

3-MANIFOLD GROUPS

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ABSTRACT. We summarize properties of 3-manifold groups, with a particular focus on the consequences of the recent results of Ian Agol, Jeremy Kahn, Vladimir Markovic and Dani Wise.

INTRODUCTION

In this paper we give an overview of properties of fundamental groups of compact 3-manifolds. This class of groups sits between the class of fundamental groups of surfaces, which for the most part are well understood, and the class of fundamental groups of higher dimensional manifolds, which are very badly understood for the simple reason that given any finitely presented group π and any $n \geq 4$, there exists a closed n -manifold with fundamental group π . (See [CZi93, Theorem 5.1.1] or [ST80, Section 52] for a proof.) This basic fact about high-dimensional manifolds is the root of many problems; for example, the unsolvability of the isomorphism problem for finitely presented groups [Ad55, Rab58] implies that closed manifolds of dimensions greater than three cannot be classified (see [Mav58, Mav60]).

The study of 3-manifold groups is also of great interest since for the most part, 3-manifolds are determined by their fundamental groups. More precisely, a closed, irreducible, non-spherical 3-manifold is uniquely determined by its fundamental group (see Theorem 2.3).

Our account of 3-manifold groups is based on the following building blocks:

- (1) If N is an irreducible 3-manifold with infinite fundamental group, then the Sphere Theorem (see (C.1) below), proved by Papakyriakopoulos [Pap57a], implies that N is in fact an Eilenberg–Mac Lane space. It follows, for example, that $\pi_1(N)$ is torsion-free.
- (2) The work of Waldhausen [Wan68a, Wan68b] produced many results on the fundamental groups of Haken 3-manifolds, e.g., the solution to the word problem.
- (3) The Jaco–Shalen–Johannson (JSJ) decomposition [JS79, Jon79] of an irreducible 3-manifold with incompressible boundary gave insight into the subgroup structure of the fundamental groups of Haken 3-manifolds and prefigured Thurston’s Geometrization Conjecture.
- (4) The formulation of the Geometrization Conjecture and its proof for Haken 3-manifolds by Thurston [Thu82a] and in the general case by Perelman [Per02, Per03a, Per03b]. In particular, it became possible to prove that

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3-manifold groups share many properties with linear groups: they are residually finite [Hem87], they satisfy the Tits Alternative (see (C.20) and (K.2) below), etc.

- (5) The solutions to Marden’s Tameness Conjecture by Agol [Ag07] and Calegari–Gabai [CG06], combined with Canary’s Covering Theorem [Can96] implies the Subgroup Tameness Theorem (see Theorem 5.2 below), which describes the finitely generated, geometrically infinite subgroups of fundamental groups of finite-volume hyperbolic 3-manifolds. As a result, in order to understand the finitely generated subgroups of such hyperbolic 3-manifold groups, one can therefore mainly restrict attention to the geometrically finite case.
- (6) The results announced by Wise [Wis09], with proofs provided in the preprint [Wis12] (see also [Wis11]), revolutionized the field. First and foremost, they imply the Virtually Fibered Conjecture for Haken hyperbolic 3-manifolds. Wise in fact proves something stronger, namely that if N is a hyperbolic 3-manifold with an embedded geometrically finite surface then $\pi_1(N)$ is *virtually compact special*—see Section 5.3 for the definition. As well as virtual fibering, this also implies that $\pi_1(N)$ is LERF and large, and has some unexpected corollaries: for instance, $\pi_1(N)$ is linear over \mathbb{Z} .
- (7) Agol [Ag12], building on the proof of the Surface Subgroup Conjecture by Kahn–Markovic [KM09] and the aforementioned work of Wise, recently gave a proof of the Virtually Haken Conjecture. Indeed, he proves that the fundamental group of any closed hyperbolic 3-manifold is virtually compact special.

Despite the great interest in 3-manifold groups, survey papers seem to be few and far between. We refer to [Neh65], [Sta71], [Neh74], [Hem76], [Thu82a], [CZi93, Section 5], [Ki97] for some results on 3-manifold groups and lists of open questions.

The goal of this paper is to fill what we perceive as a gap in the literature, and to give an extensive overview of results on fundamental groups of compact 3-manifolds with a particular emphasis on the impact of the Geometrization Theorem of Perelman, the Tameness Theorem of Agol, Calegari–Gabai and Virtually Compact Special Theorem of Agol [Ag12], Kahn–Markovic [KM09] and Wise [Wis12]. Our approach is to summarize many of the results in several diagrams and to provide detailed references for each implication appearing in these diagrams. We will mostly consider results of a ‘combinatorial group theory’ nature that hold for fundamental groups of 3-manifolds which are either closed or have toroidal boundary. We do not make any claims to originality—all results are either already in the literature, or simple consequences of established facts, or well-known to the experts.

As with any survey, this paper reflects the tastes and biases of the authors. The following lists some of the topics which we leave basically untouched:

- (1) Fundamental groups of non-compact 3-manifolds. Note though that Scott [Sco73b] showed that given a 3-manifold M with finitely generated fundamental group, there exists a compact 3-manifold with the same fundamental group as M .

- (2) ‘Geometric’ and ‘large scale’ properties of 3-manifold groups; see, e.g., [Ge94, KaL97, KaL98, BN08, BN10, Si11] for some results in this direction. We also leave aside automaticity, formal languages, Dehn functions and combings: see, for instance, [Brd93, BrG96, Sho92, CEHLPT92].
- (3) Three-dimensional Poincaré duality groups; see, e.g., [Tho95, Dav00, Hil11] for further information.
- (4) Specific properties of fundamental groups of knot complements (known as ‘knot groups’). We note that in general, irreducible 3-manifolds with boundary are not determined by their fundamental groups, but interestingly prime knots in S^3 are in fact determined by their groups [GL89, Whn87]. Knot groups were some of the earliest and most popular examples of 3-manifold groups to be studied. We refer to [Neh65, Neh74] for a summary of some early work, and to [Str74] and [SWW10] for a sample of interesting papers specifically focussing on knot groups.
- (5) Fundamental groups of distinguished classes of 3-manifolds. For instance, arithmetic hyperbolic 3-manifold groups exhibit many special features. (See, for example, [MaR03, Lac11, Red07] for more on arithmetic 3-manifolds).
- (6) The representation theory of 3-manifolds is a substantial field in its own right, which fortunately is served well by Shalen’s survey paper [Shn02].

We conclude the paper with a discussion of some outstanding open problems in the theory of 3-manifold groups.

This paper is not intended as a leisurely introduction to 3-manifolds. Even though most terms will be defined, we will assume that the reader is already somewhat acquainted with 3-manifold topology. We refer to [Hem76, Hat, JS79, Ja80] for background material. Another gap we perceive is the lack of a post-Geometrization-Theorem 3-manifold book. We hope that somebody else will step forward and fill this gaping hole.

Conventions and notations. All spaces are assumed to be connected and compact and all groups are assumed to be finitely presented, unless it is specifically stated otherwise. All rings have an identity. If N is a 3-manifold and $S \subseteq N$ a submanifold, then we denote by $\nu S \subseteq N$ a tubular neighborhood of S .

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1. DECOMPOSITION THEOREMS

1.1. The Prime Decomposition Theorem. A 3-manifold N is called *prime* if N cannot be written as a non-trivial connected sum of two manifolds, i.e., if $N = N_1 \# N_2$, then $N_1 = S^3$ or $N_2 = S^3$. Furthermore N is called *irreducible* if every embedded S^2 bounds a 3-ball. Note that an irreducible 3-manifold is prime. Also, if N is an orientable prime 3-manifold with no spherical boundary components, then by [Hem76, Lemma 3.13] either N is irreducible or $N = S^1 \times S^2$. The following theorem is due to Kneser [Kn29], Haken [Hak61] and Milnor [Mil62] (see also [Hem76, Chapter 3]).

Theorem 1.1. (Prime Decomposition Theorem) *Let N be a compact, oriented 3-manifold with no spherical boundary components.*

- (1) *There exists a decomposition $N \cong N_1 \# \cdots \# N_r$ where the 3-manifolds N_1, \dots, N_r are oriented prime 3-manifolds.*
- (2) *If $N \cong N_1 \# \cdots \# N_r$ and $N \cong N'_1 \# \cdots \# N'_s$ where the 3-manifolds N_i and N'_i are oriented prime 3-manifolds, then $r = s$ and (possibly after reordering) there exist orientation-preserving diffeomorphisms $N_i \rightarrow N'_i$.*

*In particular, $\pi_1(N) = \pi_1(N_1) * \cdots * \pi_1(N_r)$ is the free product of fundamental groups of prime 3-manifolds.*

1.2. The Loop Theorem and the Sphere Theorem. The life of 3-manifold topology as a flourishing subject started with the proof of the Loop Theorem and the Sphere Theorem by Papakyriakopoulos. We first state the Loop Theorem.

Theorem 1.2. (Loop Theorem) *Let N be a compact 3-manifold and $F \subseteq \partial N$ a subsurface. If $\text{Ker}(\pi_1(F) \rightarrow \pi_1(N))$ is non-trivial, then there exists a proper embedding $g: (D^2, \partial D^2) \rightarrow (N, F)$ such that $g(\partial D^2)$ represents a non-trivial element in $\text{Ker}(\pi_1(F) \rightarrow \pi_1(N))$.*

A somewhat weaker version (usually called ‘Dehn’s Lemma’) of this theorem was first stated by Dehn [De10] in 1910, but Kneser [Kn29, p. 260] found a gap in the proof provided by Dehn. The Loop Theorem was finally proved by Papakyriakopoulos [Pap57a, Pap57b] building on work of Johansson [Jos35]. We refer to [Hom57, SpW58, Sta60, Wan67b, Gon99, Bin83, Jon94, AR04] and [Hem76, Chapter 4] for more details and several extensions. We now turn to the Sphere Theorem.

Theorem 1.3. (Sphere Theorem) *Let N be an orientable 3-manifold with $\pi_2(N) \neq 0$. Then N contains an embedded 2-sphere which is homotopically non-trivial.*

This theorem was proved by Papakyriakopoulos [Pap57a] under a technical assumption which was removed by Whitehead [Whd58a]. (We also refer to [Whd58b, Bat71, Gon99, Bin83] and [Hem76, Theorem 4.3] for extensions and more information.)

1.3. Preliminary observations about 3-manifold groups. The main subject of this paper are the properties of fundamental groups of compact 3-manifolds. In this section we argue that for most purposes it suffices to study the fundamental

groups of compact, orientable, irreducible 3-manifolds whose boundary is either empty or toroidal.

We start out with the following basic observation.

Observation 1.4. *Let N be a compact 3-manifold.*

- (1) *Denote by \widehat{N} the 3-manifold obtained from N by gluing 3-balls to all spherical components of ∂N . Then $\pi_1(\widehat{N}) = \pi_1(N)$.*
- (2) *If N is non-orientable, then there exists a double cover which is orientable.*

Note that free products of groups are well understood, and most properties of groups are preserved under passing to an index-two supergroup. (Note though that this is not true for all properties; for example, conjugacy separability does not in general pass to degree-two extensions [CMi77, Goa86].) In light of Theorem 1.1 and Observation 1.4, we therefore generally restrict ourselves to the study of orientable, irreducible 3-manifolds with no spherical boundary components.

An embedded surface $\Sigma \subseteq N$ with components $\Sigma_1, \dots, \Sigma_k$ is *incompressible* if for each $i = 1, \dots, k$ we have $\Sigma_i \neq S^2, D^2$ and the map $\pi_1(\Sigma_i) \rightarrow \pi_1(N)$ is injective. The following lemma is a well-known consequence of the Loop Theorem.

Lemma 1.5. *Let N be a compact 3-manifold. Then there exist 3-manifolds N_1, \dots, N_k whose boundary components are incompressible, and a free group F such that $\pi_1(N) \cong \pi_1(N_1) * \dots * \pi_1(N_k) * F$.*

Proof. By the above observation we can without loss of generality assume that N has no spherical boundary components. Let $\Sigma \subseteq \partial N$ be a component such that $\pi_1(\Sigma) \rightarrow \pi_1(N)$ is not injective. By the Loop Theorem (see Theorem 1.2) there exists a properly embedded disk $D \subseteq N$ such that the curve $\partial D \subseteq \Sigma$ is essential, i.e., does not bound an embedded disk in Σ .

Let N' be the result of capping off the spherical boundary components of $N \setminus \nu D$ by 3-balls. If N' is connected, then $\pi_1(N) \cong \pi_1(N') * \mathbb{Z}$; otherwise $\pi_1(N) \cong \pi_1(N_1) * \pi_1(N_2)$ where N_1, N_2 are the two components of N' . The lemma now follows by induction on the lexicographically ordered pair $(-\chi(\partial N), b_0(\partial N))$ since we have either that $-\chi(\partial N') < -\chi(\partial N)$ (in the case that Σ is not a torus), or that $\chi(\partial N') = \chi(\partial N)$ and $b_0(\partial N') < b_0(\partial N)$ (in the case that Σ is a torus). \square

We say that a group A is a *retract* of a group B if there exist group homomorphisms $\varphi: A \rightarrow B$ and $\psi: B \rightarrow A$ such that $\psi \circ \varphi = \text{id}_A$. In particular, in this case φ is injective and we can then view A as a subgroup of B .

Lemma 1.6. *Let N be a compact 3-manifold with non-empty boundary. Then $\pi_1(N)$ is a retract of the fundamental group of a closed 3-manifold.*

Proof. Denote by M the double of N , i.e., $M = N \cup_{\partial N} N$. Note that M is a closed 3-manifold. Let f be the canonical inclusion of N into M and let $g: M \rightarrow N$ be the map which restricts to the identity on the two copies of N in M . Clearly $g \circ f = \text{id}_N$ and hence $g_* \circ f_* = \text{id}_{\pi_1(N)}$. \square

Many properties of groups are preserved under retracts and taking free products; this way, many problems on 3-manifold groups can be reduced to the study of fundamental groups of closed 3-manifolds. Due to the important role played by

3-manifolds with toroidal boundary components we will be slightly less restrictive, and in the remainder we study fundamental groups of compact, orientable, irreducible 3-manifolds N such that the boundary is either empty or toroidal.

1.4. The JSJ decomposition and the Geometrization Theorem. In the previous section we saw that an oriented, compact 3-manifold with no spherical boundary components admits a decomposition along spheres such that the set of resulting pieces are unique up to diffeomorphism. In the following we say that a 3-manifold N is *atoroidal* if any map $T \rightarrow N$ from a torus to N which induces an embedding $\pi_1(T) \rightarrow \pi_1(N)$ can be homotoped into the boundary of N . There exist orientable irreducible 3-manifolds which cannot be cut into atoroidal pieces in a unique way (e.g., the 3-torus). Nonetheless, any orientable irreducible 3-manifold admits a canonical decomposition along tori, but to formulate this result we need the notion of a Seifert fibered manifold.

A *Seifert fibered manifold* is a 3-manifold N together with a decomposition into disjoint simple closed curves (called *Seifert fibers*) such that each Seifert fiber has a tubular neighborhood that forms a standard fibered torus. The *standard fibered torus* corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$, equipped with the natural fibering by circles. If $a > 1$, then the middle Seifert fiber is called *singular*. A compact Seifert fiber space has only a finite number of singular fibers. It is often useful to think of a Seifert fibered manifold as a circle bundle over a 2-dimensional orbifold. We refer to [Sei33, Or72, Hem76, Ja80, JD83, Sco83a, Brn93, LeR10] for further information and for the classification of Seifert fibered manifolds.

Some 3-manifolds (e.g., lens spaces) admit distinct Seifert fibered structures; generally, however, this will not be of importance to us (but see, e.g., [Ja80, Theorem VI.17]). Sometimes, later in the text, we will slightly abuse language and say that a 3-manifold is Seifert fibered if it admits the structure of a Seifert fibered manifold.

Remark.

- (1) The only orientable non-prime Seifert fibered manifold is $\mathbb{R}P^3 \# \mathbb{R}P^3$ (see, e.g., [Hat, Proposition 1.12] or [Ja80, Lemma VI.7]).
- (2) By Epstein's Theorem [Ep72], a 3-manifold N which is not homeomorphic to the solid Klein bottle admits a Seifert fibered structure if and only if it admits a foliation by circles.

The following theorem was first announced by Waldhausen [Wan69] and was proved independently by Jaco–Shalen [JS79] and Johannson [Jon79].

Theorem 1.7. (JSJ Decomposition Theorem) *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori T_1, \dots, T_k such that each component of N cut along $T_1 \cup \dots \cup T_k$ is atoroidal or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.*

In the following we refer to the tori T_1, \dots, T_k as the *JSJ tori* and we will refer to the components of N cut along $\bigcup_{i=1}^k T_i$ as the *JSJ components of N* . Let M be a JSJ component of N . After picking base points for N and M and a path connecting these base points, the inclusion $M \subseteq N$ induces a map on the level of fundamental groups. This map is injective since the tori we cut along are incompressible. (We refer to [LyS77, Chapter IV.4] for details.) We can thus view $\pi_1(M)$ as a subgroup of $\pi_1(N)$, which is well defined up to the above choices, i.e., well defined up to conjugacy. Furthermore we can view $\pi_1(N)$ as the fundamental group of a graph of groups with vertex groups the fundamental groups of the JSJ components and with edge groups the fundamental groups of the JSJ tori. We refer to [Ser80, Bas93] for more on graphs of groups.

We need the following definition due to Jaco–Shalen:

Definition. Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. The *characteristic submanifold of N* is the union of the following submanifolds:

- (1) all Seifert fibered pieces in the JSJ decomposition;
- (2) all boundary tori which cobound an atoroidal JSJ component;
- (3) all JSJ tori which do not cobound a Seifert fibered JSJ component.

The following is now a consequence of the ‘Characteristic Pair Theorem’ of Jaco–Shalen [JS79, p. 138].

Theorem 1.8. *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary which admits at least one JSJ torus. If $f: M \rightarrow N$ is a map from a Seifert fibered manifold M to N which is π_1 -injective and if $M \neq S^1 \times D^2$ and $M \neq S^1 \times S^2$, then f is homotopic to a map $g: M \rightarrow N$ such that $g(M)$ lies in a component of the characteristic submanifold of N .*

The following proposition is now an immediate consequence of Theorem 1.8. The proposition gives a useful criterion for showing that a collection of tori are in fact the JSJ tori of a given 3-manifold.

Proposition 1.9. *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. Let T_1, \dots, T_k be disjointly embedded tori in N . Suppose the following hold:*

- (1) *the components M_1, \dots, M_l of N cut along $T_1 \cup \dots \cup T_k$ are either Seifert fibered or atoroidal; and*
- (2) *if a torus T_i cobounds two Seifert fibered components M_r and M_s (where it is possible that $r = s$), then the regular fibers of M_r and M_s do not define the same element in $H_1(T_i)$.*

Then T_1, \dots, T_k are the JSJ tori of N .

We now turn to the study of atoroidal 3-manifolds. We say that a closed 3-manifold is *spherical* if it admits a complete metric of constant positive curvature. Note that fundamental groups of spherical 3-manifolds are finite; in particular spherical 3-manifolds are atoroidal.

In the following we say that a compact 3-manifold is *hyperbolic* if its interior admits a complete metric of constant negative curvature -1 . The following theorem is due to Mostow [Mos68] in the closed case and due to Prasad [Pra73] and Marden [Man74] independently in the case of non-trivial boundary. (See also [Rat06, Chapter 11] for a proof due to Gromov.)

Theorem 1.10. (Mostow–Prasad–Marden Rigidity Theorem) *Let M and N be finite volume hyperbolic 3-manifolds. Any isomorphism $\pi_1(M) \rightarrow \pi_1(N)$ is induced by a unique isometry $M \rightarrow N$.*

This theorem implies in particular that the geometry of finite volume hyperbolic 3-manifolds is determined by their topology. This is not the case if we drop the finite-volume condition. More precisely, the Ending Lamination Theorem states that hyperbolic 3-manifolds with finitely generated fundamental groups are determined by their topology and by their ‘ending laminations’. The Ending Lamination Theorem was conjectured by Thurston [Thu82a] and was proved by Brock–Canary–Minsky [BCM04, Miy10]. We also refer to [Miy94, Miy03, Miy06, Ji12] for more background information and to [Bow11a, Bow11b] for an alternative approach.

A hyperbolic 3-manifold has finite volume if and only if it is either closed or has toroidal boundary (see [Thu79, Theorem 5.11.1] or [Bon02, Theorem 2.9]). Since in this paper we are mainly interested in 3-manifolds with empty or toroidal boundary, we henceforth restrict ourselves to hyperbolic 3-manifolds with finite volume. We will therefore work with the following understanding.

Convention. Unless we say explicitly otherwise, in the remainder of the paper, a hyperbolic 3-manifold is always understood to have finite volume.

With this convention, hyperbolic 3-manifolds are atoroidal; in fact, the following slightly stronger statement holds (see [Man74, Proposition 6.4], [Thu79, Proposition 5.4.4] and also [Sco83a, Corollary 4.6]):

Theorem 1.11. *Let N be a hyperbolic 3-manifold. If $\Gamma \subseteq \pi_1(N)$ is abelian and not cyclic, then there exists a boundary torus S and $h \in \pi_1(N)$ such that*

$$\Gamma \subseteq h \pi_1(S) h^{-1}.$$

The Elliptization Theorem and the Hyperbolization Theorem together imply that every atoroidal 3-manifold is either spherical or hyperbolic. Both theorems were conjectured by Thurston [Thu82a, Thu82b] and the Hyperbolization Theorem was proved by Thurston for Haken manifolds (see [Thu86b, Thu86c, Mor84] and see also [Su81, Mc92, Ot96, Ot98, Ot01, Ka01]). The full proof of both theorems was first given by Perelman in his seminal papers [Per02, Per03a, Per03b] building on earlier work of R. Hamilton [Hamc82, Hamc95, Hamc99]. We refer to [MT07] for full details and to [CZ06a, CZ06b], [KIL08] and [BBBMP10] for further information on the proof. Finally we refer to [Mil03, An04, Be06, Mc11] for expository accounts.

Theorem 1.12. (Elliptization Theorem) *Every closed, orientable 3-manifold with finite fundamental group is spherical.*

It is well known that S^3 equipped with the canonical metric is the only spherical simply connected 3-manifold. It follows that the Elliptization Theorem implies the Poincaré Conjecture: the 3-sphere S^3 is the only simply connected, closed 3-manifold. It also follows that a 3-manifold N is spherical if and only if it is the quotient of S^3 by a finite group, which acts freely and isometrically. In particular, we can view $\pi_1(N)$ as a finite subgroup of $\mathrm{SO}(4)$ which acts freely on S^3 . This last statement gives rise to a complete classification of the finite groups that arise as the fundamental groups of 3-manifolds: any such group is either cyclic, or is a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order. We refer to [Or72, Chapter 1, Theorem 1] and [Or72, Chapter 2, Theorem 2] for details and for the complete list of 3-manifolds with finite fundamental groups. (We also refer to [Mil57], [Lee73] and [Tho79] for some ‘pre-Geometrization’ results on the classification of finite fundamental groups of 3-manifolds.) Finally, note that spherical 3-manifolds are in fact Seifert fibered (see [Sco83b] or [Bon02, Theorem 2.8]).

We now turn to atoroidal 3-manifolds with infinite fundamental groups.

Theorem 1.13. (Hyperbolization Theorem) *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. If N is atoroidal and $\pi_1(N)$ is infinite, then N is either Seifert fibered or hyperbolic.*

Combining the JSJ Decomposition Theorem with the Elliptization Theorem and the Hyperbolization Theorem we now obtain the following:

Theorem 1.14. (Geometrization Theorem) *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori T_1, \dots, T_k such that each component of N cut along $T_1 \cup \dots \cup T_k$ is hyperbolic or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.*

Remark. There exists also a version of the Geometrization Theorem for non-orientable 3-manifolds; we refer to [Bon02, Conjecture 4.1] for the statement.

The following theorem says that the JSJ decomposition behaves well under passing to finite covers.

Theorem 1.15. *Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. Let $N' \rightarrow N$ be a finite cover. Then N' is irreducible and the pre-images of the JSJ tori of N under the projection map are the JSJ tori of N' . Furthermore N' is hyperbolic (respectively Seifert fibered) if and only if N is hyperbolic (respectively Seifert fibered).*

The fact that N' is again irreducible follows from the Equivariant Sphere Theorem (see [MSY82] and see also [Duw85, Ed86, JR89]). The other statements are straightforward consequences of Proposition 1.9 and the Hyperbolization Theorem. Alternatively we refer to [MS86, p. 290] and [JR89] for details.

Finally, the following theorem, which is an immediate consequence of Proposition 1.9, often allows us to reduce proofs to the closed case:

Theorem 1.16. *Let N be an irreducible, orientable, compact 3-manifold with non-trivial toroidal boundary. We denote the boundary tori by S_1, \dots, S_k and we denote the JSJ tori by T_1, \dots, T_l . Let $M = N \cup_{\partial N} N$ be the double of N along the boundary. Then the two copies of T_i for $i = 1, \dots, l$ together with the S_i which bound hyperbolic components are the JSJ tori for M .*

1.5. The geometric decomposition. The decomposition in Theorem 1.15 can be viewed as somewhat *ad hoc* ('Seifert fibered vs. hyperbolic'). The geometric point of view introduced by Thurston gives rise to an elegant reformulation of Theorem 1.14. Thurston introduced the notion of a *geometry of a 3-manifold* and of a *geometric 3-manifold*. We will now give a quick summary of the definitions and the most relevant results. We refer to the expository papers by Scott [Sco83a] and Bonahon [Bon02] and to Thurston's book [Thu97] for proofs and further references.

A *3-dimensional geometry X* is a smooth, simply connected 3-manifold which is equipped with a smooth, transitive action of a Lie group G by diffeomorphisms on X , with compact point stabilizers. The Lie group G is called the *group of isometries* of X . A *geometric structure* on a 3-manifold N is a diffeomorphism from the interior of N to X/π , where π is a discrete subgroup of G acting freely on X . The geometry X is said to *model* N , and N is said to *admit an X -structure*, or just to be an *X -manifold*. There are also two technical conditions, which rule out redundant examples of geometries: the group of isometries is required to be maximal among Lie groups acting transitively on X with compact point stabilizers; and X is required to have a compact model.

Thurston showed that, up to a certain equivalence, there exist precisely eight 3-dimensional geometries that model compact 3-manifolds. These geometries are: the 3-sphere, Euclidean 3-space, hyperbolic 3-space, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, the universal cover $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ of $\mathrm{SL}(2, \mathbb{R})$, and two further geometries called Nil and Sol. We refer to [Sco83a] for details. Note that spherical and hyperbolic manifolds are precisely the type of manifolds we introduced in the previous section. (It is well known that a 3-manifold equipped with a complete spherical metric has to be closed.) A 3-manifold is called *geometric* if it is an X -manifold for some geometry X .

The following theorem summarizes the relationship between Seifert fibered manifolds and geometric 3-manifolds.

Theorem 1.17. *Let N be an orientable, compact 3-manifold with empty or toroidal boundary. We assume that $N \neq S^1 \times D^2$, $N \neq S^1 \times S^1 \times I$, and that N does not equal the twisted I -bundle over the Klein bottle (i.e., the total space of the unique non-trivial interval bundle over the Klein bottle). Then N is a Seifert fibered manifold if and only if N admits a geometric structure based on one of the following geometries: the 3-sphere, Euclidean 3-space, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, Nil.*

We refer to [Bon02, Theorem 4.1] and [Bon02, Theorems 2.5, 2.7, 2.8] for the proof and for references (see also [Sco83a, Theorem 5.3] and [FM10, Lecture 31]). (Note that in [Bon02] the geometries $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ and Nil are referred to as $H^2 \widetilde{\times} E^1$ and $E^2 \widetilde{\times} E^1$.)

By a *torus bundle* we mean a fiber bundle over S^1 with fiber the 2-torus T . The action of the monodromy on $H_1(T; \mathbb{Z})$ defines an element in $\mathrm{SL}(2, \mathbb{Z})$. If the monodromy is orientation preserving, then it follows from an elementary linear algebraic argument that one of the following occurs:

- (1) the monodromy matrix is $\pm \mathrm{id}$, or
- (2) it is non-diagonalizable but has eigenvalue ± 1 , or
- (3) it has two distinct real eigenvalues.

In the first case we say that the monodromy is *abelian*, in the second case we say it is *nilpotent* and in the remaining case we say it is *Anosov*. By [Sco83a] the monodromy is abelian or nilpotent if and only if N is Seifert fibered. The following theorem (see [Sco83a, Theorem 5.3] or [Dub88]) now gives a complete classification of Sol-manifolds.

Theorem 1.18. *Let N be a compact, orientable 3-manifold. Then N is a Sol-manifold if and only if N is either the double of the twisted I -bundle over the Klein bottle or N is a torus bundle with Anosov monodromy.*

The following is now the ‘geometric version’ of Theorem 1.14 (see [Mor05, Conjecture 2.2.1] and [FM10]).

Theorem 1.19. (Geometric Decomposition Theorem) *Let N be an orientable, compact, irreducible 3-manifold with empty or toroidal boundary. We assume that $N \neq S^1 \times D^2$, $N \neq S^1 \times S^1 \times I$, and that N does not equal the twisted I -bundle over the Klein bottle. Then there exists a collection of disjointly embedded incompressible surfaces S_1, \dots, S_k which are either tori or Klein bottles, such that each component of N cut along $S_1 \cup \dots \cup S_k$ is geometric. Furthermore, any such collection with a minimal number of components is unique up to isotopy.*

We will quickly outline the existence of such a decomposition, assuming Theorems 1.14, 1.17 and 1.18.

Proof. Let N be an orientable, irreducible 3-manifold with empty or toroidal boundary such that $N \neq S^1 \times D^2$, $N \neq S^1 \times S^1 \times I$, and such that N does not equal the twisted I -bundle over the Klein bottle. By Theorem 1.14 there exists a minimal collection of disjointly embedded incompressible tori T_1, \dots, T_k such that each component of N cut along $T_1 \cup \dots \cup T_k$ is either hyperbolic or Seifert fibered. We denote the components of N cut along $T_1 \cup \dots \cup T_k$ by M_1, \dots, M_r . Note that $M_i \neq S^1 \times D^2$ since the JSJ tori are incompressible and since $N \neq S^1 \times D^2$. Now suppose that one of the M_i is $S^1 \times S^1 \times I$. By the minimality of the number of tori and by our assumption that $N \neq S^1 \times S^1 \times I$, it follows easily that N is a torus bundle with a non-trivial JSJ decomposition. By Theorem 1.18 and the discussion preceding it we see that N is a Sol manifold, hence already geometric.

In view of Theorem 1.18 we can assume that N is not the double of the twisted I -bundle over the Klein bottle. For any i such that the JSJ torus T_i bounds a twisted I -bundle over the Klein bottle we now replace T_i by the Klein bottle which is the core of the twisted I -bundle.

It is now straightforward to verify (using Theorem 1.17) that the resulting collection of tori and Klein bottles decomposition has the required properties. \square

Remark. The proof of Theorem 1.19 also shows how to obtain the decomposition postulated by Theorem 1.19 from the decomposition given by Theorem 1.14. Let N be an orientable, compact, irreducible 3-manifold with empty or toroidal boundary. If N is a Sol-manifold, then N has one JSJ torus, namely a surface fiber, but N is geometric. Now suppose that N is not a Sol-manifold. Denote by T_1, \dots, T_l the JSJ tori of N . We assume that they are ordered such that T_1, \dots, T_r are precisely the tori which do not bound twisted I -bundles over a Klein bottle. For $i = r + 1, \dots, l$, each T_i cobounds a twisted I -bundle over a Klein bottle K_i and a hyperbolic JSJ component. The decomposition of Theorem 1.19 is then given by $T_1 \cup \dots \cup T_r \cup K_{r+1} \cup \dots \cup K_l$.

Remark. Let Σ be a compact surface with $\chi(\Sigma) < 0$ and let φ be a self-diffeomorphism of Σ . The Nielsen–Thurston classification says that φ is either

- (1) of finite order, or
- (2) reducible (i.e., it leaves a multiset of disjoint curves invariant), or
- (3) it is pseudo-Anosov.

(This trichotomy should be viewed as analogous to the trichotomy for self-diffeomorphisms of a torus stated before Theorem 1.18.) We refer to [Nie44], [BIC88], [Thu88], [FM12] and [FLP79] for details. Thurston determined the geometric structure of the mapping torus N of φ in terms of the type of φ as follows (see [Thu86c] and [Ot96, Ot01]).

- (1) If φ is of finite order, then N admits an $\mathbb{H}^2 \times \mathbb{R}$ structure.
- (2) If φ is reducible, then N admits a non-trivial JSJ decomposition.
- (3) If φ is pseudo-Anosov, then N is hyperbolic.

Before we continue our discussion of geometric 3-manifolds we introduce a definition. Given a property \mathcal{P} of groups we say that a group π is *virtually* \mathcal{P} if π admits a (not necessarily normal) subgroup of finite index that satisfies \mathcal{P} .

In Table 1 we summarize some of the key properties of geometric 3-manifolds. Given a geometric 3-manifold, the first column lists the geometry type, the second describes the fundamental group of N and the third describes the topology of N (or a finite-sheeted cover).

If the geometry is neither Sol nor hyperbolic, then by Theorem 1.17 the manifold N is Seifert fibered. One can think of a Seifert fibered manifold as an S^1 -bundle over an orbifold. We denote by χ the orbifold Euler characteristic of the base orbifold and we denote by e the Euler number. We refer to [Sco83a] for the precise definitions.

We now give the references for Table 1. We refer to [Sco83a, p. 478] for the last two columns. For the first three rows we refer to [Sco83a, p. 449, p. 457, p. 448]. We refer to [Sco83a, p. 467] for details regarding Nil and we refer to [Bon02, Theorem 2.11] and [Sco83a, Theorem 5.3] for details regarding Sol. Finally we refer to [Sco83a, p. 459, p. 462, p. 448] for details regarding the last three geometries. The fact that the fundamental group of a hyperbolic 3-manifold does not contain a non-trivial abelian normal subgroup will be shown in Theorem 3.5.

If N is a non-spherical Seifert fibered manifold, then the Seifert fiber subgroup is infinite cyclic and normal in $\pi_1(N)$ (see [JS79, Lemma II.4.2] for details). It now follows from the above table that the geometry of a geometric manifold can

Geometry	Fundamental group	Topology	χ	e
Spherical	π is finite	finitely covered by S^3	> 0	$\neq 0$
$S^2 \times \mathbb{R}$	$\pi = \mathbb{Z}$ or π is the infinite dihedral group	N or a double cover equals $S^1 \times S^2$	> 0	$= 0$
Euclidean	π is virtually \mathbb{Z}^3	N finitely covered by $S^1 \times S^1 \times S^1$	0	0
Nil	π is virtually nilpotent but not virtually \mathbb{Z}^3	N finitely covered by a torus bundle with nilpotent monodromy	0	$\neq 0$
Sol	π is solvable but not virtually nilpotent	N or a double cover is a torus bundle with Anosov monodromy		
$\mathbb{H}^2 \times \mathbb{R}$	π is virtually a product $\mathbb{Z} \times F$ with F a non-cyclic free group	N finitely covered by $S^1 \times \Sigma$ where Σ is a surface with $\chi(\Sigma) < 0$	< 0	0
$\widetilde{\mathrm{SL}(2, \mathbb{R})}$	π is a non-split extension of a non-cyclic free group F by \mathbb{Z}	N finitely covered by a non-trivial S^1 -bundle over a surface Σ with $\chi(\Sigma) < 0$	< 0	$\neq 0$
hyperbolic	π infinite and π does not contain a non-trivial abelian normal subgroup	N is atoroidal		

TABLE 1. Geometries of 3-manifolds.

be read off from its fundamental group. In particular, if a 3-manifold admits a geometric structure, then the type of that geometric structure is unique (see also [Sco83a, Theorem 5.2] and [Bon02, Section 2.5]). Furthermore, note that some geometries are very rare: there exist only finitely many 3-manifolds with Euclidean geometry or $S^2 \times \mathbb{R}$ geometry [Sco83a]. Finally, note that if N is a geometric 3-manifold with non-trivial boundary, then the geometry is either $\mathbb{H}^2 \times \mathbb{R}$ or hyperbolic.

1.6. 3-manifolds with (virtually) solvable fundamental group. The above discussion can be used to classify the abelian, nilpotent and solvable groups which appear as fundamental groups of 3-manifolds.

Theorem 1.20. *Let N be an orientable, non-spherical 3-manifold which is either closed or has toroidal boundary. Then the following are equivalent:*

- (1) $\pi_1(N)$ is solvable;
- (2) $\pi_1(N)$ is virtually solvable;
- (3) N is one of the following six types of manifolds:
 - (a) $N = \mathbb{R}P^3 \# \mathbb{R}P^3$;

- (b) $N = S^1 \times D^2$;
- (c) $N = S^1 \times S^2$;
- (d) N admits a finite solvable cover which is a torus bundle;
- (e) $N = S^1 \times S^1 \times I$, where I is the standard interval $I = [0, 1]$;
- (f) N is the twisted I -bundle over the Klein bottle.

Before we prove the theorem, we state a useful lemma.

Lemma 1.21. *Let π be a group. If π decomposes non-trivially as an amalgamated free product $\pi = A *_C B$, then π contains a non-cyclic free subgroup unless $[A : C], [B : C] \leq 2$. Similarly, if π decomposes non-trivially as an HNN extension $\pi = A *_C$, then π contains a non-cyclic free subgroup unless one of the inclusions of C into A is an isomorphism.*

The proof of the lemma is a standard application of Bass–Serre theory [Ser80]. We are now ready to prove the theorem.

Proof of Theorem 1.20. The implication (1) \Rightarrow (2) is obvious. Note that the group $\pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3) = \mathbb{Z}/2 * \mathbb{Z}/2$ is isomorphic to the infinite dihedral group, so is solvable. It is clear that if N is one of the remaining types (b)–(f) of 3-manifolds, then $\pi_1(N)$ is also solvable. This shows (3) \Rightarrow (1).

Finally, assume that (2) holds. We will show that (3) holds. Let A and B be two non-trivial groups. By Lemma 1.21, $A * B$ contains a non-cyclic free group (in particular it is not virtually solvable) unless $A = B = \mathbb{Z}/2$. Note that by the Elliptization Theorem, any 3-manifold M with $\pi_1(M) \cong \mathbb{Z}/2$ is diffeomorphic to $\mathbb{R}P^3$. It follows that if N is a compact 3-manifold with solvable fundamental group, then either $N = \mathbb{R}P^3 \# \mathbb{R}P^3$ or N is prime.

Since $S^1 \times S^2$ is the only orientable prime 3-manifold which is not irreducible we can henceforth assume that N is irreducible. Now let N be an irreducible 3-manifold which is either closed or has toroidal boundary and such that $\pi = \pi_1(N)$ is infinite and solvable. We furthermore assume that $N \neq S^1 \times S^1 \times I$ and that N does not equal the twisted I -bundle over the Klein bottle. It now follows from Theorem 1.19 that $\pi_1(N)$ is the fundamental group of a graph of groups where the vertex groups are fundamental groups of geometric 3-manifolds. By Lemma 1.21, $\pi_1(N)$ contains a non-cyclic free group unless the 3-manifold is already geometric. If N is geometric, then it follows from the discussion preceding this theorem that N is either a Euclidean manifold, a Sol-manifold or a Nil-manifold, and that N is finitely covered by a torus bundle. It follows from the discussion of these geometries in [Sco83a] that the finite cover is in fact a finite solvable cover. (Alternatively we could have applied [EM72, Theorems 4.5 and 4.8], [EM72, Corollary 4.10] and [Tho79, Section 5] for a proof of the theorem which does not require the full Geometrization Theorem but only requires the Elliptization Theorem.) \square

Remark. If N is a compact 3-manifold with nilpotent fundamental group, then it follows from the proof of the above theorem that N is either a spherical, a Euclidean, or a Nil-manifold. Using the discussion of these geometries in [Sco83a] one can then determine the list of nilpotent groups which can appear as fundamental groups of compact 3-manifolds.

This list of nilpotent groups was already determined ‘pre-Geometrization’ by Thomas [Tho68, Theorem N] for the closed case and by Evans–Moser [EM72, Theorem 7.1] in the general case.

Remark. In Table 2 we give the complete list of all compact 3-manifolds with abelian fundamental groups. The table can be obtained in a straightforward

abelian group π	compact 3-manifolds with fundamental group π
\mathbb{Z}	$S^2 \times S^1, D^2 \times S^1$ (the twisted sphere bundle over the circle)
\mathbb{Z}^3	$S^1 \times S^1 \times S^1$
\mathbb{Z}/n	the lens spaces $L(n, m)$, $m \in \{1, \dots, n\}$ with $(n, m) = 1$
$\mathbb{Z} \oplus \mathbb{Z}$	$S^1 \times S^1 \times I$
$\mathbb{Z} \oplus \mathbb{Z}/2$	$S^1 \times \mathbb{R}P^2$

TABLE 2. Abelian fundamental groups of 3-manifolds.

fashion from the Prime Decomposition Theorem and the Geometrization Theorem. The fact that the groups in the table are indeed the only abelian groups that appear as fundamental groups of compact 3-manifolds is in fact a classical ‘pre-Geometrization’ result. The list of abelian fundamental groups of closed 3-manifolds was first determined by Reidemeister [Rer36] and in the general case by Epstein ([Ep61a, Theorem 3.3] and [Ep61b, Theorem 9.1]). (See also [Hem76, Theorems 9.12 and 9.13].)

2. THE CLASSIFICATION OF 3-MANIFOLDS BY THEIR FUNDAMENTAL GROUPS

In this section we will discuss the degree to which the fundamental group determines a 3-manifold and its topological properties. By Moise’s Theorem [Moi52, Moi77] (see also [Bin59, Hama76, Shn84]) any topological 3-manifold also admits a smooth structure, and two 3-manifolds are homeomorphic if and only if they are diffeomorphic. We can therefore freely go back and forth between the topological and the smooth categories.

Remark.

- (1) By work of Cerf [Ce68] and Hatcher [Hat83] (see also [Lau85]), given any closed 3-manifold M the map $\text{Diff}(M) \rightarrow \text{Homeo}(M)$ between the space of diffeomorphisms of M and the space of homeomorphisms of M is in fact a weak homotopy equivalence.
- (2) Bing [Bin52] gives an example of a continuous involution on S^3 with fixed point set a wild $S^2 \subset S^3$. In particular, this involution cannot be smoothed.

2.1. Closed 3-manifolds and fundamental groups. It is well known that closed, compact surfaces are determined by their fundamental groups, and compact surfaces with boundary are determined by their fundamental groups together

with the number of boundary components. In 3-manifold theory a similar, but more subtle, picture emerges.

One quickly notices that there are three ways to construct pairs of closed, orientable, non-diffeomorphic 3-manifolds with isomorphic fundamental groups.

- (A) Consider lens spaces $L(p_1, q_1)$ and $L(p_2, q_2)$. They are diffeomorphic if and only if $p_1 = p_2$ and $q_1 q_2^{\pm 1} \equiv \pm 1 \pmod{p_i}$, but their fundamental groups are isomorphic if and only if $p_1 = p_2$.
- (B) Let M and N be two oriented 3-manifolds. Denote by \overline{N} the same manifold as N but with opposite orientation. Then $\pi_1(M \# N) \cong \pi_1(M \# \overline{N})$ but if N does not admit an orientation reversing diffeomorphism, then $M \# N$ and $M \# \overline{N}$ are not diffeomorphic.
- (C) Let M_1, M_2 and N_1, N_2 be 3-manifolds with $\pi_1(M_i) \cong \pi_1(N_i)$ and such that M_1 and N_1 are not diffeomorphic. Then $\pi_1(M_1 \# M_2) \cong \pi_1(N_1 \# N_2)$ but in general $M_1 \# M_2$ is not diffeomorphic to $N_1 \# N_2$.

The first statement was proved by Reidemeister [Rer35] (see also [Mil66] and [Hat, Section 2.1]). The other two statements follow from the uniqueness of the prime decomposition. In the subsequent discussion we will see that (A), (B) and (C) form in fact a complete list of methods for finding examples of pairs of closed, orientable, non-diffeomorphic 3-manifolds with isomorphic fundamental groups.

Recall that it follows from Theorem 1.1 that, given a compact, orientable 3-manifold, the fundamental group is isomorphic to a free product of fundamental groups of prime 3-manifolds. The Kneser Conjecture implies that the converse holds.

Theorem 2.1. (Kneser Conjecture) *Let N be a compact, orientable 3-manifold with incompressible boundary. If there exists an isomorphism $\pi_1(N) \cong \Gamma_1 * \Gamma_2$, then there exist compact, orientable 3-manifolds N_1 and N_2 with $\pi_1(N_i) \cong \Gamma_i$ and $N \cong N_1 \# N_2$.*

The Kneser Conjecture was first proved by Stallings [Sta59a, Sta59b] in the closed case, and by Heil [Hei72] in the bounded case. (We also refer to [Ep61c] and [Hem76, Section 7] for details.)

The following theorem is a consequence of the Geometrization Theorem, the Mostow–Prasad Rigidity Theorem, work of Waldhausen [Wan68b, Corollary 6.5] and Scott [Sco83b, Theorem 3.1] and classical work on spherical 3-manifolds (see [Or72, p. 113]).

Theorem 2.2. *Let N and N' be two orientable, closed, prime 3-manifolds with $\pi_1(N) \cong \pi_1(N')$. Then either N and N' are homeomorphic, or N and N' are both lens spaces.*

(We refer to [Hei69] for an extension of Waldhausen's result to the non-orientable case.) Summarizing, Theorems 1.1, 2.1 and 2.2 show that fundamental groups determine closed 3-manifolds up to orientation of the prime factors and up to the indeterminacy arising from lens spaces. More precisely, we have the following theorem.

Theorem 2.3. *Let N and N' be two closed, oriented 3-manifolds with isomorphic fundamental groups. Then there exist natural numbers p_1, \dots, p_r , q_1, \dots, q_r and q'_1, \dots, q'_r and oriented manifolds N_1, \dots, N_s and N'_1, \dots, N'_s such that the following three conditions hold:*

(1) *we have homeomorphisms*

$$\begin{aligned} N &\cong L(p_1, q_1) \# \dots \# L(p_r, q_r) \# N_1 \# \dots \# N_s \text{ and} \\ N' &\cong L(p_1, q'_1) \# \dots \# L(p_r, q'_r) \# N'_1 \# \dots \# N'_s; \end{aligned}$$

(2) *N_i and N'_i are homeomorphic (but possibly with opposite orientations); and*

(3) *for $i = 1, \dots, r$ we have $q'_i \equiv \pm q_i^{\pm 1} \pmod{p_i}$.*

2.2. 3-manifolds with boundary, and peripheral structures. By Theorem 2.2, orientable, prime 3-manifolds with infinite fundamental groups are determined by their fundamental groups if they are closed. The same conclusion does not hold if we allow boundary. For example, if K is the trefoil knot with an arbitrary orientation, then $S^3 \setminus \nu(K \# K)$ and $S^3 \setminus \nu(K \# -K)$ (i.e., the exteriors of the granny knot and the square knot) have isomorphic fundamental groups, but the spaces are not homeomorphic (which can be seen by studying the Blanchfield form [Bl57] which in turn can be studied using Levine–Tristram signatures, see [Ke73, Lev69, Tri69]). (Note though that prime knots in S^3 are in fact determined by their groups [GL89, Whn87].)

We will need the following definition to formulate the classification theorem.

Definition. Let N be a 3-manifold with incompressible boundary. The fundamental group of $\pi_1(N)$ together with the set of conjugacy classes of subgroups determined by the boundary components is called the *peripheral structure* of N .

We now have the following theorem.

Theorem 2.4. *Let N and N' be two compact, orientable, irreducible 3-manifolds with non-spherical, non-trivial incompressible boundary.*

- (1) *If there exists an isomorphism $\pi_1(N) \rightarrow \pi_1(N')$, then N can be turned into N' using finitely many ‘Dehn flips’.*
- (2) *There exist only finitely many compact, orientable, irreducible 3-manifolds with non-spherical, non-trivial incompressible boundary such that the fundamental group is isomorphic to $\pi_1(N)$.*
- (3) *If there exists an isomorphism $\pi_1(N) \rightarrow \pi_1(N')$ which sends the peripheral structure of N isomorphically to the peripheral structure of N' , then N and N' are homeomorphic.*

The first two statements of the theorem were proved by Johannson [Jon79, Theorem 29.1 and Corollary 29.3]. We refer to [Jon79, Section X] for the definition of Dehn flips. (See also [Swp80] for a proof of the second statement.) The third statement was proved by Waldhausen. We refer to [Wan68b] and [JS76] for details.

2.3. Properties of 3-manifolds and their fundamental groups. In the previous section we saw that irreducible, orientable closed 3-manifolds with infinite fundamental groups are determined by their fundamental groups. We furthermore saw that the fundamental group determines the fundamental groups of the prime factors of a given compact, orientable 3-manifold. It is interesting to ask which topological properties of 3-manifolds can be ‘read off’ from the fundamental group.

Let $N \neq S^1 \times D^2$ be a compact, irreducible 3-manifold which is not a line bundle. Let $\text{Diff}_0(N)$ be the identity component of the group $\text{Diff}(N)$ of diffeomorphisms of N . The quotient $\text{Diff}(N)/\text{Diff}_0(N)$ is denoted by $\mathcal{M}(N)$. Furthermore, given a group π , we denote by $\text{Out}(\pi)$ the group of outer automorphisms of π (i.e., the quotient of the group of isomorphisms of π by the subgroup of inner automorphisms of π). It follows from the Rigidity Theorem, from Waldhausen [Wan68a, Corollary 7.5] and from the Geometrization Theorem, that the canonical map

$$\mathcal{M}(N) \rightarrow \{\varphi \in \text{Out}(\pi) : \varphi \text{ preserves the peripheral structure}\}$$

is an isomorphism. If N is a closed 3-manifold which is not Seifert fibered, then it follows from the Rigidity Theorem, from Zimmermann [Zim82, Satz 0.1] and from the Geometrization Theorem that any finite subgroup of $\text{Out}(\pi_1(N))$ can be represented by a finite group of diffeomorphisms of N (see also [HT87]). The case of Seifert fibered 3-manifolds is somewhat more complicated and is treated by Zieschang and Zimmermann [ZZ82, Zim79] and Raymond [Ray80] (see also [RaS77, HT78, HT83]).

We now give a few more situations in which topological information can be ‘directly’ obtained from the fundamental group.

- (1) Let N be a compact 3-manifold and $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ a non-trivial class. Stallings [Sta62] (together with the resolution of the Poincaré Conjecture) showed that ϕ is a fibered class (i.e., can be realized by a fibration $N \rightarrow S^1$) if and only if $\text{Ker}\{\phi : \pi_1(N) \rightarrow \mathbb{Z}\}$ is finitely generated. (If $\pi_1(N)$ is a one-relator group the latter condition can be verified easily using Brown’s criterion, see [Bro87, Mas06, Dun01] for details.)
- (2) If $\pi_1(N)$ admits a non-trivial decomposition as a graph of groups (e.g., as an amalgamated product or an HNN extension), then this decomposition gives rise to a decomposition along incompressible surfaces of N with the same underlying graph. (Some care is needed here: the edge and vertex groups of the new decomposition may be different from the edge and vertex groups of the original decomposition.) We refer to Epstein [Ep61c], Waldhausen [Wan67a] and Culler–Shalen [CS83, Proposition 2.3.1] for details.
- (3) If N is a geometric 3-manifold, then by the discussion of Section 1.5 the geometry of N is determined by the properties of $\pi_1(N)$.
- (4) If N is a 3-manifold with boundary and with $H_2(N; \mathbb{Z}) = 0$, i.e., N is the exterior of a knot in a rational homology sphere, then Calegari [Cal09, Proof of Proposition 4.4] gives a group-theoretic interpretation of the Thurston norm of N . (See [Thu86a] and Section 8.4 for a definition

of the Thurston norm. Note though that there does not seem to be a good group-theoretical equivalent to the Thurston norm for general 3-manifolds.)

- (5) Scott and Swarup gave an algebraic characterization of the JSJ decomposition of a compact, orientable 3-manifold with incompressible boundary [SS01] (see also [SS03]).

In general though it is difficult to obtain topological information about N by just applying group-theoretical methods to $\pi_1(N)$. For example, given a closed 3-manifold N it is obvious that the minimal number $r(N)$ of generators of $\pi_1(N)$ is a lower bound on the Heegaard genus $g(N)$ of N . It has been known for a while that $r(N) \neq g(N)$ for graph manifolds [BuZ84, Zie88, ScW07]. In contrast it follows from work of Souto [Sou08] and Namazi–Souto [NS09] that $r(N) = g(N)$ for ‘sufficiently complicated’ hyperbolic 3-manifolds (see also [Mas06] for further examples). Recently Li [Li11] showed that there exist also hyperbolic 3-manifolds with $r(N) < g(N)$. We refer to [Shn07] for background information.

3. CENTRALIZERS

3.1. The centralizer theorems. Let π be a group. The *centralizer* of a subset $X \subseteq \pi$ is defined to be the subgroup

$$C_\pi(X) := \{g \in \pi : gx = xg \text{ for all } x \in X\}.$$

Determining the centralizers is often one of the key steps in understanding a group. In the world of 3-manifold groups, thanks to the Geometrization Theorem, an almost complete picture emerges. In this section we will only consider 3-manifolds to which Theorem 1.14 applies, i.e., 3-manifolds with empty or toroidal boundary which are compact, orientable and irreducible. But many of the results of this section also generalize fairly easily to fundamental groups of compact 3-manifolds in general, using the arguments of Sections 1.1 and 1.3.

The following theorem reduces the determination of centralizers to the case of Seifert fibered manifolds.

Theorem 3.1. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. We write $\pi = \pi_1(N)$. Let $g \in \pi$ be non-trivial. If $C_\pi(g)$ is non-cyclic, then one of the following holds:*

- (1) *there exists a JSJ torus T and $h \in \pi$ such that $g \in h \pi_1(T) h^{-1}$ and such that*

$$C_\pi(g) = h \pi_1(T) h^{-1};$$

- (2) *there exists a boundary component S and $h \in \pi$ such that $g \in h \pi_1(S) h^{-1}$ and such that*

$$C_\pi(g) = h \pi_1(S) h^{-1};$$

- (3) *there exists a Seifert fibered component M and $h \in \pi$ such that $g \in h \pi_1(M) h^{-1}$ and such that*

$$C_\pi(g) = h C_{\pi_1(M)}(h^{-1}gh)h^{-1}.$$

Remark. Note that one could formulate the theorem more succinctly: if g is non-trivial and $C_\pi(g)$ is non-cyclic, then there exists a component C of the characteristic submanifold and $h \in \pi$ such that $g \in h \pi_1(C) h^{-1}$ and such that

$$C_\pi(g) = h C_{\pi_1(C)} (h^{-1} g h) h^{-1}.$$

We will provide a short proof of Theorem 3.1 which makes use of the deep results of Jaco–Shalen and Johannson and of the Geometrization Theorem for non-Haken manifolds. Alternatively the theorem can be proved using the Geometrization Theorem much more explicitly—we refer to [Fr11] for details.

Proof. We first consider the case that N is hyperbolic. In Section 6 we will see that we can view $\pi = \pi_1(N)$ as a discrete, torsion-free subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Note that the centralizer of any non-trivial matrix in $\mathrm{PSL}(2, \mathbb{C})$ is abelian and isomorphic to either \mathbb{Z} or \mathbb{Z}^2 ; this can be seen easily using the Jordan normal form of such a matrix. Now let $g \in \pi \subseteq \mathrm{PSL}(2, \mathbb{C})$ be non-trivial. If $C_\pi(g)$ is not infinite cyclic, then it is a free abelian group of rank two. It now follows from Theorem 1.11 that either (1) or (2) holds.

If N is Seifert fibered, then the theorem is trivial. It follows from Theorem 1.14 that it remains to consider the case where N admits a non-trivial JSJ decomposition. In that case N is in particular Haken (see Section 6 for the definition) and the theorem follows from [JS79, Theorem VI.1.6] (see also [JS78, Theorem 4.1] and [Jon79, Proposition 32.9]). \square

We now turn to the study of centralizers in Seifert fibered manifolds. Let N be a Seifert fibered manifold with a given Seifert fiber structure. Then there exists a projection map $p: N \rightarrow B$ where B is the base orbifold. We denote by $B' \rightarrow B$ the orientation cover; note that this is either the identity or a 2-fold cover. Following [JS79] we refer to $(p_*)^{-1}(\pi_1(B'))$ as the *canonical subgroup* of $\pi_1(N)$. If f is a regular Seifert fiber of the Seifert fibration, then we refer to the subgroup of $\pi_1(N)$ generated by f as the *Seifert fiber subgroup*. Recall that if N is non-spherical, then the Seifert fiber subgroup is infinite cyclic and normal. (Note that the fact that the Seifert fiber subgroup is normal implies in particular that it is well defined, and not just up to conjugacy.)

Remark. The definition of the canonical subgroup and of the Seifert fiber subgroup depend on the Seifert fiber structure. By [Sco83a, Theorem 3.8] (see also [OVZ67] and [JS79, II.4.11]) a Seifert fibered manifold N admits a unique Seifert fiber structure unless N is either covered by S^3 , $S^2 \times \mathbb{R}$, or the 3-torus, or if either $N = S^1 \times D^2$, N is an I -bundle over the torus or the Klein bottle.

The following theorem, together with Theorem 3.1, now classifies centralizers of compact, orientable, irreducible 3-manifolds with empty or toroidal boundary.

Theorem 3.2. *Let N be an irreducible, orientable, non-spherical Seifert fibered manifold with a given Seifert fiber structure. Let $g \in \pi = \pi_1(N)$ be a non-trivial element. Then the following hold:*

- (1) *if g lies in the Seifert fiber group, then $C_\pi(g)$ equals the canonical subgroup;*
- (2) *if g does not lie in the Seifert fiber group, then the intersection of $C_\pi(g)$ with the canonical subgroup is abelian—in particular, $C_\pi(g)$ admits an abelian subgroup of index at most two;*

- (3) if g does not lie in the canonical subgroup, then $C_\pi(g)$ is infinite cyclic.

Proof. The first statement is [JS79, Proposition II.4.5]. The second and the third statement follow from [JS79, Proposition II.4.7]. \square

Let N be an irreducible, orientable, non-spherical Seifert fibered manifold. It follows immediately from the theorem that if g does not lie in the Seifert fiber group of a Seifert fiber structure, then $C_\pi(g)$ is isomorphic to one of \mathbb{Z} , $\mathbb{Z} \oplus \mathbb{Z}$, or the fundamental group of a Klein bottle. (See [JS78, p. 82] for details.)

3.2. Consequences of the centralizer theorems. Let π be a group and $g \in \pi$. We say $h \in \pi$ is a *root* of g if a power of h equals g . We denote by $\text{roots}_\pi(g)$ the set of all roots of g in π . Following [JS79, p. 32] we say that $g \in \pi$ has *trivial root structure* if $\text{roots}_\pi(g)$ lies in a cyclic subgroup of π . We say that $g \in \pi$ has *nearly trivial root structure* if $\text{roots}_\pi(g)$ lies in a subgroup of π which admits an abelian subgroup of index at most two.

Theorem 3.3. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Let $g \in \pi = \pi_1(N)$.*

- (1) *If g does not have trivial root structure, then there exists a Seifert fibered JSJ component M of N and $h \in \pi$ such that g lies in $h\pi_1(M)h^{-1}$ and*

$$\text{roots}_\pi(g) = h \text{roots}_{h^{-1}gh}(\pi_1(M)) h^{-1}.$$

- (2) *If N is Seifert fibered and if $g \in \pi_1(N)$ does not have nearly trivial root structure, then hgh^{-1} lies in a Seifert fiber group of N .*
- (3) *If N is Seifert fibered and if $g \in \pi_1(N)$ lies in the Seifert fiber group, then all roots of hgh^{-1} are conjugate to an element represented by a power of a singular Seifert fiber of N .*

If the Seifert fibered manifold N does not contain any embedded Klein bottles, then by [JS79, Addendum II.4.14] we get the following strengthening of conclusion (2): either $g \in \pi_1(N)$ has trivial root structure, or g is conjugate to an element in a Seifert fiber group of N .

Proof. Note that the roots of g necessarily lie in $C_\pi(g)$. The theorem now follows immediately from Theorem 3.1 and from [JS79, Proposition II.4.13]. \square

Remark. Let N be a 3-manifold. Kropholler [Kr90, Proposition 1] (see also [Ja75] and [Shn01]) showed, without using the Geometrization Theorem, that if $x \in \pi_1(N)$ is an element of infinite order such that x^n is conjugate to x^m , then $m = \pm n$. This fact also follows immediately from Theorem 3.3.

Given a group π we say that an element g is *divisible by an integer n* if there exists an h with $g = h^n$. We now obtain the following corollary to Theorem 3.3.

Corollary 3.4. *Let N be a compact, orientable, irreducible, non-spherical 3-manifold with empty or toroidal boundary. Then $\pi_1(N)$ does not contain elements which are infinitely divisible, i.e., divisible by infinitely many integers.*

Remark. For Haken 3-manifolds this result had been proved in [EJ73, Shn75, Ja75] (see also [Wan69] and [Fe76a, Fe76b, Fe76c]).

As we saw earlier, the fundamental group of a non-spherical Seifert fibered manifold admits a normal infinite cyclic subgroup, namely the Seifert fiber group. The following consequence of Theorem 3.1 shows that the converse holds.

Theorem 3.5. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ admits a normal infinite cyclic subgroup, then N is Seifert fibered.*

This was shown by Casson–Jungreis [CJ94] and independently by Gabai [Ga92] before the Geometrization Theorem was proved. (We also refer to [Wan67a, Wan68a], [GH75] and [Ja80, Theorem VI.24] for partial results, [Bow04] for an alternative proof, and [Mai03] and [Whn92] for extensions to orbifolds and to the non-orientable case.) If N is a compact orientable 3-manifold with non-empty boundary, then by [JS79, Lemma II.4.8] a more precise conclusion holds: if $\pi_1(N)$ admits a normal infinite cyclic subgroup Γ , then Γ is the Seifert fiber group for some Seifert fibration of N .

Proof of Theorem 3.5. Suppose $\pi = \pi_1(N)$ admits a normal infinite cyclic subgroup G . Recall that $\text{Aut } G$ is canonically isomorphic to $\mathbb{Z}/2$. The conjugation action of π on G defines a homomorphism $\varphi: \pi \rightarrow \text{Aut } G = \mathbb{Z}/2$. We write $\pi' = \text{Ker}(\varphi)$. Clearly $\pi' = C_\pi(G)$. It follows immediately from Theorem 3.1 that either N is Seifert fibered, or $\pi' = \mathbb{Z}$ or $\pi' = \mathbb{Z}^2$. But the latter case also implies that N is either a solid torus, an I -bundle over the torus or an I -bundle over the Klein bottle. In particular N is again Seifert fibered. \square

Given a group π and an element $g \in \pi$, the set of conjugacy classes of g is in a canonical bijection with the set $\pi/C_g(\pi)$. We thus easily obtain the following corollary to Theorem 3.1.

Theorem 3.6. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If N is not a Seifert fibered manifold, then the number of conjugacy classes is infinite for any $g \in \pi_1(N)$.*

This result was proved (in slightly greater generality) by de la Harpe–Préaux [dlHP07] using different methods. We refer to [dlHP07] for an application of this result to the von Neumann algebra $W_\lambda^*(\pi_1(N))$.

The following was shown by Hempel [Hem87, p. 390] without using the Geometrization Theorem.

Theorem 3.7. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary, and let S be a boundary component. Then $\pi_1(S)$ is a maximal abelian subgroup of $\pi_1(N)$.*

Proof. The result is well known to hold for Seifert fibered manifolds. The general case follows immediately from Theorem 3.1. \square

Recall that a subgroup A of a group π is called *malnormal* if $A \cap gAg^{-1} = 1$ for all $g \in \pi \setminus A$.

Theorem 3.8. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary.*

- (1) Let S be a boundary component. If the JSJ component which contains S is hyperbolic, then $\pi_1(S)$ is a malnormal subgroup of $\pi_1(N)$.
- (2) Let T be a JSJ torus. If both of the JSJ components abutting T are hyperbolic, then $\pi_1(T)$ is a malnormal subgroup of $\pi_1(N)$.

The first statement was proved by de la Harpe–Weber [dlHW11, Theorem 3] and can be viewed as a strengthening of the previous theorem. We refer to [Fr11] for an alternative proof. The second statement can be proved using the same techniques.

Given a group π we define an *ascending sequence of centralizers of length m* to be a sequence of subgroups of the form:

$$C_\pi(g_1) \subsetneq C_\pi(g_2) \subsetneq \cdots \subsetneq C_\pi(g_m).$$

We define $m(\pi)$ to be the maximal length of an ascending sequence of centralizers. Note that if $m(\pi) < \infty$, then π satisfies in particular property Max-c (maximal condition on centralizers); see [Kr90] for details. If N is a compact, orientable, irreducible, non-spherical 3-manifold with empty or toroidal boundary, then it follows from Theorems 3.1 and 3.2 that $m(\pi_1(N)) \leq 3$. It follows from [Kr90, Lemma 5] that $m(\pi_1(N)) \leq 16$ for any spherical N . It now follows from [Kr90, Lemma 4.2], combined with the basic facts of Sections 1.1 and 1.3 and some elementary arguments that $m(\pi_1(N)) \leq 17$ for any compact 3-manifold. We refer to [Kr90] for an alternative proof of this fact which does not require the Geometrization Theorem. We also refer to [Hil06] for a different approach.

We finish this section by illustrating how the results discussed so far can be used to quickly determine all 3-manifolds whose fundamental groups have a given interesting group-theoretic property. As an example we describe all 3-manifold groups which are CA and CSA. A group is said to be *CA* (short for *centralizer abelian*) if the centralizer of any non-identity element is abelian. Equivalently, a group is CA if and only if the intersection of any two distinct maximal abelian subgroups is trivial, if and only if “commuting” is an equivalence relation on the set of non-identity elements. For this reason, CA groups are also sometimes called “commutative transitive groups” (or CT groups, for short).

Lemma 3.9. *Let π be a CA group and $g \in \pi$, $g \neq 1$. If $C_\pi(g)$ is infinite cyclic, then $C_\pi(g)$ is self-normalizing.*

Proof. Suppose $C := C_\pi(g)$ is infinite cyclic. Let x be a generator for C , and let $y \in \pi$ such that $yC = Cy$. Then $xyx^{-1} = x^{\pm 1}$ and hence $y^2xy^{-2} = x$. Thus x commutes with y^2 , and since y^2 commutes with y , we obtain that x commutes with y . Hence y commutes with g and thus $y \in C_\pi(g) = C$. \square

The class of CSA groups was introduced by Myasnikov and Remeslennikov [MyR96] as a natural (in the sense of first-order logic, universally axiomatizable) generalization of torsion-free word-hyperbolic groups (see Section 5.4 below). A group is said to be *CSA* (short for *conjugately separated abelian*) if all of its maximal abelian subgroups are malnormal. Alternatively, a group is CSA if and only if the centralizer of every non-identity element is abelian and self-normalizing. (As a consequence, every subgroup of a CSA group is again CSA.) It is easy to

see that $\text{CSA} \Rightarrow \text{CA}$. There are CA groups which are not CSA, e.g., the infinite dihedral group, see [MyR96, Remark 5]. However, for 3-manifold groups, we have:

Corollary 3.10. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary, and suppose $\pi = \pi_1(N)$ is non-abelian. Then the following are equivalent:*

- (1) *Every JSJ component of N is hyperbolic.*
- (2) *π is CA.*
- (3) *π is CSA.*

Proof. We only need to show $(1) \Rightarrow (3)$ and $(2) \Rightarrow (1)$. Suppose all JSJ components of N are hyperbolic. Then by Theorem 3.1, the centralizer $C_\pi(g)$ of each $g \neq 1$ in π is abelian (so π is CA). It remains to show that each such $C_\pi(g)$ is self-normalizing. If $C_\pi(g)$ is cyclic, then this follows from the preceding lemma, and if $C_\pi(g)$ is not cyclic by Theorems 3.1 and 3.8. This shows $(1) \Rightarrow (3)$. The implication $(2) \Rightarrow (1)$ follows easily from Theorem 3.2, (1) and the fact that subgroups of CA groups are CA. \square

4. CONSEQUENCES OF THE GEOMETRIZATION THEOREM

In this section we summarize, in Diagram 1, various results on 3-manifold groups which do not rely on the work of Agol, Kahn–Markovic and Wise. Many of these results do, however, rely on the Geometrization Theorem. In light of the discussion in Sections 1.1, 1.3 and 1.6 we will concentrate on the study of fundamental groups of compact, orientable, irreducible 3-manifolds N with empty or toroidal boundary and such that $\pi_1(N)$ is neither finite nor solvable. (Note that this implies that the boundary is incompressible, since the only irreducible 3-manifold with compressible, toroidal boundary is $S^1 \times D^2$.)

We first give some of the definitions which we will use in Diagram 1. The definitions are roughly in the order that they appear in the diagram.

- (A.1) A space X is the Eilenberg–Mac Lane space for a group π , written as $X = K(\pi, 1)$, if $\pi_1(X) \cong \pi$ and if $\pi_i(X) = 0$ for $i \geq 2$.
- (A.2) The deficiency of a finite presentation $\langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ of a group is defined to be $k - l$. The *deficiency of a finitely presented group* π is defined to be the maximum over the deficiencies of all finite presentations of π . Note that some authors use the negative of this quantity.
- (A.3) A group is called *coherent* if every finitely generated subgroup is also finitely presented.
- (A.4) Given a space X and a homomorphism $\alpha: \pi_1(X) \rightarrow \Gamma$ the L^2 -Betti numbers $b_i^{(2)}(X, \alpha)$ were introduced by Atiyah [At76]; we refer to [Lü02] for details of the definition. If the group homomorphism is the identity map, then we just write $b_i^{(2)}(X) = b_i^{(2)}(X, \text{id})$.
- (A.5) Let N be a 3-manifold N and R be an integral domain. A cofinal tower of regular finite covers of N is a sequence of regular finite covers

$$\cdots \rightarrow \tilde{N}_2 \rightarrow \tilde{N}_1 \rightarrow \tilde{N}_0 = N$$

such that $\bigcap_{i \in \mathbb{N}} \pi_1(\tilde{N}_i) = 1$. If the limit

$$\lim_{i \rightarrow \infty} \frac{b_1(\tilde{N}_i; R)}{[N : \tilde{N}_i]}$$

exists for any cofinal tower of regular finite covers of N , and if all the limits agree, then we denote this unique limit by

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; R)}{[N : \tilde{N}]}.$$

- (A.6) We denote by $\overline{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} .
- (A.7) The *Frattini subgroup* $\Phi(\pi)$ of a group π is the intersection of all maximal subgroups. If π does not admit a maximal subgroup, then we define $\Phi(\pi) = \pi$. (By an elementary argument, the Frattini subgroup of π also agrees with the intersection of all normal subgroups which are not strictly contained in a proper normal subgroup of π .)
- (A.8) Let R be a (commutative) ring. We say that a group π is *linear over R* if π admits a faithful representation $\pi \rightarrow \mathrm{GL}(n, R)$ for some n . Note that if π injects into $\mathrm{GL}(n, R)$ for some n , then π also admits an injective homomorphism $\pi \rightarrow \mathrm{SL}(n+1, R)$.
- (A.9) Let N be a 3-manifold. By a surface in N we will always mean an orientable, properly embedded, compact surface. Note that if N is orientable, then a surface in our sense will always be two-sided.
 - (a) A surface $\Sigma \subset N$ is called *separating* if $N \setminus \Sigma$ is disconnected.
 - (b) The 3-manifold N is *Haken* (or *sufficiently large*) if N is irreducible, orientable, compact, and if N admits an embedded incompressible surface.
 - (c) By a *non-fiber surface* we mean an incompressible, connected surface in N which is not the fiber of a fibration $N \rightarrow S^1$.
 - (d) We say a surface Σ in N is *separable* if Σ is connected and such that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable. (See (A.18) for the definition of a separable subgroup.)
- (A.10) A group is *large* if it contains a finite-index subgroup which admits an epimorphism onto a non-cyclic free group.
- (A.11) Given $k \in \mathbb{N}$ we refer to

$$\mathrm{coker}\{H_1(\partial N; \mathbb{Z}) \rightarrow H_1(N; \mathbb{Z})\}$$

as the *non-peripheral homology of N* . Note that if N has non-peripheral homology of rank k , then any finite cover of N has non-peripheral homology of rank at least k . We say that a 3-manifold N is *homologically large* if given any $k \in \mathbb{N}$ there exists a finite regular cover N' of N which has non-peripheral homology of rank at least k .

- (A.12) Given a group π and an integral domain R with quotient field Q we write $vb_1(\pi; R) = \infty$ if for any k there exists a finite-index (not necessarily normal) subgroup π' of π such that

$$\mathrm{rank}_R(H_1(\pi'; R)) := \dim_Q(H_1(\pi'; Q)) \geq k.$$

In that case we say that π has infinite virtual first R -Betti number. Given a 3-manifold N we write $vb_1(N; R) = \infty$ if $vb_1(\pi_1(N); R) = \infty$. We will sometimes write $vb_1(N) = vb_1(N; \mathbb{Z})$.

Note that if N is an irreducible, non-spherical, compact 3-manifold with empty or toroidal boundary such that $vb_1(\pi; R) = \infty$, then for any k there exists also a finite-index *normal* subgroup π' of π such that $\text{rank}_R(H_1(\pi'; R)) \geq k$. Indeed, if $\text{char}(R) = 0$, then this follows from elementary group-theoretic arguments, and if $\text{char}(R) \neq 0$, then this follows from [Lac09, Theorem 5.1], since the Euler characteristic of $N = K(\pi_1(N), 1)$ is zero. (Here we used that our assumptions on N imply that $N = K(\pi_1(N), 1)$ —see (C.1).)

- (A.13) A group π is called *indicable* if π admits an epimorphism onto \mathbb{Z} . A group π is called *locally indicable* if any finitely generated subgroup of π admits an epimorphism onto \mathbb{Z} .
- (A.14) A group π is called *left-orderable* if it admits a strict total ordering “ $<$ ” which is left-invariant, i.e., it has the property that if $g, h, k \in \pi$ with $g < h$, then $kg < kh$. A group π is called *bi-orderable* if it admits a strict total ordering which is left- and right-invariant.
- (A.15) Given a property \mathcal{P} of groups we say that a group π is *virtually* \mathcal{P} if π admits a finite-index subgroup (not necessarily normal) which satisfies \mathcal{P} .
- (A.16) Given a class \mathcal{P} of groups we say that a group π is *residually* \mathcal{P} if given any non-trivial $g \in \pi$ there exists a surjective homomorphism $\alpha: \pi \rightarrow G$ onto a group $G \in \mathcal{P}$ and such that $\alpha(g)$ is non-trivial. A case of particular importance is when \mathcal{P} is the class of finite groups, in which case π is said to be *residually finite*. Another important case is when \mathcal{P} is the class of finite p -groups for p a prime (that is, the class of groups of p -power order), in which case π is said to be *residually* p .
- (A.17) The *profinite topology* on a group π is the coarsest topology with respect to which every homomorphism from π to a discrete finite group is continuous. Note that π is residually finite if and only if the profinite topology on π is Hausdorff. Similarly, the pro- p topology on π is the coarsest topology with respect to which every homomorphism from π to a finite p -group is continuous.
- (A.18) Let π be a group. We say that a subset S is *separable* if S is closed in the profinite topology on π —equivalently, for any $g \in \pi \setminus S$, there exists a homomorphism $\alpha: \pi \rightarrow G$ to a finite group such that $\alpha(g) \notin \alpha(S)$. The group π is called *locally extended residually finite (LERF)* (or *subgroup separable*) if any finitely generated subgroup is separable, and π is *AERF* (or *abelian subgroup separable*) if any finitely generated abelian subgroup of π is separable.
- (A.19) Let π be a group. We say that π is *double-coset separable* if given any two finitely generated subgroups $A, B \subseteq \pi$ and any $g \in \pi$ the subset $AgB \subset \pi$ is separable. Note that AgB is separable if and only if $(g^{-1}Ag)B$ is separable, and therefore to prove double-coset separability it suffices to show that products of finitely generated subgroups are separable.
- (A.20) Let π be a group and Γ a subgroup. We say that π *induces the full profinite topology on* Γ if the restriction of the profinite topology on π to Γ is the full

profinite topology on Γ —equivalently, for any finite-index subgroup $\Gamma' \subseteq \Gamma$ there exists a finite-index subgroup $\pi' \subseteq \pi$ such that $\pi' \cap \Gamma \subseteq \Gamma'$.

- (A.21) Let N be an orientable, irreducible 3-manifold with empty or toroidal boundary. We will say that N is *efficient* if the graph of groups corresponding to the JSJ decomposition is efficient, i.e., if the following hold:
- (a) $\pi_1(N)$ induces the full profinite topology on the fundamental groups of the JSJ tori and of the JSJ pieces; and
 - (b) the fundamental groups of the JSJ tori and the JSJ pieces, viewed as subgroups of $\pi_1(N)$, are separable.

We refer to [WZ10] for details.

- (A.22) Let π be a finitely presentable group. We say that the *word problem for π is solvable* if given any finite presentation for π there exists an algorithm which can determine whether or not a given word in the generators is trivial. Similarly, the *conjugacy problem for π is solvable* if given any finite presentation for π there exists an algorithm to determine whether or not any two given words in the generators represent conjugate elements of π . We refer to [CZi93, Section D.1.1.9] for details. (Note that by [Mil92, Lemma 2.2] the word problem and the conjugacy problem is solvable for one finite presentation if and only if it is solvable for every finite presentation.)
- (A.23) A group is called *Hopfian* if it is not isomorphic to a proper quotient of itself.

Before we move on to Diagram 1 we state a few conventions which we apply in the diagram.

- (B.1) In Diagram 1 we mean by N an irreducible, compact, orientable 3-manifold such that its boundary consists of a (possibly empty) collection of tori. We furthermore assume throughout Diagram 1 that $\pi := \pi_1(N)$ is neither solvable nor finite. Note that without these extra assumptions some of the implications do not hold. For example not every Seifert fibered manifold N admits a finite cover N' with $b_1(N') > 1$, but this is the case if π is furthermore neither solvable nor finite.
- (B.2) Arrow (7) splits into three arrows, this means that precisely one of the three possible conclusion holds.
- (B.3) Red arrows indicate that the conclusion holds *virtually*, e.g., if N is a Seifert fibered space such that $\pi_1(N)$ is neither finite nor solvable, then N contains virtually an incompressible torus.
- (B.4) If a property \mathcal{P} of groups is written in green, then the following conclusion always holds: If N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary such that the fundamental group of a finite (not necessarily regular) cover of N has Property \mathcal{P} , then $\pi_1(N)$ also has Property \mathcal{P} .

In most cases it is clear that the properties in Diagram 1 written in green satisfy this condition. Note that it follows from Theorem 1.15 that if N' is a finite cover of an irreducible, compact, orientable 3-manifold N with empty or toroidal boundary, then N' is hyperbolic (Seifert fibered, admits non-trivial JSJ decomposition) if and only if N has the same property.

- (B.5) Note that a concatenation of red and black arrows which leads to a green property means that the initial group also has the green property.
- (B.6) An arrow with a condition next to it indicates that this conclusion only holds if this extra condition is satisfied.

Finally we give one last disclaimer: the diagram is meant as a guide to the precise statements in the text and in the literature; it should not be used as a reference in its own right.

Diagram 1 now summarizes several consequences of the Geometrization Theorem.

We now give the justifications for the implications of Diagram 1. In the subsequent discussion we strive for maximal generality, in particular, unless we say otherwise, we will only assume that N is a connected 3-manifold. We will give the required references and arguments for the general case. In particular each justification can be read independently of all the other steps. We will also give further information and background material to put the statements in context.

- (C.1) Let N be an irreducible, orientable 3-manifold with infinite fundamental group. It follows from the irreducibility of N and the Sphere Theorem (see Theorem 1.3) that $\pi_2(N) = 0$. Since $\pi_1(N)$ is infinite, it follows from the Hurewicz theorem that $\pi_i(N) = 0$ for any $i > 2$, i.e., N is an Eilenberg–Mac Lane space.
- (C.2) Let N be an irreducible, orientable 3-manifold with infinite fundamental group. By (C.1) we have $N = K(\pi_1(N), 1)$. Since the Eilenberg–Mac Lane space is finite-dimensional it follows from standard arguments that $\pi_1(N)$ is torsion-free. (Indeed, if $g \in \pi_1(N)$ is an element of finite order k , then consider $G = \langle g \rangle \subseteq \pi_1(N)$ and denote by \tilde{N} the corresponding covering space of N . Then the 3-manifold \tilde{N} is an Eilenberg–Mac Lane space for \mathbb{Z}/k , hence $H_*(\mathbb{Z}/k; \mathbb{Z}) = H_*(\tilde{N}; \mathbb{Z})$, but the only cyclic group with finite homology is the trivial group.)
- (C.3) Let N be an irreducible 3-manifold with empty or toroidal boundary and with infinite fundamental group. By (C.1) this implies that N is an Eilenberg–Mac Lane space. It now follows from work of Epstein (see [Ep61a, Lemmas 2.2 and 2.3] and [Ep61a, Theorem 2.5]) that the deficiency of $\pi_1(N)$ equals $1 - b_3(N)$.
- (C.4) Scott [Sco73b] (see also [Sco73a], [Sta77] and [RS90]) showed that if N is any 3-manifold such that $\pi_1(N)$ is finitely generated, then N admits a compact submanifold M such that $\pi_1(M) \rightarrow \pi_1(N)$ is an isomorphism. In particular it follows that $\pi_1(N)$ is finitely presented. It now follows easily that the fundamental group of any 3-manifold is coherent.
- (C.5) The Geometrization Theorem (see Theorem 1.14) implies that any irreducible, orientable, compact 3-manifold N with empty or toroidal boundary satisfies one of the following:
 - (a) N is Seifert fibered, or
 - (b) N is hyperbolic, or
 - (c) N admits an incompressible torus.
- (C.6) An orientable hyperbolic 3-manifold N admits a faithful discrete representation $\pi_1(N) \rightarrow \text{Isom}^+(\mathbb{H}^3)$ where $\text{Isom}^+(\mathbb{H}^3)$ denotes the orientation preserving isometries of 3-dimensional hyperbolic space. There is a well known identification of $\text{Isom}^+(\mathbb{H}^3)$ with $\text{PSL}(2, \mathbb{C})$, which thus gives rise to a faithful discrete representation $\pi_1(N) \rightarrow \text{PSL}(2, \mathbb{C})$. It is a consequence of the Rigidity Theorem that this representation is unique up to conjugation and complex conjugation. Another consequence of rigidity is that there exists in fact a faithful discrete representation $\pi_1(N) \rightarrow \text{PSL}(2, \overline{\mathbb{Q}})$ over the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} (see [MaR03, Corollary 3.2.4]). Thurston (see [Shn02, Section 1.6]) showed that the representation $\rho: \pi_1(N) \rightarrow \text{PSL}(2, \overline{\mathbb{Q}})$ lifts in fact to a faithful, discrete representation $\tilde{\rho}: \pi_1(N) \rightarrow \text{SL}(2, \overline{\mathbb{Q}})$.

The set of lifts of $\rho: \pi_1(N) \rightarrow \mathrm{PSL}(2, \overline{\mathbb{Q}})$ to a representation $\tilde{\rho}: \pi_1(N) \rightarrow \mathrm{SL}(2, \overline{\mathbb{Q}})$ is in a natural one-to-one correspondence with the set of Spin-structures on N ; we refer to [Cu86] and [MFP11, Section 2] for details.

If T is a boundary component, then $\tilde{\rho}(\pi_1(T))$ is a discrete subgroup isomorphic to \mathbb{Z}^2 . It follows that, up to conjugation, we have

$$\tilde{\rho}(\pi_1(T)) \subseteq \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} : \varepsilon \in \{-1, 1\}, a \in \overline{\mathbb{Q}} \right\}.$$

By [Cal06, Corollary 2.4] we have $\mathrm{tr}(\tilde{\rho}(a)) = -2$ if $a \in \pi_1(T)$ is represented by a curve on T which cobounds a surface in N .

- (C.7) Long–Reid [LoR98] showed that if π is isomorphic to a subgroup of $\mathrm{SL}(2, \overline{\mathbb{Q}})$ and if π is the fundamental group of a compact, orientable, non-spherical 3-manifold, then π is residually finite simple. (Note that the assumption that π is a non-spherical 3-manifold group is necessary since not all subgroups of $\mathrm{SL}(2, \overline{\mathbb{Q}})$ are residually simple.)

We refer to [But10] for more results on 3-manifolds (virtually) surjecting onto finite simple groups.

- (C.8) It is easy to see that the Frattini subgroup of a residually simple group is trivial.
- (C.9) The fundamental group of a Seifert fibered manifold is well known to be linear over \mathbb{Z} . We will provide a proof in Theorem 8.7 which was suggested to us by Boyer.
- (C.10) Let N be a Seifert fibered manifold. It follows from Lemma 8.8 that N is finitely covered by an S^1 -bundle over a connected orientable surface F . If $\pi_1(N)$ is neither solvable nor finite, then $\chi(F) < 0$. The surface F thus admits an essential curve c , the S^1 -bundle over the curve c is an incompressible torus.
- (C.11) Let N be an orientable, compact 3-manifold which admits an incompressible torus T . (Note that T could be any incompressible boundary torus.) By [LN91, Theorem 2.1] (see also (C.28) for a more general statement) the subgroup $\pi_1(T) \subseteq \pi_1(N)$ is separable, i.e., T is a separable surface. If $\pi_1(N)$ is furthermore not solvable, then the torus is not a fiber surface.
- (C.12) Let N be a compact, connected irreducible 3-manifold with non-empty incompressible boundary. Cooper–Long–Reid [CLR97, Theorem 1.3] (see also [Rat87], [But04] and [Lac07a]) have shown that in that case either N is covered by $S^1 \times S^1 \times I$ or $\pi_1(N)$ is large.

Now let N be a closed 3-manifold. Let Σ be a separable non-fiber surface, i.e., Σ is a connected incompressible surface in N which is not a fiber surface. By Stallings' Theorem [Sta62] (see also [Hem76, Theorem 10.5]) there exists a $g \in \pi_1(N \setminus \nu\Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(\Sigma)$ is separable by assumption, we can separate g from $\pi_1(\Sigma)$. A standard argument shows that in the corresponding finite cover N' of N the preimage of Σ consists of at least two, non null-homologous and non-homologous orientable surfaces. Any two such surfaces give rise to an epimorphism from $\pi_1(N')$ onto a free group with two generators. (See also [LoR05, Proof of Theorem 3.2.4].)

- (C.13) Let N be a compact 3-manifold with trivial or toroidal boundary. Let $\varphi: \pi_1(N) \rightarrow F$ be an epimorphism onto a non-cyclic free group. We claim that $\pi_1(N)$ is homologically large, i.e., we claim that given any $k \in \mathbb{N}$ there exists a finite cover N' of N such that

$$\text{rank}_{\mathbb{Z}} \text{coker}\{H_1(\partial N'; \mathbb{Z}) \rightarrow H_1(N'; \mathbb{Z})\} \geq k.$$

Indeed, let $k \in \mathbb{N}$. We denote by S_1, \dots, S_m (respectively T_1, \dots, T_n) the boundary components of N which have the property that φ restricted to the boundary torus is trivial (respectively non-trivial). Note that the image of $\pi_1(T_i) \subseteq F$ is a non-trivial infinite cyclic group generated by some $a_i \in F$. We now pick a prime number p with $p \geq 2n + k$. Since free groups are residually p , we can find an epimorphism $\alpha: F \rightarrow P$ where P is a p -group with the property that $\alpha(a_i)$ is non-trivial for $i = 1, \dots, n$. We write $F' = \text{Ker}(\alpha)$ and we denote by $q: N' \rightarrow N$ the covering of N corresponding to $\alpha \circ \varphi$. Note that if S' is any boundary component of N' which covers one of the S_i , then $\pi_1(S') \rightarrow \pi_1(N') \rightarrow F'$ is the trivial map. Using this observation we now calculate that

$$\begin{aligned} & \text{rank}_{\mathbb{Z}} \text{coker}\{H_1(\partial N'; \mathbb{Z}) \rightarrow H_1(N'; \mathbb{Z})\} \\ & \geq \text{rank}_{\mathbb{Z}} \text{coker}\{H_1(\partial N'; \mathbb{Z}) \rightarrow H_1(F'; \mathbb{Z})\} \\ & \geq b_1(F') - \sum_{i=1}^n b_1(q^{-1}(T_i)) \\ & \geq b_1(F') - 2 \sum_{i=1}^n b_0(q^{-1}(T_i)) \\ & \geq |P|(b_1(F) - 1) + 1 - 2n \frac{|P|}{p} \\ & \geq |P| - 2n \frac{|P|}{p} \\ & = \frac{|P|}{p}(p - 2n) \\ & \geq k. \end{aligned}$$

(See also [CLR97, Corollary 2.9] for a related argument.)

- (C.14) If N is an irreducible, compact, orientable 3-manifold with empty or toroidal boundary and such that $b_1(N) \geq 2$, then a straightforward Thurston-norm argument (see [Thu86a] or [CC03, Corollary 10.5.11]) shows that either N is a torus bundle (in which case $\pi_1(N)$ is solvable), or N admits a homologically essential non-fiber surface. Note that this surface is necessarily non-separating.

Note that since Σ is homologically essential it follows from standard arguments (e.g., using Stallings' Theorem [Sta62]) that Σ does in fact not lift to the fiber of a fibration in any finite cover.

- (C.15) Howie [How82, Proof of Theorem 6.1] (see also [HoS85, Lemma 2]) has shown that if N is an orientable, compact, irreducible 3-manifold and $\Gamma \subseteq \pi_1(N)$ a finitely generated subgroup of infinite index, then $b_1(\Gamma) \geq 1$. Furthermore a standard transfer argument shows that if G is a finite index subgroup of a group H , then $b_1(G) \geq b_1(H)$. Combining these two facts it follows that if N is an orientable, compact, irreducible 3-manifold with $b_1(N) \geq 1$, then

any finitely generated subgroup Γ of $\pi_1(N)$ has the property that $b_1(\Gamma) \geq 1$, i.e., $\pi_1(N)$ is locally indicable.

- (C.16) Burns–Hale [BHa72] have shown that a locally indicable group is left-orderable. Note that left-orderability is not a ‘green property’, i.e., there exist compact 3-manifolds with non-left-orderable fundamental groups which admit left-orderable finite-index subgroups (see [BRW05] and [DPT05]).
- (C.17) Let $N \neq S^1 \times D^2$ be an irreducible, compact, orientable 3-manifold. If N has toroidal boundary, then each boundary component is incompressible and hence N is Haken. If N is closed and $b_1(N) \geq 1$, then $H_2(N; \mathbb{Z})$ is non-trivial. Let Σ be an oriented surface representing a non-trivial element in $H_2(N; \mathbb{Z})$. Since N is irreducible we can assume that Σ has no spherical components. Among all such surfaces we take a surface of maximal Euler characteristic. It now follows from the Loop Theorem (see Theorem 1.2) that any component of such a surface is incompressible, in particular N is Haken. (See also [Hem76, Lemma 6.6] for a proof.)
- (C.18) Let N be an irreducible, compact, orientable 3-manifold with empty or toroidal boundary. If N is Haken and if N is not a closed Seifert fibered manifold, then it follows from the work of Allenby–Boler–Evans–Moser–Tang [ABEMT79, Theorem 2.9 and Theorem 4.7] that the Frattini group is trivial.
If N is a closed Seifert fibered manifold with infinite fundamental group, then its Frattini group is a (possibly trivial) subgroup of the infinite cyclic subgroup generated by a regular Seifert fiber (see [ABEMT79, Lemma 4.6]).
- (C.19) Evans–Moser [EM72, Corollary 4.10] showed that if N is an irreducible Haken 3-manifold such that $\pi_1(N)$ is non-solvable, then $\pi_1(N)$ contains a non-cyclic free group.
- (C.20) Tits [Tit72] showed that a group which is linear over \mathbb{C} is either virtually solvable or contains a non-cyclic free group; this dichotomy is commonly referred to as the Tits Alternative. (Recall that in Diagram 1 we assumed that π is neither finite nor solvable, it follows from Theorem 1.20 that π is not virtually solvable.)

The combination of the above and of (C.19) shows that the fundamental group of a compact 3-manifold with empty or toroidal boundary is either virtually solvable or it contains a non-cyclic free group. This dichotomy is a weak version of the Tits alternative for 3-manifold groups. We refer to (K.2) for a stronger version of the Tits alternative for 3-manifold groups. We also refer to [Par92, ShW92, KZ07] for several ‘pre-geometrization’ results on the Tits alternative.

- (C.21) A group which contains a non-abelian free group is non-amenable. Indeed, it is well known that any subgroup and any finite-index supergroup of an amenable group is also amenable. On the other hand, non-cyclic free groups are not amenable.
- (C.22) A consequence of the Lubotzky Alternative (cf. [LuS03, Window 9, Corollary 18]) asserts that a finitely generated group which is linear over \mathbb{C} either is virtually solvable or, for any prime p , has infinite virtual first \mathbb{F}_p -Betti number (see also [Lac09, Theorem 1.3] and [Lac11, Section 3]).

We refer to [CE10, Example 5.7], [ShW92], [Lac09], [Wal09] and [Lac11, Section 4] for more on the growth of \mathbb{F}_p -Betti numbers of finite covers of hyperbolic 3-manifolds. We also refer to [Me90] for a ‘pre-Perelman’ result regarding the \mathbb{F}_p -homology of finite covers of 3-manifolds.

- (C.23) Let N be a Seifert fibered manifold. Niblo [Nib92] showed that $\pi_1(N)$ is double-coset separable. In particular $\pi_1(N)$ is LERF.

It follows from work of Hall [Hal49] that fundamental groups of Seifert fibered spaces with boundary are LERF. Scott [Sco78, Sco85] showed that fundamental groups of closed Seifert fibered spaces are LERF. We also refer to [BBS84, Nib90, Tre90, Lop94, Gi97, Wil07, BaC12, Pat12] for alternative proofs and extensions of Scott’s theorem.

- (C.24) Let N be any compact 3-manifold. In [AF10] it is shown that, for all but finitely many primes p , the group $\pi_1(N)$ is virtually residually p .

Note that if N is a graph manifold (i.e., if all its JSJ components are Seifert fibered manifolds), then by [AF10, Proposition 2] a slightly stronger statement holds: for any prime p the group $\pi_1(N)$ is virtually residually p . Also note that for hyperbolic 3-manifolds, or more generally for 3-manifolds N such that $\pi_1(N)$ is linear over \mathbb{C} , it follows from [We73, Theorem 4.7] that for almost all primes p the group $\pi_1(N)$ is virtually residually p .

- (C.25) The subsequent, well-known argument in (H.2) can be used to show that a group which is virtually residually p is also residually finite.

The fact that fundamental groups of compact 3-manifolds are residually finite was first shown by Hempel [Hem87] and Thurston [Thu82a, Theorem 3.3].

- (C.26) Mal’cev [Mal40] (see also [Mal65, Theorem VII]) showed that a finitely generated residually finite group is Hopfian.

- (C.27) We refer to [LyS77, Theorem IV.4.6] for a proof of the fact that finitely presented groups which are residually finite have solvable word problem.

In fact, a more precise statement can be made: the fundamental group of a compact 3-manifold has an exponential Dehn function; see [CEHLPT92] for details. Also note that Waldhausen [Wan68b] solved the word problem for fundamental groups of 3-manifolds which are virtually Haken.

- (C.28) E. Hamilton [Hamb01] showed that the fundamental group of any compact, orientable 3-manifold is AERF. See also [LN91] and [AH99] for earlier results.

- (C.29) The conjugacy problem has been solved for all 3-manifolds with incompressible boundary by Préaux [Pre05, Pre06], building on ideas of Sela [Sel93].

- (C.30) In a group for which the conjugacy problem is solvable, the solution applied to the conjugacy class of the trivial element also gives a solution to the word problem.

- (C.31) Wilton–Zalesskii [WZ10, Theorem A] showed that closed prime orientable 3-manifolds are efficient. If N is a prime 3-manifold with toroidal boundary, then we denote by W the result of gluing exteriors of hyperbolic knots to the boundary components of N . It follows from Proposition 1.9 that the JSJ tori of W consist of the JSJ tori of N and the boundary tori of N . Since W is efficient by [WZ10, Theorem A] it now follows that N is also efficient. See also [AF10, Chapter 5] for a discussion of the question whether

closed prime orientable 3-manifolds are, for all but finitely many primes p , virtually p -efficient. (Here p -efficiency is the natural analogue of efficiency for the pro- p -topology; cf. [AF10, Section 5.1].)

- (C.32) Lott–Lück [LL95, Theorem 0.1] (see also [Lü02, Section 4.2]) showed that if N is a compact irreducible non-spherical 3-manifold with empty or toroidal boundary, then $b_i^{(2)}(N) = b_i^{(2)}(\pi) = 0$ for any i . We refer to [LL95] and [Lü02, Section 4.2] for the calculation of L^2 -Betti numbers of any compact 3-manifold.
- (C.33) It follows from work of Lück [Lü94, Theorem 0.1] that given a topological space X and a homomorphism $\pi_1(X) \rightarrow \Gamma$ to a residually finite group Γ , the L^2 -Betti numbers $b_i^{(2)}(X, \pi_1(X) \rightarrow \Gamma)$ can be viewed as a limit of ordinary Betti numbers of finite covers of X . Combining this result with (C.32) we see that if N is a compact irreducible non-spherical 3-manifold with empty or toroidal boundary, then

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; \mathbb{Z})}{[N : \tilde{N}]} = 0.$$

We refer to [BeG04] for a related result on towers of finite index irregular covers.

Remark. In Diagram 1, statements (C.1)–(C.4) do not rely on the Geometrization Theorem. Statement (C.5) is a variation on the Geometrization Theorem, whereas statements (C.6)–(C.23) gain their relevance from the Geometrization Theorem. Finally, the general statements (C.24)–(C.33) rely directly on the Geometrization Theorem. In particular the results of Hempel [Hem87] and E. Hamilton [Hamb01] were proved for 3-manifolds ‘for which geometrization works’; by the work of Perelman these results then hold in the above generality.

There are a few arrows and results on 3-manifold groups which can be proved using the Geometrization Theorem, and which we left out of the diagrams:

- (D.1) Let N be a compact, orientable, irreducible 3-manifold. Kojima [Koj87] and Luecke [Lue88] first showed that if N contains an incompressible, non-boundary parallel torus and if $\pi_1(N)$ is not solvable, then $vb_1(N; \mathbb{Z}) = \infty$. In spirit their proof is rather similar to the steps we provide.
- (D.2) Let N be a compact 3-manifold with incompressible boundary, with no spherical boundary components, which is not a product on a boundary component. (i.e., there does not exist a component \S of ∂N such that $N \cong S \times [0, 1]$.) It follows from standard arguments (e.g., boundary subgroup separability, see (L.3) for details) that for any k there exists a finite cover $\tilde{N} \rightarrow N$ such that \tilde{N} has at least k boundary components. In particular a Poincaré duality argument immediately implies that $vb_1(N; \mathbb{Z}) = \infty$.
- (D.3) Wilton [Wil08, Corollary 2.10] determined the closed 3-manifolds N such that $\pi_1(N)$ is residually free. In particular, it is shown that if N is a prime, orientable 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is residually free, then N is the product of a circle with a connected surface.
- (D.4) A group is called *co-Hopfian* if it is not isomorphic to any proper subgroup of itself. Note that the fundamental groups of some compact 3-manifolds,

such as the 3-torus or a torus knot, are not co-Hopfian. But ‘generically’ fundamental groups of compact 3-manifolds tend to be co-Hopfian. We refer to [GW92, GW94, GLW94], [WW94, WY99], [PV00] and [BGHM10] for details and related results.

- (D.5) Boyer–Rolfsen–Wiest [BRW05, Corollary 1.6] showed that the fundamental groups of Seifert fibered manifolds are virtually bi-orderable. Perron–Rolfsen [PR03, PR06] have shown that fundamental groups of many fibered 3-manifolds (i.e., 3-manifolds which fiber over S^1) are bi-orderable. In the other direction, Smythe (see [Neh74, p. 228]) proved that the fundamental group of the trefoil complement is not bi-orderable. In particular, not all fundamental groups of fibered 3-manifolds are bi-orderable. We also refer to Clay–Rolfsen [CR10] for many more examples. Their methods also provide examples of fibered hyperbolic 3-manifolds with non bi-orderable fundamental groups.
- (D.6) If N is a Seifert fibered manifold, then the proof of [AF10, Proposition 4.16] shows that $\pi_1(N)$ admits a finite-index subgroup which is residually torsion-free nilpotent.

By (G.24) and (G.26) this gives an alternative proof that fundamental groups of Seifert fibered manifolds are virtually bi-orderable.

- (D.7) The lower central series of 3-manifold groups was studied by Cochran–Orr [CO98]. Teichner [Tei97] showed that if the lower central series of the fundamental group of a closed 3-manifold stabilizes, then the maximal nilpotent quotient is the fundamental group of a closed 3-manifold (and such groups were determined in [Tho68]).
- (D.8) The fact that Seifert fibered manifolds admit a geometric structure can in most cases be used to give an alternative proof of the fact that their fundamental groups are linear over \mathbb{C} . More precisely, if N admits a geometry X , then $\pi_1(N)$ is a discrete subgroup of $\text{Isom}(X)$. By [Bo] the isometry groups of the following geometries are subgroups of $\text{GL}(4, \mathbb{R})$: spherical geometry, $S^2 \times \mathbb{R}$, Euclidean geometry, Nil, Sol and hyperbolic geometry. Furthermore, the fundamental group of an $\mathbb{H}^2 \times \mathbb{R}$ -manifold is a subgroup of $\text{GL}(5, \mathbb{R})$. On the other hand, the isometry group of $\widetilde{\text{SL}(2, \mathbb{R})}$ is not linear (see, e.g., [Di77, p. 170]).

Also note that groups which are virtually polycyclic are linear over \mathbb{Z} by the Auslander–Swan Theorem (see [Swn67, Au67]) and (H.4). This implies in particular that fundamental groups of Sol-manifolds are linear over \mathbb{Z} .

- (D.9) The *Whitehead group* $Wh(\pi)$ of a group π is defined as the quotient of $K_1(\mathbb{Z}[\pi])$ by $\pm\pi$. Here $K_1(\mathbb{Z}[\pi])$ is the abelianization of $\lim_{n \rightarrow \infty} \text{GL}(n, \mathbb{Z}[\pi])$, i.e., it is the abelianization of the direct limit of the general linear groups over $\mathbb{Z}[\pi]$. We refer to [Mil66] for details.

The Whitehead group of the fundamental group of a compact, orientable, non-spherical irreducible 3-manifold is trivial. This follows from the Geometrization Theorem together with the work of Farrell–Jones [FJ86], Waldhausen [Wan78], Farrell–Hsiang [FH81] and Plotnick [Pl80].

Two homotopy equivalent manifolds M and M' are simple homotopy equivalent if $Wh(\pi_1(M'))$ is trivial. It follows in particular that two compact,

orientable, non-spherical irreducible 3-manifolds which are homotopy equivalent are in fact simple homotopy equivalent. On the other hand, homotopy equivalent lens spaces are not necessarily simple homotopy equivalent. We refer to [Mil66, Coh73, Rou11] and [Ki97, p. 119] for more details.

Bartels–Farrell–Lück [BFL11] showed that the fundamental group of any 3-manifold N satisfies the Farrell–Jones Conjecture. This gives another proof that $\text{Wh}(N)$ is trivial. It also implies that if $\pi_1(N)$ is torsion-free, then it satisfies the Kaplansky Conjecture, i.e., the group ring $\mathbb{Z}[\pi_1(N)]$ has only two idempotents, namely 0 and 1.

- (D.10) We say that a group has Property U if it contains uncountably many maximal subgroups of infinite index. Margulis–Soifer [MaS81, Theorem 4] showed that every finitely generated group which is linear over \mathbb{C} and not virtually solvable has Property U . Using the fact that free groups are linear, one can use this result to show that in fact any large group also has Property U . Tracing through Diagram 1 now implies that the fundamental group of any orientable, compact, aspherical 3-manifold N with empty or toroidal boundary has Property U , unless $\pi_1(N)$ is solvable.

If π is the fundamental group of a hyperbolic 3-manifold, then it follows from [GSS10, Corollary 1.2] that any maximal subgroup of infinite index is in fact infinitely generated.

- (D.11) Romanovskiĭ [Rom69] and Burns [Bur71] showed that the free product of finitely many LERF groups is again LERF. This shows that the fundamental group of a compact 3-manifold is LERF, if the fundamental groups of the prime manifolds in its prime decomposition are LERF.
- (D.12) Dimca–Suciu [DS09] (see also [Kot10] and [BMS11]) showed that a group which is the fundamental group both of a closed 3-manifold and of a closed Kähler manifold, is necessarily finite.

5. THE WORK OF AGOL, KAHN–MARKOVIC, AND WISE

The Geometrization Theorem resolves the Poincaré Conjecture and, more generally, the classification of 3-manifolds with finite fundamental group. For 3-manifolds with infinite fundamental groups, the Geometrization Theorem can be viewed as asserting that the key problem is to understand the hyperbolic case.

In this section we first discuss the Tameness Theorem, proved independently by Agol [Ag07] and by Calegari–Gabai [CG06], which implies an essential dichotomy for finitely generated subgroups of hyperbolic 3-manifolds. We then turn to the Virtually Compact Special Theorem of Agol [Ag12], Kahn–Markovic [KM09] and Wise [Wis12]. This theorem, together with the Tameness Theorem and further work of Agol [Ag08] and Haglund [Hag08] and work of many others, resolves many hitherto intractable questions about hyperbolic 3-manifolds.

5.1. The Tameness Theorem. Agol [Ag07] and Calegari–Gabai [CG06] proved independently in 2004 the following theorem, which was first conjectured by Marden [Man74] in 1974:

Theorem 5.1. (Tameness Theorem) *Let N be a hyperbolic 3-manifold, not necessarily of finite volume. If $\pi_1(N)$ is finitely generated, then N is topologically tame, i.e., N is homeomorphic to the interior of a compact 3-manifold.*

We refer to [Cho06, Som06, Can08, Ga09, Bow10, Man07] for further details regarding the statement and alternative approaches to the proof. We especially refer to [Can08, Section 6] for a detailed discussion of earlier results leading towards the proof of the Tameness Theorem.

In the context of this paper, the main application is the Subgroup Tameness Theorem below. In order to formulate the theorem we need a few more definitions.

- (1) A *surface group* is the fundamental group of a closed, orientable surface of genus at least one.
- (2) We refer to [KAG86, p. 4] and [KAG86, p. 10] for the definition of a *quasi-Fuchsian surface group* and the definition of a *geometrically finite Kleinian group*. (Note that a geometrically finite Kleinian group is necessarily finitely generated.) Now let N be a hyperbolic 3-manifold. We can identify $\pi_1(N)$ with a discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ which is well defined up to conjugation (see [Shn02, Section 1.6] and (C.6)). We say that a subgroup $\Gamma \subseteq \pi_1(N) \subseteq \mathrm{SL}(2, \mathbb{C})$ is *geometrically finite* if $\Gamma \subseteq \mathrm{SL}(2, \mathbb{C})$ is a geometrically finite Kleinian group. We refer to [Bow93] for a discussion of various different equivalent definitions of ‘geometrically finite’. We say that a surface $\Sigma \subset N$ is *geometrically finite* if Σ is incompressible and if the subgroup $\pi_1(\Sigma) \subseteq \pi_1(N)$ is geometrically finite.
- (3) We say that a 3-manifold N is *fibered* if there exists a fibration $N \rightarrow S^1$. By a *surface fiber* in a 3-manifold N we mean the fiber of a fibration $N \rightarrow S^1$. We say that $\Gamma \subseteq \pi_1(N)$ is a *surface fiber subgroup* if there exists a surface fiber Σ such that $\Gamma = \pi_1(\Sigma)$. We say $\Gamma \subseteq \pi_1(N)$ is a *virtual surface fiber subgroup* if N admits a finite cover $N' \rightarrow N$ such that $\Gamma \subseteq \pi_1(N')$ and such that Γ is a surface fiber subgroup of N' .

We can now state the Subgroup Tameness Theorem which follows from combining the Tameness Theorem with Canary’s Covering Theorem (see [Can94, Section 4], [Can96] and [Can08, Corollary 8.1]):

Theorem 5.2. (Subgroup Tameness Theorem) *Let N be a hyperbolic 3-manifold and let $\Gamma \subseteq \pi_1(N)$ be a finitely generated subgroup. Then either*

- (1) Γ is a virtual surface fiber group, or
- (2) Γ is geometrically finite.

The importance of the Subgroup Tameness Theorem will become fully apparent in Sections 6 and 7.

5.2. The Virtually Compact Special Theorem. In his landmark 1982 article [Thu82a], Thurston posed twenty-four questions, which illustrated the limited understanding of hyperbolic 3-manifolds at that point. These questions guided research into hyperbolic 3-manifolds in the following years. Huge progress towards answering these questions has been made since. For example Perelman’s proof of the Geometrization Theorem answered Thurston’s Question 1 and the proof

by Agol and Calegari–Gabai of the Tameness Theorem answered Question 5 on Thurston’s list.

By early 2012, all but five of Thurston’s questions had been answered. Of the open problems, Question 23 plays a special role. Thurston conjectured that not all volumes of hyperbolic 3-manifolds are rationally related. This is a very difficult question which in nature is much closer to deep problems in number theory than to topology or differential geometry. We now list the remaining four questions (with the original numbering):

Questions 5.3. (Thurston, 1982)

- (15) *Are fundamental groups of hyperbolic 3-manifolds LERF?*
- (16) *Is every hyperbolic 3-manifold virtually Haken?*
- (17) *Does every hyperbolic 3-manifold have a finite-sheeted cover with positive first Betti number?*
- (18) *Is every hyperbolic 3-manifold virtually fibered?*

(It is clear that a positive answer to Question 18 implies a positive answer to Question 17 and in (C.17) we saw that a positive answer to Question 17 implies a positive answer to Question 16.) There has been a tremendous effort to resolve these four questions over the last three decades. (See Section 5.7 for an overview of previous results.) Nonetheless, progress has been slow for the better part of the period. In fact opinions on Question 18 were split. Regarding this particular question, Thurston himself famously wrote ‘this dubious-sounding question seems to have a definite chance for a positive answer’ [Thu82a, p. 380].

A stunning burst of creativity over the last five years has lead to the following theorem, which was proved by Agol [Ag12], Kahn–Markovic [KM09] and Wise [Wis12], with major contributions from Agol–Groves–Manning [Ag12], Bergeron–Wise [BW09], Haglund–Wise [HaW08, HaW12], Hsu–Wise [HsW12] and Sageev [Sag95, Sag97].

Theorem 5.4. (Virtually Compact Special Theorem) *If N is a hyperbolic 3-manifold, then $\pi_1(N)$ is virtually compact special.*

Remark.

- (1) We will give the definition of ‘virtually compact special’ in Section 5.3. In that section we will also state the theorem of Haglund and Wise (see Corollary 5.9) which gives an alternative formulation of the Virtually Compact Special Theorem in terms of subgroups of Right-Angled Artin Groups.
- (2) In the case that N is closed and admits a geometrically finite surface, a proof was first given by Wise [Wis12]. Wise also gave a proof in the case that N has non-trivial boundary (see Theorem 5.15). Finally, if N is closed and does not admit a geometrically finite surface, then the decisive ingredients of the proof were given by the work of Kahn–Markovic [KM09] and Agol [Ag12]. The latter builds heavily on the ideas and results of [Wis12]. See Diagram 2 for further details.

We will discuss the consequences of the Virtually Compact Special Theorem in detail in Section 6, but as an *amuse-bouche* we mention that it gives an affirmative

answer to Thurston's Questions 15 to 18. More precisely, Theorem 5.4 together with the Tameness Theorem, work of Haglund [Hag08], Haglund–Wise [HaW08] and Agol [Ag08] implies the following corollary.

Corollary 5.5. *If N is a hyperbolic 3-manifold, then*

- (1) $\pi_1(N)$ is LERF;
- (2) N is virtually Haken;
- (3) $vb_1(N) = \infty$; and
- (4) N is virtually fibered.

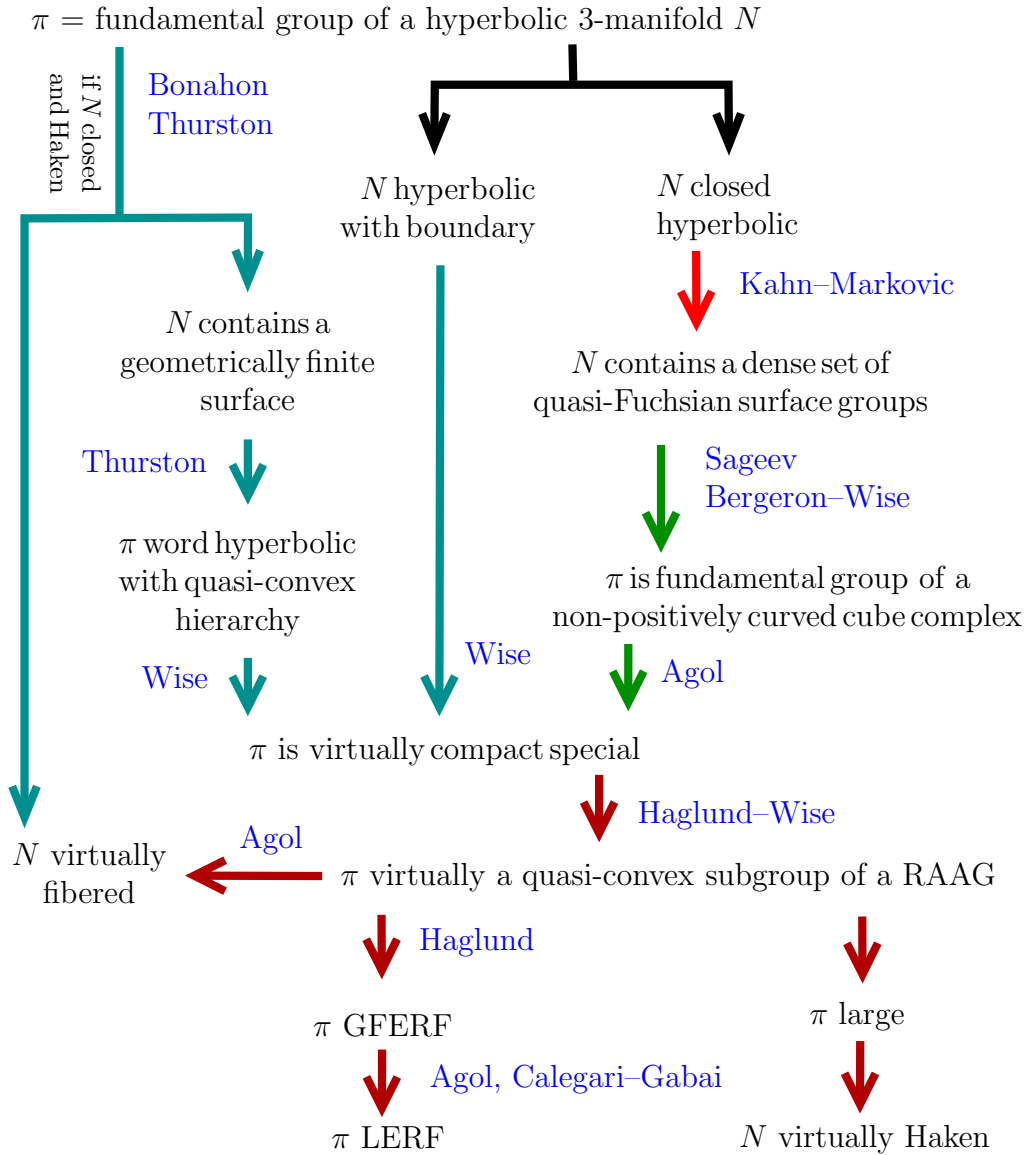


Diagram 2. The Virtually Compact Special Theorem.

Diagram 2 summarizes the various contributions to the proof of Theorem 5.4. The diagram can also be viewed as a guide to the next sections. More precisely we use the following color code.

- (1) Turquoise arrows correspond to Section 5.4.
- (2) The red arrow is treated in Section 5.5.
- (3) The green arrows are covered in Section 5.6.
- (4) Finally, the brown arrows correspond to the consequences of Theorem 5.4. They are treated in detail in Section 6.

5.3. Special cube complexes. The idea of applying non-positively curved cube complexes to the study of 3-manifolds originated with the work of Sageev [Sag95]. Haglund and Wise’s definition of a *special* cube complex was a major step forward, and sparked the recent surge of activity [HaW08]. In this section, we give rough definitions that are designed to give a flavor of the material. The reader is referred to [HaW08] for a precise treatment. For most applications, Corollary 5.8 or Corollary 5.9 can be taken as a definition.

A *cube complex* X is a cell complex in which each cell is a cube and the attaching maps are combinatorial isomorphisms. We also impose the condition, whose importance was brought to the fore by Gromov, that X should admit a locally CAT(0) (i.e., non-positively curved) metric. One of the attractions of cube complexes is that this condition can be phrased purely combinatorially. Note that the link of a vertex in a cube complex naturally has the structure of a simplicial complex.

Theorem 5.6. (Gromov’s Link Condition) *A cube complex X admits a non-positively curved metric if and only if the link of each vertex is flag. Recall that a simplicial cube complex is flag if every subcomplex Y that is isomorphic to the boundary of an n -simplex (for $n \geq 2$) is the boundary of an n -simplex in X .*

This theorem is due originally to Gromov [Gro87]. See also [BH99] for a proof (Theorem 5.20), as well as many more details about CAT(0) metric spaces and cube complexes. The next definition is due to Salvetti [Sal87].

Example (Salvetti complexes). Let Σ be any graph. We build a cube complex S_Σ as follows:

- (1) S_Σ has a single 0-cell x_0 ;
- (2) S_Σ has one (oriented) 1-cell e_v for each vertex v of Σ ;
- (3) S_Σ has a square 2-cell with boundary reading $e_u e_v \bar{e}_u \bar{e}_v$ whenever u and v are joined by an edge in Σ ;
- (4) for $n > 2$, the n -skeleton is defined inductively—attach an n -cube to any subcomplex isomorphic to the boundary of n -cube which does not already bound an n -cube.

It is an easy exercise to check that S_Σ satisfies Gromov’s Link Condition and hence is non-positively curved.

Definition. The fundamental group of the Salvetti complex S_Σ is the *Right-Angled Artin Group* (RAAG) A_Σ . Let $V = (v_1, \dots, v_k)$ be the vertex set of Σ . The

corresponding RAAG is defined as

$$A_\Sigma = \langle v_1, \dots, v_k : [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are connected by an edge} \rangle.$$

Note: the definition of A_Σ specifies a certain generating set.

Right angled Artin groups were introduced by Baudisch [Bah81] under the name semi-free groups, but they are also sometimes referred to as *graph groups* or *free partially commutative groups*. We refer to [Cha07] for a very readable survey paper on RAAGs.

Cube complexes have natural immersed codimension-one subcomplexes, called *hyperplanes*. If an n -cube C in X is identified with $[-1, 1]^n$, then a hyperplane of C is any intersection of C with a coordinate hyperplane of \mathbb{R}^n . We then glue together hyperplanes in adjacent cubes whenever they meet, to get the hyperplanes of $\{Y_i\}$ of X , which naturally immerse into X . Pulling back the cubes in which the cells of Y_i land defines an interval bundle N_i over Y_i , which also has a natural immersion $\iota_i: N_i \rightarrow X$. This interval bundle has a natural boundary ∂N_i , which is a 2-to-1 cover of Y_i , and we let $N_i^o = N_i \setminus \partial N_i$. Using this language, we can write down a short list of pathologies for hyperplanes in cube complexes.

- (1) A hyperplane Y_i is *one-sided* if $N_i \rightarrow Y_i$ is not a product bundle. Otherwise it is *two-sided*.
- (2) A hyperplane Y_i is *self-intersecting* if $\iota_i: Y_i \rightarrow X$ is not an injection.
- (3) A hyperplane Y_i is *directly self-osculating* if there are distinct vertices x, y in the same component of ∂N_i such that $\iota_i(x) = \iota_i(y)$ but, for some small neighborhoods $B_\varepsilon(x)$ and $B_\varepsilon(y)$, the restriction of ι_i to $(B_\varepsilon(x) \sqcup B_\varepsilon(y)) \cap N_i^o$ is an injection.
- (4) A distinct pair of hyperplanes Y_i, Y_j is *inter-osculating* if they both intersect and osculate; that is, the map $Y_i \sqcup Y_j \rightarrow X$ is not an embedding and there are vertices $x \in \partial N_i$ and $y \in \partial N_j$ such that $\iota_i(x) = \iota_j(y)$ but, for some small neighborhoods $B_\varepsilon(x)$ and $B_\varepsilon(y)$, the restriction of $\iota_i \sqcup \iota_j$ to $(B_\varepsilon(x) \cap N_i^o) \sqcup (B_\varepsilon(y) \cap N_j^o)$ is an injection.

In Figure 1 we give a schematic illustration of directly self-osculating and inter-osculating hyperplanes in a cube complex.

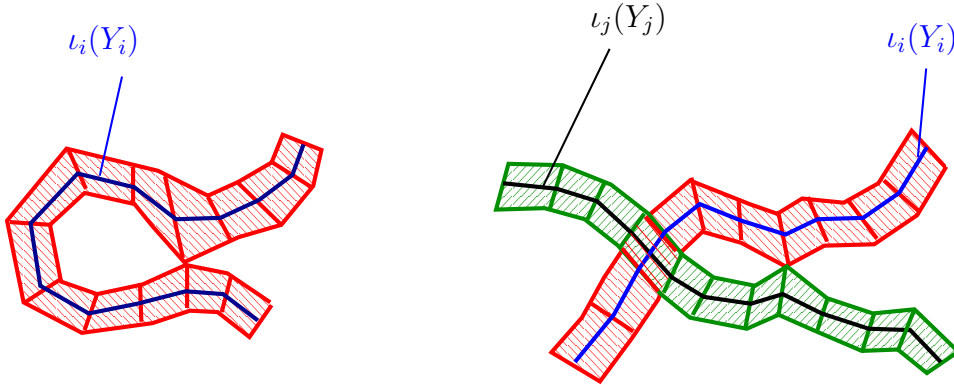


Figure 1. Directly self-osculating and inter-osculating hyperplanes.

Definition (Haglund–Wise [HaW08]). A cube complex X is *special* if none of the above pathologies occur. (In fact, we have given the definition of *A-special* from [HaW08]. Their definition of a *special* cube complex is slightly less restrictive. However, these two definitions agree up to passing to finite covers, so the two notions of ‘virtually special’ coincide.)

Definition. The *hyperplane graph* of a cube complex X is the graph $\Sigma(X)$ with vertex-set equal to the hyperplanes of X , and with two vertices joined by an edge if and only if the corresponding hyperplanes intersect.

If every hyperplane of X is two-sided, then there is a natural *typing map* $\phi_X: X \rightarrow S_{\Sigma(X)}$, which we now describe. Each 0-cell of X maps to the unique 0-cell x_0 of $S_{\Sigma(X)}$. Each 1-cell e crosses a unique hyperplane Y_e of X ; ϕ_X maps e to the 1-cell e_{Y_e} of $S_{\Sigma(X)}$ that corresponds to the hyperplane Y_e , and the two-sidedness hypothesis ensures that orientations can be chosen consistently. Finally, ϕ_X is defined inductively on higher dimensional cubes: a higher-dimensional cube C is mapped to the unique cube of $S_{\Sigma(X)}$ with boundary $\phi_X(\partial C)$.

The key observation of [HaW08] is that pathologies (2)–(4) above correspond exactly to the failure of the map ϕ_X to be a local isometry. We now sketch the argument. For each 0-cell x of X , the typing map ϕ_X induces a map of links $\phi_{X*}: \text{lk}(x) \rightarrow \text{lk}(x_0)$. This map ϕ_{X*} embeds $\text{lk}(x)$ as an isometric subcomplex of $\text{lk}(x_0)$. Indeed, if ϕ_x identifies two 0-cells of $\text{lk}(x)$, then we have a self-intersection or a direct self-osculation; likewise, if there are 0-cells u, v of $\text{lk}(x)$ that are not joined by an edge but $\phi_{X*}(u)$ and $\phi_{X*}(v)$ are joined by an edge in $\text{lk}(x_0)$, then there is an inter-osculation.

This is one direction of [HaW08, Theorem 4.2]:

Theorem 5.7. (Haglund–Wise) *A non-positively curved cube complex X is special if and only if there is a local isometry $\phi: X \rightarrow S_{\Sigma}$ for Σ some graph.*

The other direction of the theorem is a straightforward consequence of the results of [Hag08].

Lifting the local isometry ϕ to universal covers, there is a genuine isometric embedding of universal covers $\tilde{X} \hookrightarrow \tilde{S}_{\Sigma}$. In particular, the map ϕ induces an injection $\phi_*: \pi_1 X \rightarrow A_{\Sigma}$. Therefore, the fundamental group of a special cube complex is a subgroup of a Right-Angled Artin Group. On the other hand, a covering space of a special cube complex is itself a special cube complex. Theorem 5.7 therefore yields a characterization of subgroups of Right-Angled Artin Groups.

Definition (Special group). A group is called (*compact*) *special* if it is the fundamental group of a non-positively curved (*compact*) special cube complex.

Corollary 5.8. *A group is special if and only if it is a subgroup of a Right-Angled Artin Group.*

See (G.3) below for the proof.

Arbitrary subgroups of RAAGs may exhibit quite wild behavior. However, if the cube complex X is compact, then $\pi_1 X$ turns out to be a quasi-convex subgroup of a RAAG, and hence much better behaved.

Definition. Let X be a geodesic metric space. A subspace Y is said to be *quasi-convex* if there exists $\kappa \geq 0$ such that any geodesic in X with endpoints in Y is contained within the κ -neighborhood of Y .

Definition. Let π be a group with a fixed generating set S . A subgroup $H \subseteq \pi$ is said to be *quasi-convex* if it is a quasi-convex subspace of $\text{Cay}_S(\pi)$, the Cayley graph of π with respect to the generating set S .

Note that in general the notion of quasi-convexity depends on the choice of generating set S . Recall, however, that the definition of a RAAG specifies a generating set; we will always take the given choice of generating set when we talk about a quasi-convex subgroup of a RAAG.

Corollary 5.9. *A group is compact special if and only if it is a quasi-convex subgroup of a Right-Angled Artin Group.*

See (G.4) below for the proof.

5.4. Haken hyperbolic 3-manifolds: Wise's theorem. In this subsection, we discuss Wise's proof that closed, Haken hyperbolic 3-manifolds are virtually fibered. The starting point for Wise's work is the following theorem of Bonahon [Bon86] and Thurston, which is a special case of the Tameness Theorem.

Theorem 5.10. (Bonahon–Thurston) *Let N be a closed hyperbolic 3-manifold and let $\Sigma \subseteq N$ be an incompressible connected surface. Then either*

- (1) Σ lifts to a surface fiber in a finite cover, or
- (2) Σ is geometrically finite.

In particular, a closed hyperbolic Haken manifold is either virtually fibered or admits a geometrically finite surface. Also, note that by the argument of (C.14) and by Theorem 5.10, any 3-manifold with $b_1(N) \geq 2$ admits a geometrically finite surface.

Let N be a closed, hyperbolic 3-manifold that contains a geometrically finite surface. Thurston proved that N in fact admits a hierarchy of geometrically finite surfaces (see [Can94, Theorem 2.1]). In order to link up with Wise's results we need to recast Thurston's result in the language of geometric group theory.

Definition. A group is called *word-hyperbolic* if it acts properly discontinuously and cocompactly by isometries on a Gromov-hyperbolic space. This notion was introduced by Gromov [Gro81, Gro87]. See [BH99, Section III.Γ.2], and the references therein, for details.

When π is word-hyperbolic, the quasi-convexity of a subgroup of π does not depend on the choice of generating set [BH99, Corollary III.Γ.3.6], so we may speak unambiguously of a quasi-convex subgroup of a word-hyperbolic group.

Next, we introduce the class \mathcal{QH} of groups with a quasi-convex hierarchy.

Definition. The class \mathcal{QH} is defined to be the smallest class of finitely generated groups that is closed under isomorphism and satisfies the following properties.

- (1) $1 \in \mathcal{QH}$.

- (2) If $A, B \in \mathcal{QH}$ and the inclusion map $C \hookrightarrow A *_C B$ is a quasi-isometric embedding, then $A *_C B \in \mathcal{QH}$.
- (3) If $A \in \mathcal{QH}$ and the inclusion map $C \hookrightarrow A *_C$ is a quasi-isometric embedding, then $A *_C \in \mathcal{QH}$.

By, for instance, [BH99, Corollary III.Γ.3.6], a finitely generated subgroup of a word-hyperbolic group is quasi-isometrically embedded if and only if it is quasi-convex, which justifies the terminology.

The next proposition now makes it possible to go from hyperbolic 3-manifolds to the purely group-theoretic realm.

Proposition 5.11. *Let N be a closed hyperbolic 3-manifold. Then*

- (1) $\pi = \pi_1(N)$ is word hyperbolic;
- (2) a subgroup of π is geometrically finite if and only if it is quasi-convex;
- (3) if N has a hierarchy of geometrically finite surfaces, then $\pi_1(N) \in \mathcal{QH}$.

Proof. For the first statement, note that \mathbb{H}^3 is Gromov-hyperbolic and so the fundamental groups of closed hyperbolic manifolds are word-hyperbolic (see [BH99] for details). We refer to [Swp93, Theorem 1.1 and Proposition 1.3] and also [KaS96, Theorem 2] for proofs of the second statement. The third statement follows from the second statement. \square

We thus obtain the following reinterpretation of the aforementioned theorem of Thurston:

Theorem 5.12. (Thurston) *If N is a closed, hyperbolic 3-manifold containing a geometrically finite surface, then $\pi_1(N)$ is word-hyperbolic and $\pi_1(N) \in \mathcal{QH}$.*

The main theorem of [Wis12], Theorem 13.3, concerns word-hyperbolic groups with a quasi-convex hierarchy.

Theorem 5.13. (Wise) *Every word-hyperbolic group in \mathcal{QH} is virtually compact special.*

We immediately obtain the following corollary.

Corollary 5.14. *If N is a closed hyperbolic 3-manifold that contains a geometrically finite surface, then $\pi_1(N)$ is virtually compact special.*

Wise also proved a generalization of Theorem 5.13 to the case of certain relatively word-hyperbolic groups, from which he deduces the corresponding result in the cusped case [Wis12, Theorem 16.28 and Corollary 14.16].

Theorem 5.15. (Wise) *If N is a non-compact hyperbolic 3-manifold of finite volume, then $\pi_1(N)$ is virtually compact special.*

5.5. Quasi-Fuchsian surface subgroups: the work of Kahn and Markovic. As discussed in Section 5.4, Wise's work applies to hyperbolic 3-manifolds with a geometrically finite hierarchy. A non-Haken 3-manifold, on the other hand, has no hierarchy by definition. Likewise, although Haken hyperbolic 3-manifolds without a geometrically finite hierarchy are virtually fibered by Theorem 5.10, Thurston's Questions 15 (LERF), as well as other important open problems such as largeness, do not follow from Wise's theorems in this case.

The starting point for dealing with hyperbolic 3-manifolds without a geometrically finite hierarchy is provided by Kahn and Markovic's proof of the Surface Subgroup Conjecture. More precisely, as a key step towards answering Thurston's question in the affirmative, Kahn–Markovic [KM09] showed that the fundamental group of any closed hyperbolic 3-manifold contains a surface group. In fact they proved a significantly stronger statement. In order to state the precise theorem of Kahn and Markovic we need two more definitions:

- (1) In the following we will need the notion of a *quasi-Fuchsian surface group*. We refer to [KAG86, p. 4] and [KAG86, p. 10] for the definition. Note that a surface group is quasi-Fuchsian if and only if it is geometrically finite [Oh02, Lemma 4.66].
- (2) We fix an identification of $\pi_1(N)$ with a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$. We say that N *contains a dense set of quasi-Fuchsian surface groups* if for each great circle C of $\partial\mathbb{H}^3 = S^2$ there exists a sequence of π_1 -injective immersions $\iota : \Sigma_i \rightarrow N$ of surfaces Σ_i such that the following hold:
 - (a) for each i the group $\iota_*(\pi_1(\Sigma_i))$ is a quasi-Fuchsian surface group,
 - (b) the sequence $\partial\Sigma_i \subset \partial\mathbb{H}^3$ converges to C in the Hausdorff metric.

We can now state the theorem of Kahn–Markovic [KM09]. (Note that this particular formulation is [Be12, Théorème 5.3].)

Theorem 5.16. (Kahn–Markovic) *Every closed hyperbolic 3-manifold contains a dense set of quasi-Fuchsian surface groups.*

5.6. Agol's theorem. Let N be a closed hyperbolic 3-manifold. In the previous section we saw that Kahn–Markovic showed that N contains a dense set of quasi-Fuchsian surface groups. The following theorem of Bergeron–Wise [BW09], building extensively on work of Sageev [Sag95, Sag97] then makes it possible to approach hyperbolic 3-manifolds via non-positively curved cube complexes.

Theorem 5.17. (Sageev, Bergeron–Wise) *Let N be a closed hyperbolic 3-manifold which contains a dense set of quasi-Fuchsian surface groups. Then $\pi_1(N)$ is also the fundamental group of a compact non-positively curved cube complex.*

The following theorem was conjectured by Wise [Wis12] and proved recently by Agol [Ag12].

Theorem 5.18. (Agol) *Let π be word-hyperbolic and the fundamental group of a compact, non-positively curved cube complex. Then π is virtually compact special.*

The proof of Theorem 5.18 relies heavily on the results of the appendix to [Ag12], which are due to Agol, Groves and Manning. The results of this appendix extend the techniques of [AGM09] to word-hyperbolic groups with torsion, and combine them with the Wise's Malnormal Special Quotient Theorem [Wis12, Theorem 12.3].

Note that the combination of Theorems 5.16, 5.17 and 5.18 now implies Theorem 5.4 for closed hyperbolic 3-manifolds.

5.7. Summary of previous research. Questions 15 to 18 of Thurston have been a central area of research in 3-manifold topology over the last 30 years. The study of these questions lead to various other questions and conjectures. Perhaps the most important of these is the Lubotzky–Sarnak Conjecture (see [Lub96a]) that there is no closed hyperbolic 3-manifold N such that $\pi_1(N)$ has Property (τ) . (We refer to [Lub94] for the definition of Property (τ) .)

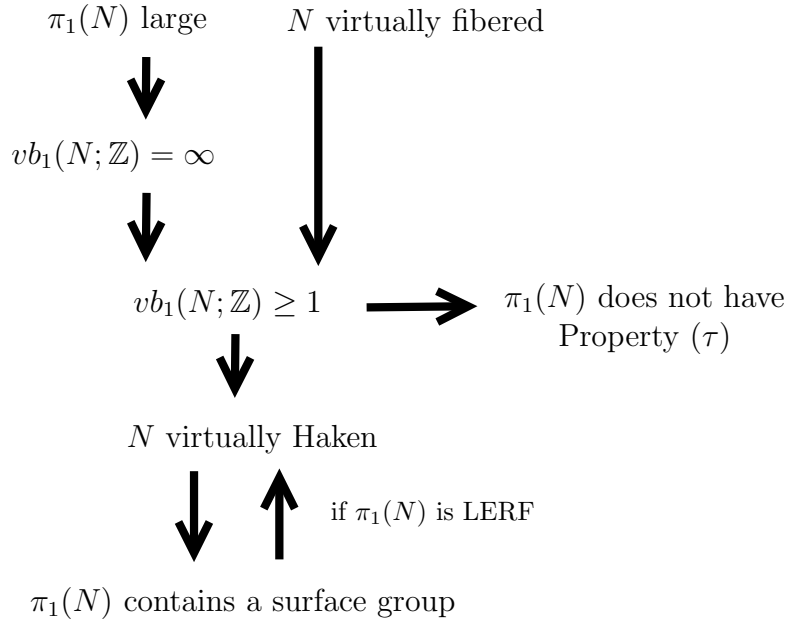


Diagram 3. Virtual properties of 3-manifolds.

In Diagram 3 we list various (virtual) properties of 3-manifold groups and logical implications between them. Some of the implications are obvious, and two implications follow from (C.13) and (C.17). Also note that if a 3-manifold N contains a surface group, then it admits a π_1 -injective map $\pi_1(\Sigma) \rightarrow \pi_1(N)$ of a closed surface Σ with genus at least one. If $\pi_1(N)$ is furthermore LERF, then there exists a finite cover of N such that the immersion lifts to an embedding (see [Sco78, Lemma 1.4] for details). Finally note that if $vb_1(N; \mathbb{Z}) \geq 1$, then by [Lub96a, p. 444] the group $\pi_1(N)$ does not have Property (τ) .

We will now survey some of the work in the past on Thurston’s questions and the properties of Diagram 3. The literature is so extensive that we cannot hope to achieve completeness. Beyond the summary below we also refer to the survey papers by Long–Reid [LoR05] and Lackenby [Lac11] for further details and references.

We arrange this survey by grouping references under the question that they address.

Question 5.19. (Surface Subgroup Conjecture) *Let N be a closed hyperbolic 3-manifold. Does $\pi_1(N)$ contain a (quasi-Fuchsian) surface group?*

The following papers attack Question 5.19.

- (1) Cooper–Long–Reid [CLR94, Theorem 1.5] showed that if N is a closed hyperbolic 3-manifold which fibers over S^1 , then there exists a π_1 -injective immersion of a quasi-Fuchsian surface into N . We note one important consequence: if N is any hyperbolic 3-manifold such that $\pi_1(N)$ is LERF and contains a surface subgroup, then $\pi_1(N)$ is large (cf. 12).
- (2) Li [Li02], Cooper–Long [CoL01] and Wu [Wu04] showed that in many cases the Dehn surgery on a hyperbolic 3-manifold contains a surface group.
- (3) Lackenby [Lac10] showed that closed arithmetic hyperbolic 3-manifolds contain surface groups.

Question 5.20. (Virtually Haken Conjecture) *Is every closed hyperbolic 3-manifold virtually Haken?*

Here is a summary of approaches towards the Virtually Haken Conjecture.

- (1) Thurston [Thu79] showed that all but finitely many Dehn fillings of the Figure 8 knot complement are not Haken. There has therefore been a considerable interest in studying the Virtually Haken Conjecture for fillings of 3-manifolds N . We refer to Aitchison–Rubinstein [AR99b], Aitchison–Matsumoti–Rubinstein [AMR97, AMR99], Baker [Ba88, Ba89, Ba90, Ba91], Boyer–Zhang [BrZ00], Cooper–Long [CoL99] (building on [FF98]), Cooper–Walsh [CoW06a, CoW06b], Hempel [Hem90], Kojima–Long [KL88], Masters [Mas00, Mas07], Masters–Menasco–Zhang [MMZ04, MMZ09], Morita [Moa86] and Zhang [Zh05] for work in this direction.
- (2) Hempel [Hem82, Hem84, Hem85a] and Wang [Wag90] studied the Virtually Haken Conjecture for 3-manifolds which admit an orientation reversing involution.
- (3) Long [Lo87] showed that if N is a hyperbolic 3-manifold which admits a totally geodesic immersion of a closed surface, then N is virtually Haken.
- (4) We refer to Millson [Mis76], Clozel [Cl87], Labesse–Schwermer [LaS86], Li–Millson [LM93], Rajan [Raj04], Reid [Red07] and Schwermer [Sch04, Sch10] for details of approaches to the Virtually Haken Conjecture for arithmetic hyperbolic 3-manifolds using number theoretic methods.
- (5) Reznikov [Rez97] gave a careful study of hyperbolic 3-manifolds N with $vb_1(N) = 0$.
- (6) Experimental evidence towards the validity of the conjecture was provided by Dunfield–Thurston [DnT03].
- (7) We refer to Lubotzky [Lub96b] and Lackenby [Lac06, Lac07b, Lac09] for work towards the stronger conjecture, that fundamental groups of hyperbolic 3-manifolds are large.

Question 5.21. *Let N be a hyperbolic 3-manifold. Is $\pi_1(N)$ LERF?*

The following papers gave evidence for an affirmative answer to Question 5.21. Note that, by the Subgroup Tameness Theorem, $\pi_1(N)$ is LERF if and only if every geometrically finite subgroup is separable, i.e., $\pi_1(N)$ is *GFERF*. See (G.9) for details.

- (1) The first examples of hyperbolic 3-manifolds with LERF fundamental groups were given by Gitik [Gi99b].
- (2) Agol–Long–Reid [ALR01] showed that geometrically finite subgroups of Bianchi groups are separable.
- (3) Wise [Wis06] showed that the fundamental group of the Figure 8 knot complement is LERF.
- (4) Agol–Groves–Manning [AGM09] showed that fundamental groups of hyperbolic 3-manifolds are LERF if every word-hyperbolic group is residually finite.
- (5) After the definition of special complexes was given in [HaW08], it was shown that various classes of hyperbolic 3-manifolds had virtually special fundamental groups, and hence were LERF (and virtually fibered). The following were shown to be virtually compact special:
 - (a) ‘standard’ arithmetic 3-manifolds [BHW11];
 - (b) certain branched covers of the figure-eight knot [Be08];
 - (c) manifolds built from gluing all-right ideal polyhedra, such as augmented link complements [CDW09].

Question 5.22. (Lubotzky–Sarnak Conjecture) *Let N be a closed hyperbolic 3-manifold. Is it true that $\pi_1(N)$ does not have Property (τ) ?*

The following represents some of the major work on the Lubotzky–Sarnak Conjecture. We also refer to [Lac11, Section 7] and [LZ03] for further details.

- (1) Lubotzky [Lub96a] stated the conjecture and proved that certain arithmetic 3-manifolds have positive virtual first Betti number, extending the above-mentioned work of Millson [Mis76] and Clozel [Cl87].
- (2) Lackenby [Lac06] showed that the Lubotzky–Sarnak Conjecture, together with a conjecture about Heegaard gradients, implies the Virtually Haken Conjecture.
- (3) Long–Lubotzky–Reid [LLuR08] proved that the fundamental group of every hyperbolic 3-manifold has Property (τ) with respect to some cofinal family of finite-index normal subgroups.
- (4) Lackenby–Long–Reid [LaLR08b] proved that if the fundamental group of a hyperbolic 3-manifold N is LERF, then $\pi_1(N)$ does not have Property (τ) .

Questions 5.23. *Let N be a hyperbolic 3-manifold with $b_1(N) \geq 1$.*

- (1) *Does N admit a finite cover N' with $b_1(N') \geq 2$?*
- (2) *Is $vb_1(N) = \infty$?*
- (3) *Is $\pi_1(N)$ large?*

The virtual Betti numbers of hyperbolic 3-manifolds in particular were studied by the following authors:

- (1) Cooper–Long–Reid [CLR97, Theorem 1.3] have shown that if N is a compact, irreducible 3-manifold with non-trivial incompressible boundary, then either N is covered by a product $N = S^1 \times S^1 \times I$, or $\pi_1(N)$ is large (see also [Rat87], [But04] and [Lac07a]).
- (2) Cooper–Long–Reid [CLR07, Theorem 1.3], Venkataramana [Ve08] and Agol [Ag06], showed that if N is an arithmetic 3-manifold with $vb_1(N) \geq$

- 1, then $vb_1(N) = \infty$. In fact by further work of Lackenby–Long–Reid [LaLR08a] it follows that $\pi_1(N)$ is large.
- (3) Masters [Mas02, Corollary 1.2] showed that if N is a fibered 3-manifold such that the genus of the fiber is 2, then $vb_1(N; \mathbb{Z}) = \infty$.
 - (4) Kionke–Schwermer [KiS12] showed that certain arithmetic hyperbolic 3-manifolds admit a cofinal sequence of finite index covers with rapid growth of first Betti numbers.
 - (5) Cochran and Masters [CMA06] studied the growth of Betti numbers in abelian covers of 3-manifolds with Betti number equal to two or three.
 - (6) Koberda [Kob12] gives a detailed study of Betti numbers of finite covers of fibered 3-manifolds.

Question 5.24. (Virtually Fibered Conjecture) *Is every hyperbolic 3-manifold virtually fibered?*

The following papers deal with the Virtually Fibered Conjecture:

- (1) An affirmative answer to the question was given for specific classes of 3-manifolds, e.g., certain knot and link complements, by Agol–Boyer–Zhang [ABZ08], Aitchison–Rubinstein [AR99a], DeBlois [DeB10], Gabai [Ga86], Guo–Zhang [GZ09], Reid [Red95], Leininger [Ler02] and Walsh [Wah05].
- (2) Button [But05] gave computational evidence towards an affirmative answer to the Virtually Fibered Conjecture.
- (3) Long–Reid [LoR08b] (see also Dunfield–Ramakrishnan [DR10] and [Ag08, Theorem 7.1]) showed that arithmetic hyperbolic 3-manifolds which are fibered admit in fact finite covers with arbitrarily many fibered faces in the Thurston norm ball.
- (4) Lackenby [Lac06] (see also [Lac11]) gave an approach to the Virtually Fibered Conjecture using ‘Heegaard gradients’. This approach was further developed by Lackenby [Lac04], Maher [Mah05] and Renard [Ren09]. Also note that Renard [Ren11] gave another approach to the conjecture.
- (5) Agol [Ag08] showed that 3-manifolds with virtually RFRS fundamental groups are virtually fibered. (See (E.4) for the definition of RFRS.) The first examples of 3-manifolds with virtually RFRS fundamental groups were given by Agol [Ag08], Bergeron [Be08], Bergeron–Haglund–Wise [BHW11] and Chesebro–DeBlois–Wilton [CDW09].

6. CONSEQUENCES OF THE VIRTUALLY COMPACT SPECIAL THEOREM

In this section we summarize various consequences of the Virtually Compact Special Theorem of Agol, Kahn–Markovic and Wise. As in Section 4 we present the results in a diagram.

We start out with the further definitions which are needed for Diagram 4. Again the definitions are roughly in the order that they appear in the diagram.

- (E.1) A compact orientable irreducible 3-manifold with empty or toroidal boundary is called a *graph manifold* if all JSJ components are Seifert fibered manifolds.
- (E.2) We say that a group π *virtually retracts* onto a subgroup $A \subseteq \pi$ if there exists a finite-index subgroup $\pi' \subseteq \pi$ that contains A and a homomorphism

$g: \pi' \rightarrow A$ such that $g(a) = a$ for any $a \in A$. In this case, we say that A is a *virtual retract* of π .

- (E.3) For N a hyperbolic 3-manifold, we say that $\pi_1(N)$ is GFERF if all geometrically finite subgroups are separable.
- (E.4) A group π is called *residually finite rationally solvable* (RFRS) if there exists a filtration of π by subgroups $\pi = \pi_0 \supseteq \pi_1 \supseteq \pi_2 \cdots$ such that:
 - (1) $\bigcap_i \pi_i = \{1\}$;
 - (2) π_i is a normal, finite-index subgroup of π , for any i ;
 - (3) for any i the map $\pi_i \rightarrow \pi_i/\pi_{i+1}$ factors through $\pi_i \rightarrow H_1(\pi_i; \mathbb{Z})/\text{torsion}$.
- (E.5) A group π is called *potent* if for any non-trivial $g \in \pi$ and any $n \in \mathbb{Z}$ there exists an epimorphism $\alpha: \pi \rightarrow G$ onto a finite group G such that $\alpha(g)$ has order n .
- (E.6) A subgroup of π is called *characteristic* if it is preserved by any automorphism of π . A group π is called *characteristically potent*, if given any non-trivial $g \in \pi$ and any $n \in \mathbb{N}$ there exists a finite index characteristic subgroup $\pi' \subseteq \pi$ such that g has order n in π/π' .
- (E.7) A group π is called *weakly characteristically potent* if for any non-trivial $g \in \pi$ there exists an $r \in \mathbb{N}$ such that for any $n \in \mathbb{Z}$ there exists a characteristic finite-index subgroup $\pi' \subseteq \pi$ such that $g\pi'$ has order rn in π/π' .
- (E.8) A group π is called *poly-free* if it admits a finite sequence of subgroups

$$\pi = \Gamma_0 \triangleright \Gamma_1 \triangleright \Gamma_2 \triangleright \cdots \triangleright \Gamma_k = \{1\}$$

such that for any $i \in \{0, \dots, k-1\}$ the quotient group Γ_i/Γ_{i+1} is a (not necessarily finitely-generated) free group.

- (E.9) A group π is called *conjugacy separable* if for any two non-conjugate elements $g, h \in \pi$ there exists an epimorphism $\alpha: \pi \rightarrow G$ onto a finite group G such that $\alpha(g)$ and $\alpha(h)$ are not conjugate. A group π is called *hereditarily conjugacy separable* if any (not necessarily normal) finite-index subgroup of π is conjugacy separable.
- (E.10) Let π be a torsion-free group. We say that a collection of elements

$$g_1, \dots, g_n \in \pi$$

is *independent* if distinct pairs of elements do not have conjugate non-trivial powers; that is, if there are $k, l \in \mathbb{Z} \setminus \{0\}$ and $c \in \pi$ with $cg_i^k c^{-1} = g_j^l$, then $i = j$. The group π is called *omnipotent* if given any independent collection

$$g_1, \dots, g_n \in \pi$$

there exists $k \in \mathbb{N}$ such that for any $l_1, \dots, l_n \in \mathbb{N}$ there exists a homomorphism $\alpha: \pi \rightarrow G$ to a finite group G such that the order of $\alpha(g_i) \in G$ is kl_i . This definition was introduced by Wise in [Wis00].

- (E.11) We say that a 3-manifold N with empty or toroidal boundary is *non-positively curved* (or *n.p.c.*) if the interior of N admits a complete non-positively curved Riemannian metric.
- (E.12) A group π has the *finitely generated intersection property* (or *f.g.i.p.* for short) if the intersection of any two finitely generated subgroups of π is also finitely generated.

Diagram 4 is supposed to be read in the same vein as Diagram 1. For the reader's convenience we recall some of the conventions.

- (F.1) In Diagram 4 we mean by N an irreducible, compact, orientable 3-manifold such that its boundary consists of a (possibly empty) collection of tori. We furthermore assume throughout Diagram 4 that $\pi := \pi_1(N)$ is neither solvable nor finite. Note that without these extra assumptions some of the implications do not hold. For example the fundamental group of the 3-torus T is a RAAG, but $\pi_1(T)$ is not large.
- (F.2) In the diagram the top arrow splits into several arrows. In this case exactly one of the possible three conclusions holds.
- (F.3) Red arrows indicate that the conclusion holds *virtually*, e.g., if π is RFRS, then N is virtually fibered.
- (F.4) If a property \mathcal{P} of groups is written in green, then the following conclusion always holds: If N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary such that the fundamental group of a finite (not necessarily regular) cover of N has Property \mathcal{P} , then $\pi_1(N)$ also has Property \mathcal{P} . In (H.1) to (H.7) below we will show that the various properties written in green do indeed have the above property.
- (F.5) Note that a concatenation of red and black arrows which leads to a green property means that the initial group also has the green property.
- (F.6) An arrow with a condition next to it indicates that this conclusion only holds if this extra condition is satisfied.
- (F.7) The dashed arrow in the top center indicates that this is a conjecture. To underline that this implication is a conjecture we also decorate this arrow by two question marks.

$N =$ irreducible, orientable, compact 3-manifold with empty or toroidal boundary
such that $\pi = \pi_1(N)$ is neither finite nor solvable

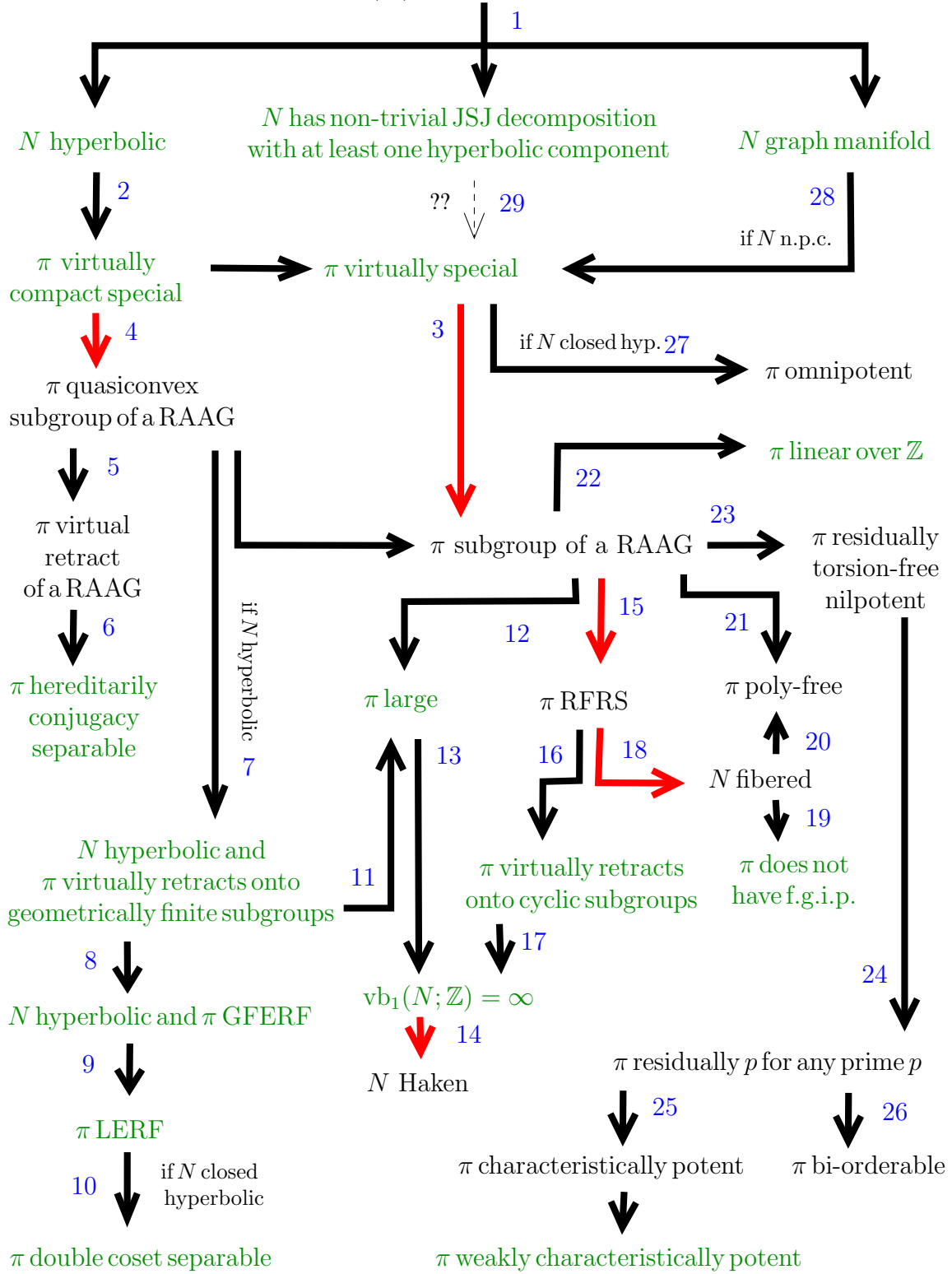


Diagram 4. Consequences of the Virtually Compact Special Theorem.

We now give the justifications for the implications of Diagram 4. As in Diagram 1 we strive for maximal generality. Unless we say otherwise, we will therefore only assume that N is a connected 3-manifold and each justification can be read independently of all the other steps.

- (G.1) Let N be an irreducible, orientable, compact 3-manifold with empty or toroidal boundary. It follows from the Geometrization Theorem (see Theorem 1.14) that N is either hyperbolic, or a graph manifold or it admits a non-trivial JSJ decomposition with at least one hyperbolic JSJ component.
- (G.2) Let N be a hyperbolic 3-manifold, then the Virtually Compact Special Theorem of Agol [Ag12], Kahn–Markovic [KM09] and Wise [Wis12] implies that $\pi_1(N)$ is virtually compact special. We refer to Section 5 for details.
- (G.3) Haglund–Wise [HaW08] showed that if a group π is special, then π admits a subgroup of finite index which is a subgroup of a RAAG. Indeed, suppose that π is the fundamental group of a special cube complex X . In the terminology of Haglund and Wise, X has a finite-sheeted *A-special* covering space \hat{X} [HaW08, Proposition 3.10]. There is a graph Σ and a local isometry $\phi: \hat{X} \rightarrow S_\Sigma$ by [HaW08, Theorem 4.2]. The induced map on universal covers $\tilde{\phi}: \tilde{X} \rightarrow \tilde{S}_\Sigma$ is then an isometry onto a convex subcomplex of \tilde{S}_Σ [HaW08, Lemma 2.11]. It follows that ϕ_* , the induced map on fundamental groups, is injective.

Note that the converse is also true, i.e., any group which is a subgroup of a RAAG is in fact special. Indeed, if π is a subgroup of a RAAG A_Σ , then π is the fundamental group of a covering space X of the Salvetti complex S_Σ . The Salvetti complex is special and so, by [HaW08, Corollary 3.8], X is also special.

- (G.4) Suppose that π is the fundamental group of a compact special cube complex X . Just as in (G.3), there is a finite-sheeted *A-special* cover \hat{X} of X , a graph Σ and a map of universal covers $\tilde{\phi}: \tilde{X} \rightarrow \tilde{S}_\Sigma$ that maps \tilde{X} isometrically onto a convex subcomplex of \tilde{S}_Σ . Because $\pi_1(\hat{X})$ acts cocompactly on \tilde{X} , it follows from [Hag08, Corollary 2.29] that $\phi_*\pi_1(\hat{X})$ is a quasi-convex subgroup of $\pi_1(S_\Sigma) = A_\Sigma$.

Again, the converse is also true, i.e., any group which is a quasi-convex subgroup of a RAAG is in fact compact special. Indeed, as in (G.3), if π is a subgroup of a RAAG A_Σ , then π is the fundamental group of some covering space X of the Salvetti complex S_Σ . By [Hag08, Corollary 2.29], π acts cocompactly on some convex subcomplex \tilde{Y} of the universal cover of S_Σ . The quotient $Y = \tilde{Y}/\pi$ is a locally convex, compact subcomplex of X and so, by [HaW08, Corollary 3.9], Y is special.

- (G.5) Haglund [Hag08, Theorem F] showed that quasi-convex subgroups of RAAGs are virtual retracts. In fact, he proved that quasi-convex subgroups of Right-Angled Coxeter Groups are virtual retracts, generalizing earlier results of Scott [Sco78] (the reflection group of the right-angled hyperbolic pentagon) and Agol–Long–Reid [ALR01] (reflection groups of arbitrary right-angled hyperbolic polyhedra).

- (G.6) Minasyan [Min09] has shown that any RAAG is hereditarily conjugacy separable. It follows immediately that virtual retracts of RAAGs are hereditarily conjugacy separable.

In (I.1) we will see that this result of Minasyan is a key ingredient in the proof of Hamilton–Wilton–Zalesskii [HWZ11] that the fundamental group of any orientable irreducible closed 3-manifold is conjugacy separable.

- (G.7) Suppose that N is a hyperbolic 3-manifold of finite volume and $\pi = \pi_1(N)$ is a quasi-convex subgroup of a RAAG A_Σ . Let Γ be a geometrically finite subgroup of π . The idea is that Γ should be a quasi-convex subgroup of A_Σ . One could then apply [Hag08, Theorem F] to deduce that Γ is a virtual retract of A_Σ and hence of π . However, it is not true in full generality that a quasi-convex subgroup of a quasi-convex subgroup is again quasi-convex, and so a careful argument is needed. In the closed case, the required technical result is [Hag08, Corollary 2.29]. In the cusped case, in fact it turns out that Γ may not be a quasi-convex subgroup of A_Σ . Nevertheless, it is possible to circumvent this difficulty. We now give detailed references.

If N is closed, then π is word-hyperbolic and Γ is a quasi-convex subgroup of π (see (K.18)). The group π acts by isometries on \tilde{S}_Σ , the universal cover of the Salvetti complex of A_Σ . Fix a base 0-cell $x_0 \in \tilde{S}_\Sigma$. The 1-skeleton of \tilde{S}_Σ is precisely the Cayley graph of A_Σ with respect to its standard generating set, and so, by hypothesis, the orbit $\pi.x_0$ is a quasi-convex subset of $\tilde{S}_\Sigma^{(1)}$. By [Hag08, Corollary 2.29], π acts cocompactly on some convex subcomplex $\tilde{X} \subseteq \tilde{S}_\Sigma$. Using the Morse Lemma for geodesics in hyperbolic spaces [BH99, Theorem III.D.1.7], the orbit $\Gamma.x_0$ is a quasi-convex subset of $\tilde{X}^{(1)}$. Using [Hag08, Corollary 2.29] again, it follows that Γ acts cocompactly on a convex subcomplex $\tilde{Y} \subseteq \tilde{X}$. The complex \tilde{Y} is also a convex subcomplex of \tilde{S}_Σ , which by a final application of [Hag08, Corollary 2.29] implies that $\Gamma.x_0$ is a quasi-convex subset of $\tilde{S}_\Sigma^{(1)}$, or, equivalently, that Γ is a quasi-convex subgroup of A_Σ . Hence, by [Hag08, Theorem F], Γ is a virtual retract of A_Σ and hence of π .

If N is not closed, then π is not word-hyperbolic, but in any case it is *relatively hyperbolic* and Γ is a *relatively quasi-convex* subgroup (see (K.18) below for a reference). One can show in this case that Γ is again a virtual retract of A_Σ and hence of π . The argument is rather more involved than the argument in the word-hyperbolic case; in particular, it is not necessarily true that Γ is a quasi-convex subgroup of A_Σ . See [CDW09, Theorem 1.3] for the details. The proof again relies on [Hag08, Theorem F] together with work of Manning–Martinez–Pedrosa [MMP10].

- (G.8) The following well-known argument shows that a virtual retract G of a residually finite group π is separable (cf. [Hag08, Section 3.4]). Let $\rho: \pi_0 \rightarrow G$ be a retraction onto G from a subgroup π_0 of finite index in π . Define a map $\delta: \pi_0 \rightarrow \pi_0$ by $g \mapsto g^{-1}\rho(g)$. It is easy to check that δ is continuous in the profinite topology on π_0 , and so $G = \delta^{-1}(1)$ is closed. That is to say, G is separable in π_0 , and hence in π . In particular, if N is a compact 3-manifold,

then $\pi = \pi_1(N)$ is residually finite by (C.25), and so if π virtually retracts onto geometrically finite subgroups, then π is GFERF.

- (G.9) Now let N be a hyperbolic 3-manifold such that $\pi = \pi_1(N)$ is GFERF and let $\Gamma \subseteq \pi$ be a finitely generated subgroup. We want to show that Γ is separable. By the Subgroup Tameness Theorem (see Theorem 5.2) and by our assumption we only have to deal with the case that Γ is a virtual surface fiber group. But an elementary argument shows that in that case Γ is separable (see, e.g., (K.11) for more details).
- (G.10) Let π be a group which is word hyperbolic with every quasi-convex subgroup separable. Minasyan [Min06] showed that any product of finitely many quasi-convex subgroups is separable. If N is a closed hyperbolic 3-manifold and $\pi_1(N)$ is GFERF, then because the quasi-convex subgroups are precisely the geometrically finite subgroups (see (K.18)), it follows that any product of geometrically finite subgroups is separable. A direct argument using part (ii) of [Nib92, Proposition 2.2] shows that any product of any subgroup with a virtual surface fiber group is separable. Therefore, by the Subgroup Tameness Theorem (see Theorem 5.2) the group $\pi_1(N)$ is double-coset separable.

It is expected that the analogue of Minasyan's theorem holds in the relatively hyperbolic setting, in which case the same argument would yield double-coset separability for GFERF fundamental groups of cusped hyperbolic manifolds. Note that separability of double cosets of abelian subgroups of finite-volume Kleinian groups was proved in [HWZ11].

- (G.11) Let N be a hyperbolic 3-manifold such that $\pi = \pi_1(N)$ virtually retracts onto any geometrically finite subgroup. Let $F \subseteq \pi$ be a geometrically finite non-cyclic free subgroup, such as a Schottky subgroup. (That every non-elementary Kleinian group contains a Schottky subgroup was first observed by Myrberg [Myr41].) Then by assumption there exists a finite index subgroup of π with an epimorphism onto F . This shows fundamental groups of hyperbolic 3-manifolds are large.
- (G.12) Antolín–Minasyan [AM11, Corollary 1.6] showed that a subgroup of a Right-Angled Artin Group is either free abelian of finite rank or maps onto a non-cyclic free group. This implies directly the fact that if N is a 3-manifold and if $\pi_1(N)$ is virtually special, then either $\pi_1(N)$ is either virtually solvable or $\pi_1(N)$ is large. (Recall that in the diagram we assume throughout that $\pi_1(N)$ is neither finite nor solvable, it follows from Theorem 1.20 that $\pi_1(N)$ is also not virtually solvable.)
- (G.13) We already saw in (C.13) that a group π which is large is homologically large, in particular it has the property that $vb_1(\pi; \mathbb{Z}) = \infty$.
- (G.14) We already saw in (C.17) that if an irreducible, compact 3-manifold satisfies $vb_1(N; \mathbb{Z}) \geq 1$, then N is virtually Haken. (Furthermore, we saw in (C.15) and (C.16) that $\pi_1(N)$ is virtually locally indicable and virtually left-orderable.)
- (G.15) Agol [Ag08, Theorem 2.2] showed that a RAAG is virtually RFRS. It is clear that the subgroup of a RFRS group is again RFRS.

A close inspection of Agol's proof using [DJ00, Section 1] in fact implies that a RAAG is already RFRS. We will not make use of this fact.

- (G.16) If π is RFRS, then it follows easily from the definition that, given any cyclic subgroup $\langle g \rangle$, there exists a finite-index subgroup π' such that $g \in \pi'$ and such that g represents a non-trivial element $[g]$ in the torsion-free abelian group $H' := H_1(\pi'; \mathbb{Z})/\text{torsion}$. There exists a finite-index subgroup H'' of H' which contains g and such that g represents a primitive element in H'' . In particular there exists a homomorphism $\varphi : H'' \rightarrow \mathbb{Z}$ such that $\varphi(g) = 1$. It now follows that

$$\text{Ker}\{\pi' \rightarrow H'/H''\} \rightarrow H'' \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{1 \mapsto g} \langle g \rangle$$

is a virtual retraction onto the cyclic group generated by g .

In Proposition 8.10 we will show that there exist Seifert fibered manifolds, and also graph manifolds with non-trivial JSJ decomposition, such that their fundamental groups do not virtually retract onto all cyclic subgroups.

- (G.17) Let π be an infinite torsion-free group which is not virtually abelian and which retracts virtually onto its cyclic subgroups. An elementary argument using the transfer map shows that $vb_1(\pi; \mathbb{Z}) = \infty$ (see, e.g., [LoR08a, Theorem 2.14] for a proof).
- (G.18) Let N be a compact, irreducible 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is RFRS. Agol [Ag08, Theorem 5.1] showed that N is virtually fibered. In fact Agol proved a more refined statement. If $\phi \in H^1(N; \mathbb{Q})$ is a non-fibered class, then there exists a finite solvable cover $p: N' \rightarrow N$ (in fact a cover which corresponds to one of the π_i in the definition of RFRS) such that $p^*(\phi) \in H^1(N'; \mathbb{Q})$ lies on the boundary of a fibered cone of the Thurston norm ball of N' . (We refer to [Thu86a] and Section 8.4 for background on the Thurston norm and fibered cones.)

Let N be an irreducible 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is virtually RFRS and if N is not a graph manifold, then we will show in Proposition 8.15 (see also [Ag08, Theorem 7.2] for the hyperbolic case) that there exist finite covers of N with arbitrarily many inequivalent fibered faces.

- (G.19) Let N be a virtually fibered 3-manifold such that $\pi_1(N)$ is not virtually solvable. Jaco–Evans [Ja80, p. 76] showed that $\pi_1(N)$ does not have the f.g.i.p.

Combining this result with the ones above and with work of Soma [Som92], we obtain the following: Let N be a compact 3-manifold with empty or toroidal boundary. Then $\pi_1(N)$ has the f.g.i.p. if and only if $\pi_1(N)$ is finite or solvable. Indeed, if $\pi_1(N)$ is finite or solvable, then $\pi_1(N)$ is virtually polycyclic and so every subgroup is finitely generated. (See [Som92] for details.) If N is Seifert fibered and $\pi_1(N)$ is neither finite nor solvable, then $\pi_1(N)$ does not have the f.g.i.p. (See again [Som92] for details.) If N is hyperbolic, then it follows from the combination of (G.2), (G.3), (G.15), (G.18) and the above mentioned result of Jaco and Evans, that $\pi_1(N)$ does not have the f.g.i.p. Finally, if N has non-trivial JSJ decomposition, then by the above already the fundamental group of a JSJ component does not have the f.g.i.p., hence $\pi_1(N)$ does not have the f.g.i.p.

We refer to [Hem85b] for examples of 3-manifolds with non-toroidal boundary which have the f.g.i.p.

- (G.20) If N is a fibered 3-manifold, then there exists an epimorphism $\phi: \pi_1(N) \rightarrow \mathbb{Z}$ such that $\text{Ker}(\phi) = \pi_1(\Sigma)$, where Σ is a compact surface. If Σ has boundary, then $\pi_1(\Sigma)$ is free and $\pi_1(N)$ is poly-free. If Σ is closed, then the kernel of any epimorphism $\pi_1(N) \rightarrow \mathbb{Z}$ is a free group. It follows easily that again $\pi_1(N)$ is poly-free.
- (G.21) Hermiller–Šunić [HeS07, Theorem A] have shown that any RAAG is poly-free. It is clear that any subgroup of a poly-free group is also poly-free.
- (G.22) Hsu and Wise [HsW99] showed that any RAAG is linear over \mathbb{Z} (see also [Hu94] and [DJ00]). The idea of the proof is that any RAAG embeds in a right angled Coxeter group, but these are known to be linear over \mathbb{Z} (see for example [Bou81, Chapitre V, § 4, Section 4]).
- (G.23) The lower central series (π_n) of a group π is defined inductively via $\pi_1 := \pi$ and $\pi_{n+1} = [\pi, \pi_n]$. If π is a RAAG, then the lower central series (π_n) intersects to the trivial group and the successive quotients π_n/π_{n+1} are free abelian groups. This was proved by Duchamp–Krob [DK92] (see also [Dr83, Section III]). This implies that any RAAG (and hence any subgroup of a RAAG) is residually torsion-free nilpotent.
- (G.24) Gruenberg [Gru57, Theorem 2.1] showed that every torsion-free nilpotent group is residually p for any prime p .
- (G.25) Any group π which is residually p for all primes p , is characteristically potent (see for example [BM06, Proposition 2.2]). We refer to [ADL10, Section 10] for more information and references on potent groups.
- (G.26) Rhemtulla [Rh73] showed that a group which is residually p for infinitely many primes p is bi-orderable.

Note that the combination of (G.23) and (G.24) with [Rh73] implies that RAAGs are bi-orderable. This result was also proved directly by Duchamp–Thibon [DbT92].

- (G.27) Theorem 14.26 of [Wis12] asserts that word-hyperbolic groups (in particular fundamental groups of closed hyperbolic 3-manifolds) which are virtually special are omnipotent.

Wise observes in [Wis00, Corollary 3.15] that if π is an omnipotent, torsion-free group and if $g, h \in \pi$ are two elements with g not conjugate to $h^{\pm 1}$, then there exists an epimorphism $\alpha: \pi \rightarrow G$ to a finite group, such that the orders of $\alpha(g)$ and $\alpha(h)$ are different. This can be viewed as strong form of conjugacy separability for pairs of elements g, h with g not conjugate to $h^{\pm 1}$.

Wise also states that a corresponding result holds in the cusped case [Wis12, Remark 14.27]. However, it is not the case that cusped hyperbolic manifolds necessarily satisfy the definition of omnipotence given in (E.10). Indeed, it is easy to see that \mathbb{Z}^2 is not omnipotent (see [Wis00, Remark 3.3]), and also that a retract of an omnipotent group is omnipotent. However, there are many examples of cusped hyperbolic 3-manifolds N such that $\pi_1(N)$ retracts onto a cusp subgroup (see, for instance, (G.7)). Therefore, the fundamental group of such a 3-manifold N is not omnipotent.

- (G.28) Let N be an aspherical graph manifold. Then $\pi_1(N)$ is virtually special if and only if N is non-positively curved. Indeed, Liu [Liu11] showed this if

N has a non-trivial JSJ decomposition. The case that N is a Seifert fibered 3-manifold follows ‘by inspection.’ More precisely, let N be an aspherical Seifert fibered 3-manifold. If $\pi = \pi_1(N)$ is virtually special, then by (G.3), (G.15 and (G.16) π virtually retracts onto infinite cyclic subgroups. It now follows easily that π is virtually special if and only if its underlying geometry is either Euclidean or $\mathbb{H} \times \mathbb{R}$. On the other hand it is well-known (see, e.g., [Leb95]) that these are precisely the geometries of aspherical Seifert fibered 3-manifolds which support a non-positively curved metric.

By Leeb [Leb95] a graph manifold with boundary is non-positively curved. Liu thus showed in particular that fundamental groups of graph manifolds with boundary are virtually special; this was also obtained by Przytycki–Wise [PW11]. Note that there exist closed Seifert fibered manifolds, and also graph manifolds with non-trivial JSJ decompositions, that are not virtually fibered (see, e.g., [Nem96]), and hence are neither virtually special nor non-positively curved (see also [BK96a, BK96b], [Leb95] and [BS05]). Finally note that there also exist fibered graph manifolds which are not virtually special; for instance, the fundamental groups of non-trivial torus bundles are not virtually RFRS by (G.17) and (G.16) (cf. [Ag08, p. 271]), also see [Liu11, Section 2.2] for examples with a non-trivial JSJ decomposition.

- (G.29) Let N be a compact orientable 3-manifolds with empty or toroidal boundary which is not a graph manifold. It is an interesting question whether Wise’s approach generalizes to N , i.e., whether $\pi_1(N)$ is virtually special, or virtually compact special.

Note that Leeb [Leb95] has shown that N is in fact non-positively curved. An affirmative answer to the above question and the results of Liu and Leeb (see (G.28)) would thus imply the following beautiful statement: the fundamental group of a compact orientable aspherical 3-manifold N with empty or toroidal boundary is virtually special if and only if N is non-positively curved.

Most of the ‘green properties’ are either green by definition or for elementary reasons. We thus will only justify the following statements.

- (H.1) Let π be any group. Long–Reid [LoR05, Proof of Theorem 4.1.4] (or [LoR08a, Proof of Theorem 2.10]) showed that the ability to retract onto linear subgroups of a group π extends to finite index supergroups of π . We get the following conclusions:
- (a) Since cyclic subgroups are linear it follows that if a finite index subgroup of π retracts onto cyclic subgroups, then π also retracts onto cyclic subgroups.
 - (b) If N is hyperbolic and if N admits a finite cover N' such that $\pi' = \pi_1(N')$ retracts onto geometrically finite subgroups, then it follows from the above and from the linearity of $\pi = \pi_1(N)$, that π also retracts onto geometrically finite subgroups.
- (H.2) It is straightforward to prove that a group π with a residually finite subgroup π' of finite index is also residually finite. Indeed, if $g \in \pi$ is a non-trivial element, then there is a subgroup π'' of π' , having finite index in π ,

which does not contain g . The action of π on left cosets of π'' provides a homomorphism to a finite group $\text{Sym}(\pi/\pi'')$ which does not kill g .

Similarly one can show that if π admits a subgroup of finite index which is LERF (respectively GFERF if $\pi = \pi_1(N)$ is the fundamental group of a hyperbolic 3-manifold), then π itself is LERF (respectively GFERF).

- (H.3) Niblo [Nib92, Proposition 2.2] showed that if $\pi' \subseteq \pi$ is a finite-index subgroup of a group π , then π is double-coset separable if and only if π' is double-coset separable.
- (H.4) Let R be a commutative ring and π be a group which is linear over R . Suppose that π is a subgroup of finite index of a group π' . Let $\alpha: \pi \rightarrow \text{GL}(n, R)$ be a faithful representation. Then π' acts faithfully on $R[\pi'] \otimes_{R[\pi]} R^n \cong R^{n[\pi':\pi]}$ by left-multiplication. It follows that π' is also linear over R .
- (H.5) Let π be a group which admits a finite-index subgroup π' which is weakly characteristically potent. Since subgroups of weakly characteristically potent groups are weakly characteristically potent we can by a standard argument assume that π' is in fact a characteristic finite-index subgroup of π . Now let $g \in \pi$. We denote by $k \in \mathbb{N}$ the minimal number such that $g^k \in \pi'$. Since π' is weakly characteristically potent there exists an $r' \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there exists a characteristic finite-index subgroup $\pi_n \subseteq \pi'$ such that $g^k \pi_n$ has order rn in π'/π_n . We now let $r = r'k$. Note that $\pi_n \subseteq \pi$ is normal since $\pi_n \subseteq \pi'$ is characteristic. Clearly $g^{rn} = 1 \in \pi/\pi_n$. Furthermore, if m is such that $g^m \in \pi_n$, then $g^m \in \pi'$, hence $m = km'$. It now follows easily that m divides $rn = kr'n$. Finally note that π_n is characteristic in π since $\pi_n \subseteq \pi'$ and $\pi' \subseteq \pi$ are characteristic. This shows that π is also weakly characteristically potent.
- (H.6) In Theorem 8.1 we will show that if N is an irreducible 3-manifold with empty or toroidal boundary that admits a subgroup $K \subseteq \pi_1(N)$ of finite index which is hereditarily conjugacy separable, then $\pi_1(N)$ is also hereditarily conjugacy separable.

The following gives a list of further results and alternative arguments which we left out of Diagram 4.

- (I.1) Hamilton–Wilton–Zalesskii [HWZ11] showed that if N is an orientable irreducible closed 3-manifold such that the fundamental group of every JSJ piece is conjugacy separable, then $\pi_1(N)$ is conjugacy separable. By doubling along the boundary and appealing to Lemma 1.6, the same result holds for compact, irreducible 3-manifolds with toroidal boundary.

It follows from (G.2), (G.4), (G.5), (G.6)) and (H.6) that fundamental groups of hyperbolic 3-manifolds are conjugacy separable. Furthermore fundamental groups of Seifert fibered manifolds are conjugacy separable (see [RSZ98], [Mao07] and [AKT05, AKT10]). It now follows from the aforementioned result of Wilton and Zalesskii that the fundamental group of any orientable, irreducible 3-manifold with empty or toroidal boundary is conjugacy separable.

Finally note that if a finitely presented group is conjugacy separable (see (E.9) for the definition), then the argument of [LyS77, Theorem IV.4.6] also

shows that the conjugacy problem is solvable. The above results therefore give another solution to the Conjugacy Problem first solved by Pr  aux (see (C.29)).

- (I.2) Let π be a group. We denote its profinite completion by $\widehat{\pi}$. The group π is called *good* if the map $H^*(\widehat{\pi}; A) \rightarrow H^*(\pi; A)$ is an isomorphism for any finite π -module A (see [Ser97, D.2.6 Exercise 2]). Wilton–Zalesskii [WZ10] showed that fundamental groups of graph manifolds are good. Cavendish [Cav11], building on the results of Wise, showed that fundamental groups of all compact 3-manifolds are good.
- (I.3) Let N be an irreducible, non-spherical compact, orientable 3-manifold with empty or toroidal boundary. Tracing through the arguments of Diagram 1 and Diagram 4 shows that $vb_1(N) \geq 1$. It follows from [Lub96a, p. 444] that the group $\pi_1(N)$ does not have Property (τ) . (We refer to [LZ03] and [Lub96a] for the definition of Property (τ)).

This answers in particular the Lubotzky–Sarnak Conjecture (see [Lub96a] and [Lac11] for details) in the affirmative which states that there exists no hyperbolic 3-manifold such that its fundamental group has Property (τ) .

Note that a group which does not have Property (τ) also does not have Kazhdan’s Property (T) , see, e.g., [Lub96a, p. 444] for details and see [BHV08] for background on Kazhdan’s Property (T) . This shows that the fundamental group of a compact, orientable, irreducible, non-spherical 3-manifold with empty or toroidal boundary does not satisfy Kazhdan’s Property (T) . This result was first obtained by Fujiwara [Fu99].

- (I.4) It follows from Diagram 4 that the combination of the results of Liu [Liu11] (see (G.28 and see also [PW11]), Wise and Agol [Ag08] implies that non-positively curved graph manifolds (e.g., graph manifolds with boundary, see [Leb95]) are virtually fibered. Wang–Yu [WY97] proved directly that graph manifolds with boundary are virtually fibered (see also [Nem96]), and Svetlov [Sv04] proved that non-positively curved graph manifolds are virtually fibered.
- (I.5) Baudisch [Bah81] showed that if Γ is a 2-generator subgroup of a RAAG, then Γ is either a free abelian group or a free group.
- (I.6) Fundamental groups of graph manifolds are in general not LERF (see, e.g., [BKS87, Mat97a, Mat97b, RW98, NW01]). Indeed, there are finitely generated subgroups of graph-manifold groups that are not contained in any proper subgroup of finite index [NW98]. On the other hand, Przytycki–Wise [PW11, Theorem 1.1] have shown that if N is a graph manifold and Σ is an oriented incompressible surface which is embedded in N , then $\pi_1(\Sigma)$ is separable in $\pi_1(N)$.
- (I.7) Several results on fundamental groups of hyperbolic 3-manifolds with non-trivial boundary can be deduced from the closed case. (Recall that, according to our convention, we only consider hyperbolic 3-manifolds of finite volume.) More precisely the following hold:
 - (a) Every hyperbolic 3-manifold N has a closed hyperbolic Dehn filling M , and so $\pi_1(N)$ surjects onto $\pi_1(M)$. In particular, the fact that the fundamental group of every closed hyperbolic 3-manifold is large gives a

- new proof of the theorem of Cooper–Long–Reid that the same is true for fundamental groups of hyperbolic 3-manifolds with boundary [CLR97].
- (b) Further, it follows from the work of Groves–Manning [GM08, Corollary 9.7] or Osin [Os07, Theorem 1.1] that given any hyperbolic 3-manifold N with boundary and given any finite set $A \subseteq \pi_1(N)$, there exists a hyperbolic Dehn filling M of N such that the induced map $\pi_1(N) \rightarrow \pi_1(M)$ is injective when restricted to A .
 - (c) Manning–Martinez–Pedroza [MMP10, Proposition 5.1] showed that if the fundamental groups of all closed hyperbolic 3-manifolds are LERF, then the fundamental groups of all hyperbolic 3-manifolds with boundary are also LERF.

7. SUBGROUPS OF 3-MANIFOLD GROUPS

In this section we collect properties of finitely generated infinite-index subgroups of 3-manifold groups in a diagram. The study of 3-manifold groups and the study of their subgroups go hand in hand, and the content of this section therefore partly overlaps with the results mentioned in the previous sections.

Most of the definitions required for understanding Diagram 5 have been introduced above. We therefore need to introduce only the following new definitions.

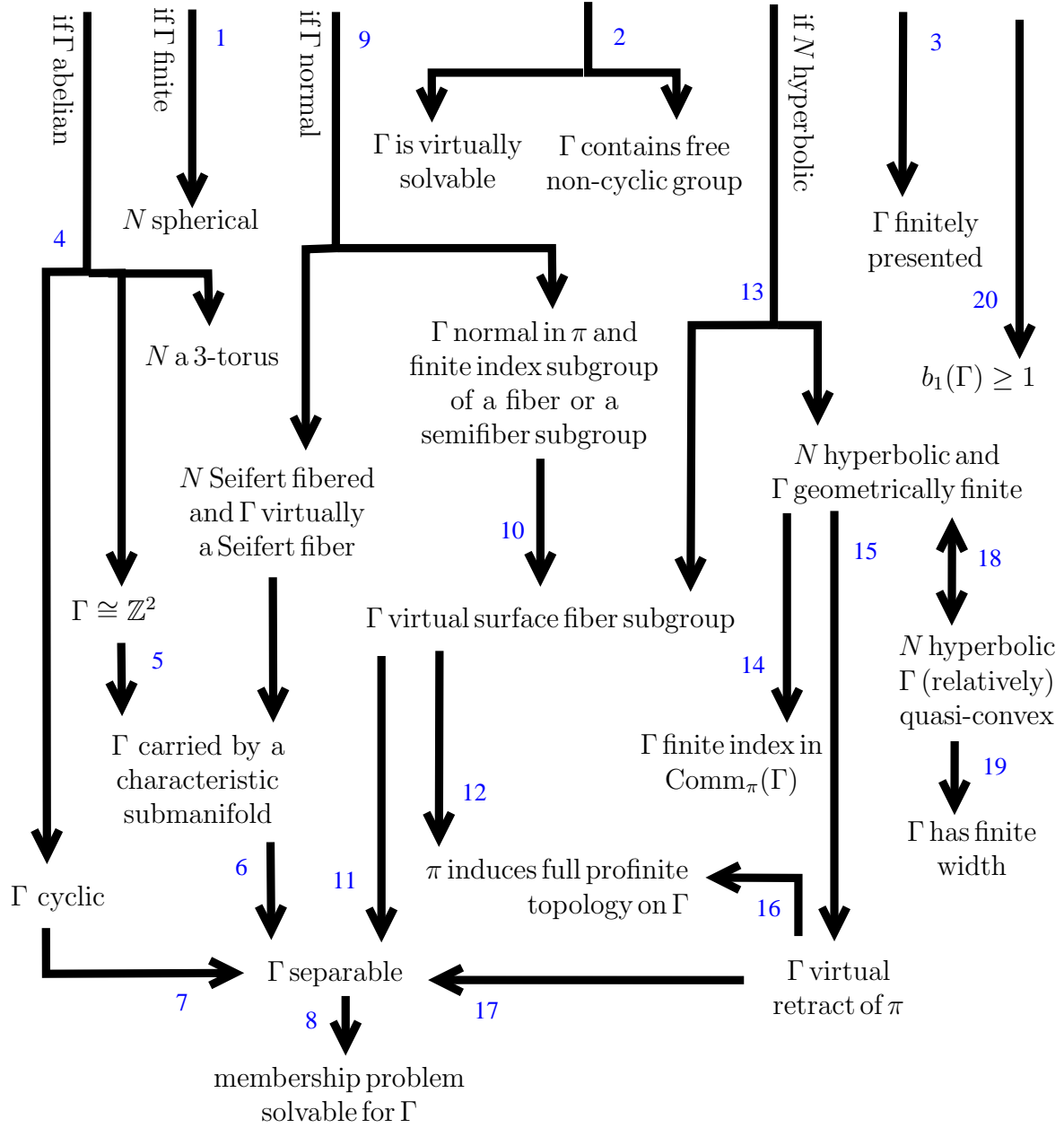
- (J.1) Let N be a 3-manifold. Let $\Gamma \subseteq \pi_1(N)$ be a subgroup and $X \subseteq N$ a connected subspace. We say that Γ is *carried by* X if Γ is a subgroup of $\text{Im}\{\pi_1(X) \rightarrow \pi_1(N)\}$ (up to conjugation).
- (J.2) Let π be a finitely generated group and Γ a finitely generated subgroup. We say that the *membership problem is solvable for* Γ if, given a finite generating set g_1, \dots, g_k for π , there exists an algorithm which can determine whether or not an input word w in the generators g_1, \dots, g_k defines an element of Γ .
- (J.3) Let N be a 3-manifold. We say that a connected compact surface $\Sigma \subseteq N$ is a *semifiber* if N is the union of two twisted I -bundles over the non-orientable surface Σ along their S^0 -bundles. (Note that in the literature usually the surface given by the S^0 -bundle is referred to as a ‘semifiber’.) Note that if $\Sigma \subseteq N$ is a semifiber, then there exists a double cover $p: \tilde{N} \rightarrow N$ such that $p^{-1}(\Sigma)$ consists of two components, each of which is a surface fiber.
- (J.4) Let Γ be a subgroup of a group π . The *width* of Γ was defined in [GMRS98]. We say that $g_1, \dots, g_n \in \pi$ are *essentially distinct* if $\Gamma g_i \neq \Gamma g_j$ whenever $i \neq j$. Conjugates of Γ by essentially distinct elements are called *essentially distinct conjugates*. The *width* of Γ in π is the maximal $n \in \mathbb{N} \cup \{\infty\}$ such that there exists a collection of n essentially distinct conjugates of Γ with the property that the intersection of any two elements of the collection is infinite. The width of Γ is 1 if Γ is malnormal. If Γ is normal and infinite, then the width of Γ equals its index.
- (J.5) Let Γ be a subgroup of a group π . We define the commensurator subgroup of Γ to be the subgroup

$$\text{Comm}_\pi(\Gamma) := \{g \in \pi : \Gamma \cap g\Gamma g^{-1} \text{ has finite index in } \Gamma\}.$$

As in Diagrams 1 and 4 we use the convention that if an arrow splits into several arrows, then exactly one of the possible conclusions holds. Furthermore, if an

arrow is decorated with a condition, then the conclusion holds if that condition is satisfied.

Γ finitely generated non-trivial subgroup of $\pi = \pi_1(N)$ of infinite index,
 N is an irreducible compact orientable 3-manifold with empty or toroidal boundary



In Diagram 5 we put several restrictions on the 3-manifold N we consider. Below, in the justifications for the arrows in Diagram 5, we will only assume that N is connected, and we will not put any other blanket restrictions on N . Before

we give the justifications we point out that only (K.15) depends on the Virtually Compact Special Theorem.

- (K.1) If N is a compact, irreducible 3-manifold with empty or toroidal boundary that admits a non-trivial finite subgroup, then it follows from the Sphere Theorem proved by Papakyriakopoulos [Pap57a] that N is spherical. See (C.2) for details.
- (K.2) Let N be any compact 3-manifold and let Γ be a finitely generated subgroup of $\pi = \pi_1(N)$. Then one of the following holds:
 - (a) either Γ is virtually solvable, or
 - (b) Γ contains a non-cyclic free subgroup.
 (In other words, π satisfies the ‘Tits Alternative’.) Indeed, if Γ is a finitely generated subgroup of $\pi = \pi_1(N)$, then by Scott’s core theorem (C.4), applied to the covering of N corresponding to $\Gamma \subset \pi_1(N)$, there exists a compact 3-manifold M with $\pi_1(M) = \Gamma$. Suppose that $\pi_1(M)$ is not virtually solvable. It follows easily from Theorem 1.1, Lemma 1.5, Lemma 1.21 combined with (C.19) and (C.20) that $\pi_1(M)$ contains a non-cyclic free subgroup.
- (K.3) Scott [Sco73b] proved that any finitely generated 3-manifold group is also finitely presented. See (C.4) for more information.
- (K.4) Let N be a compact, irreducible, orientable 3-manifold with empty or toroidal boundary. Let $\Gamma \subseteq \pi_1(N)$ be an abelian subgroup. It follows either from Theorems 3.1 and 3.2, or alternatively from the remark after the proof of Theorem 1.20 and Scott’s core theorem (C.4), that Γ is either cyclic or $\Gamma \cong \mathbb{Z}^2$ or $\Gamma \cong \mathbb{Z}^3$. In the latter case it follows from the discussion in Section 1.6 that N is the 3-torus.
- (K.5) Let N be an irreducible, orientable, compact, irreducible 3-manifold with empty or toroidal boundary and let $\Gamma \subseteq \pi = \pi_1(N)$ be a subgroup isomorphic to \mathbb{Z}^2 . Then there exists a singular map $f: T \rightarrow N$ from the 2-torus to N such that $f_*(\pi_1(T)) = \Gamma$. It now follows from Theorem 1.8 that Γ is carried by a characteristic submanifold.
 The above statement is also known as the ‘Torus Theorem.’ It was announced by Waldhausen [Wan69] and the first proof was given by Feustel [Fe76a, Fe76b]. We refer to [Wan69, CF76, Sco80] for information on the closely related ‘Annulus Theorem.’ Both theorems can be viewed as predecessors of the Characteristic Pair Theorem.
- (K.6) Let N be a compact orientable 3-manifold with empty or toroidal boundary. Let $M \subseteq N$ be a characteristic submanifold and let $\Gamma \subseteq \pi_1(M)$ be a finitely generated subgroup. Then Γ is separable in $\pi_1(M)$ by Scott’s theorem [Sco78] (see also (C.23)), and $\pi_1(M)$ is separable in $\pi_1(N)$ by Wilton–Zalesskii [WZ10, Theorem A] (see also (C.31)). It follows that $\Gamma \subseteq \pi_1(N)$ is separable. Note that the same argument also generalizes to hyperbolic JSJ components with LERF fundamental groups. More precisely, if M is a hyperbolic JSJ component of N such that $\pi_1(M)$ is LERF and if $\Gamma \subseteq \pi_1(M)$ is a finitely generated subgroup, then $\Gamma \subseteq \pi_1(N)$ is separable.
- (K.7) E. Hamilton [Hamb01] showed that the fundamental group of any compact, orientable 3-manifold is abelian subgroup separable. In particular, any cyclic subgroup is separable. See also (C.28) for more information.

If $\pi_1(N)$ is virtually RFRS and Γ is an infinite cyclic subgroup of $\pi_1(N)$, then it follows from (G.16) that Γ is a virtual retract of $\pi_1(N)$, which by (K.17) gives another proof that Γ is separable.

- (K.8) The argument of [LyS77, Theorem IV.4.6] can be used to show that if $\Gamma \subseteq \pi$ is a finitely generated separable subgroup of a finitely presented group π , then the membership problem for Γ is solvable.
- (K.9) Let N be a compact orientable 3-manifold that admits a normal finitely generated non-trivial subgroup Γ of infinite index. It follows from work of Hempel–Jaco [HJ72, Theorem 3], the resolution of the Poincaré Conjecture, Theorem 3.5, and (K.3), that one of the following conclusions hold:
 - (a) N is Seifert fibered and Γ is a subgroup of the Seifert fiber subgroup, or
 - (b) N fibers over S^1 with surface fiber Σ and Γ is a finite-index subgroup of $\pi_1(\Sigma)$, or
 - (c) N is the union of two twisted I -bundles over a compact connected surface Σ which meet in the corresponding S^0 -bundles and Γ is a finite-index subgroup of $\pi_1(\Sigma)$.
- (K.10) Let N be a compact 3-manifold. Let Γ be a normal subgroup of $\pi_1(N)$ which is also a finite-index subgroup of $\pi_1(\Sigma)$, where Σ is a surface fiber of a fibration $N \rightarrow S^1$. We consider $\tilde{\pi} := \mathbb{Z} \ltimes \Gamma \subset \mathbb{Z} \ltimes \pi_1(\Sigma) = \pi_1(N)$. Note that $\tilde{\pi}$ is a subgroup, since $\Gamma \subseteq \pi_1(N)$ is normal. We denote by \tilde{N} the finite cover of N corresponding to $\tilde{\pi}$. It is clear that Γ is a surface fiber subgroup of \tilde{N} .
- (K.11) Let N be a compact orientable 3-manifold with empty or toroidal boundary and let Σ be a fiber of a fibration $N \rightarrow S^1$. Then $\pi_1(N) \cong \mathbb{Z} \ltimes \pi_1(\Sigma)$ and $\pi_1(\Sigma) \subseteq \pi_1(N)$ is therefore separable. It follows easily that if $\Gamma \subset \pi_1(N)$ is a virtual surface fiber subgroup, then $\Gamma \subseteq \pi_1(N)$ is separable.
- (K.12) Let N be a compact orientable 3-manifold with empty or toroidal boundary and let Σ be a fiber of a fibration $N \rightarrow S^1$. Then $\pi_1(N) \cong \mathbb{Z} \ltimes \pi_1(\Sigma)$ where $1 \in \mathbb{Z}$ acts by some $\Phi \in \text{Aut}(\pi_1(\Sigma))$. Now let $\Gamma \subseteq \pi_1(\Sigma)$ be a finite-index subgroup. Because $\pi_1(\Sigma)$ is finitely generated, there are only finitely many subgroups of index $[\pi_1(\Sigma) : \Gamma]$, and so $\Phi^n(\Gamma) = \Gamma$ for some n . Now $\tilde{\pi} := n\mathbb{Z} \ltimes \Gamma$ is a subgroup of finite index in $\pi_1(N)$ such that $\tilde{\pi} \cap \pi_1(\Sigma) = \Gamma$. This shows that π induces the full profinite topology on the surface fiber subgroup $\pi_1(\Sigma)$. It follows easily that π also induces the full profinite topology on any virtual surface fiber subgroup.
- (K.13) Let N be a hyperbolic 3-manifold. The Subgroup Tameness Theorem (see Theorem 5.2) asserts that if $\Gamma \subseteq \pi_1(N)$ is a finitely generated subgroup, then Γ is either geometrically finite or Γ is a virtual surface fiber subgroup.
- (K.14) Let N be a hyperbolic 3-manifold and let $\Gamma \subseteq \pi = \pi_1(N)$ be a geometrically finite subgroup of infinite index. Then Γ has finite index in $\text{Comm}_\pi(\Gamma)$. We refer to [Can08, Theorem 8.7] for a proof (see also [KaS96] and [Ar01, Theorem 2]), and we refer to [Ar01, Section 5] for more results in this direction.

If $\Gamma \subseteq \pi_1(N)$ is a virtual surface fiber subgroup, then $\text{Comm}_\pi(\Gamma)$ is easily seen to be a finite-index subgroup of π , so Γ has infinite index in its commensurator. The commensurator thus gives another way to formulate the dichotomy of (K.13).

- (K.15) Let N be a hyperbolic 3-manifold and $\Gamma \subseteq \pi_1(N)$ be a geometrically finite subgroup. In (G.7) we saw that it follows from the Virtually Compact Special Theorem of Agol, Kahn-Markovic and Wise that Γ is a virtual retract of $\pi_1(N)$.

Note that if on the other hand $\Gamma \subseteq \pi_1(N)$ is a virtual surface fiber subgroup and if the monodromy of the fibration does not have finite order (e.g., if N is hyperbolic), then it is straightforward to see that Γ is not a virtual retract of $\pi_1(N)$. We thus obtain one more way to formulate the dichotomy of (K.13).

- (K.16) It is easy to prove that every group induces the full profinite topology on each of its virtual retracts.
- (K.17) Let π be a residually finite group (e.g., a 3-manifold group, see (C.25)). If $\Gamma \subseteq \pi$ is a virtual retract, then the subgroup Γ is also separable in π . See (G.8) for details.
- (K.18) Let N be a closed, hyperbolic 3-manifold. As mentioned in Proposition 5.11, it follows that $\pi = \pi_1(N)$ is word-hyperbolic, and a subgroup of π is geometrically finite if and only if it is quasi-convex (see [Swp93, Theorem 1.1 and Proposition 1.3] and also [KaS96, Theorem 2]).

If N has toroidal boundary, then $\pi = \pi_1(N)$ is not word-hyperbolic, but it is hyperbolic *relative to its collection of peripheral subgroups*. By [Hr10, Corollary 1.3], a subgroup Γ of π is geometrically finite if and only if it is *relatively quasi-convex*. The reader is referred to [Hr10] for thorough treatments of the various definitions of relative hyperbolicity and of relative quasi-convexity, as well as proofs of their equivalence.

- (K.19) Let π be the fundamental group of a hyperbolic 3-manifold N and let Γ be a geometrically finite subgroup. By (K.18) this means that Γ is a relatively quasi-convex subgroup of π . The main result of [GMRS98] shows that the width of Γ is finite when N is closed (so π is word-hyperbolic), and the general case follows from [HrW09].

If, on the other hand, $\Gamma \subseteq \pi_1(N)$ is a virtual surface fiber subgroup, then the width of Γ is infinite. The width thus gives another way to formulate the dichotomy of (K.13).

- (K.20) Let N be a compact, orientable, irreducible 3-manifold and let $\Gamma \subseteq \pi_1(N)$ be a subgroup of infinite index. The argument of the proof of [How82, Theorem 6.1] shows that $b_1(\Gamma) \geq 1$. See also (C.15).

We conclude this section with a few more results and references about subgroups of hyperbolic 3-manifold groups.

- (L.1) Let N be a hyperbolic 3-manifold and let $\Gamma \subseteq \pi = \pi_1(N)$ be a subgroup generated by two elements x and y . Jaco-Shalen [JS79, Theorem VI.4.1] showed that if $x, y \in \pi_1(N)$ do not commute and if the subgroup $\langle x, y \rangle \subseteq \pi_1(N)$ has infinite index, then Γ is a free group. For closed hyperbolic 3-manifolds this result was generalized by Gitik [Gi99a, Theorem 1].
- (L.2) Let Γ be a finitely generated subgroup of a group π . We say Γ is *tight* in π if for any $g \in \pi$ there exists an n such that $g^n \in \Gamma$. Clearly a finite index subgroup of π is tight. It follows from the Subgroup Tameness Theorem (see Theorem 5.2) that any tight subgroup of $\pi_1(N)$ is of finite index. For N

with non-trivial toroidal boundary, this was first shown by Canary [Can94, Theorem 6.2].

- (L.3) Let N be an orientable, irreducible compact 3-manifold with (not necessarily toroidal) boundary. Let X be a connected, incompressible subsurface of the boundary of N . Long–Niblo [LN91, Theorem 1] showed that then $\pi_1(X) \subseteq \pi_1(N)$ is separable.
- (L.4) Soma [Som91] proved various results on the intersections of conjugates of virtual surface fiber subgroups.
- (L.5) Elkalla [El84] proved several technical results about subnormal subgroups of 3-manifold groups.
- (L.6) We refer to [WW94, WY99] and [BGHM10, Section 7] for results on finite-index subgroups of 3-manifold groups.

8. PROOFS

In this section we collect the proofs of several statements that were mentioned in the previous sections.

8.1. Conjugacy separability. It is immediate that a subgroup of a residually finite group is itself residually finite, and it is also easy to prove that a group with a residually finite subgroup of finite index is itself residually finite. In contrast, the property of conjugacy separability (see (E.9)) is more delicate. Goryaga gave an example of a non-conjugacy-separable group with a conjugacy separable subgroup of finite index [Goa86]. In the other direction, Martino–Minasyan constructed examples of conjugacy separable groups with non-conjugacy-separable subgroups of finite index [MM09].

For this reason, one defines a group to be *hereditarily conjugacy separable* if every finite-index subgroup is conjugacy separable. We will now show that, in the 3-manifold context, one can apply a criterion of Chagas–Zalesskii [ChZ10] to prove that hereditary conjugacy separability passes to finite extensions.

Theorem 8.1. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary and suppose that $K \subseteq \pi = \pi_1(N)$ is a subgroup of finite index. If K is hereditarily conjugacy separable, then so is π .*

We may assume that N does not admit Sol geometry, as polycyclic groups are known to be conjugacy separable by a theorem of Remeslennikov [Rev69]. Furthermore, we may assume that K is normal, corresponding to a regular covering map $N' \rightarrow N$ of finite degree. In particular, K is also the fundamental group of a compact, orientable, irreducible 3-manifold with toroidal boundary. For $g \in \pi$, we summarize the structure of the centralizer $C_\pi(g)$ in the following proposition, which is an immediate consequence of Theorems 3.1 and 3.2.

Proposition 8.2. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary that does not admit Sol geometry and let $g \in \pi = \pi_1(N)$ be non-trivial. Then either $C_\pi(g)$ is free abelian or there is a Seifert fibered piece N of the JSJ decomposition of N such that $C_\pi(g)$ is a subgroup of index at most two in $\pi_1(N)$.*

We will now prove three lemmas about centralizers. These enable us to apply a result of Chagas–Zaleskii [ChZ10] to finish the proof.

Lemma 8.3. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary that does not admit Sol geometry and let $g \in \pi = \pi_1(N)$ be non-trivial. The centralizer $C_\pi(g)$ is conjugacy separable.*

Proof. If $C_\pi(g)$ is free abelian, then this is clear. Otherwise, $C_\pi(g)$ is the fundamental group of a Seifert fibered manifold, which is conjugacy separable by a theorem of Martino [Mao07]. \square

For a group G , \widehat{G} denotes the profinite completion of G , and for a subgroup $H \subseteq G$, \overline{H} denotes the closure of H in \widehat{G} .

Lemma 8.4. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary that does not admit Sol geometry and let $g \in \pi = \pi_1(N)$. The canonical map $\widehat{C_\pi(g)} \rightarrow \overline{C_\pi(g)}$ is an isomorphism.*

Proof. We need to prove that the profinite topology on π induces the full profinite topology on $C_\pi(g)$. To this end, it is enough to prove that every finite-index subgroup H of $C_\pi(g)$ is separable in π .

If $C_\pi(g)$ is free abelian, then so is H , so H is separable by the main theorem of [Hamb01]. Therefore, suppose $C_\pi(g)$ is a subgroup of index at most two in $\pi_1(M)$, where M is a Seifert fibered vertex space of N , so H is a subgroup of finite index in $\pi_1(M)$. By (C.31), the group π induces the full profinite topology on $\pi_1(M)$ and $\pi_1(M)$ is separable in π . It follows that H is separable in π . \square

The final condition is a direct consequence of Proposition 3.2 and Corollary 12.2 of [Min09], together with the hypothesis that K is hereditarily conjugacy separable.

Lemma 8.5. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary that does not admit Sol geometry and let $g \in \pi = \pi_1(N)$. The inclusion $\overline{C_\pi(g)} \rightarrow C_{\widehat{\pi}}(g)$ is surjective.*

These lemmas enable us to apply the following useful criterion of Chagas and Zaleskii.

Proposition 8.6 ([ChZ10, Proposition 2.1]). *Let π be a finitely generated group containing a conjugacy separable normal subgroup K of finite index. Let $a \in \pi$ be an element such that there exists a natural number m with $a^m \in K$ and the following conditions hold:*

- (1) $C_\pi(a^m)$ is conjugacy separable;
- (2) $\widehat{C_K(a^m)} = \overline{C_K(a^m)} = C_{\widehat{K}}(a^m)$.

Then whenever $b \in \pi$ is not conjugate to a , there is a homomorphism f from π to a finite group such that $f(a)$ is not conjugate to $f(b)$.

We are now in a position to prove Theorem 8.1.

Proof of Theorem 8.1. By replacing K with the intersection of its conjugates, we may assume that K is normal. Let $a, b \in \pi$ be non-conjugate and let m

be non-zero with $a^m \in K$. By Lemma 8.3, the centralizer $C_\pi(a^m)$ is conjugacy separable. Let N' be a finite-sheeted covering space of N with $K = \pi_1(N')$. Lemma 8.4 applied to $K = \pi_1(N')$ shows that the first equality in condition (2) of Proposition 8.6 holds and similarly Lemma 8.5 shows that the second equality holds. Therefore, Proposition 8.6 applies to show that there is a homomorphism from π to a finite group under which the images of a and b are non-conjugate. Therefore, π is conjugacy separable. \square

8.2. Fundamental groups of Seifert fibered manifolds are linear over \mathbb{Z} . In this section we will give a proof (due to Boyer) of the following theorem.

Theorem 8.7. *Let N be a Seifert fibered manifold. Then $\pi_1(N)$ is linear over \mathbb{Z} .*

Before we prove Theorem 8.7 we consider the following two lemmas. The first lemma is well known, but we include the proof for completeness' sake.

Lemma 8.8. *If N is a Seifert fibered manifold, then N is finitely covered by an S^1 -bundle over an orientable connected surface F .*

Proof. We first consider the case that N is closed. Denote by B the base orbifold of the Seifert fibered manifold N . If B is a 'good' orbifold in the sense of [Sco83a, p. 425], then B is finitely covered by an orientable connected surface F . This cover $F \rightarrow B$ gives rise to a map $Y \rightarrow N$ of Seifert fibered manifolds. Since the base orbifold of the Seifert fibered manifold Y is a surface it follows that Y is in fact an S^1 -bundle over F .

The 'bad' orbifolds are classified in [Sco83a, p. 425], and in the case of base orbifolds the only two classes of bad orbifolds which can arise are $S^2(p)$ and $S^2(p, q)$ (see [Sco83a, p. 430]). The former arises from the lens space $L(p, 1)$ and the latter from the lens space $L(p, q)$. But lens spaces are covered by S^3 which is an S^1 -bundle over the sphere.

Now consider the case that N has boundary. We consider the double $M = N \cup_{\partial N} N$. Note that M is again a Seifert fibered manifold. By the above there exists a finite-sheeted covering map $p : Y \rightarrow M$ such that Y is an S^1 -bundle over a surface and p preserves the Seifert fibers. It now follows that $p^{-1}(N) \subseteq Y$ is a sub-Seifert fibered manifold. In particular any component of $p^{-1}(N)$ is also an S^1 -bundle over a surface. \square

Remark. Let N be any orientable, compact, irreducible 3-manifold with empty or toroidal boundary. A useful generalization of Lemma 8.8 says that N admits a finite cover such that all JSJ components of N' which are Seifert fibered are in fact S^1 -bundles over a surface. We refer to [Hem87, Hamb01, AF10] for details.

Lemma 8.9. *Suppose N is an S^1 -bundle over an orientable surface F . Then $\pi_1(N)$ is linear over \mathbb{Z} .*

Proof. We first recall that surface groups are linear over \mathbb{Z} . Indeed, if F is a sphere or a torus, then this is obvious. If F has boundary, then $\pi_1(N)$ is a free group and hence embeds into $\mathrm{SL}(2, \mathbb{Z})$. Finally, if F is closed, then Newman [New85] showed that there exists an embedding $\pi_1(F) \rightarrow \mathrm{SL}(8, \mathbb{Z})$. (Alternatively, Scott [Sco78, Section 3] showed that $\pi_1(F)$ is a subgroup of a right angled Coxeter group on 5 generators, and hence by [Bou81, Chapitre V, § 4, Section 4] we can

in fact embed $\pi_1(F)$ into $\mathrm{SL}(5, \mathbb{Z})$. Also [DSS89] contains a proof that surface groups embed into RAAGs, and hence are linear over \mathbb{Z} by [HsW99].)

We now turn to the proof of the lemma. We first consider the case that N is a trivial S^1 -bundle, i.e., $N \cong S^1 \times F$. But then $\pi_1(N) = \mathbb{Z} \times \pi_1(F)$ is the direct product of \mathbb{Z} with a surface group, so $\pi_1(N)$ is \mathbb{Z} -linear by the above.

If N has boundary, then F also has boundary and we obtain $H^2(F; \mathbb{Z}) = 0$, so the Euler class of the S^1 -bundle $N \rightarrow F$ is trivial. We therefore conclude that in this case, N is a trivial S^1 -bundle.

Now assume that N is a non-trivial S^1 -bundle. By the above this implies that F is a closed surface. If $F = S^2$, then the long exact sequence in homotopy theory shows that $\pi_1(N)$ is cyclic, hence linear. If $F \neq S^2$, then it follows again from the long exact sequence in homotopy theory that the subgroup $\langle t \rangle$ of $\pi_1(N)$ generated by a fiber is normal and infinite cyclic, and that we have a short exact sequence

$$1 \rightarrow \langle t \rangle \rightarrow \pi_1(N) \rightarrow \pi_1(F) \rightarrow 1.$$

Let $e \in H^2(F; \mathbb{Z}) \cong \mathbb{Z}$ be the Euler class of F . A presentation for $G := \pi_1(N)$ is given by

$$G = \left\langle a_1, b_1, \dots, a_r, b_r, t : \prod_{i=1}^r [a_i, b_i] = t^e, t \text{ central} \right\rangle.$$

Let G_e be the subgroup of G generated by the a_i, b_i and t^e . It is straightforward to check that the assignment

$$\rho(a_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(b_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(t^e) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho(a_i) = \rho(b_i) = \mathrm{id} \quad \text{for } i \geq 2$$

yields a representation $\rho: G_e \rightarrow \mathrm{SL}(3, \mathbb{Z})$ such that $\rho(t^e)$ has infinite order.

Now let σ be the composition $G_e \rightarrow G = \pi_1(N) \rightarrow \pi_1(F) \rightarrow \mathrm{SL}(n, \mathbb{Z})$, where the last homomorphism is a faithful representation of $\pi_1(F)$, which exists by the above. Then

$$\rho \times \sigma: G_e \rightarrow \mathrm{SL}(3, \mathbb{Z}) \times \mathrm{SL}(n, \mathbb{Z}) \subseteq \mathrm{SL}(n+3, \mathbb{Z})$$

is an embedding. This shows that the finite-index subgroup G_e of G is \mathbb{Z} -linear. By (H.4) this implies that G is also linear over \mathbb{Z} . \square

We can now provide a proof of Theorem 8.7.

Proof of Theorem 8.7. Let N be a Seifert fibered manifold. By Lemma 8.8, N is finitely covered by a 3-manifold N' which is an S^1 -bundle over an orientable surface F . It now follows from Lemma 8.9 that $\pi_1(N')$ is linear over \mathbb{Z} , and so $\pi_1(N)$ is also linear over \mathbb{Z} by (H.4). \square

8.3. Non-virtually-fibered graph manifolds. There exist non-fibered Seifert fibered manifolds, and also non-fibered graph manifolds with non-trivial JSJ decomposition (see, e.g., [Nem96]). The following proposition shows that such examples also have the property that their fundamental groups do not virtually retract onto cyclic subgroups.

Proposition 8.10. *Let N be a non-spherical graph manifold which is not virtually fibered. Then $\pi_1(N)$ does not virtually retract onto all its cyclic subgroups.*

Proof. Let N be a non-spherical graph manifold such that $\pi_1(N)$ virtually retracts onto all its cyclic subgroups. We will show that N is virtually fibered.

Note that N is finitely covered by a manifold such that any JSJ component is an S^1 -bundle over a surface (see the remark after Lemma 8.8). We can therefore without loss of generality assume that N itself is already of that form.

We first consider the case that N is a Seifert fibered manifold, i.e., that N is an S^1 -bundle over a surface Σ . The assumption that N is non-spherical implies that the regular fiber generates an infinite cyclic subgroup of $\pi_1(N)$. It is well known, and straightforward to see, that if $\pi_1(N)$ retracts onto this infinite cyclic subgroup, then N is a product $S^1 \times \Sigma$; in particular, N is fibered.

We now consider the case that N has a non-trivial JSJ decomposition. We denote the JSJ pieces of N by M_v , where v ranges over some index set V . By hypothesis, each M_v is an S^1 -bundle over a surface with boundary, so each M_v is in fact a product. We denote by f_v the Seifert fiber of M_v . Note that each f_v generates an infinite cyclic subgroup of $\pi_1(N)$.

Since $\pi_1(N)$ virtually retracts onto cyclic subgroups, for each v we can find a finite-sheeted covering space \tilde{N}_v of N such that $\tilde{\pi}_v = \pi_1(\tilde{N}_v)$ retracts onto $\langle f_v \rangle$. In particular, the image of f_v is non-trivial in $H_1(\tilde{N}_v; \mathbb{Z})/\text{torsion}$. Let \tilde{N} be any regular finite-sheeted cover of N that covers every \tilde{N}_v . (For instance, $\pi_1(\tilde{N})$ could be the intersection of all the conjugates of $\bigcap_{v \in V} \tilde{\pi}_v$.)

Let \tilde{f} be a Seifert fiber of a JSJ component of \tilde{N} . Up to the action of the deck group of $\tilde{N} \rightarrow N$, \tilde{f} covers the lift of some f_v in \tilde{N}_v . It follows that \tilde{f} is non-trivial in $H_1(\tilde{N}; \mathbb{Z})/\text{torsion}$.

Therefore, there exists a homomorphism $\phi: H_1(\tilde{N}; \mathbb{Z}) \rightarrow \mathbb{Z}$ which is non-trivial on the Seifert fibers of all JSJ components of \tilde{N} . Since each JSJ component is a product, the restriction of ϕ to each JSJ component of \tilde{N} is a fibered class. By [EN85, Theorem 4.2], we conclude that \tilde{N} fibers over S^1 . \square

8.4. (Fibered) faces of the Thurston norm ball of finite covers. Let N be a compact, orientable 3-manifold. We say that $\phi \in H^1(N; \mathbb{R})$ is *fibered*, if ϕ can be represented by a non-degenerate closed 1-form. Note that by [Tis70] an integral class $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ is fibered if and only if there exists a fibration $p: N \rightarrow S^1$ such that the induced map $p_*: \pi_1(N) \rightarrow \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ .

The *Thurston norm* of $\phi \in H^1(N; \mathbb{Z})$ is defined as

$$\|\phi\|_T = \min \{ \chi_-(\Sigma) : \Sigma \subseteq N \text{ properly embedded surface dual to } \phi \}.$$

Here, given a surface Σ with connected components $\Sigma_1 \cup \dots \cup \Sigma_k$, we define $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$. Thurston [Thu86a] (see also [CC03, Chapter 10]) proved the following results:

- (1) $\|-\|_T$ defines a seminorm on $H^1(N; \mathbb{Z})$ which can be extended to a seminorm $\|-\|_T$ on $H^1(N; \mathbb{R})$.

- (2) The norm ball

$$\{\phi \in H^1(N; \mathbb{R}) : \|\phi\|_T \leq 1\}$$

is a finite-sided rational polyhedron.

- (3) There exist open top-dimensional faces F_1, \dots, F_k of the Thurston norm ball such that

$$\{\phi \in H^1(N; \mathbb{R}) : \phi \text{ fibered}\} = \bigcup_{i=1}^k \mathbb{R}^+ F_i.$$

These faces are called the *fibered faces* of the Thurston norm ball.

The Thurston norm ball is evidently symmetric in the origin. We say that two faces F and G are *equivalent* if $F = \pm G$. Note that a face F is fibered if and only if $-F$ is fibered.

The Thurston norm is degenerate in general, e.g., for 3-manifolds with homologically essential tori. On the other hand the Thurston norm of a hyperbolic 3-manifold is non-degenerate, since a hyperbolic 3-manifold admits no homologically essential surfaces of non-negative Euler characteristic.

We start out with the following fact.

Proposition 8.11. *Let $p: M \rightarrow N$ be a finite cover. Then $\phi \in H^1(N; \mathbb{R})$ is fibered if and only if $p^*\phi \in H^1(M; \mathbb{R})$ is fibered. Furthermore*

$$\|p^*\phi\|_T = [M : N] \cdot \|\phi\|_T \text{ for any class } \phi \in H^1(N; \mathbb{R}).$$

In particular, the map $p^: H^1(N; \mathbb{R}) \rightarrow H^1(M; \mathbb{R})$ is, up to a scale factor, an isometry and it maps fibered cones into fibered cones.*

The first statement is an immediate consequence of Stallings' Fibered Theorem [Sta62], and the second statement follows from work of Gabai [Ga83, Corollary 6.18].

We can now prove the following proposition.

Proposition 8.12. *Let N be an irreducible 3-manifold with empty or toroidal boundary, which is not a graph manifold. Then given any $k \in \mathbb{N}$ there exists a finite cover M of N such that the Thurston norm ball of M has at least k inequivalent faces.*

If N is closed and hyperbolic, then the proposition relies on the Virtually Compact Special Theorem (Theorem 5.4). In the other cases it follows from the work of Cooper–Long–Reid [CLR97] and classical facts on the Thurston norm.

Proof. We first suppose that N is hyperbolic. It follows from (G.2), (G.4), (G.12) and (C.13) that N admits a finite cover M with $b_1(M) \geq k$. Since the Thurston norm of a hyperbolic 3-manifold is non-degenerate it follows that the Thurston norm ball of M has at least $2^{k-1} \geq k$ inequivalent faces.

We now suppose that N is not hyperbolic. By assumption there exists a hyperbolic JSJ component X which is hyperbolic and which necessarily has non-trivial boundary. It follows from [CLR97, Theorem 1.3] (see also (C.12)) that $\pi_1(N)$ is large and hence by (C.13) that there exists a finite cover \tilde{X} with non-peripheral homology of rank at least k .

A standard argument, using (C.31), now shows that there exists a finite cover M of N which admits a hyperbolic JSJ component Y which covers \tilde{X} . An elementary argument shows that Y also has non-peripheral homology of rank at least k . We consider $p: H_2(Y; \mathbb{R}) \rightarrow H_2(Y, \partial Y; \mathbb{R})$ and $V := \text{Im } p$. Using Poincaré Duality, the Universal Coefficient Theorem and the information on the non-peripheral homology, we see that $\dim(V) \geq k$.

We now consider $q: H_2(Y; \mathbb{R}) \rightarrow H_2(M, \partial M; \mathbb{R})$ and $W := \text{Im } q$. Since p is the composition of q and the restriction map $H_2(M, \partial M; \mathbb{R}) \rightarrow H_2(Y, \partial Y; \mathbb{R})$ we see that $\dim W \geq \dim V \geq k$. Since N is hyperbolic it follows that the Thurston norm of Y is non-degenerate, in particular it is non-degenerate on V . By [EN85, Proposition 3.5] the Thurston norm of $p_*\phi$ in Y agrees with the Thurston norm of $q_*\phi$ in M . It thus follows that the Thurston norm of M is non-degenerate on W , in particular the Thurston norm ball of M has at least $2^{k-1} \geq k$ inequivalent faces. \square

We say that $\phi \in H^1(N; \mathbb{R})$ is *quasi-fibered* if ϕ lies on the closure of a fibered cone of the Thurston norm ball of N . We can now formulate Agol's Virtually Fibered Theorem (see [Ag08, Theorem 5.1]).

Theorem 8.13. (Agol) *Let N be an irreducible, compact 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is virtually RFRS. Then given any $\phi \in H^1(N; \mathbb{R})$ there exists a finite cover $p: M \rightarrow N$ such that $p^*\phi$ is quasi-fibered.*

The following is now a straightforward consequence of Agol's theorem.

Proposition 8.14. *Let N be an irreducible, compact 3-manifold with empty or toroidal boundary such that the Thurston norm ball of N has at least k inequivalent faces. If $\pi_1(N)$ is virtually RFRS and if N is not a graph manifold, then given any $k \in \mathbb{N}$ there exists a finite cover M of N such that the Thurston norm ball of M has at least k inequivalent fibered faces.*

The proof of the proposition is precisely that of [Ag08, Theorem 7.2]. We therefore give just a very quick outline of the proof.

Proof. We pick classes ϕ_i ($i = 1, \dots, k$) in $H^1(N; \mathbb{R})$ which lie in k inequivalent faces. For $i = 1, \dots, k$ we then apply Theorem 8.13 to the class ϕ_i and we obtain a finite cover $\tilde{N}_i \rightarrow N$ such that the pull-back of ϕ_i is quasi-fibered.

We now denote by $p: M \rightarrow N$ the cover corresponding to $\bigcap \pi_1(\tilde{N}_i)$. It follows from Proposition 8.11 that pull-backs of quasi-fibered classes are quasi-fibered, and that pull-backs of inequivalent faces of the Thurston norm ball lie on inequivalent faces of the Thurston norm ball. It thus follows that $p^*\phi_1, \dots, p^*\phi_k$ lie on closures of inequivalent fibered faces of M , i.e., M has at least k inequivalent fibered faces. \square

The following proposition is now an immediate consequence of Propositions 8.12 and 8.14.

Proposition 8.15. *Let N be an irreducible, compact 3-manifold with empty or toroidal boundary which is not a graph manifold. If $\pi_1(N)$ is virtually RFRS, then there exists a finite cover M of N such that the Thurston norm ball of M has at least k inequivalent fibered faces.*

Remark.

- (1) Let N be an irreducible, compact 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is virtually RFRS but not virtually abelian. According to [Ag08, Theorem 7.2] the manifold N admits finite covers with arbitrarily many inequivalent faces in the Thurston norm ball. At this level of generality the statement does not hold. As an example consider the product manifold $N = S^1 \times \Sigma$. Any finite cover M of N is again a product; in particular the Thurston norm ball of M has just two faces.
- (2) It is an interesting question to find criteria which determine whether or not a given graph manifold has virtually arbitrarily many faces in the Thurston norm ball.

9. SUMMARY AND OPEN QUESTIONS

9.1. Recapitulation. In Sections 1.1 and 1.3 we saw that for most purposes it suffices to study fundamental groups of compact, orientable, irreducible 3-manifolds with empty or toroidal boundary. Furthermore, we saw in Section 1.5 that fundamental groups of spherical 3-manifolds, i.e., 3-manifolds with finite fundamental groups, are completely classified, and we saw in Theorem 1.20 that 3-manifolds with virtually solvable fundamental groups are well understood. It is interesting to note, however, that solvable fundamental groups of 3-manifolds in some sense have ‘worse’ properties than fundamental groups of hyperbolic 3-manifolds. In fact in contrast to the picture we developed in Diagram 4 for hyperbolic 3-manifold groups, we have the following lemma:

Lemma 9.1. *Let N be a Sol-manifold and let $\pi = \pi_1(N)$. Then*

- (1) π is not virtually RFRS,
- (2) π is not virtually special,
- (3) π does not admit a finite-index subgroup which is residually p for all primes p , and
- (4) π does not virtually retract onto all its cyclic subgroups.

The first statement was shown by Agol [Ag08, p. 271], the second is an immediate consequence of the first statement and (G.15), the third is proved in [AF11], and the fourth statement follows easily from the fact that any finite cover N' of a Sol-manifold is a Sol-manifold again and so $b_1(N') = 1$, contradicting (G.17).

In the following we will thus restrict ourselves to 3-manifolds in the class

$$\mathcal{A} = \begin{cases} \text{class of all irreducible, orientable, compact 3-manifolds} \\ N \text{ with empty or toroidal boundary such that } \pi_1(N) \text{ is} \\ \text{neither finite nor solvable.} \end{cases}$$

More precisely, we will summarize some of the properties of 3-manifolds in the following subclasses of \mathcal{A} :

\mathcal{SFS} = class of all Seifert fibered $N \in \mathcal{A}$,

\mathcal{G} = $\left\{ \begin{array}{l} \text{class of all } N \in \mathcal{A} \text{ which are graph manifolds} \\ \text{and not Seifert fibered,} \end{array} \right.$

\mathcal{VS} = class of all $N \in \mathcal{A}$ such that $\pi_1(N)$ is virtually special,

\mathcal{VCS} = class of all $N \in \mathcal{A}$ such that $\pi_1(N)$ is virtually compact special,

\mathcal{H} = class of all $N \in \mathcal{A}$ which are hyperbolic 3-manifolds.

	Property of N	\mathcal{SFS}	\mathcal{G}	\mathcal{H}	\mathcal{VS}	\mathcal{VCS}
1	π linear over \mathbb{C}	Yes	?	Yes	Yes	Yes
2	π linear over \mathbb{Z}	Yes	?	Yes	Yes	Yes
3	π is virtually residually torsion-free nilpotent	Yes	?	Yes	Yes	Yes
4	π virtually bi-orderable	Yes	?	Yes	Yes	Yes
5	π residually finite simple	?	?	Yes	?	?
6	π conjugacy separable	Yes	Yes	Yes	Yes	Yes
7	π LERF	Yes	No	Yes	No	No
8	π large	Yes	Yes	Yes	Yes	Yes
9	N virtually fibered	No	No	Yes	Yes	Yes
10	π virtually RFRS	No	No	Yes	Yes	Yes
11	π virtually retracts onto infinite cyclic subgroups	No	No	Yes	Yes	Yes

TABLE 3. Properties of 3-manifold groups.

In Table 3 we summarize some properties of 3-manifold groups. Note that ‘No’ means that there exists at least one manifold in this class for which the property does not hold. The properties marked by ‘Yes’ are all covered in Sections 4 and 6.

In the following we will justify all the ‘No’s in Table 3 and we give a few remarks:

- (3) By (G.24) and (G.26) a positive answer to (3) implies that (4) holds.
- (4) Boyer–Rolfsen–Wiest [BRW05, Question 1.10] asked whether 3-manifold groups are virtually bi-orderable. We refer to [BRW05] for more information on bi-orderability of 3-manifold groups.
- (7) It is known by work of Burns–Karrass–Solitar [BKS87] (see also [Mat97a, Mat97b], [RW98]) that there exist examples of graph manifolds with boundary which are not subgroup separable. In fact there exist graph manifolds with virtually compact special fundamental groups which are not LERF. For instance, the non-LERF link group exhibited in [NW01, Theorem 1.3] is the Right-Angled Artin Group defined by the graph with four vertices that is homeomorphic to the interval. Niblo and Wise also

showed [NW98] that there exist graph manifolds which admit finitely generated subgroups which are not even engulfed, i.e., not even contained in a proper finite-index subgroup.

- (9) There exist closed aspherical Seifert fibered manifolds and closed graph manifolds with non-trivial JSJ decomposition which are not virtually fibered (see [Nem96]). In particular, by (G.18) there exist manifolds in \mathcal{S} and \mathcal{G} such that their fundamental groups are not virtually RFRS. We refer to [Liu11] for a classification of graph manifolds with virtually RFRS fundamental groups.
- (11) In Proposition 8.10 we saw that there exist Seifert fibered manifolds and graph manifolds with non-trivial JSJ decomposition such that their fundamental groups do not retract onto all their infinite cyclic subgroups. On the other hand, by (G.16) and (H.1), groups which are virtually RFRS retract onto all their infinite cyclic subgroups.

9.2. 3-manifolds with a non-trivial JSJ decomposition. By the Virtually Compact Special Theorem of Agol, Kahn–Markovic and Wise, we now have a very good understanding of hyperbolic 3-manifolds. It seems reasonable to conjecture that the fundamental group of any irreducible, orientable 3-manifold with empty or toroidal boundary that contains at least one hyperbolic piece in its JSJ decomposition is virtually special. Recall that by Leeb [Leb95] any such 3-manifold is non-positively curved. In light of Liu’s results (see (G.28)) we can reformulate this conjecture in the following succinct way.

Conjecture 9.2. *Let N be a compact, aspherical 3-manifold with empty or toroidal boundary. Then N is non-positively curved if and only if $\pi_1(N)$ is virtually special.*

As we saw in Section 6, a proof of Conjecture 9.2 would imply that if N is an aspherical, non-positively curved 3-manifold with empty or toroidal boundary, then $\pi_1(N)$ is virtually a subgroup of a RAAG; in particular $\pi_1(N)$ is RFRS and N is virtually fibered, $\pi_1(N)$ is linear over \mathbb{Z} , and $\pi_1(N)$ is virtually residually torsion-free nilpotent. On the other hand, to deduce separability results for subgroups of $\pi_1(N)$, one needs something stronger.

To make this precise, fix π a virtually special group. We will call a subgroup Γ *well behaved* if there is a subgroup π_0 of finite index in π and an embedding ϕ of π_0 into a Right-Angled Artin Group A_Σ such that $\phi(\pi_0 \cap \Gamma)$ is quasi-convex. A well behaved subgroup is a virtual retract, and in particular separable (cf. (G.7)). One would therefore like to exhibit a large, intrinsically defined class of well behaved subgroups of any virtually special 3-manifold group π . In the remainder of this subsection, we propose the class of *fully relatively quasi-convex subgroups* as a class of well behaved subgroups, and outline one possible reason why they could be well behaved.

We work in the context of relatively hyperbolic groups. The following theorem follows quickly from [Dah03, Theorem 0.1].

Theorem 9.3 ([Dah03]). *Let N be a compact, irreducible 3-manifold with empty or toroidal boundary. Let M_1, \dots, M_k be the maximal graph manifold pieces of the JSJ decomposition of N , let S_1, \dots, S_l be the tori in the boundary of N that*

adjoin a hyperbolic piece and let T_1, \dots, T_m be the tori in the JSJ decomposition of M that separate two (not necessarily distinct) hyperbolic pieces of the JSJ decomposition. The fundamental group of N is hyperbolic relative to the set of parabolic subgroups

$$\{H_i\} = \{\pi_1(M_p)\} \cup \{\pi_1(S_q)\} \cup \{\pi_1(T_r)\}.$$

A graph manifold group is hyperbolic relative to itself. This reflects the heuristic fact that many group-theoretic pathologies of 3-manifold groups arise in graph manifolds.

There is a notion of a *relatively quasi-convex* subgroup of a relatively hyperbolic group; see (G.7) and the references mentioned there for more details. A subgroup Γ of a relatively hyperbolic group π is called *fully relatively quasi-convex* if it is relatively quasi-convex and, furthermore, for each i , $\Gamma \cap H_i$ is either trivial or a subgroup of finite index in H_i . The class of fully relatively quasi-convex subgroups is a plausible candidate for a large intrinsically defined class of well behaved subgroups of π .

Conjecture 9.4. *Let N be a compact, aspherical 3-manifold with empty or toroidal boundary which is non-positively curved. Let Γ be a subgroup of $\pi = \pi_1(N)$ that is fully relatively quasi-convex with respect to the natural relatively hyperbolic structure on π . Then Γ is well behaved; that is, there is a subgroup π_0 of finite index in π and an embedding ϕ of π_0 into a Right-Angled Artin Group A_Σ such that $\phi(\pi_0 \cap \Gamma)$ is combinatorially quasi-convex.*

This is related to the following, more succinct, conjecture.

Conjecture 9.5. *Let N be a compact, irreducible 3-manifold in which no torus of the JSJ decomposition bounds a Seifert fibered 3-manifold on both sides. Then $\pi_1(N)$ is LERF.*

Note that, under the hypotheses of Conjecture 9.5, the parabolic subgroups of $\pi_1(N)$ in the relatively hyperbolic structure are LERF. See [LoR01] for some evidence towards this conjecture.

Conjecture 9.4 would follow from an affirmative answer to the following extension of Conjecture 9.2 (cf. [CDW09, Theorem 5.8]).

Question 9.6. *Let N be a compact, aspherical 3-manifold with empty or toroidal boundary which is non-positively curved. Is $\pi_1(N)$ virtually compact special?*

It may very well be that the answer to Question 9.6 is negative; indeed, the techniques of [Liu11] and [PW11] do not give compact cube complexes. If this is the case, then it is nevertheless extremely desirable to either prove Conjecture 9.4 or to exhibit a different intrinsically defined class of subgroups that are well behaved.

9.3. Non-non-positively curved 3-manifolds. The above discussion shows that a clear picture of the properties of aspherical non-positively curved 3-manifolds is emerging. The ‘last frontier,’ oddly enough, seems to be the study of 3-manifolds which are not non-positively curved. We summarize some known properties in the following theorem.

Theorem 9.7. *Let N be an aspherical 3-manifold with empty or toroidal boundary which does not admit a non-positively curved metric. Then the following hold:*

- (1) N is a closed graph manifold;
- (2) $\pi_1(N)$ is conjugacy separable;
- (3) for any prime p , the group $\pi_1(N)$ is virtually residually p .

The first statement was proved by Leeb [Leb95] and the other two statements are known to hold for fundamental groups of all graph manifolds by [WZ10] and [AF10], respectively.

We saw in Lemma 9.1 and Proposition 8.10 that there are many desirable properties which fundamental groups of Sol-manifolds and some graph manifolds do not have. Also, recall that there are graph manifolds which are not virtually fibered (cf. (G.28)). We can nonetheless pose the following question.

Questions 9.8. *Let N be an aspherical 3-manifold with empty or toroidal boundary which does not admit a non-positively curved metric.*

- (1) *Is $\pi_1(N)$ linear over \mathbb{C} ?*
- (2) *Is $\pi_1(N)$ linear over \mathbb{Z} ?*
- (3) *If $\pi_1(N)$ is not solvable, does $\pi_1(N)$ admit a finite-index subgroup which is residually p for any prime p ?*
- (4) *Is $\pi_1(N)$ virtually bi-orderable?*

9.4. Other questions. We conclude this paper with several questions about fundamental groups of 3-manifolds which are (at least on the surface) not related to the discussion of the previous subsections.

9.4.1. Linear representations. We start out with the following question asked by Thurston [Ki97, Problem 3.33]:

Question 9.9. *Does every finitely generated 3-manifold group have a faithful representation in $\mathrm{GL}(4, \mathbb{R})$?*

Note that $\mathrm{SL}(2, \mathbb{C})$ injects into $\mathrm{GL}(4, \mathbb{R})$, so the answer is positive for hyperbolic 3-manifolds. But we are not even aware of a proof that there is an n such that the fundamental group of any Seifert fibered manifold embeds in $\mathrm{GL}(n, \mathbb{R})$; we refer to (D.8) for more information.

The following was conjectured by Luo [Luo12, Conjecture 1].

Conjecture 9.10. *Let N be a compact 3-manifold. Given any non-trivial $g \in \pi_1(N)$, there exists a finite commutative ring R and a homomorphism $\alpha: \pi_1(N) \rightarrow \mathrm{SL}(2, R)$ such that $\alpha(g)$ is non-trivial.*

9.4.2. Groups which are residually simple. Long-Reid [LoR98] showed that the fundamental group of any hyperbolic 3-manifold is residually simple. On the other hand, some finite fundamental groups of 3-manifolds are not residually simple (e.g., $\mathbb{Z}/4$) and any non-abelian solvable group is not residually simple. We are not aware of any other examples of 3-manifold groups which are not residually finite simple. We therefore pose the following question:

Question 9.11. *Let N be an irreducible 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is neither finite nor solvable. Is $\pi_1(N)$ residually finite simple?*

9.4.3. *The group ring of 3-manifold groups.* We now turn to the study of group rings. If π is a torsion-free group, then the Zero Divisor Conjecture (see, e.g., [Lü02, Conjecture 10.14]) asserts that the group $\mathbb{Z}[\pi]$ has no non-trivial zero divisors. This conjecture is still wide open, in particular it is not even known for 3-manifold groups. For future reference we record this special case of the Zero Divisor Conjecture:

Conjecture 9.12. *Let N be an aspherical 3-manifold with empty or toroidal boundary. Then $\mathbb{Z}[\pi_1(N)]$ has no non-trivial zero divisors.*

Let Γ now be any torsion-free group. Then $\mathbb{Z}[\Gamma]$ has no non-trivial zero divisors if one of the following holds:

- (1) Γ is elementary amenable (e.g., solvable-by-finite),
- (2) Γ is locally indicable, or
- (3) Γ is left orderable.

We refer to [KLM88, Theorem 1.4], [RZ98, Proposition 6], [Lin93] and [Hig40] for the proofs. It is clear that if a group Γ is residually a group for which the Zero Divisor Conjecture holds, then it also holds for Γ . It now follows that Conjecture 9.12 holds if either of the following two questions is answered in the affirmative:

Questions 9.13. *Let N be an aspherical 3-manifold with empty or toroidal boundary.*

- (1) *Is $\pi_1(N)$ residually torsion-free elementary amenable?*
- (2) *Is $\pi_1(N)$ residually torsion-free left orderable?*

A related question arises when one studies Ore localizations (see, e.g., [Lü02, Section 8.2.1] for a survey). If Γ contains a non-cyclic free group, then $\mathbb{Z}[\Gamma]$ does not admit an Ore localization (see, e.g., [Lin06, Proposition 2.2]). On the other hand, if Γ is an amenable group, then $\mathbb{Z}[\Gamma]$ admits an Ore localization $\mathbb{K}(\Gamma)$ (see, e.g., [Ta57] and [DLMSY03, Corollary 6.3]). If Γ satisfies furthermore the Zero Divisor Conjecture, then the natural map $\mathbb{Z}[\Gamma] \rightarrow \mathbb{K}(\Gamma)$ is injective. We can then view $\mathbb{Z}[\Gamma]$ as a subring of the skew field $\mathbb{K}(\Gamma)$ and $\mathbb{K}(\Gamma)$ is flat over $\mathbb{Z}[\Gamma]$.

Let N be an aspherical 3-manifold with empty or toroidal boundary. It seems reasonable to ask whether the group ring $\mathbb{Z}[\pi_1(N)]$ is residually a skew field which is flat over $\mathbb{Z}[\pi_1(N)]$. Note that an affirmative answer would follow from an affirmative answer to either of the two questions asked in Question 9.13 (with the second one slightly modified to ‘torsion-free amenable left orderable’). Maps from $\mathbb{Z}[\pi_1(N)]$ to skew fields played a major role in the work of Cochran–Orr–Teichner [COT03], Cochran [Coc04] and Harvey [Har05].

9.4.4. *Potence.* Recall that a group π is called *potent* if for any non-trivial $g \in \pi$ and any $n \in \mathbb{N}$ there exists an epimorphism $\alpha: \pi \rightarrow G$ onto a finite group G such that $\alpha(g)$ has order n . As we saw above, many 3-manifold groups are virtually

potent. It is also straightforward to see that fundamental groups of fibered 3-manifolds are potent. Also, Shalen [Shn11] proved the following result: Let π be the fundamental group of a hyperbolic 3-manifold and let $n > 2$ be an integer. Then there exist finitely many conjugacy classes C_1, \dots, C_m in π such that for any $g \notin C_1 \cup \dots \cup C_m$ there exists a homomorphism $\alpha: \pi_1(N) \rightarrow G$ onto a finite group G such that $\alpha(g)$ has order n .

The following question naturally arises:

Question 9.14. *Let N be an aspherical 3-manifold with empty or toroidal boundary. Is $\pi_1(N)$ potent?*

9.4.5. *Left-orderability and L -spaces.* Let N be an irreducible 3-manifold with empty or toroidal boundary. By (C.15) and (C.16) above, if $b_1(N) \geq 1$, then $\pi_1(N)$ is left-orderable (see also [BRW05] for a different approach). On the other hand if $b_1(N) = 0$, i.e., if N is a rational homology sphere, then there is presently no good criterion for determining whether $\pi_1(N)$ is left-orderable or not. Before we formulate the subsequent conjecture we recall that a rational homology sphere N is called an *L -space* if the total rank of its Heegaard Floer homology $\widehat{HF}(N)$ equals $|H_1(N; \mathbb{Z})|$. We refer to the foundational papers of Ozsváth–Szabó [OS04a, OS04b] for details.

The following conjecture was formulated by Boyer–Gordon–Watson [BGW11, Conjecture 3]:

Conjecture 9.15. *Let N be an irreducible rational homology sphere. Then $\pi_1(N)$ is left-orderable if and only if N is not an L -space.*

We refer to [BGW11] for background and we refer to [Pet09, BGW11, CIW10, CIW11, CLW11, LW11, CT11, LeL11] for evidence towards an affirmative answer. We also refer to [CD03, RoS10] for the interaction between left orderability and foliations.

9.4.6. *Groups of weight 1.* A group π is said to be of *weight 1* if π admits a normal generator, i.e., if there exists a $g \in \pi$ such that the smallest normal subgroup containing g equals π .

If N is obtained by Dehn surgery along a knot $K \subseteq S^3$, then the image of the meridian of K is a normal generator of $\pi_1(N)$, i.e., $\pi_1(N)$ has weight 1. The converse does not hold, i.e., there exist closed 3-manifolds N such that $\pi_1(N)$ has weight 1, but which are not obtained by Dehn surgery along a knot $K \subseteq S^3$. For example, if $N = P_1 \# P_2$ is the connected sum of two copies of the Poincaré homology sphere P , then $\pi_1(N)$ is normally generated by $a_1 a_2$, where $a_1 \in \pi_1(P_1)$ is an element of order 3 and $a_2 \in \pi_1(P_2)$ is an element of order 5. On the other hand it follows from [CGLS87, Theorem 2.0.3] that N cannot be obtained from Dehn surgery along a knot $K \subseteq S^3$.

The following question, which is still wide open, is a variation of a question asked by Cochran (see [GS87, p. 550]).

Question 9.16. *Let N be a closed, orientable, irreducible 3-manifold such that $\pi_1(N)$ has weight 1. Is N the result of Dehn surgery along a knot $K \subseteq S^3$?*

9.4.7. *Poincaré duality groups.* It is natural to ask whether there is an intrinsic, group-theoretic characterization of 3-manifold groups. Given $n \in \mathbb{N}$, Johnson–Wall [JW72] introduced the notion of an n -dimensional *Poincaré duality group* (usually just referred to as a PD_n -group). The fundamental group of any closed, orientable, aspherical n -manifold is a PD_n -group. Now suppose that π is a PD_n -group. If $n = 1$ and $n = 2$, then π is the fundamental group of a closed, orientable, aspherical n -manifold. Davis [Dav98, Theorem C] showed that for any $n \geq 4$ there exists a finitely generated PD_n -group which is not finitely presented, in particular it is not the fundamental group of an aspherical closed n -manifold. The following question though is still open:

Question 9.17. *Let $n \geq 3$. Is every finitely presented PD_n group the fundamental group of a closed, orientable, aspherical n -manifold?*

We refer to [Tho95, Dav00, Hil08, Hil11] for more information in the case $n = 3$ and for known results.

9.4.8. *Homology of finite regular covers.* Let N be an irreducible, non-spherical 3-manifold with empty or toroidal boundary. Then we saw in (C.32) that

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; \mathbb{Z})}{[\tilde{N} : N]} = 0.$$

It is natural to ask what is the limit behavior of torsion in the homology of finite covers. The following question has been raised by several authors (see, e.g., [BV10], [Lü02, Question 13.73] and [Lü12, Conjecture 1.12]).

Question 9.18. *Let N be an irreducible 3-manifold with empty or toroidal boundary. We denote by $\text{vol}(N)$ the sum of the volumes of the hyperbolic pieces in the JSJ decomposition of N . Does the following equality hold?*

$$\limsup_{\tilde{N}} \frac{1}{[\tilde{N} : N]} \ln |\text{Tor } H_1(\tilde{N}; \mathbb{Z})| = \frac{1}{6\pi} \text{vol}(N)$$

(Here we take the limit over all finite regular covers \tilde{N} of N .) Or more optimistically, does the following equality hold

$$\lim_{i \rightarrow \infty} \frac{1}{[\tilde{N}_i : N]} \ln |\text{Tor } H_1(\tilde{N}_i; \mathbb{Z})| = \frac{1}{6\pi} \text{vol}(N)$$

for any cofinal tower of regular finite index covers $\{N_i\}_{i \in \mathbb{N}}$?

We refer to [CS08, CDS09, ACS10, CS11] for results relating the homology of a hyperbolic 3-manifold to the hyperbolic volume. An attractive approach to the question is the result of Lück–Schick [LüS99] that $\text{vol}(N)$ can be expressed in terms of a certain L^2 -torsion of N . The relationship between L^2 -torsion and growth of torsion homology is explored by Silver–Williams [SW02a, SW02b], Kitano–Morifuji–Takasawa [KMT03], Le [Le10] and Raimbault [Rai10].

Note that an affirmative answer would imply that the order of torsion in the homology of a hyperbolic 3-manifold grows exponentially by going to finite covers. To the best of our knowledge even the following much weaker question is still open:

Question 9.19. *Let N be a hyperbolic 3-manifold. Does N admit a finite cover \tilde{N} with $\text{Tor } H_1(\tilde{N}; \mathbb{Z}) \neq 0$?*

It is also interesting to study the behavior of the \mathbb{F}_p -Betti numbers in finite covers and the number of generators of the first homology group in finite covers. Little seems to be known about these two problems (but see [LLS11] for some partial results regarding the former problem). It is an interesting question whether

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; \mathbb{F}_p)}{[\tilde{N} : N]} = \lim_{\tilde{N}} \frac{b_1(\tilde{N}; \mathbb{Z})}{[\tilde{N} : N]}$$

for any compact 3-manifold. We also refer to [Lac11] for further questions on \mathbb{F}_p -Betti numbers in finite covers.

Given a 3-manifold N , the behavior of the homology in a cofinal nested sequence of finite-sheeted covers \tilde{N}_i can depend on the particular choice of sequence. For example, F. Calegari–Dunfield [CD06] together with Boston–Ellenberg [BE06] showed that there exists a closed hyperbolic 3-manifold and a cofinal nested sequence of finite-sheeted covers \tilde{N}_i such that $b_1(\tilde{N}_i) = 0$, but with $vb_1(N) > 0$. Another instance of this phenomenon can be seen in [LLuR08].

9.4.9. Fibered faces. Let N be a hyperbolic 3-manifold such that $\pi_1(N)$ is virtually RFRS. Agol [Ag08, Theorem 7.2] shows that N admits finite covers with an arbitrarily large number of fibered faces. We conclude this survey paper with the following question on virtual fiberedness:

Question 9.20. *Does every irreducible non-positively curved 3-manifold admit a finite cover such that all faces of the Thurston norm ball are fibered?*

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