Transactions on Modelling and Simulation vol 3, © 1993 WIT Press, www.witpress.com, ISSN 1743-355X

# 3-D magnetic field of coils with an open metallic core in boundary-integral approach

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## ABSTRACT

Boundary-integral approach to 3-D stationary magnetic fields exited by coils and/or current-carrying rings is presented. Metallic elements placed variously in neighbours of the coils are considered like inner boundaries for outer magnetic field. The boundary-integral model is formulated for indirect boundary quantities: monopole and/or dipole surface densities of the 'magnetic charge'. The author's BIMS package has been performed. Some results of computation of the magnetic field in an open magnetic core put inside or outside of the coils, axially or not, are presented. The original post-processing of the package affords possibilities to accomplish the particular design, rather untypical in engineering practice. This is a matter, among other things, that the coil inductance influenced by both the magnetic flux in the open core and the surface currents on screen plates can be precisely calculated.

## INTRODUCTION

Boundary-integral model is to be effectively applied in order to study some parts of electric devices consisting of coils, current-carrying rings and metallic elements placed in the neighbours of them. The 3-D patterns that may be generally devoid of symmetry are considered. The boundary-integral model has been formulated for indirect boundary quantities: $\sigma_m$ monopole surface density of the 'magnetic charge' (in teslas) and $\tau_m$ dipole surface density of the 'magnetic charge' (in teslas metres) generating single- and double-layer potentials respectively - see *Brebbia & co-authors* [11]. Both these quantities have fictitious nature but it has been proved that using them is useful for the model conception and effective for the computer code algorithms. The idea of the use of magnetic surface densities like the boundary variables was presented by *Tozoni* [8] as a secondary sources method on the form of two alternative models of stationary fields. The author observed, following the *Courant &*

*Gilbert* [1] fundamentals and *Jaswon* [2] formulations, that one common model obeying the secondary sources approach may be created in which both types of the magnetic charge surface densities are applied. This idea was comprehensively presented in [3] and shortly marked in [4]. Same test results pertaining to the distribution of the magnetic flux density on an open magnetic core have been reported in [6].

The monopole surface density $\sigma_m$ is defined in any point P on the boundary $\Gamma_f$ formed by the linear ferromagnetic body characterized by relative magnetic constant $\mu_{rf} \gg 1$. This quantity can be realized by supposition that a vary thin transition layer has being close against the body surface. Assuming that $\mu_r$ varies continuously from 1 to $\mu_{rf}$ crosswise the layer we gather that for infinite thin layer the monopole surface density of the magnetic charge may be imagined. The defining formula for $\sigma_m$ uses the normal magnetic strength at two points $P_-$ and $P_+$ at both sides of $\Gamma_f$

$$\sigma_m(P) = -\mu_0 (1 - \frac{1}{\mu_{rf}}) H_\perp(P_-) = -\mu_0 (\mu_{rf} - 1) H_\perp(P_+) \quad (1)$$

expressing correctly the interface conditions for normal components of the magnetic field strength. In similar way the interface conditions for tangential components of the magnetic flux density in any point P on the boundary $\Gamma_d$ formed by pseudo-diamagnetic medium of $\mu_{rd} \ll 1$, are described by the surface gradient of $\tau_m$

$$\nabla_\| \tau_m(P) = (1 - \mu_{rd}) \boldsymbol{B}_\|(P_-) = (\frac{1}{\mu_{rd}} - 1) \boldsymbol{B}_\|(P_+) \quad (2)$$

The boundary variables obey the second kind Fredholm's equations set that with the simplest embedded gauge has the following form

$$\begin{aligned} \sigma_m(P) + \int_{\Gamma_f} K_f(P,Q)\, \sigma_m(Q)\, dQ + \int_{\Gamma_d} K_f'(P,Q)\, \tau_m(Q)\, dQ &= -u_\sigma(P) \\ \tau_m(P) - \int_{\Gamma_f} K_d'(Q,P)\, \sigma_m(Q)\, dQ - \int_{\Gamma_d} K_d(P,Q)\, \tau_m(Q)\, dQ &= +u_\tau(P) \end{aligned} \quad (3)$$

where the kernels are given by formulae (4) and (5). They are expressed as relevant partial derivatives of the position vector $r_{QP}$ with respect to $n$ and $\nu$ symbolizing the local co-ordinates externally normal to the boundaries at the points P (field point) and Q (source point) respectively

$$K_f(P,Q) = \frac{\lambda_f}{2\pi} \frac{\partial}{\partial n} |\boldsymbol{r}_{QP}|^{-1} \quad ; \quad K_f'(P,Q) = \frac{\lambda_f}{2\pi} \frac{\partial^2}{\partial n \partial \nu} |\boldsymbol{r}_{QP}|^{-1} \quad (4)$$

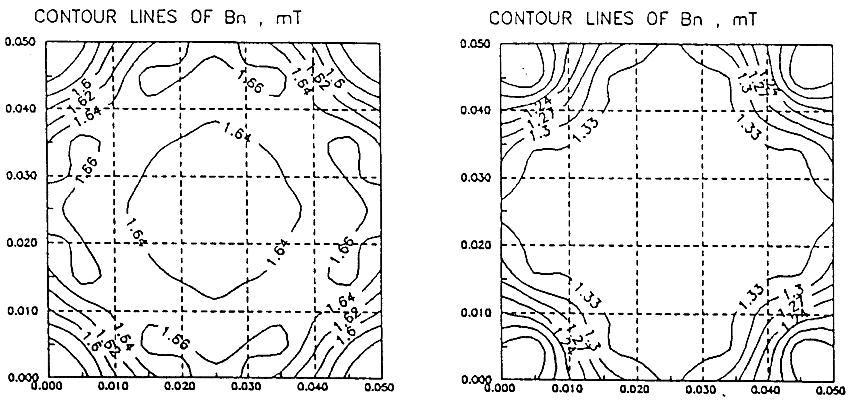
$$K_d'(P,Q) = \frac{\lambda_d}{2\pi}|r_{QP}|^{-1} \quad ; \quad K_d(P,Q) = \frac{\lambda_d}{2\pi}\frac{\partial}{\partial v}|r_{QP}|^{-1} \tag{5}$$

where the permeability factors $\lambda_f$ and $\lambda_d$ in (4) and (5) are

$$\lambda_f = \frac{1-\mu_{rf}}{1+\mu_{rf}} \quad ; \quad \lambda_d = \frac{1-\mu_{rd}}{1+\mu_{rd}} \tag{6}$$

and it is realized that for the ideal ferromagnetic medium $\mu_{rf} \to \infty$ what implies that $\lambda_f \to -1$ and for ideal diamagnetic medium (modelling supra-conducting screens) $\mu_{rd} \to 0$ implying $\lambda_d \to 1$. The right-hand-side terms of equations (3) $u_o(P)=2\mu_0\lambda_f H_\perp(P)$ and $u_r(P)=2\mu_0\lambda_d\psi_m(P)$ are to be determined on the positions of the core and screen boundaries with the assumption that the same core and screen elements are removed from the system under consideration.

The authors have performed the package termed BIMS based on the above magnetostatic model appropriated for design the structures composed of coils, cores and screens. The chosen computational results are shown in fig.1. The distribution of $\sigma_m$ has been computed like the solution of equations (3). The magnetic flux distribution calculated accordingly to the secondary sources approach as the joint effect of both sources: the given ring current exciting the magnetic flux in free space and $\sigma_m$ appointing $\mu_0 H_\perp$ accordingly to formula (1) under assumption that $\mu_{rf}$ = 1000. The results have been submitted to a kriging treatment to form the contour lines of $B_\perp$.




(a) (b)

Figure 1: Magnetic flux density on a front wall of the open magnetic cores of lengths: (a) - 20 mm, (b) - 25 mm excited by 20 A carrying by the ring of average radius equal to 35,5 mm and the ring body cross-section 2,5; 0,3 mm.

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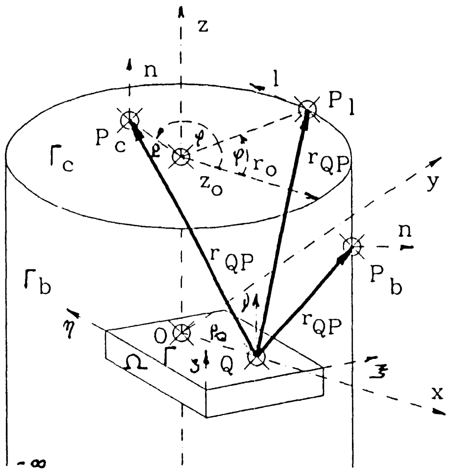
## MAGNETIC FLUX DUE TO BOUNDARY QUANTITIES

The intrinsic component of the ring-shaped coil inductance can be approximately calculated accordingly to commonly known algorithms. In author's paper [5] some preciseness pertaining to the inductance of the conducting ring has been reported. Let us develop this problem when the screen and cores are present in the neighbours of rings or ring-shaped coils, with reference to the BIMS package. The post-processing codes of the package are appropriated for engineering design. Besides the quantities directly obtained by solution of equations (3) some integral quantities are need for practice. Let us now to fix the main subject of the paper upon the algorithm for computing the magnetic flux of the ring influenced by the monopole and/or dipole surface densities of the magnetic charge. Fig.2 shows the outline of the object under consideration.




$$P_l(r_0\cos\varphi, r_0\sin\varphi, z_0)$$
$$e_l = - e_x\sin\varphi + e_y\cos\varphi$$

$$P_c(\varrho\cos\varphi, \varrho\sin\varphi, z_0)$$
$$e_n = e_z$$

$$P_b(r_0\cos\varphi, r_0\sin\varphi, z)$$
$$e_n = e_x\cos\varphi + e_y\sin\varphi$$

$$Q(\rho_Q, 0, 0)$$
$$e_\nu = e_\zeta$$

Figure 2: Model configuration

### Magnetic flux due to dipole density

The magnetic flux $\Phi_c$ has to be determined through the surface $\Gamma_c$ inside the circle $\mathcal{L}_c$ of radius $r_0$. The flux is generated by $\tau_m$ distributed on the surface $\Gamma_d$ of the metallic element $\Omega$ situated at some distance from $\mathcal{L}_c$. The surface distribution of $\tau_m$ is seen as being previously determined. Considering the rectangles mesh on the whole $\Gamma_d$ we determine $mn^2$ points $Q_i$, where: m - number of rectangles, n - order of the applied Legendre's polynomial determining the *Gauss*' quadrature. Let to each point $Q_i \in \Gamma_d$ the elementary magnetic dipole moment $j_i = j_i e_\nu$ (in teslas metres cube) be attributed that has obviously one non zero component, externally normal to $\xi$ $\eta$ plane

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$$j_i = \frac{S_i}{4} w_{jk} \tau_m(\xi_j, \eta_k) \tag{7}$$

where: $S_i$ - surface of the mesh rectangle, $w_{jk}$ - *Gauss'* quadrature weights, $\xi_j$, $\eta_k$ - local co-ordinates on the rectangle. The total magnetic flux $\Phi_c$ on $\Gamma_c$ shall be the sum of $mn^2$ elementary fluxes $\Phi_{ci}$ involved by $j_i$. Relating to the $x,y,z$ co-ordinate system marked on Fig. 2 we determine $Q_i$ like the source point $Q_i(\rho_Q,0,0) \in \Gamma_d$ and the field point $P_l(r_0\cos\varphi,\ r_0\sin\varphi,\ z_0) \in \mathcal{L}_c$, then the following Stokes' theorem is valid

$$\Phi_{ci} = \int_{\mathcal{L}_c} \mathbf{A}\cdot d\mathbf{l} = \int_0^{2\pi} A_\varphi r_0 \, d\varphi \tag{8}$$

where the magnetic vector potential is expressed

$$\mathbf{A} = \frac{1}{4\pi} \frac{\mathbf{r}_{PQ} \times j_i \mathbf{e}_\nu}{|\mathbf{r}_{QP}|^3} \tag{9}$$

The unit vector normal to $\Gamma_d$ at $Q_i$ is

$$\mathbf{e}_\nu = c_x \mathbf{e}_x + c_y \mathbf{e}_y + c_z \mathbf{e}_z \tag{10}$$

where $c_x$ , $c_y$ , $c_z$ - direction cosines between $\nu$-axis and the $x,\ y,\ z$ co-ordinate axes respectively. After necessary substitutions and formal operations the tangential component of the magnetic vector potential is to be written

$$A_\varphi = \frac{j_i}{4\pi} \frac{c_y z_0 \sin\varphi + (c_x z_0 + c_z \rho_Q)\cos\varphi - c_z r_0}{[(r_0\cos\varphi - \rho_Q)^2 + (r_0 \sin\varphi)^2 + z_0^2]^{3/2}} \tag{11}$$

that leads up to two parts of integral formulae (8) for $\Phi_{ci} = \Phi_{i1} + \Phi_{i2}$. The first term comprising both the second and third members of the numerator of formula (11) can be transformed into a general complete elliptical integral that using the *Press & co-authors* [10] notation

$$\int_0^{\frac{\pi}{2}} \frac{(\cos^2\psi + b\sin^2\psi)\, d\psi}{(\cos^2\psi + p\sin^2\psi)\sqrt{1 - k\sin^2\psi}} = \mathrm{cel}(k_c, p, a, b)$$

is to written

$$\Phi_{i1} = F_{i1} \mathrm{cel}(k_c, p, a, b) \tag{12}$$

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where: $F_{i1} = \dfrac{j_i r_0 [c_z(\rho_Q + r_0) + c_x z_0]}{\pi[(r_0+\rho_Q)^2 + z_0^2]^{3/2}}$ ; $k^2 = 1 - k_c^2 = \dfrac{4 r_0 \rho_Q}{(r_0+\rho_Q)^2 + z_0^2}$

and $p = k_c^2$ ; $a = 1$ ; $b = \dfrac{c_z(r_0 - \rho_Q) - c_x z_0}{c_z(r_0+\rho_Q) + c_x z_0}$ . Thus, the relevant computer code for formula (12) is to be simply performed. The sin-member of the numerator of formula (11) is a pseudo-elliptic integral of exact analytical solution. Expressing this integral in the notation similar to (12) we have

$$\int_0^{\frac{\pi}{2}} \frac{\sin\psi\cos\psi\, d\psi}{(\cos^2\psi + p\sin^2\psi)\sqrt{1 - k\sin^2\psi}} = \mathrm{pel}(k,p)$$

and finely

$$\Phi_{i2} = F_{i2}\,\mathrm{pel}(k,p) \tag{13}$$

where: $F_{i2} = \dfrac{2 j_i r_0 c_j z_0}{\pi[(r_0+\rho_Q)^2 + z_0^2]^{3/2}}$ and both $k$ and $p=1\text{-}k^2$ like in formula (12) what gives $\mathrm{pel}(k,p) = \dfrac{1}{k}\left(\dfrac{1}{\sqrt{1-k^2}} - 1\right)$ . In many practical configurations is $c_j = 0$ implying $\Phi_{i2} = 0$ but it is the case of $\Gamma_d$ either parallel or perpendicular to $\Gamma_c$. The use of formulae (12)-(13) is not restricted by the position of the source point $Q_i$ except it is really adjacent to $\mathcal{L}_c$ that does rather not happen in practice.

Magnetic flux due to monopole density

The use of formula (8) to compute the magnetic flux generated by $\sigma_m$ of the previously calculated distribution is hardly possible because the magnetic vector potential is not simply expressible by the monopole surface density of the magnetic charge. Let us present the algorithm idea leading up to an effective computing this magnetic flux and, in consequence, the inductance of the coil under consideration. We refer ones again to Fig. 2.

Let the surface distribution of $\sigma_m$ on the walls $\Gamma_f$ of a ferromagnetic core element be calculated by the BIMS package or it is determined in any other way. Consider the rectangle mesh on the whole $\Gamma_f$ and determine $mn^2$ *Gauss'* quadrature points $Q_i$ for each of them we should, refereeing to formula (7), attribute the elementary magnetic charge $Q_{mi}$ (in teslas metres squared) as follows

$$Q_{mi} = \frac{S_i}{4} w_{jk} \sigma_m(\xi_j, \eta_k) \tag{14}$$

The cylindrical surface $\Gamma_b$ based on $\mathscr{L}_c$ and expanded to minus infinity is marked in Fig. 2. Providing foı various possible positions of $\Omega$ we should consider two magnetic fluxes: $\Phi_c$ through $\Gamma_c$ and $\Phi_b$ through $\Gamma_b$, both joint with the magnetic charge placed inside $\Gamma_b \bigcap \Gamma_c$ by the *Gauss'* law: $Q_m = \Phi_b + \Phi_c$. To express both of them in dependance upon the elementary magnetic charge we have

$$\Phi_{bi} = \frac{Q_{mi}}{4\pi} \int_0^{2\pi} \int_{z_0}^{-\infty} \frac{\boldsymbol{r}_{QP} \cdot \boldsymbol{e}_n}{|\boldsymbol{r}_{QP}|^3} r_0 \, dz \, d\varphi \quad ; \quad \Phi_{ci} = \frac{Q_{mi}}{4\pi} \int_0^{2\pi} \int_0^{r_0} \frac{\boldsymbol{r}_{QP} \cdot \boldsymbol{e}_n}{|\boldsymbol{r}_{QR}|^3} \rho \, d\rho \, d\varphi \tag{15}$$

where $P_b$, $P_c$ and $e_n$ for relevant fluxes are marked in fig. 2. The first formula of (15) is preferable; it may be integrated almost everywhere except, owing to singularity, the points $Q_i \in \Gamma_b$. Then, with regard to the *Gauss'* law the following result for $\Phi_{ci}$ is obtained

$$\Phi_{ci} = Q_{mi} - \Phi_{bi} = \frac{Q_{mi}}{2} \begin{cases} 1 - k_b \mathrm{cel}(k_c, p, a, b) & , \text{ if } \rho_Q < r_0 \\ k_b \mathrm{cel}(k_c, p, a, b) & , \text{ if } \rho_Q > r_0 \end{cases} \tag{16}$$

where the factor

$$k_b = \frac{2 r_0 z_0}{\pi (r_0 + \rho_Q) \sqrt{(r_0 + \rho_Q)^2 + z_0^2}} \tag{17}$$

and $\mathrm{cel}(k_c,p,a,b)$ - general complete elliptic integral of the following parameters: $k_c^2 = 1 - 4r_0\rho_Q/[(r_0+\rho_Q)^2+z_0^2]$ ; $b=(r_0-\rho_Q)/(r_0+\rho_Q)$ ; $a=1$ ; $p=b^2$.

In the case when formula (16) is not applicable ($\rho_Q = r_0$) the second integral of (15) has got to be used for direct determination of $\Phi_c$. After the relevant integral operations the following resulting formula is obtained

$$\Phi_{ci} = \frac{Q_{mi}}{2} [1 + k_d \mathrm{nel}(k_c, p, a, b)] \tag{18}$$

where $\mathrm{nel}(k_c,p,a,b)$ is special non-elliptic integral of the same $k_c^2$ like one in formula (16). The integral $\mathrm{nel}(k_c,p,a,b)$ differs from $\mathrm{cel}(k_c,p,a,b)$ since in the term under the sign of integration the member $\cos^2 2\psi + p\sin^2 2\psi$ is present in the denominator of nel, before $\cos^2\psi p\sin^2\psi$ occurring in cel. Other parameters are: $b=[\rho_Q(r_0-\rho_Q)-z^2]/[\rho_Q(r_0+\rho_Q)+z_0]$ ; $a=-1$. The factor before $\mathrm{nel}(k_c,p,a,b)$ is

$$k_d = \frac{2\,[\rho_Q(\rho_Q + r_0) + z_0^2]}{\pi\, z_0\sqrt{(\rho_Q + r_0)^2 + z_0^2}} \tag{19}$$

In the BIMS package post-processing code the integral hel($k_c,p,a,b$) has been approximated by the *Weddle's* rule based upon the 6-order *Newton-Cotes'* quadrature - see *Korn & Korn* [8], especially appropriated at 96 points of a circular argument.

## COMPUTATIONAL EXPERIMENTS

The above presented algorithms haves been tested to compute the contribution of $\sigma_m$ and $\tau_m$ to the total inductance of conducting rings.

The first example pertains to the conducting ring of average radius equal to 100 mm. The radius of circular section of the ring body is 1 mm. The inductance of the ring in free space calculated by the use of *author's* [5] program is just 0,387 $\mu$H. An open magnetic core on the form of ferro-magnetic box of square cross-section is placed symmetrically inside the ring. The core length is just 140 mm and three transversal sizes of it are considered: 50 × 50, 60 × 60 and 70 × 70 mm. For the relative magnetic constant of the core $\mu_{rf}$=1000 the following contributions have been obtained: 0,028; 0,049 and 0,074 $\mu$H increasing the ring inductance by 7,2; 12,6 and 19,1 % respectively. The following results have been obtained when the magnetic core was replaced for an electromagnetic screen formed by the metallic non-ferromagnetic box of high conductivity. Supposing that the frequency of the ring current is sufficiently high we apply an almost ideal diamagnetic model of $\mu_{rd}$=0.001 to determine the dipole surface density of the magnetic charge on the box walls. Thus, the equivalent effect of the induced surface current can be effectively calculated. For the screen sizes identical to the core ones we obtained the following contributions: 0,018; 0,021; and 0,026 $\mu$H decreasing the ring inductance by 4,6; 5,4 and 6,2 % respectively.

The second example concerns the above ring equipped with outer magnetic core formed by the ferromagnetic box of sizes 240 × 240 × 100 mm. The ring is placed 40, 30 and 20 mm apart parallel to the largest box wall. The computed contributions to the ring inductance are 0,180; 0,269 and 0,448 $\mu$H thus, it is considerably augmented of 46,5; 69,5 and 115,7 % respectively. For the diamagnetic model of the electromagnetic screen of the same sizes the magnetic dipole contributions are 0,109; 0,174 and 0,303 $\mu$H decreasing the ring inductance by 24,3; 38,8 and 67,6 %. When the ring is placed perpendicularly to the core wall in the distance of 120 and 110 mm than the only slight influence of the core is observed, namely 0,022 and 0,024 $\mu$H that means the increasing of the core inductance of 5,7 and 6,2 %. For the screen replacing the core we have 0.0155 and 0,0216 $\mu$H determining the decrease in the ring inductance by 3,5 and 4,8 %.

## CONCLUSIONS

As well the above mentioned test examples as the other design results have proved that both the indirect boundary-integral approach based on the use of the secondary sources method [3],[4] and the inductance determination studied more thoroughly [5]-[7] are perfectly suitable to perform the computer codes. The BIMS package for all his test features enabled us to confirm the general correctness of the theoretical approach and revealed to be useful for design practice.

## ACKNOWLEDGEMENTS

The author is grateful to the *Polish State Committee for Scientific Research* for financial support for grant no 307969101, which made this research possible.

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