3-Connected Line Graphs of Triangular Graphs are Panconnected and 1-Hamiltonian

H. J. Broersma
H. J. Veldman
TWENTE UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF APPLIED MATHEMATICS
P.O. BOX 217
7500 AE ENSCHEDE
THE NETHERLANDS

ABSTRACT

A graph is k-triangular if each edge is in at least k triangles. Triangular is a synonym for 1-triangular. It is shown that the line graph of a triangular graph of order at least 4 is panconnected if and only if it is 3-connected. Furthermore, the line graph of a k-triangular graph is k-hamiltonian if and only if it is $(k + 2)$-connected $(k \geq 1)$. These results generalize work of Clark and Wormald and of Lesniak-Foster. Related results are due to Oberly and Sumner and to Kanetkar and Rao.

1. PRELIMINARIES

We use [1] for basic terminology and notation, and consider simple graphs only. Let G be a graph. We will often identify a trail in G with the subgraph induced by its edges. Hence a subgraph T of G is a trail if and only if T is connected and at most two vertices of T have odd degree in T. An edge e of G is *dominated* by the trail T if e is incident with at least one vertex of T; $B(T)$ denotes the set of edges of G dominated by T and we write $b(T)$ for $|B(T)|$. A *dominating trail* or D-trail of G is a trail that dominates all edges of G, while a *spanning trail* or S-trail contains all vertices of G. A *circuit* is a nontrivial

Journal of Graph Theory, Vol. 11, No. 3, 399–407 (1987)
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closed trail. In particular, an S-circuit is the same as a spanning eulerian subgraph; in the literature, graphs with an S-circuit are often called *supereulerian*. We will speak of *line graphs* instead of edge graphs; the line graph of G is denoted $L(G)$. G is *k-triangular* if every edge of G is contained in at least k triangles ($k \geq 1$); G is *triangular* if G is 1-triangular. G is *panconnected* if G is connected and, for every pair (u, v) of distinct vertices of G, there exists a (u, v)-path of length k for each k with $d(u, v) \leq k \leq |V(G)| - 1$. G is *k-hamiltonian* if $G - U$ is hamiltonian for every subset U of $V(G)$ with $0 \leq |U| \leq k$ ($k \geq 0$). G is *locally connected* (*locally k-connected*) if, for each vertex v of G, the neighborhood $N(v)$ induces a connected (k-connected) subgraph ($k \geq 1$).

The following characterization of hamiltonian line graphs was obtained in [4].

Theorem 1. (Harary and Nash-Williams [4]). The line graph $L(G)$ of a graph G is hamiltonian if and only if either G has a D-circuit or G is isomorphic to $K_{1,s}$ for some $s \geq 3$.

The following lemmas, including those stated without proof, are easily established.

Lemma 2. Let e and f be distinct edges of a graph G and T a trail in G connecting an end of e and an end of f such that $e,f \notin E(T)$. Then $L(G)$ contains an (e,f)-path of length k for each k with $|E(T)| + 1 \leq k \leq b(T) - 1$.

Lemma 3. If G is a connected and k-triangular graph, then $L(G)$ is $(k + 1)$-connected ($k \geq 1$).

Proof. If G is connected and k-triangular, then G is $(k + 1)$-edge-connected and hence $L(G)$ is $(k + 1)$-connected. ▮

Lemma 4. If G is a nontrivial connected triangular graph, then G contains an S-circuit.

Proof. Assuming the contrary, let C be a circuit of G of maximum order, uv an edge of G with $u \notin V(C)$ and $v \in V(C)$, and D a triangle containing uv. Then the circuit C' with $E(C') = [E(C) \cup E(D)] - [E(C) \cap E(D)]$ contradicts the choice of C. ▮

Lemma 5. If G is a k-triangular graph, then $L(G)$ is locally k-connected ($k \geq 1$).

2. MAIN RESULTS

Theorem 6. Let G be a triangular graph of order at least 4. Then $L(G)$ is panconnected if and only if $L(G)$ is 3-connected.

Proof. Let G be a triangular graph with $|V(G)| \geq 4$. It is well known that every panconnected graph of order at least 4 is 3-connected. Hence, if $L(G)$ is panconnected, then, since $|V[L(G)]| \geq 4$, $L(G)$ is 3-connected.

Conversely, assume $L(G)$ is 3-connected, but not panconnected. Then, by Lemma 2, there exists a pair (e,f) of distinct edges of G and a smallest integer m with $d_{L(G)}(e,f) \leq m \leq |E(G)| - 1$ such that

(1) There is no trail T' between an end of e and an end of f with $|E(T')| + 1 \leq m \leq b(T') - 1$ and $e,f \notin E(T')$.

Obviously, $m > d_{L(G)}(e,f)$ and there is a trail T between an end of e and an end of f with $|E(T)| + 1 \leq m - 1 = b(T) - 1$ and $e,f \notin E(T)$. We additionally assume that

(2) T has maximum length under the given conditions.

Let $e = u_1u_2$ and $f = v_1v_2$, where u_1 is the origin of T and v_1 the terminus of T. We make a number of observations.

(3) If exactly one of the edges e and f has both ends on T, then no triangle of G contains both e and f.

Assuming the contrary to (3), suppose, e.g., there is a triangle D of G with $V(D) = \{u_1, u_2, v_2\}$, where $u_1, u_2 \in V(T)$, $v_2 \notin V(T)$ and $v_1 = u_i$ for $i = 1$ or $i = 2$. If v_2 is incident with an edge in $E(G) - B(T)$, then the trail T' with $E(T') = E(T) \cup \{u_{3-i}v_2\}$contradicts (1), while otherwise T' contradicts (2).

(4) G has no triangle D containing at most one of the edges e and f and satisfying $V(D) \cap V(T) \neq \emptyset$ and $E(D) \cap E(T) = \emptyset$.

Assuming the contrary to (4), the trail T' with $E(T') = [E(T) \cup E(D)] - [E(D) \cap \{e,f\}]$ contradicts (1) or (2).

(5) G has no triangle containing neither of the edges e and f and exactly one edge of T.

Assuming the contrary to (5), let D be a triangle of G with $E(D) \cap \{e,f\} = \emptyset$ and $E(D) \cap E(T) = \{g\}$. Then the trail T' with $E(T') = (E(T) \cup E(D)) - \{g\}$ contradicts (1) or (2).

(6) If some triangle of G contains
(a) a vertex incident with an edge in $E(G) - B(T)$, and
(b) an edge g of T, and
(c) exactly one of the edges e and f,
then g is a cut-edge of T.

Assuming the contrary to (6), suppose, e.g., there is a triangle D of G with $V(D) = \{u_1, u_2, u\}$, where u_2 is incident with an edge in $E(G) - B(T)$, $u_1u \in E(T)$, $u_2u \neq f$ and u_1u is not a cut-edge of T. Then the trail T' with $E(T') = [E(T) \cup \{u_2u\}] - \{u_1u\}$ contradicts (1).

(7) T is nontrivial.

Assume that T is trivial, so that $u_1 = v_1$. Let D be a triangle containing u_1. By (4), D contains both e and f, and is hence uniquely determined. It follows that $d(u_1) = 2$ and $u_2v_2 \in E(G)$. But then the trail u_2v_2 contradicts (1).

Since $b(T) = m \leq |E(G)| - 1$, $G - V(T)$ contains a nontrivial component H. By (7) and the fact that $L(G)$ is 3-connected, the edge cut $[V(H), V(T)]$ contains at least three edges. Let g be an edge in $[V(H), V(T)]$ with $g \notin \{e,f\}$ and D a triangle containing g. By (3), D does not contain both e and f. Hence, by (4), D contains an edge h of T. By (5), therefore, D contains exactly one of the edges e and f. Now (6) implies that h is a cut edge of T.

Assume, e.g., that $V(D) = \{u_1, u_2, u\}$, so that $g = u_2u$ and $h = u_1u$. In particular, e has exactly one end on T. We make another observation.

(8) f has exactly one end on T.

Assuming that f has both ends on T, $[V(H), V(T)]$ contains an edge $g' \notin \{e, g\}$. Let D' be a triangle containing g'. Just like D, D' contains e and an edge of T, h' say, which is a cut-edge of T. But then the origin u_1 of T is incident with two cut-edges of T, which is impossible.

Let T_1 be the component of $T - h$ containing u_1 and let T_2 be the other component of $T - h$, so that the unique vertex of T incident with f is in T_2. We make one more observation.

(9) There exists no edge vu_1 with $v \in V(G) - [V(T) \cup \{u_2\}]$.

Assuming the contrary to (9), let D' be a triangle containing vu_1. By (4), D' contains an edge of T. Hence $e \notin E(D')$ and, by (5), $f \in E(D')$. It follows that $V(D') = \{v, u_1, u\}$ and $f = uv$. But then the trail T' with $E(T') = E(T) \cup \{u_2u, u_1v\}$ contradicts (1).

T_1 cannot be trivial, otherwise, in view of (9), the trail T' with $E(T') = [E(T) \cup \{u_2u\}] - \{u_1u\}$ contradicts (1). Hence T_1 is a circuit. Since $|V(T_1)| \neq 1 \neq |\overline{V(T_1)}|$ and $L(G)$ is 3-connected, $|[V(T_1), \overline{V(T_1)}]| \geq 3$. Hence $[V(T_1), \overline{V(T_1)}]$ contains an edge xy with $x \in V(T_1)$ and $xy \notin \{u_1u, u_1u_2\}$. Since h is a cut-edge of T, $xy \notin E(T)$. Let D' be a triangle containing xy. We now establish the theorem by deriving contradictions in all possible cases.

Case 1. $x \neq u_1$.

Then x is incident with neither of the edges e and f, so that D' contains at most one of the edges e and f. By (4) and (8), $f \notin E(D')$. Two possibilities remain.

Case 1.1. $e \in E(D')$.

Then $V(D') = \{x, u_1, u_2\}$ and, by (4), $xu_1 \in E(T_1)$. (6) now implies that xu_1 is a cut-edge of T, contradicting the fact that T_1, being a circuit, is 2-edge-connected.

Case 1.2. $e \notin E(D')$.

Then, by (4) and (5), $|E(D') \cap E(T)| = 2$. It follows that $V(D') = \{x, u_1, u\}$ and $xu_1 \in E(T_1)$. Now the trail T' with $E(T') = (E(T) \cup \{u_2u, ux\}) - \{xu_1\}$ contradicts (1).

Case 2. $x = u_1$.

Then, by (9), $y \in V(T_2)$. By (4) and (8), it is impossible that D' contains exactly one of the edges e and f. Two possibilities remain.

Case 2.1. D' contains both e and f.

Then $V(D') = \{u_1, u_2, y\}$ and $f = yu_2$. Now the trail T' with $E(T') = [E(T) \cup \{u_2u, yu_1\}] - \{uu_1\}$ contradicts (1).

Case 2.2. D' contains neither e nor f.

Then, by (4) and (5), $|E(D') \cap E(T)| = 2$. It follows that $V(D') = \{u_1, u, y\}$ and $uy \in E(T_2)$. Now the trail T' with $E(T') = [E(T) \cup \{u_2u, u_1y\}] - \{uy\}$ contradicts (1). ∎

Corollary 7. (Clark and Wormald [3]). If G is a connected 2-triangular graph, then $L(G)$ is panconnected.

Proof Combine Lemma 3 and Theorem 6. ∎

Corollary 8. (Clark and Wormald [3]). If G is a 2-connected triangular graph, then $L(G)$ is panconnected.

Proof. Let $[S, \bar{S}]$ be an edge-cut of G with $|S| \neq 1 \neq |\bar{S}|$. Since G is 2-connected, $[S, \bar{S}]$ contains two nonadjacent edges. Since G is triangular, $[S, \bar{S}]$ must contain a third edge. It follows that $L(G)$ is 3-connected and we are done by Theorem 6. ∎

Let G be a connected graph with $\delta(G) \geq 3$. Then $L(G)$ is triangular. Furthermore, it is easily shown that $L[L(G)]$ is 3-connected if and only if G has no cut vertex of degree 3 (cf. [6]). Thus Theorem 6 has the following consequence also:

Corollary 9. Let G be a connected graph with $\delta(G) \geq 3$. Then $L[L(G)]$ is panconnected if and only if G contains no cut vertex of degree 3.

The next result shows that, in Theorem 6, "panconnected" may be replaced by "1-hamiltonian."

Theorem 10. Let G be a k-triangular graph ($k \geq 1$). Then $L(G)$ is k-hamiltonian if and only if $L(G)$ is $(k + 2)$-connected.

Proof. It suffices to prove the theorem for $k = 1$; the proof is then completed by induction on k, using Theorem 1 and Lemma 4. Since 1-hamiltonian graphs are necessarily 3-connected, it remains to establish sufficiency.

Let G be a triangular graph and assume $L(G)$ is 3-connected, but not 1-hamiltonian. By Theorem 1 and Lemma 4, $L(G)$ is hamiltonian, so there is a vertex e of $L(G)$ such that $L(G) - e$ is nonhamiltonian. e is an edge of G and, by Theorem 1, $G - e$ has no D-circuit. Let C be a circuit of $G - e$ such that

(10) C has maximum order among all circuits of $G - e$.

We make the following observation:

(11) $G - e$ has no cycle C_0 satisfying
$V(C_0) \cap V(C) \neq \emptyset \neq V(C_0) \cap [V(G) - V(C)]$ and $|E(C_0) \cap E(C)| \leq 1$.

Assuming the contrary to (11), the circuit C' with $E(C') = [E(C) \cup E(C_0)] - [E(C) \cap E(C_0)]$ contradicts (10).

Since C is not a D-circuit, $G - V(C)$ has a nontrivial component H. Since $L(G)$ is 3-connected, $|[V(H), V(C)]| \geq 3$. Hence there exist two distinct triangles D_1 and D_2 with $E(D_i) \cap [V(H), V(C)] \neq \emptyset$ ($i = 1, 2$). By (11), both D_1 and D_2 contain e. $(D_1 \cup D_2) - e$ is a cycle of length 4, which, by (11), has two edges in common with C. It follows that e has exactly one end on C, v say. Let u be the other end of e and v_i the unique vertex in $V(D_i) - \{u, v\}$, so that $vv_i \in E(C)$ ($i = 1, 2$).

We now show that D_1 and D_2 are the only triangles of G containing edges of $D_1 \cup D_2$. It then follows that every nontrivial component of $G - [E(D_1) \cup E(D_2)]$ is triangular and hence, by Lemma 4, contains an S-circuit. The proof is then completed by noting that the union of $(D_1 \cup D_2) - e$ with S-circuits of the nontrivial components of $G - [E(D_1) \cup E(D_2)]$ is an S-circuit of $G - e$, contradicting the fact that not even $G - e$ has a D-circuit.

Suppose G contains a triangle D with $D \neq D_1, D_2$ and $E(D) \cap [E(D_1) \cup E(D_2)] \neq \emptyset$. By (11), D contains neither of the edges uv_1 and uv_2. We derive contradictions in two cases.

Case 1. D contains e.

Let v_3 be the vertex in $V(D) - \{u, v\}$. From (11) we deduce that $vv_3 \in E(C)$. Let $F = \{vv_1, vv_2, vv_3\}$. F contains two edges f and g such that $C - \{f, g\}$ is a connected subgraph of G: if all edges of F belong to the same block of C, then f and g are arbitrarily chosen from F, whereas in the opposite case f and g are chosen in different blocks of C. If, e.g., $f = vv_1$ and $g = vv_2$, then it follows that the subgraph C' of G with $V(C') = V(C) \cup \{u\}$ and $E(C') = [E(C) \cup \{uv_1, uv_2\}] - \{vv_1, vv_2\}$ is a circuit that contradicts (10).

Case 2. D contains vv_1 or vv_2.

Suppose, e.g., D contains vv_1. By (11), v_1 and v_2 are nonadjacent, so D contains a vertex w with $w \notin V(D_1) \cup V(D_2)$. Again by (11), at least one of the edges vw and v_1w is in $E(C)$. If $vw \in E(C)$ and $v_1w \notin E(C)$, then the circuit with edge set $(E(C) \cup \{uv_1, uv_2, v_1w\}) - \{vv_2, vw\}$ contradicts (10). If $vw \notin E(C)$ and $v_1w \in E(C)$, then the circuit with edge set $[E(C) \cup \{uv_1, uv_2, vw\}] - \{vv_2, v_1w\}$ contradicts (10). Finally, if $vw \in E(C)$ and $v_1w \in E(C)$, then (10) is violated by the circuit with edge set $[E(C) \cup \{uv_1, uv_2\}] - \{vv_1, vv_2\}$. ∎

Corollary 11. (Lesniak-Foster [6]). Let G be a connected graph with $\delta(G) \geq 3$. Then $L[L(G)]$ is $(\delta(G) - 3)$-hamiltonian.

Proof. $L(G)$ is connected and $[\delta(G) - 2]$-triangular, so that, by Lemma 3, $L[L(G)]$ is $[\delta(G) - 1]$-connected. The proof is completed by applying Theorem 1 and Lemma 4 in case $\delta(G) = 3$ and Theorem 10 in case $\delta(G) \geq 4$. ∎

Note that Theorem 10 also implies that, if G is a connected graph with $\delta(G) \geq 3$ such that $L[L(G)]$ is not $[\delta(G) - 2]$-hamiltonian, then $L[L(G)]$ is not $\delta(G)$-connected.

The next consequence of Theorem 10 is completely analogous to Corollary 9.

Corollary 12. (Lesniak-Foster [6]). Let G be a connected graph with $\delta(G) \geq 3$. Then $L[L(G)]$ is 1-hamiltonian if and only if G contains no cut vertex of degree 3.

Via a trivial variation on the proof of Theorem 10 we obtain the following result:

Theorem 13. Let G be a k-triangular graph ($k \geq 1$). Then the following statements are equivalent:

(i) $G - F$ contains an S-circuit for every subset F of $E(G)$ with $0 \leq |F| \leq k$;
(ii) G is $(k + 2)$-edge-connected.

3. RELATED RESULTS AND CONJECTURES

Oberly and Sumner [7] have shown that every connected, locally connected graph of order at least 3 containing no induced $K_{1,3}$ is hamiltonian. Via induction on k one immediately obtains the following generalization:

Theorem 14. If G is a connected, locally k-connected graph of order at least 3 containing no induced $K_{1,3}$, then G is $(k - 1)$-hamiltonian $(k \geq 1)$.

In view of Lemma 5 and the fact that no line graph contains an induced $K_{1,3}$, Corollary 11 is a consequence of Theorem 14, too. Likewise, Corollary 7 is also implied by the following result:

Theorem 15. (Kanetkar and Rao [5]). If G is a connected, locally 2-connected graph containing no induced $K_{1,3}$, then G is panconnected.

By Lemma 5, Theorem 6 is a special case of the following conjectured improvement on Theorem 15:

Conjecture 16. Let G be a connected, locally connected graph of order at least 4 containing no induced $K_{1,3}$. Then G is panconnected if and only if G is 3-connected.

The next conjecture is analogous to Conjecture 16.

Conjecture 17. Let G be a connected, locally k-connected graph containing no induced $K_{1,3}$ $(k \geq 1)$. Then G is k-hamiltonian if and only if G is $(k + 2)$-connected.

Conjecture 17 is more general than Theorem 10. In [2] it was shown that every connected, locally k-connected graph is $(k + 1)$-connected $(k \geq 1)$. Hence Conjecture 17 also generalizes Theorem 14 for $k \geq 2$. Again, it suffices to prove Conjecture 17 for $k = 1$.

Finally, consider the following statement:

(*) Let G be a k-triangular graph $(k \geq 1)$. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s + 2)$-connected.

By Theorem 10, (*) is true for $s \leq k$. For given k, it would be interesting to know for which values of s (*) holds. If, e.g., it were shown that, for each $k \geq 2$, (*) holds for $s = 2k$, then the following result of Lesniak-Foster [6] would be generalized: if G is a 2-connected graph with $\delta(G) \geq 4$, then $L[L(G)]$ is $[2\delta(G) - 4]$-hamiltonian. Similarly, improvements on Conjecture 17 may be possible.

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