. \vec{\gamma}()$.$) : matter stress tensor as measured by \mathcal{E}$

$$
T=S+\underline{n} \otimes \boldsymbol{p}+\boldsymbol{p} \otimes \underline{n}+E \underline{n} \otimes \underline{n}
$$

## Spatial coordinates and shift vector


$\left(x^{i}\right)=\left(x^{1}, x^{2}, x^{3}\right)$ coordinates on $\Sigma_{t}$
$\left(x^{i}\right)$ vary smoothly between
neighbouring hypersurfaces $\Rightarrow$ $\left(x^{\alpha}\right)=\left(t, x^{1}, x^{2}, x^{3}\right)$ well behaved coordinate system on M
Associated natural basis:

$$
\begin{aligned}
& \partial_{t}:=\frac{\partial}{\partial t} \\
& \partial_{i}:=\frac{\partial}{\partial x^{i}}, \quad i \in\{1,2,3\}
\end{aligned}
$$

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$$

$\left\langle\mathbf{d} t, \boldsymbol{\partial}_{t}\right\rangle=1 \Rightarrow \boldsymbol{\partial}_{t}$ Lie-drags the hypersurfaces $\Sigma_{t}$, as $\boldsymbol{m}:=N \boldsymbol{n}$ does. The difference between $\boldsymbol{\partial}_{t}$ and $\boldsymbol{m}$ is called the shift vector and is denoted $\boldsymbol{\beta}$ :

$$
\partial_{t}=: \boldsymbol{m}+\boldsymbol{\beta}
$$

Notice: $\boldsymbol{\beta}$ is tangent to $\Sigma_{t}: \boldsymbol{n} \cdot \boldsymbol{\beta}=0$

## Metric tensor in terms of lapse and shift

Components of $\beta$ w.r.t. $\left(x^{i}\right): \beta=: \beta^{i} \partial_{i}$ and $\underline{\beta}=: \beta_{i} \mathbf{d} x^{i}$
Components of $\boldsymbol{n}$ w.r.t. $\left(x^{\alpha}\right)$ :
$n^{\alpha}=\left(\frac{1}{N},-\frac{\beta^{1}}{N},-\frac{\beta^{2}}{N},-\frac{\beta^{3}}{N}\right)$ and $n_{\alpha}=(-N, 0,0,0)$
Components of $\boldsymbol{g}$ w.r.t. $\left(x^{\alpha}\right)$ :

$$
g_{\alpha \beta}=\left(\begin{array}{cc}
g_{00} & g_{0 j} \\
g_{i 0} & g_{i j}
\end{array}\right)=\left(\begin{array}{cc}
-N^{2}+\beta_{k} \beta^{k} & \beta_{j} \\
\beta_{i} & \gamma_{i j}
\end{array}\right)
$$

or equivalently $g_{\mu \nu} d x^{\mu} d x^{\nu}=-N^{2} d t^{2}+\gamma_{i j}\left(d x^{i}+\beta^{i} d t\right)\left(d x^{j}+\beta^{j} d t\right)$
Components of the inverse metric:

$$
g^{\alpha \beta}=\left(\begin{array}{cc}
g^{00} & g^{0 j} \\
g^{i 0} & g^{i j}
\end{array}\right)=\left(\begin{array}{cc}
-\frac{1}{N^{2}} & \frac{\beta^{j}}{N^{2}} \\
\frac{\beta^{i}}{N^{2}} & \gamma^{i j}-\frac{\beta^{i} \beta^{j}}{N^{2}}
\end{array}\right)
$$

Relation between the determinants : $\sqrt{-g}=N \sqrt{\gamma}$

## $3+1$ Einstein system

Thanks to the Gauss, Codazzi and Ricci equations
reminder , the Einstein equation is equivalent to the system

- $\left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \gamma_{i j}=-2 N K_{i j} \quad$ kinematical relation $\boldsymbol{K}=-\frac{1}{2} \mathcal{L}_{n} \gamma$
- $\left(\frac{\partial}{\partial t}-\mathcal{L}_{\boldsymbol{\beta}}\right) K_{i j}=-D_{i} D_{j} N+N\left\{R_{i j}+K K_{i j}-2 K_{i k} K^{k}{ }_{j}\right.$
$\left.+4 \pi\left[(S-E) \gamma_{i j}-2 S_{i j}\right]\right\} \quad$ dynamical part of Einstein equation
- $R+K^{2}-K_{i j} K^{i j}=16 \pi E \quad$ Hamiltonian constraint
- $D_{j} K^{j}{ }_{i}-D_{i} K=8 \pi p_{i} \quad$ momentum constraint


## The full PDE system

## Supplementary equations:

$$
\begin{aligned}
& D_{i} D_{j} N=\frac{\partial^{2} N}{\partial x^{i} \partial x^{j}}-\Gamma^{k}{ }_{i j} \frac{\partial N}{\partial x^{k}} \\
& D_{j} K^{j}{ }_{i}=\frac{\partial K^{j}{ }_{i}}{\partial x^{j}}+\Gamma^{j}{ }_{j k} K^{k}{ }_{i}-\Gamma^{k}{ }_{j i} K^{j}{ }_{k} \\
& D_{i} K=\frac{\partial K}{\partial x^{i}} \\
& \mathcal{L}_{\boldsymbol{\beta}} \gamma_{i j}=\frac{\partial \beta_{i}}{\partial x^{j}}+\frac{\partial \beta_{j}}{\partial x^{i}}-2 \Gamma^{k}{ }_{i j} \beta_{k} \\
& \mathcal{L}_{\boldsymbol{\beta}} K_{i j}=\beta^{k} \frac{\partial K_{i j}}{\partial x^{k}}+K_{k j} \frac{\partial \beta^{k}}{\partial x^{i}}+K_{i k} \frac{\partial \beta^{k}}{\partial x^{j}} \\
& R_{i j}=\frac{\partial \Gamma^{k}{ }_{i j}}{\partial x^{k}}-\frac{\partial \Gamma^{k}{ }_{i k}}{\partial x^{j}}+\Gamma^{k}{ }_{i j} \Gamma^{l}{ }_{k l}-\Gamma^{l}{ }_{i k} \Gamma^{k}{ }_{l j} \\
& R=\gamma^{i j} R_{i j} \\
& \Gamma^{k}{ }_{i j}=\frac{1}{2} \gamma^{k l}\left(\frac{\partial \gamma_{l j}}{\partial x^{i}}+\frac{\partial \gamma_{i l}}{\partial x^{j}}-\frac{\partial \gamma_{i j}}{\partial x^{l}}\right)
\end{aligned}
$$

## History of 3+1 formalism

- G. Darmois (1927): 3+1 Einstein equations in terms of $\left(\gamma_{i j}, K_{i j}\right)$ with $\alpha=1$ and $\beta=0$ (Gaussian normal coordinates)
- A. Lichnerowicz (1939) : $\alpha \neq 1$ and $\beta=0$ (normal coordinates)
- Y. Choquet-Bruhat (1948) : $\alpha \neq 1$ and $\beta \neq 0$ (general coordinates)
- R. Arnowitt, S. Deser \& C.W. Misner (1962) : Hamiltonian formulation of GR based on a $3+1$ decomposition in terms of $\left(\gamma_{i j}, \pi^{i j}\right)$ NB: spatial projection of Einstein tensor instead of Ricci tensor in previous works
- J. Wheeler (1964) : coined the terms lapse and shift
- J.W. York (1979) : modern 3+1 decomposition based on spatial projection of Ricci tensor


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## GR as a 3-dimensional dynamical system

3+1 Einstein system $\Longrightarrow$ Einstein equation $=$ time evolution of tensor fields
$(\gamma, \boldsymbol{K})$ on a single 3-dimensional manifold $\Sigma$
(Wheeler's geometrodynamics (1964))

No time derivative of $N$ nor $\beta$ : lapse and shift are not dynamical variables (best seen on the ADM Hamiltonian formulation)
This reflects the coordinate freedom of GR
choice of foliation $\left(\Sigma_{t}\right)_{t \in \mathbb{R}} \Longleftrightarrow$ choice of lapse function $N$ choice of spatial coordinates $\left(x^{i}\right) \Longleftrightarrow$ choice of shift vector $\boldsymbol{\beta}$

## Constraints

The dynamical system has two constraints:

- $R+K^{2}-K_{i j} K^{i j}=16 \pi E \quad$ Hamiltonian constraint
- $D_{j} K^{j}{ }_{i}-D_{i} K=8 \pi p_{i} \quad$ momentum constraint

Similar to $\boldsymbol{D} \cdot \boldsymbol{B}=0$ and $\boldsymbol{D} \cdot \boldsymbol{E}=\rho / \epsilon_{0}$ in Maxwell equations for the electromagnetic field

## Cauchy problem

The first two equations of the $3+1$ Einstein system
4 reminder
can be put in the form of a Cauchy problem:

$$
\begin{equation*}
\frac{\partial^{2} \gamma_{i j}}{\partial t^{2}}=F_{i j}\left(\gamma_{k l}, \frac{\partial \gamma_{k l}}{\partial x^{m}}, \frac{\partial \gamma_{k l}}{\partial t}, \frac{\partial^{2} \gamma_{k l}}{\partial x^{m} \partial x^{n}}\right) \tag{1}
\end{equation*}
$$

Cauchy problem: given initial data at $t=0: \gamma_{i j}$ and $\frac{\partial \gamma_{i j}}{\partial t}$, find a solution for $t>0$

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\end{equation*}
$$

Cauchy problem: given initial data at $t=0: \gamma_{i j}$ and $\frac{\partial \gamma_{i j}}{\partial t}$, find a solution for $t>0$
But this Cauchy problem is subject to the constraints

- $R+K^{2}-K_{i j} K^{i j}=16 \pi E \quad$ Hamiltonian constraint
- $D_{j} K^{j}{ }_{i}-D_{i} K=8 \pi p_{i} \quad$ momentum constraint


## Preservation of the constraints

Thanks to the Bianchi identities, it can be shown that if the constraints are satisfied at $t=0$, they are preserved by the evolution system (1)

## Existence and uniqueness of solutions

## Question:

Given a set $\left(\Sigma_{0}, \gamma, \boldsymbol{K}, E, \boldsymbol{p}\right)$, where $\Sigma_{0}$ is a three-dimensional manifold, $\gamma$ a Riemannian metric on $\Sigma_{0}, \boldsymbol{K}$ a symmetric bilinear form field on $\Sigma_{0}, E$ a scalar field on $\Sigma_{0}$ and $p$ a 1-form field on $\Sigma_{0}$, which obeys the constraint equations, does there exist a spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{T})$ such that $(\boldsymbol{g}, \boldsymbol{T})$ fulfills the Einstein equation and $\Sigma_{0}$ can be embedded as an hypersurface of $\mathscr{M}$ with induced metric $\gamma$ and extrinsic curvature $\boldsymbol{K}$ ?

## Answer:

- the solution exists and is unique in a vicinity of $\Sigma_{0}$ for analytic initial data (Cauchy-Kovalevskaya theorem) (Darmois 1927, Lichnerowicz 1939)
- the solution exists and is unique in a vicinity of $\Sigma_{0}$ for generic (i.e. smooth) initial data (Choquet-Bruhat 1952)
- there exists a unique maximal solution (Choquet-Bruhat \& Geroch 1969)


## Outline

## (1) The 3+1 foliation of spacetime

(2) 3+1 decomposition of Einstein equation
(3) The Cauchy problem
(4) Conformal decomposition

## Conformal metric

Introduce on $\Sigma_{t}$ a metric $\tilde{\gamma}$ conformally related to the induced metric $\gamma$ :

$$
\gamma_{i j}=\Psi^{4} \tilde{\gamma}_{i j}
$$

$\psi$ : conformal factor
Inverse metric:

$$
\gamma^{i j}=\psi^{-4} \tilde{\gamma}^{i j}
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## Motivations:

- the gravitational field degrees of freedom are carried by conformal equivalence classes (York 1971)
- the conformal decomposition is of great help for preparing initial data as solution of the constraint equations


## Conformal connection

$\tilde{\gamma}$ Riemannian metric on $\Sigma_{t}$ : it has a unique Levi-Civita connection associated to it: $\tilde{D} \tilde{\gamma}=0$
Christoffel symbols: $\tilde{\Gamma}^{k}{ }_{i j}=\frac{1}{2} \tilde{\gamma}^{k l}\left(\frac{\partial \tilde{\gamma}_{l j}}{\partial x^{i}}+\frac{\partial \tilde{\gamma}_{i l}}{\partial x^{j}}-\frac{\partial \tilde{\gamma}_{i j}}{\partial x^{l}}\right)$
Relation between the two connections:
$D_{k} T^{i_{1} \ldots i_{p}}{ }_{j_{1} \ldots j_{q}}=\tilde{D}_{k} T^{i_{1} \ldots i_{p}}{ }_{j_{1} \ldots j_{q}}+\sum_{r=1}^{p} C^{i_{r}}{ }_{k l} T^{i_{1} \ldots l \ldots i_{p}}{ }_{j_{1} \ldots j_{q}}-\sum_{r=1}^{q} C^{l}{ }_{k j_{r}} T_{j_{1} \ldots i_{1} \ldots j_{q}}^{i_{1} \ldots i_{p}}$
with $C^{k}{ }_{i j}:=\Gamma^{k}{ }_{i j}-\tilde{\Gamma}^{k}{ }_{i j}$
One finds

$$
C^{k}{ }_{i j}=2\left(\delta^{k}{ }_{i} \tilde{D}_{j} \ln \psi+\delta^{k}{ }_{j} \tilde{D}_{i} \ln \psi-\tilde{D}^{k} \ln \psi \tilde{\gamma}_{i j}\right)
$$

Application: divergence relation: $D_{i} v^{i}=\Psi^{-6} \tilde{D}_{i}\left(\Psi^{6} v^{i}\right)$

## Conformal decomposition of the Ricci tensor

From the Ricci identity:

$$
R_{i j}=\tilde{R}_{i j}+\tilde{D}_{k} C^{k}{ }_{i j}-\tilde{D}_{i} C^{k}{ }_{k j}+C^{k}{ }_{i j} C^{l}{ }_{l k}-C^{k}{ }_{i l} C^{l}{ }_{k j}
$$

In the present case this formula reduces to
$R_{i j}=\tilde{R}_{i j}-2 \tilde{D}_{i} \tilde{D}_{j} \ln \psi-2 \tilde{D}_{k} \tilde{D}^{k} \ln \psi \tilde{\gamma}_{i j}+4 \tilde{D}_{i} \ln \psi \tilde{D}_{j} \ln \psi-4 \tilde{D}_{k} \ln \psi \tilde{D}^{k} \ln \psi \tilde{\gamma}_{i j}$
Scalar curvature :
where $R:=\gamma^{i j} R_{i j}$ and $\tilde{R}:=\tilde{\gamma}^{i j} \tilde{R}_{i j}$

## Conformal decomposition of the extrinsic curvature

- First step: traceless decomposition:

$$
K^{i j}=: A^{i j}+\frac{1}{3} K \gamma^{i j}
$$

with $\gamma_{i j} A^{i j}=0$

- Second step: conformal decomposition of the traceless part:

$$
A^{i j}=\Psi^{\alpha} \tilde{A}^{i j}
$$

with $\alpha$ to be determined

## "Time evolution" scaling $\alpha=-4$

Time evolution of the 3-metric $\subset$ reminder: $\left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \gamma^{i j}=2 N K^{i j}$

- trace part: $\left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \ln \psi=\frac{1}{6}\left(\tilde{D}_{i} \beta^{i}-N K-\frac{\partial}{\partial t} \ln \tilde{\gamma}\right)$
- traceless part : $\left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \tilde{\gamma}^{i j}=2 N \Psi^{4} A^{i j}+\frac{2}{3} \tilde{D}_{k} \beta^{k} \tilde{\gamma}^{i j}$


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This suggests to introduce

$$
\tilde{A}^{i j}:=\Psi^{4} A^{i j} \quad \text { (Nakamura 1994) }
$$

$\Longrightarrow$ momentum constraint becomes

$$
\tilde{D}_{j} \tilde{A}^{i j}+6 \tilde{A}^{i j} \tilde{D}_{j} \ln \psi-\frac{2}{3} \tilde{D}^{i} K=8 \pi \psi^{4} p^{i}
$$

## "Momentum-constraint" scaling $\alpha=-10$

Momentum constraint: $D_{j} K^{i j}-D^{i} K=8 \pi p^{i}$
Now $D_{j} K^{i j}=D_{j} A^{i j}+\frac{1}{3} D^{i} K$ and

$$
\begin{aligned}
D_{j} A^{i j} & =\tilde{D}_{j} A^{i j}+C^{i}{ }_{j k} A^{k j}+C^{j}{ }_{j k} A^{i k} \\
& =\tilde{D}_{j} A^{i j}+2\left(\delta^{i}{ }_{j} \tilde{D}_{k} \ln \psi+\delta^{i}{ }_{k} \tilde{D}_{j} \ln \psi-\tilde{D}^{i} \ln \psi \tilde{\gamma}_{j k}\right) A^{k j}+6 \tilde{D}_{k} \ln \psi A^{i k} \\
& =\tilde{D}_{j} A^{i j}+10 A^{i j} \tilde{D}_{j} \ln \psi-2 \tilde{D}^{i} \ln \psi \underbrace{\tilde{\gamma}_{j k} A^{j k}}_{=0}
\end{aligned}
$$

Hence $D_{j} A^{i j}=\Psi^{-10} \tilde{D}_{j}\left(\Psi^{10} A^{i j}\right)$

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$$

Hence $D_{j} A^{i j}=\Psi^{-10} \tilde{D}_{j}\left(\Psi^{10} A^{i j}\right)$
This suggests to introduce

$$
\hat{A}^{i j}:=\Psi^{10} A^{i j} \quad \text { (Lichnerowicz 1944) }
$$

$\Longrightarrow$ momentum constraint becomes

$$
\tilde{D}_{j} \hat{A}^{i j}-\frac{2}{3} \psi^{6} \tilde{D}^{i} K=8 \pi \Psi^{10} p^{i}
$$

## Hamiltonian constraint as the Lichnerowicz equation

Hamiltonian constraint: $R+K^{2}-K_{i j} K^{i j}=16 \pi E$
Now $R=\Psi^{-4} \tilde{R}-8 \Psi^{-5} \tilde{D}_{i} \tilde{D}^{i} \Psi$ and $K_{i j} K^{i j}=\psi^{-12} \hat{A}_{i j} \hat{A}^{i j}+\frac{K^{2}}{3}$
so that

$$
\tilde{D}_{i} \tilde{D}^{i} \Psi-\frac{1}{8} \tilde{R} \Psi+\frac{1}{8} \hat{A}_{i j} \hat{A}^{i j} \Psi^{-7}+\left(2 \pi E-\frac{1}{12} K^{2}\right) \Psi^{5}=0
$$

This is Lichnerowicz equation (or Lichnerowicz-York equation).

## Summary: conformal $3+1$ Einstein system

Version $\alpha=-4$ (Shibata \& Nakamura 1995):

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \Psi=\frac{\Psi}{6}\left(\tilde{D}_{i} \beta^{i}-N K-\frac{\partial}{\partial t} \ln \tilde{\gamma}\right) \\
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \tilde{\gamma}^{i j}=2 N \tilde{A}^{i j}+\frac{2}{3} \tilde{D}_{k} \beta^{k} \tilde{\gamma}^{i j} \\
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) K=-\Psi^{-4}\left(\tilde{D}_{i} \tilde{D}^{i} N+2 \tilde{D}_{i} \ln \Psi \tilde{D}^{i} N\right) \\
& +N\left[4 \pi(E+S)+\tilde{A}_{i j} \tilde{A}^{i j}+\frac{K^{2}}{3}\right] \\
& \left(\begin{array}{l}
\left.\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \tilde{A}^{i j}=\Psi^{-4}\left[N\left(\tilde{R}^{i j}-2 \tilde{D}^{i} \tilde{D}^{j} \ln \Psi\right)-\tilde{D}^{i} \tilde{D}^{j} N\right]+\cdots \\
\left\{\begin{array}{l}
\tilde{D}_{i} \tilde{D}^{i} \Psi-\frac{1}{8} \tilde{R} \Psi+\left(\frac{1}{8} \tilde{A}_{i j} \tilde{A}^{i j}-\frac{1}{12} K^{2}+2 \pi E\right) \Psi^{5}=0
\end{array}\right.
\end{array} \tilde{D}_{j}^{i j}+6 \tilde{A}^{i j} \tilde{D}_{j} \ln \Psi-\frac{2}{3} \tilde{D}^{i} K=8 \pi \Psi^{4} p^{i}\right.
\end{aligned}
$$

## Summary: conformal $3+1$ Einstein system

Version $\alpha=-10$ :

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \Psi=\frac{\psi}{6}\left(\tilde{D}_{i} \beta^{i}-N K-\frac{\partial}{\partial t} \ln \tilde{\gamma}\right) \\
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \tilde{\gamma}^{i j}=2 N \tilde{A}^{i j}+\frac{2}{3} \tilde{D}_{k} \beta^{k} \tilde{\gamma}^{i j} \\
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) K=-\psi^{-4}\left(\tilde{D}_{i} \tilde{D}^{i} N+2 \tilde{D}_{i} \ln \psi \tilde{D}^{i} N\right) \\
& +N\left[4 \pi(E+S)+\tilde{A}_{i j} \tilde{A}^{i j}+\frac{K^{2}}{3}\right] \\
& \left(\frac{\partial}{\partial t}-\mathcal{L}_{\beta}\right) \tilde{A}^{i j}=\psi^{-4}\left[N\left(\tilde{R}^{i j}-2 \tilde{D}^{i} \tilde{D}^{j} \ln \psi\right)-\tilde{D}^{i} \tilde{D}^{j} N\right]+\cdots \\
& \left\{\begin{array}{l}
\tilde{D}_{i} \tilde{D}^{i} \Psi-\frac{1}{8} \tilde{R} \Psi+\frac{1}{8} \hat{A}_{i j} \hat{A}^{i j} \Psi^{-7}+\left(2 \pi E-\frac{1}{12} K^{2}\right) \Psi^{5}=0 \\
\tilde{D}_{j} \hat{A}^{i j}-\frac{2}{3} \Psi^{6} \tilde{D}^{i} K=8 \pi \Psi^{10} p^{i}
\end{array}\right.
\end{aligned}
$$

