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3-Nets realizing a group in a projective plane

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Abstract In a projective plane $PG(2,\mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, we give a complete classification of 3-nets realizing a finite group. An infinite family, due to Yuzvinsky (Compos. Math. 140:1614–1624, 2004), arises from plane cubics and comprises 3-nets realizing cyclic and direct products of two cyclic groups. Another known infinite family, due to Pereira and Yuzvinsky (Adv. Math. 219:672–688, 2008), comprises 3-nets realizing dihedral groups. We prove that there is no further infinite family. Urzúa's 3-nets (Adv. Geom. 10:287–310, 2010) realizing the quaternion group of order 8 are the unique sporadic examples.

If p is larger than the order of the group, the above classification holds in characteristic $p>0$ apart from three possible exceptions Alt_4, Sym_4, and Alt_5.

Motivation for the study of finite 3-nets in the complex plane comes from the study of complex line arrangements and from resonance theory; see (Falk and Yuzvinsky in Compos. Math. 143:1069–1088, 2007; Miguel and Buzunáriz in Graphs Comb. 25:469–488, 2009; Pereira and Yuzvinsky in Adv. Math. 219:672–688, 2008; Yuzvin-

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sky in Compos. Math. 140:1614–1624, 2004; Yuzvinsky in Proc. Am. Math. Soc. 137:1641–1648, 2009).

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1 Introduction

In a projective plane a 3-*net* consists of three pairwise disjoint classes of lines such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. If one of the classes has finite size, say n, then the other two classes also have size n, called the *order* of the 3-net.

The notion of a 3-net comes from classical differential geometry via the combinatorial abstraction of the notion of a 3-web. There is a long history about finite 3-nets in combinatorics related to affine planes, latin squares, loops, and strictly transitive permutation sets. In this paper we deal with 3-nets in a projective plane $PG(2,\mathbb{K})$ over an algebraically closed field $\mathbb{K}$ that are coordinatized by a group. Such a 3-net, with line classes $\mathcal{A},\mathcal{B},\mathcal{C}$ and coordinatizing group $G=(G,\cdot)$, is equivalently defined by a triple of bijective maps from G to $(\mathcal{A},\mathcal{B},\mathcal{C})$, say

$$\alpha : G\to\mathcal{A},\qquad \beta : G\to\mathcal{B},\qquad \gamma : G\to\mathcal{C}$$

such that $a\cdot b=c$ if and only if $\alpha(a),\beta(b),\gamma(c)$ are three concurrent lines in $PG(2,\mathbb{K})$ for any $a,b,c\in G$. If this is the case, the 3-net in $PG(2,\mathbb{K})$ is said to *realize* the group G. In recent years, finite 3-nets realizing a group in the complex plane have been investigated in connection with complex line arrangements and resonance theory; see [4, 13, 15, 17, 18].

In the present paper, combinatorial methods are used to investigate finite 3-nets realizing a group. Since key examples, such as algebraic 3-nets and tetrahedron type 3-nets, arise naturally in the dual plane of $PG(2,\mathbb{K})$, it is convenient to work with the dual concept of a 3-net.

Formally, a *dual* 3-*net* of order n in $PG(2,\mathbb{K})$ consists of a triple $(\Lambda_1,\Lambda_2,\Lambda_3)$ with $\Lambda_1,\Lambda_2,\Lambda_3$ pairwise disjoint point sets of size n, called *components*, such that every line meeting two distinct components meets each component in precisely one point. A dual 3-net $(\Lambda_1,\Lambda_2,\Lambda_3)$ realizing a group is *algebraic* if its points lie on a plane cubic and is of *tetrahedron type* if its components lie on the six sides (diagonals) of a nondegenerate quadrangle in such a way that $\Lambda_i=\Delta_i\cup\Gamma_i$ with Δ_i and Γ_i lying on opposite sides for $i=1,2,3$.

The goal of this paper is to prove the following classification theorem.

Theorem 1 *In the projective plane $PG(2,\mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p\ge 0$, let $(\Lambda_1,\Lambda_2,\Lambda_3)$ be a dual 3-net of order $n\ge 4$ that realizes a group G. If either $p=0$ or $p>n$, then one of the following holds.*

(I) *G is either cyclic or the direct product of two cyclic groups, and $(\Lambda_1,\Lambda_2,\Lambda_3)$ is algebraic.*
(II) *G is dihedral, and $(\Lambda_1,\Lambda_2,\Lambda_3)$ is of tetrahedron type.*

(III) *G is the quaternion group of order* 8.
(IV) *G has order* 12 *and is isomorphic to* Alt_4.
(V) *G has order* 24 *and is isomorphic to* Sym_4.
(VI) *G has order* 60 *and is isomorphic to* Alt_5.

A computer aided exhaustive search shows that if $p = 0$, then (IV) (and hence (V), (VI)) does not occur; see [14].

Theorem 1 shows that every realizable finite group can act in $PG(2, \mathbb{K})$ as a projectivity group. This confirms Yuzvinsky's conjecture for $p = 0$.

The proof of Theorem 1 uses some previous results due to Yuzvinsky [18], Urzúa [16], and Blokhuis, Korchmáros, and Mazzocca [2].

Our notation and terminology are standard; see [8]. In view of Theorem 1, $\mathbb{K}$ denotes an algebraically closed field of characteristic p where either $p = 0$ or $p \geq 5$, and any dual 3-net in the present paper is supposed to have order n with $n < p$ whenever $p > 0$.

2 Some useful results on plane cubics

A nice infinite family of dual 3-nets realizing a cyclic group arises from plane cubics in $PG(2, \mathbb{K})$; see [17]. The key idea is to use the well-known Abelian group defined on the points of an irreducible plane cubic, recalled here in the following two propositions and illustrated in Fig. 1.

Proposition 1 [7, Theorem 6.104] *A nonsingular plane cubic $\mathcal{F}$ can be equipped with an additive group $(\mathcal{F}, +)$ on the set of all its points. If an inflection point P_0 of $\mathcal{F}$ is chosen to be the identity* 0, *then three distinct points $P, Q, R \in \mathcal{F}$ are collinear if and only if $P + Q + R = 0$. For a prime number $d \neq p$, the subgroup of $(\mathcal{F}, +)$ consisting of all elements g with $dg = 0$ is isomorphic to $C_d \times C_d$, while for $d = p$, it is either trivial or isomorphic to C_p according as $\mathcal{F}$ is supersingular or not.*

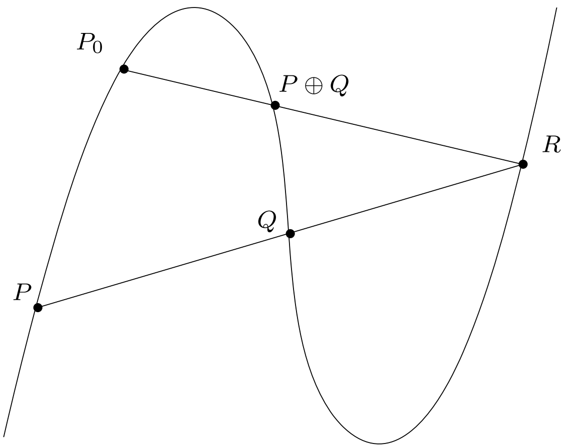

Fig. 1 Abelian group law on an elliptic curve

Proposition 2 [17, Proposition 5.6, (1)] *Let $\mathcal{F}$ be an irreducible singular plane cubic with its unique singular point U, and define the operation $+$ on $\mathcal{F} \setminus \{U\}$ in exactly the same way as on a nonsingular plane cubic. Then $(\mathcal{F}, +)$ is an Abelian group isomorphic to the additive group of $\mathbb{K}$ or to the multiplicative group of $\mathbb{K}$, according as P is a cusp or a node.*

If P is a nonsingular and noninflection point of $\mathcal{F}$, then the tangent to $\mathcal{F}$ at P meets $\mathcal{F}$ in a point P' other than P, and P' is the *tangential* point of P. Every inflection point of a nonsingular cubic $\mathcal{F}$ is the center of an involutory homology preserving $\mathcal{F}$. A classical *Lame configuration* consists of two triples of distinct lines in $PG(2, \mathbb{K})$, say ℓ_1, ℓ_2, ℓ_3 and r_1, r_2, r_3, such that no line from one triple passes through the common point of two lines from the other triple. For $1 \le j, k \le 3$, let R_{jk} denote the common point of the lines ℓ_j and r_k. There are nine such common points, and they are called the points of the Lame configuration.

Proposition 3 (Lame's Theorem) *If eight points from a Lame configuration lie on a plane cubic, then the ninth also does.*

3 3-Nets, quasigroups and loops

A latin square of order n is a table with n rows and n columns which has n^2 entries with n different elements none of them occurring twice within any row or column. If $(L, *)$ is a quasigroup of order n, then its multiplicative table, also called Cayley table, is a latin square of order n, and the converse also holds.

For two integers k, n both bigger than 1, let $(G, \cdot)$ be a group of order kn containing a normal subgroup $(H, \cdot)$ of order n. Let $\mathcal{G}$ be a Cayley table of $(G, \cdot)$. Obviously, the rows and the columns representing the elements of $(H, \cdot)$ in $\mathcal{G}$ form a latin square that is a Cayley table for $(H, \cdot)$. From $\mathcal{G}$ we may extract $k^2 - 1$ more latin squares using the cosets of H in G. In fact, for any two such cosets H_1 and H_2, a latin square $H_{1,2}$ is obtained by taking as rows (respectively columns) the elements of H_1 (respectively H_2).

Proposition 4 *The latin square $H_{1,2}$ is a Cayley table for a quasigroup isotopic to the group H.*

Proof Fix an element $t_1 \in H_1$. In $H_{1,2}$, label the row representing the element $h_1 \in H_1$ with $h_1' \in H$ where $h_1 = t_1 \cdot h_1'$. Similarly, for a fixed element $t_2 \in H_2$, label the column representing the element $h_2 \in H_2$ with $h_2' \in H$ where $h_2 = h_2' \cdot t_2$. The entries in $H_{1,2}$ come from the coset $H_1 \cdot H_2$. Now, label the entry h_3 in $H_1 \cdot H_2$ with the element $h_3' \in H$ where $h_3 = t_1 \cdot h_3' \cdot t_2$. Doing so, $H_{1,2}$ becomes a Cayley table for the subgroup $(H, \cdot)$, whence the assertion follows. □

In terms of a dual 3-net, the relationship between 3-nets and quasigroups can be described as follows. Let $(L, \cdot)$ be a loop arising from an embeddable 3-net, and consider its dual 3-net with its components $\Lambda_1, \Lambda_2, \Lambda_3$. For $i = 1, 2, 3$, the points in Λ_i are bijectively labeled by the elements of L. Let (A_1, A_2, A_3) with $A_i \in \Lambda_i$ denote the triple of the points corresponding to the element $a \in L$. With this notation, $a \cdot b = c$ holds in L if and only if the points A_1, B_2, and C_3 are collinear. In this way, points in Λ_3 are *naturally labeled* when $a \cdot b$ is the label of C_3. Let (E_1, E_2, E_3) be the triple for the unit element e of L. From $e \cdot e = e$, the points E_1, E_2, and E_3 are collinear. Since $a \cdot a = a$ only holds for $a = e$, the points A_1, A_2, A_3 are the vertices of a (nondegenerate) triangle whenever $a \neq e$. Furthermore, from $e \cdot a = a$, the points E_1, A_2, and A_3 are collinear; similarly, $a \cdot e = a$ yields that the points A_1, E_2, and A_3 are collinear. However, the points A_1, A_2, and E_3 form a triangle in general; they are collinear if and only if $a \cdot a = e$, i.e., a is an involution of L.

In some cases, it is useful to relabel the points of Λ_3 replacing the above bijection $A_3 \to a$ from Λ_3 to L by the bijection $A_3 \to a'$ where a' is the inverse of a in $(L, \cdot)$. Doing so, three points A_1, B_2, C_3 with $A_1 \in \Lambda_1$, $B_2 \in \Lambda_2$, $C_3 \in \Lambda_3$ are collinear if and only if $a \cdot b \cdot c = e$ with e being the unit element in $(L, \cdot)$. This new bijective labeling will be called a *collinear relabeling* with respect to Λ_3.

In this paper we are interested in 3-nets of $PG(2, \mathbb{K})$ that are coordinatized by a group G. If this is the case, we say that the 3-net realizes the group G. In terms of dual 3-nets where Λ_1, Λ_2, Λ_3 are the three components, the meaning of this condition is as follows: There exists a triple of bijective maps from G to $(\Lambda_1, \Lambda_2, \Lambda_3)$, say

$$\alpha : G \to \Lambda_1, \qquad \beta : G \to \Lambda_2, \qquad \gamma : G \to \Lambda_3,$$

such that $a \cdot b = c$ if and only if $\alpha(a), \beta(b), \gamma(c)$ are three collinear points for any $a, b, c \in G$.

Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net that realizes a group $(G, \cdot)$ of order kn containing a subgroup $(H, \cdot)$ of order n. Then the left cosets of H provide a partition of each component Λ_i into k subsets. Such subsets are called left H-members and denoted by $\Gamma_i^{(1)}, \ldots, \Gamma_i^{(k)}$, or simply Γ_i when this does not cause confusion. The left translation map $\sigma_g : x \mapsto x + g$ preserves every left H-member. The following lemma shows that every left H-member Γ_1 determines a dual 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ that realizes H.

Lemma 1 *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net that realizes a group $(G, \cdot)$ of order kn containing a subgroup $(H, \cdot)$ of order n. For any left coset $g \cdot H$ of H in G, let $\Gamma_1 = g \cdot H$, $\Gamma_2 = H$, and $\Gamma_3 = g \cdot H$. Then $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ that realizes H.*

Proof For any $h_1, h_2 \in H$, we have that $(g \cdot h_1) \cdot h_2 = g \cdot (h_1 \cdot h_2) = g \cdot h$ with $h \in H$. Hence, any line joining a point of Γ_1 with a point of Γ_2 meets Γ_3. □

Similar results hold for right cosets of H. Therefore, for any right coset $H \cdot g$, the triple $(\Gamma_1, \Gamma_2, \Gamma_3)$ with $\Gamma_1 = H$, $\Gamma_2 = H \cdot g$, and $\Gamma_3 = H \cdot g$ is a 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ that realizes H.

The dual 3-subnets $(\Gamma_1, \Gamma_2, \Gamma_3)$ introduced in Lemma 1 play a relevant role. When g ranges over G, we obtain k such dual 3-nets, each being called *a dual 3-net realizing the subgroup H as a subgroup of G*.

Obviously, left cosets and right cosets coincide if and only if H is a normal subgroup of G, and if this is the case, we may use the shorter term of coset.

Now assume that H is a normal subgroup of G. Take two H-members from different components, say Γ_i and Γ_j with $1 \le i < j \le 3$. By Proposition 4, there exists a member Γ_m from the remaining component Λ_m, with $1 \le m \le 3$ and $m \ne i, j$, such that $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a dual 3-net of realizing $(H, \cdot)$. Doing so, we obtain k^2 dual 3-subnets of $(\Lambda_1, \Lambda_2, \Lambda_3)$. They are all the dual 3-subnets of $(\Lambda_1, \Lambda_2, \Lambda_3)$ that realize the normal subgroup $(H, \cdot)$ as a subgroup of $(G, \cdot)$.

Lemma 2 *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net that realizes a group $(G, \cdot)$ of order kn containing a normal subgroup $(H, \cdot)$ of order n. For any two cosets $g_1 \cdot H$ and $g_2 \cdot H$ of H in G, let $\Gamma_1 = g_1 \cdot H$, $\Gamma_2 = g_2 \cdot H$, and $\Gamma_3 = (g_1 \cdot g_2) \cdot H$. Then $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ that realizes H.*

If g_1 and g_2 range independently over G, we obtain k^2 such dual 3-nets, each being called *a dual 3-net realizing the normal subgroup H as a subgroup of G.*

4 The infinite families of dual 3-nets realizing a group

A dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ with $n \ge 4$ is said to be *algebraic* if all its points lie on a (uniquely determined) plane cubic $\mathcal{F}$, called the *associated* plane cubic of $(\Lambda_1, \Lambda_2, \Lambda_3)$. Algebraic dual 3-nets fall into three subfamilies according as the plane cubic splits into three lines, or in an irreducible conic and a line, or it is irreducible.

4.1 Proper algebraic dual 3-nets

An algebraic dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is said to be *proper* if its points lie on an irreducible plane cubic $\mathcal{F}$.

Proposition 5 *Any proper algebraic dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes a group M. There is a subgroup $T \cong M$ in $(\mathcal{F}, +)$ such that each component Λ_i is a coset $T + g_i$ in $(\mathcal{F}, +)$ where $g_1 + g_2 + g_3 = 0$.*

Proof We do some computation in $(\mathcal{F}, +)$. Let $A_1, A_2, A_3 \in \Lambda_1$ be three distinct points viewed as elements in $(\mathcal{F}, +)$. First, we show that the solution of the equation in $(\mathcal{F}, +)$

$$A_1 - A_2 = X - A_3 \tag{1}$$

belongs to Λ_1. Let $C \in \Lambda_3$. By the definition of a dual 3-net, there exist $B_i \in \Lambda_2$ such that $A_i + B_i + C = 0$ for $i = 1, 2, 3$. Now choose $C_1 \in \Lambda_3$ for which $A_1 + B_2 + C_1 = 0$, and then choose $A^* \in \Lambda_1$ for which $A^* + B_3 + C_1 = 0$. Now,

$$\begin{aligned} A^* - A_3 &= -B_3 - C_1 - (-B_3 - C) = C - C_1, \\ A_1 - A_2 &= -B_2 - C_1 - (-B_2 - C) = C - C_1. \end{aligned} \tag{2}$$

Therefore, A^* is a solution of Eq. (2).

Now we are in a position to prove that Λ_1 is a coset of a subgroup of $(\mathcal{F}, +)$. For $A_0 \in \Lambda_1$, let $T_1 = \{A - A_0 \mid A \in \Lambda_1\}$. Since $(A_1 - A_0) - (A_2 - A_0) = A_1 - A_2$, Eq. (2) ensures the existence of $A^* \in \Lambda_1$ for which $A_1 - A_2 = A^* - A_0$ whenever $A_1, A_2 \in \Lambda_1$. Hence, $(A_1 - A_0) - (A_2 - A_0) \in T_1$. From this we have that T_1 is a subgroup of $(\mathcal{F}, +)$, and therefore Λ_1 is a coset $T + g_1$ of T_1 in $(\mathcal{F}, +)$.

Similarly, $\Lambda_2 = T_2 + g_2$ and $\Lambda_3 = T_3 + g_3$ with some subgroups T_2, T_3 of $(\mathcal{F}, +)$ and elements $g_2, g_3 \in (\mathcal{F}, +)$. It remains to show that $T_1 = T_2 = T_3$. The line through the points g_1 and g_2 meets Λ_3 in a point $t^* + g_3$. Replacing g_3 with $g_3 + t^*$ allows us to assume that $g_1 + g_2 + g_3 = 0$. Then three points $g_i + t_i$ with $t_i \in T_i$ are collinear if and only if $t_1 + t_2 + t_3 = 0$. For $t_3 = 0$, this yields $t_2 = -t_1$. Hence, every element of T_2 is in T_1, and the converse also holds. Hence, $T_1 = T_2$. Now, $t_3 = -t_1 - t_2$ yields that $T_3 = T_1$. Therefore, $T = T_1 = T_2 = T_3$ and $\Lambda_i = T + g_i$ for $i = 1, 2, 3$. This shows that $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes a group $M \cong T$. □

4.2 Triangular dual 3-nets

An algebraic dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is *regular* if the components lie on three lines, and it is either of *pencil type* or *triangular* according as the three lines are either concurrent or they are the sides of a triangle.

Lemma 3 *Every regular dual* 3*-net of order n is triangular.*

Proof Assume that the components of a regular dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ lie on three concurrent lines. Using homogeneous coordinates in $PG(2, \mathbb{K})$, these lines are assumed to be those with equations $Y = 0$, $X = 0$, $X - Y = 0$, respectively, so that the line of equation $Z = 0$ meets each component. Therefore, the points in the components may be labeled so that

$$\Lambda_1 = \{(1, 0, \xi) \mid \xi \in L_1\}, \qquad \Lambda_2 = \{(0, 1, \eta) \mid \eta \in L_2\},$$
$$\Lambda_3 = \{(1, 1, \zeta) \mid \zeta \in L_3\},$$

with L_i subsets of $\mathbb{K}$ containing 0. By a straightforward computation, three points $P = (1, 0, \xi)$, $Q = (0, 1, \eta)$, $R = (1, 1, \zeta)$ are collinear if and only if $\zeta = \xi + \eta$. Therefore, $L_1 = L_2 = L_3$, and $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes a subgroup of the additive group of $\mathbb{K}$ of order n. Therefore, n is a power of p. But this contradicts the hypothesis $p > n$. □

For a triangular dual 3-net, the (uniquely determined) triangle whose sides contain the components is called the *associated* triangle.

Proposition 6 *Every triangular dual* 3*-net realizes a cyclic group isomorphic to a multiplicative subgroup of* $\mathbb{K}$.

Proof Using homogeneous coordinates in $PG(2, \mathbb{K})$, the vertices of the triangle are assumed to be the points $O = (0, 0, 1)$, $X_\infty = (1, 0, 0)$, $Y_\infty = (0, 1, 0)$. For $i = 1, 2, 3$, let ℓ_i denote the fundamental line with equation $Y = 0$, $X = 0$, $Z = 0$, respectively. Therefore, the points in the components lie on the fundamental lines, and they may be labeled in such a way that

$$\Lambda_1 = \{(\xi, 0, 1) \mid \xi \in L_1\}, \qquad \Lambda_2 = \{(0, \eta, 1) \mid \eta \in L_2\},$$

$$\Lambda_3 = \{(1, -\zeta, 0) \mid \zeta \in L_3\},$$

with L_i subsets of $\mathbb{K}^*$ of a given size n. With this setting, three points $P = (\xi, 0, 1)$, $Q = (0, \eta, 1)$, $R = (1, -\zeta, 0)$ are collinear if and only if $\xi\zeta = \eta$. With an appropriate choice of the unity point of the coordinate system, both $1 \in L_1$ and $1 \in L_2$ may also be assumed. From $1 \in L_1$ we have that $L_2 = L_3$. This, together with $1 \in L_2$, implies that $L_1 = L_2 = L_3 = L$. Since $1 \in L$, L is a finite multiplicative subgroup of $\mathbb{K}$. In particular, L is cyclic. □

Remark 2 In the proof of Proposition 6, if the unity point of the coordinate system is arbitrarily chosen, the subsets L_1, L_2, and L_3 are not necessarily subgroups. Actually, they are cosets of (the unique) multiplicative cyclic subgroup H, say $L_1 = aH$, $L_2 = bH$, and $L_3 = cH$, with $ac = b$. Furthermore, since every $h \in H$ defines a projectivity $\sigma_h : x \mapsto hx$ of the projective line, and these projectivities form a group isomorphic to H, it turns out that L_i is an orbit of a cyclic projectivity group of ℓ_i of order n for $i = 1, 2, 3$.

Proposition 7 *Let* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *be a triangular dual* 3*-net. Then every point of* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *is the center of a unique involutory homology that preserves* $(\Lambda_1, \Lambda_2, \Lambda_3)$.

Proof The point $(\xi, 0, 1)$ is the center and the line through Y_∞ and the point $(-\xi, 0, 1)$ is the axis of the involutory homology φ_ξ associated to the matrix

$$\begin{pmatrix} 0 & 0 & \xi^2 \\ 0 & -\xi & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

With the above notation, if $\xi \in aH$, then h_ξ preserves Λ_1 while it sends any point in Λ_2 to a point in Λ_3, and vice versa. Similarly, for $\eta \in bH$ and $\zeta \in cH$ where ψ_η and θ_ζ are the involutory homologies associated to the matrices

$$\begin{pmatrix} -\eta & 0 & 0 \\ 0 & 0 & \eta^2 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta \end{pmatrix}.$$ □

With the notation introduced in the proof of Proposition 6, let $\Phi_1 = \{\varphi_\xi \varphi_{\xi'} \mid \xi, \xi' \in aH\}$ and $\Phi_2 = \{\psi_\eta \psi_{\eta'} \mid \eta, \eta' \in bH\}$. Then both are cyclic groups isomorphic to H. A direct computation gives the following result.

Proposition 8 $\Phi_1 \cap \Phi_2$ *is either trivial or has order* 3.

Some useful consequences are stated in the following proposition.

Proposition 9 *Let* $\Theta = \langle \Phi_1, \Phi_2 \rangle$. *Then*

$$|\Theta| = \begin{cases} |H|^2, & \textit{when } \gcd(3, |H|) = 1; \\ \frac{1}{3}|H|^2, & \textit{when } \gcd(3, |H|) = 3. \end{cases}$$

Furthermore, Θ fixes the vertices of the fundamental triangle, and no nontrivial element of Θ fixes a point outside the sides of the fundamental triangle.

We prove another useful result.

Proposition 10 *If $(\Gamma_1, \Gamma_2, \Gamma_3)$ and $(\Sigma_1, \Sigma_2, \Sigma_3)$ are triangular dual 3-nets such that $\Gamma_1 = \Sigma_1$, then the associated triangles share the vertices on their common side.*

Proof By Remark 2, Γ_1 is the orbit of a cyclic projectivity group H_1 of the line ℓ containing Γ_1, while the two fixed points of H_1 on ℓ, say P_1 and P_2, are vertices of the triangle containing $\Gamma_1, \Gamma_2, \Gamma_3$.

The same holds for Σ_1 with a cyclic projectivity group H_2 and fixed points Q_1, Q_2. From $\Gamma_1 = \Sigma_1$ we have that the projectivity group H of the line ℓ generated by H_1 and H_2 preserves Γ_1. Let M be the projectivity group generated by H_1 and H_2.

Observe that M is a finite group since it has an orbit of finite size $n \geq 3$. Clearly, $|M| \geq n$, and the equality holds if and only if $H_1 = H_2$. If this is the case, then $\{P_1, P_2\} = \{Q_1, Q_2\}$. Therefore, for the purpose of the proof, we may assume on the contrary that $H_1 \neq H_2$ and $|M| > n$.

Now, Dickson's classification of finite subgroups of $PGL(2, \mathbb{K})$ applies to M. From that classification we have that M is one of the nine subgroups listed as (1), ..., (9) in [12, Theorem 1], where e denotes the order of the stabilizer M_P of a point P in a short M-orbit, that is, an M-orbit of size smaller than M. Observe that such an M-orbit has size $|M|/e$. There exist finitely many short M-orbits, and Σ_1 is one of them. It may be that an M-orbit is trivial as it consists of just one point.

Obviously, M is neither cyclic nor dihedral as it contains two distinct cyclic subgroups of the same order $n \geq 3$.

Also, M is not an elementary Abelian p-group E of rank ≥ 2; otherwise, we would have $|E| = |M| > n$ since the minimum size of a nontrivial E-orbit is $|E|$; see (2) in [12, Theorem 1].

From (5) in [12, Theorem 1] with $p \neq 2, 3$, the possible sizes of a short Alt_4-orbit are 4 and 6, each larger than 3. On the other hand, Alt_4 has no element of order larger than 3. Therefore, $M \ncong \mathrm{Alt}_4$ for $p \neq 2, 3$.

Similarly, from (5) in [12, Theorem 1] with $p \neq 2, 3$, the possible sizes of a short Sym_4-orbit are 6, 8, 12, each larger than 4. Since Sym_4 has no element of order larger than 4, we have $M \ncong \mathrm{Sym}_4$ for $p \neq 2, 3$.

Again, from (6) in [12, Theorem 1] with $p \neq 2, 5$, the possible sizes of a short Alt_5-orbit are 10 and 12 for $p = 3$, while 12, 20, 30 for $p \neq 2, 3, 5$. Each size exceeds 5. On the other hand, Alt_5 has no element of order larger than 5. Therefore, $M \ncong \mathrm{Alt}_5$ for $p \neq 2, 5$.

The group M might be isomorphic to a subgroup L of order qk with $k \mid (q-1)$ and $q = p^h$, $h \geq 1$. Here L is the semidirect product of the unique (elementary Abelian) Sylow p-subgroup of L by a cyclic subgroup of order k. No element in L has order larger than k when $h > 1$ and p when $h = 1$. From (7) in [12, Theorem 1], any nontrivial short L-orbit has size q. Therefore, $M \cong L$ implies that $h = 1$ and $n = p$. But this is inconsistent with the hypothesis $p > n$.

Finally, M might be isomorphic to a subgroup L such that either $L = PSL(2, q)$ or $L = PGL(2, q)$ with $q = p^h$, $h \geq 1$. No element in L has order larger than $q + 1$.

From (7) and (8) in [12, Theorem 1], any short L-orbit has size either $q+1$ or $q(q-1)$. For $q \geq 3$, if $M \cong L$ occurs, then $n = q+1 \geq p+1$, a contradiction with the hypothesis $p > n$. For $q = 2$, we have that $|L| = 6$, which is smaller than 12. Therefore, $M \ncong L$.

No possibility has arisen for M. Therefore, $\{P_1, P_2\} = \{Q_1, Q_2\}$. □

4.3 Conic-line-type dual 3-nets

An algebraic dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of *conic-line type* if two of its three components lie on an irreducible conic $\mathcal{C}$ and the third one lies on a line ℓ. All such 3-nets realize groups, and they can be described using subgroups of the projectivity group $PGL(2, \mathbb{K})$ of $\mathcal{C}$. For this purpose, some basic results on subgroups and involutions in $PGL(2, \mathbb{K})$ are useful, which essentially depend on the fact that every involution in $PGL(2, \mathbb{K})$ is a perspectivity whose center is a point outside $\mathcal{C}$ and axis is the pole of the center with respect to the orthogonal polarity arising from $\mathcal{C}$. We begin with an example.

Example 1 Take any cyclic subgroup C_n of $PGL(2, \mathbb{K})$ of order $n \geq 3$ with $n \neq p$ that preserves $\mathcal{C}$. Let D_n be the unique dihedral subgroup of $PGL(2, \mathbb{K})$ containing C_n. If j is the (only) involution in $\mathcal{Z}(D_n)$ and ℓ is its axis, then the centers of the other involutions in D_n lie on ℓ. We have n involutions in D_n other than j, and the set of the their centers is taken for Λ_1. Take a C_n-orbit $\mathcal{O}$ on $\mathcal{C}$ such that the tangent to $\mathcal{C}$ at any point in $\mathcal{O}$ is disjoint from Λ_1; equivalently, the D_n-orbit $\mathcal{Q}$ is larger than $\mathcal{O}$. Then $\mathcal{Q}$ is the union of $\mathcal{O}$ together with another C_n-orbit. Take these two C_n-orbits for Λ_2 and Λ_3, respectively. Then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is a conic-line dual 3-net which realizes C_n. It may be observed that ℓ is a chord of $\mathcal{C}$ and the multiplicative group of $\mathbb{K}$ has a subgroup of order n.

The cyclic subgroups C_n form a unique conjugacy class in $PGL(2, \mathbb{K})$. For a cyclic subgroup C_n of $PGL(2, \mathbb{K})$ of order n, the above construction provides a unique example of a dual 3-net realizing C_n. Using the classification of finite subgroups of $PGL(2, \mathbb{K})$ as in the proof of [2, Theorem 6.1], the following result can be proven. For details, see the preliminary version [10, 11].

Proposition 11 *Up to projectivities, the conic-line dual 3-nets of order n are those described in Example 1.*

A corollary of this is the following result.

Proposition 12 *A conic-line dual 3-net realizes a cyclic group C_n.*

The result below can be proven with an argument similar to that used in the proof of Proposition 10. For details, see the preliminary version [10, 11].

Proposition 13 *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ and $(\Delta_1, \Delta_2, \Delta_3)$ be two conic-line-type dual 3-nets where Γ_3 lies on the line ℓ and Δ_3 lies on the line s. If $\Gamma_1 = \Delta_1$, then $\ell = s$.*

4.4 Tetrahedron type dual 3-nets

In $PG(2,\mathbb{K})$, any nondegenerate quadrangle with its six sides (including the two diagonals) may be viewed as the projection of a tetrahedron of $PG(3,\mathbb{K})$. This suggests to call two sides of the quadrangle *opposite* if they do not have any common vertex. With this definition, the six sides of the quadrangle are partitioned into three couples of opposite sides. Let $(\Lambda_1,\Lambda_2,\Lambda_3)$ be a dual 3-net of order $2n$ containing a dual 3-subnet

$$(\Gamma_1,\Gamma_2,\Gamma_3) \tag{3}$$

of order n. Observe that $(\Lambda_1,\Lambda_2,\Lambda_3)$ contains three more dual 3-subnets of order n. In fact, for $\Delta_i=\Lambda_i\setminus\Gamma_i$, each of the triples below defines such a subnet:

$$(\Gamma_1,\Delta_2,\Delta_3),\qquad(\Delta_1,\Gamma_2,\Delta_3),\qquad(\Delta_1,\Delta_2,\Gamma_3). \tag{4}$$

Now, the dual 3-net $(\Lambda_1,\Lambda_2,\Lambda_3)$ is said to be *tetrahedron-type* if its components lie on the sides of a nondegenerate quadrangle such that Γ_i and Δ_i are contained in opposite sides for $i=1,2,3$. Such a nondegenerate quadrangle is said to be *associated* to $(\Lambda_1,\Lambda_2,\Lambda_3)$. Observe that each of the six sides of the quadrangle contains exactly one of the point-sets Γ_i and Δ_i. Moreover, each of the four dual 3-subnets listed in (3) and (4) is triangular as each of its components, called a *half-set*, lies on a side of a triangle whose vertices are also vertices of the quadrangle. Therefore, there are six half-sets in any dual 3-net of tetrahedron type.

Proposition 14 *Any tetrahedron-type dual* 3*-net realizes a dihedral group.*

Proof The associated quadrangle is assumed to be the fundamental quadrangle of the homogeneous coordinate system in $PG(2,\mathbb{K})$, so that its vertices are O, X_∞, Y_∞ together with the unity point $E=(1,1,1)$. By definition, the subnet (3) is triangular. Without loss of generality,

$$\Gamma_1=\{(\xi,0,1)\mid\xi\in L_1\},\qquad\Gamma_2=\{(0,\eta,1)\mid\eta\in L_2\},$$
$$\Gamma_3=\{(1,-\zeta,0)\mid\zeta\in L_3\},$$

where $L_1=aH, L_2=bH, L_3=cH$ are cosets of H with $ac=b$; see Remark 2. We fix such triple $\{a,b,c\}$. Observe that $(a,0,1)\in\Gamma_1$, $(0,b,1)\in\Gamma_2$, and $(1,-c,0)\in\Gamma_3$. Furthermore,

$$\Delta_1=\{(1,\alpha,1)\mid\alpha\in M_1\},\qquad\Delta_2=\{(\beta,1,1)\mid\beta\in M_2\},$$
$$\Delta_3=\{(1,1,\gamma)\mid\gamma\in M_3\},$$

with M_1, M_2, and M_3 subsets of $\mathbb{K}\setminus\{0,1\}$, each of size n. Now, a direct computation similar to those carried out in Sect. 4.2 gives the result. For details, see the preliminary version [10, 11].

An alternative approach to the proof is to lift $(\Lambda_1,\Lambda_2,\Lambda_3)$ to the fundamental tetrahedron of $PG(3,\mathbb{K})$ so that the projection π from the point $P_0=(1,1,1,1)$ on the plane $X_4=0$ returns $(\Lambda_1,\Lambda_2,\Lambda_3)$. For this purpose, it is enough to define the sets lying on the edges of the fundamental tetrahedron,

$$\begin{aligned}
&\Gamma_1'=\{(\xi,0,1,0)\mid \xi\in L_1\}, && \Gamma_2'=\{(0,\eta,1,0)\mid \eta\in L_2\},\\
&\Gamma_3'=\{(1,-\zeta,0,0)\mid \zeta\in L_3\}, && \Delta_1'=\{(0,\alpha-1,0,-1)\mid \alpha\in M_1\},\\
&\Delta_2'=\{(\beta-1,0,0,-1)\mid \beta\in M_2\}, && \Delta_3'=\{(0,0,\gamma-1,-1)\mid \gamma\in M_3\},
\end{aligned}$$

and observe that $\pi(\Gamma_i')=\Gamma_i$ and $\pi(\Delta_i')=\Delta_i$ for $i=1,2,3$. Moreover, a triple (P_1,P_2,P_3) of points with $P_i\in\Gamma_i\cup\Delta_i$ consists of collinear points if and only if their projection does. Hence, $(\Gamma_1'\cup\Gamma_2',\Gamma_3'\cup\Delta_1',\Delta_2'\cup\Delta_3')$ can be viewed as a "spatial" dual 3-net realizing the same group H. Clearly, $(\Gamma_1'\cup\Gamma_2',\Gamma_3'\cup\Delta_1',\Delta_2'\cup\Delta_3')$ is contained in the sides of the fundamental tetrahedron. We claim that these sides minus the vertices form an infinite spatial dual 3-net realizing the dihedral group $2.\mathbb{K}^*$.

To prove this, parameterize the points as follows:

$$\begin{aligned}
\Sigma_1&=\{x_1=(x,0,1,0),(\varepsilon x)_1=(0,1,0,x)\mid x\in\mathbb{K}^*\},\\
\Sigma_2&=\{y_2=(1,y,0,0),(\varepsilon y)_2=(0,0,1,y)\mid y\in\mathbb{K}^*\},\\
\Sigma_3&=\{z_3=(0,-z,1,0),(\varepsilon z)_3=(1,0,0,-z)\mid z\in\mathbb{K}^*\}.
\end{aligned}\tag{5}$$

Then,

$$\begin{aligned}
&x_1,y_2,z_3 \text{ are collinear} &&\Leftrightarrow\quad z=xy,\\
&(\varepsilon x)_1,y_2,(\varepsilon z)_3 \text{ are collinear} &&\Leftrightarrow\quad z=xy \quad\Leftrightarrow\quad \varepsilon z=(\varepsilon x)y,\\
&x_1,(\varepsilon y)_2,(\varepsilon z)_3 \text{ are collinear} &&\Leftrightarrow\quad z=x^{-1}y \quad\Leftrightarrow\quad \varepsilon z=x(\varepsilon y),\\
&(\varepsilon x)_1,(\varepsilon y)_2,z_3 \text{ are collinear} &&\Leftrightarrow\quad z=x^{-1}y \quad\Leftrightarrow\quad z=(\varepsilon x)(\varepsilon y).
\end{aligned}$$

Thus, $(\Gamma_1'\cup\Gamma_2',\Gamma_3'\cup\Delta_1',\Delta_2'\cup\Delta_3')$ is a dual 3-subnet of $(\Sigma_1,\Sigma_2,\Sigma_3)$, and H is a subgroup of the dihedral group $2.\mathbb{K}^*$. As H is not cyclic but it has a cyclic subgroup of index 2, we conclude that H is itself dihedral. □

5 Classification of low-order dual 3-nets

An exhaustive computer aided search gives the following results. For details, see [14].

Proposition 15 *Any dual* 3*-net realizing an Abelian group of order* ≤ 8 *is algebraic. The dual of Urzúa's* 3*-nets are the only dual* 3*-net that realize the quaternion group of order* 8.

Proposition 16 *Any dual* 3*-net realizing an Abelian group of order* 9 *is algebraic.*

Proposition 17 *If* $p=0$, *no dual* 3*-net realizes* Alt_4.

6 Characterizations of the infinite families

Proposition 18 *Every dual* 3*-net realizing a cyclic group is algebraic.*

Proof For $n = 3$, we have that $3n = 9$, and hence all points of the dual 3-net lie on a cubic. Therefore, $n \geq 4$ is assumed.

Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order n that realizes the cyclic group $(L, *)$. Therefore, the points of each component are labeled by I_n. After a collinear relabeling with respect to Λ_3, consider the configuration of the following nine points: $0, 1, 2$ from Λ_1, $0, 1, 2$ from Λ_2, and $n-1, n-2, n-3$ from Λ_3. For the seek of a clearer notation, the point with label a in the component Λ_m will be denoted by a_m.

The configuration presents six triples of collinear points, namely

(i) $\{0_1, 1_2, (n-1)_3\}, \{1_1, 2_2, (n-3)_3\}, \{2_1, 0_2, (n-2)_3\}$;
(ii) $\{0_1, 2_2, (n-2)_3\}, \{1_1, 0_2, (n-1)_3\}, \{2_1, 1_2, (n-3)_3\}$.

Therefore, the corresponding lines form a Lame configuration. Furthermore, the three (pairwise distinct) lines determined by the two triples in (i) can be regarded as a totally reducible plane cubic, say $\mathcal{F}_1$. Similarly, a totally reducible plane curve, say $\mathcal{F}_2$, arises from the triples in (ii). Obviously, $\mathcal{F}_1 \neq \mathcal{F}_2$. Therefore, the nine points of the above Lame configuration are the base points of the pencil generated by $\mathcal{F}_1$ and $\mathcal{F}_2$. Now, define the plane cubic $\mathcal{F}$ to be the cubic from the pencil which contains 3_1.

Our next step is to show that $\mathcal{F}$ also contains each of the points $(n-4)_3$ and 3_2. For this purpose, consider the following six triples of collinear points:

(iii) $\{1_1, 2_2, (n-3)_3\}, \{2_1, 0_2, (n-2)_3\}, \{3_1, 1_2, (n-4)_3\}$;
(iv) $\{1_1, 1_2, (n-2)_3\}, \{2_1, 2_2, (n-4)_3\}, \{3_1, 0_2, (n-3)_3\}$.

Again, the corresponding lines form a Lame configuration. Since eight of its points, namely $1_1, 2_1, 3_1, 0_2, 1_2, 2_2, (n-2)_3, (n-3)_3$ lie on $\mathcal{F}$, Lame's theorem shows that $(n-4)_3$ also lies on $\mathcal{F}$. To show that $3_2 \in \mathcal{F}$, we proceed similarly using the following six triples of collinear points:

(v) $\{0_1, 3_2, (n-3)_3\}, \{1_1, 1_2, (n-2)_3\}, \{2_1, 2_2, (n-4)_3\}$;
(vi) $\{0_1, 2_2, (n-2)_3\}, \{1_1, 3_2, (n-4)_3\}, \{2_1, 1_2, (n-3)_3\}$.

To define a Lame configuration that behaves as before, eight of its points, namely 0_1, $1_1, 2_1, 1_2, 2_2, (n-2)_3, (n-3)_3, (n-4)_3$ lie on $\mathcal{F}$, and by Lame's theorem, 3_2 also lies on $\mathcal{F}$.

This completes the proof for $n = 4$. We assume that $n \geq 5$ and show that $(n-5)_3$ lies on $\mathcal{F}$. Again, we use the above argument based on the Lame configuration of the six lines arising from the following six triples of points:

(vii) $\{1_1, 3_2, (n-4)_3\}, \{2_1, 1_2, (n-3)_3\}, \{3_1, 2_2, (n-5)_3\}$;
(viii) $\{1_1, 2_2, (n-3)_3\}, \{2_1, 3_2, (n-5)_3\}, \{3_1, 1_2, (n-4)_3\}$.

From the previous discussion, eight of these points lie on $\mathcal{F}$. Lame's theorem yields that the ninth, namely $(n-5)_3$, also lies on $\mathcal{F}$. From this we infer that also $4_1 \in \mathcal{F}$. To do this, we repeat the above argument for the Lame configuration arising from the six triples of points

(ix) $\{2_1, 2_2, (n-4)_3\}$, $\{3_1, 0_2, (n-3)_3\}$, $\{4_1, 1_2, (n-5)_3\}$;
(x) $\{2_1, 1_2, (n-3)_3\}$, $\{3_1, 2_2, (n-5)_3\}$, $\{4_1, 0_2, (n-4)_3\}$.

Again, we see that eight of these points lie on $\mathcal{F}$. Hence, the ninth, namely 4_1, also lies on $\mathcal{F}$ by Lame's theorem.

Therefore, from the hypothesis that $\mathcal{F}$ passes through the ten points

$$0_1, 1_1, 2_1, 3_1, 0_2, 1_2, 2_2, (n-1)_3, (n-2)_3, (n-3)_3,$$

we have deduced that $\mathcal{F}$ also passes through the ten points

$$1_1, 2_1, 3_1, 4_1, 1_2, 2_2, 3_2, (n-2)_3, (n-3)_3, (n-4)_3.$$

Comparing these two sets of ten points shows that the latter derives from the former shifting by $+1$ when the indices are 1 and 2, while by -1 when the indices are 3. Therefore, repeating the above argument $n-4$ times gives that all points in the dual 3-net lie on $\mathcal{F}$. □

Proposition 19 [17, Theorem 5.4] *If an Abelian group G contains an element of order ≥ 10, then every dual 3-net realizing G is algebraic.*

Proposition 20 [17, Theorem 4.2] *No dual 3-net realizes an elementary Abelian group of order 2^h with $h \geq 3$.*

Proposition 21 [2, Theorem 5.1] *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net such that at least one component lies on a line. Then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or of conic-line type.*

Lemma 4 *Let $(\Gamma_1, \Gamma_2, \Gamma_3)$ be an algebraic dual 3-net lying on a plane cubic $\mathcal{F}$. If $\mathcal{F}$ is reducible, then $(\Gamma_1, \Gamma_2, \Gamma_3)$ is either triangle or of conic-line type, according as $\mathcal{F}$ splits into three lines or into a line and an irreducible conic.*

Proposition 22 *Every dual 3-net realizing a dihedral group of order $2n$ with $n \geq 3$ is of tetrahedron type.*

Proof Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net realizing a dihedral group

$$D_n = \langle x, y \mid x^2 = y^n = 1, yx = xy^{-1} \rangle.$$

Labeling naturally the points in the components Λ_i as indicated in Sect. 3, every $u \in D_n$ defines a triple of points (u_1, u_2, u_3) where $u_i \in \Lambda_i$ for $i = 1, 2, 3$, and vice versa. Doing so, three points $u_1 \in \Lambda_1$, $v_2 \in \Lambda_2$, $w_3 \in \Lambda_3$ are collinear if and only if $uv = w$ in D_n.

Therefore, for $1 \leq i \leq n-2$, the triangle with vertices x_2, $(xy)_2$, $(xy^{-i})_3$ and that with vertices $(1)_3$, y_3, $(y^{-i})_2$ are in mutual perspective position from the point x_1. For two distinct points u_i and v_j with $u_i \in \Lambda_i$ and $v_j \in \Lambda_j$ and $1 \leq i, j \leq 3$, let

$\overline{u_i v_j}$ denote the line through u_i and v_j. By the Desargues theorem, the three diagonal points, that is, the points

$$U = \overline{(x)_2(xy)_2} \cap \overline{(1)_3(y)_3},$$
$$\left(y^i\right)_1 = \overline{(x)_2\left(xy^{-i}\right)_3} \cap \overline{\left(y^{-i}\right)_2(1)_3},$$
$$\left(y^{i+1}\right)_1 = \overline{(xy)_2\left(xy^{-i}\right)_3} \cap \overline{\left(y^{-i}\right)_2(y)_3},$$

are collinear. Hence, a line ℓ_1 contains each point $(1)_1, (y)_1, \dots, (y^{n-1})_1$ in Λ_1, that is,

$$(1)_1, (y)_1, \dots, \left(y^{n-1}\right)_1 \in \ell_1.$$

There are some more useful Desargues configurations. Indeed, the pairs of triangles with vertices

$$(x)_2, \left(xy^{-1}\right)_2, \left(y^{-i-1}\right)_3 \quad \text{and} \quad (xy)_3, (x)_3, \left(y^{-i}\right)_2,$$
$$\left(y^i\right)_2, \left(y^{i+1}\right)_2, \left(y^{i+1}\right)_3 \quad \text{and} \quad (x)_3, (xy)_3, (xy)_2,$$
$$\left(xy^i\right)_2, \left(xy^{i+1}\right)_2, \left(y^i\right)_3 \quad \text{and} \quad (x)_3, (xy)_3, (1)_2,$$
$$(1)_2, (y)_2, (x)_3 \quad \text{and} \quad \left(y^i\right)_3, \left(y^{i+1}\right)_3, \left(xy^i\right)_2, \quad \text{and}$$
$$(x)_2, (xy)_2, (1)_3 \quad \text{and} \quad \left(xy^i\right)_3, \left(xy^{i+1}\right)_3, \left(y^i\right)_2$$

are in mutual perspective position from the points

$$\left(y^{-1}\right)_1, \left(xy^{-i}\right)_1, \left(y^i\right)_1, \left(y^i\right)_1, \left(y^{-i}\right)_1,$$

respectively. Therefore, there exist five more lines $m_1, \ell_2, m_2, \ell_3, m_3$ such that

$$\left\{(x)_1, (xy)_1, \dots, \left(xy^{n-1}\right)_1\right\} \subset m_1, \qquad \left\{(1)_2, (y)_2, \dots, \left(y^{n-1}\right)_2\right\} \subset \ell_2,$$
$$\left\{(x)_2, (xy)_2, \dots, \left(xy^{n-1}\right)_2\right\} \subset m_2, \qquad \left\{(1)_3, (y)_3, \dots, \left(y^{n-1}\right)_3\right\} \subset \ell_3,$$
$$\left\{(x)_3, (xy)_3, \dots, \left(xy^{n-1}\right)_3\right\} \subset m_3.$$

By Proposition 10, the lines $\ell_1, \dots, m_3$ are the sides of a nondegenerate quadrangle, which shows that the dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of tetrahedron type. □

Remark 3 By Proposition 22, the dual 3-nets given in [15, Sect. 6.2] are of tetrahedron type.

Proposition 23 *Let G be a finite group containing a normal subgroup H of order $n \geq 3$. Assume that G can be realized by a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ and that every dual 3-subnet of $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizing H as a subgroup of G is triangular. Then H is cyclic, and $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or of tetrahedron type.*

Proof By Proposition 6, H is cyclic. Fix an H-member Γ_1 from Λ_1, and denote by ℓ_1 the line containing Γ_1. Consider all the triangles that contain some dual 3-net $(\Gamma_1, \Gamma_2^j, \Gamma_3^s)$ realizing H as a subgroup of G. By Proposition 10, these triangles have two common vertices, say P and Q, lying on ℓ_1. For the third vertex R_j of the triangle containing $(\Gamma_1, \Gamma_2^j, \Gamma_3^s)$, there are two possibilities, namely, either the side PR_j contains Γ_2^j and the side QR_j contains Γ_3^s, or vice versa. Therefore, every H-member Γ_2^j from Λ_2 (as well as every H-member Γ_3^s from Λ_3) is contained in a line passing through P or Q.

Now, replace Γ_1 by another H-orbit Γ_1^i lying in Λ_1 and repeat the above argument. If ℓ_i is the line containing Γ_1^i and P_i, Q_i denote the vertices, then again every H-member Γ_2^j from Λ_2 (as well as every H-member Γ_3^s from Λ_3) is contained in a line passing through P_i or Q_i.

Assume that $\{P, Q\} \neq \{P_i, Q_i\}$. If one of the vertices arising from Γ_1, say P, coincides with one of the vertices, say P_i, arising from Γ_1^i, then the line QQ_i must contain either Γ_2^j or Γ_3^s from each $(\Gamma_1, \Gamma_2^j, \Gamma_3^s)$. Therefore, the line QQ_i must contain every H-member from Λ_2 or every H-member from Λ_3. Hence, Λ_2 or Λ_3 lies on the line QQ_i. By Proposition 21, $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or conic-line type. The latter case cannot actually occur as Λ_1 contains Γ_1 and hence contains at least three collinear points.

Therefore, $\{P, Q\} \cap \{P_i, Q_i\} = \emptyset$ may be assumed. Then the H-members from Λ_2 and Λ_3 lie on four lines, namely PP_i, PQ_i, QP_i, QQ_i. Observe that these lines may be assumed to be pairwise distinct, otherwise Λ_2 (or Λ_3) is contained in a line, and again $(\Lambda_1, \Lambda_2, \Lambda_3)$ is triangular. Therefore, half of the H-members from Λ_2 lie on one of these four lines, say PQ_i, and half of them on QP_i. Similarly, each of the lines PP_i and QQ_i contains half from the H-members from Λ_3.

In the above argument, any H-member Γ_2 from Λ_2 may play the role of Γ_1. Therefore, there exist two lines such that each H-member from Λ_1 lies on one of them. Actually, these two lines are PQ and P_iQ_i since each of them contains an H-member from Λ_1. In this case, $(\Lambda_1, \Lambda_2, \Lambda_3)$ is of tetrahedron type. □

Since a dihedral group of order ≥ 8 has a unique cyclic subgroup of index 2 and such a subgroup is characteristic, Propositions 23 and 14 have the following corollary.

Proposition 24 *Let G be a finite group of order $n \geq 12$ containing a normal dihedral subgroup D. If G is realized by a dual 3-net, then G is itself dihedral.*

7 Dual 3-nets preserved by projectivities

Proposition 25 *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ realizing a group G. If every point in Λ_1 is the center of an involutory homology that preserves Λ_1 while interchanges Λ_2 with Λ_3, then either Λ_1 is contained in a line, or $n = 9$. In the latter case, $(\Lambda_1, \Lambda_2, \Lambda_3)$ lies on a nonsingular cubic $\mathcal{F}$ whose inflection points are the points in Λ_1.*

Proof After labeling $(\Lambda_1, \Lambda_2, \Lambda_3)$ naturally, take an element $a \in G$ and denote by φ_a the (unique) involutory homology of center A_1 that maps Λ_2 onto Λ_3. Obviously, φ_a also maps Λ_3 onto Λ_2. Moreover, $\varphi_a(X_2) = Y_3 \Longleftrightarrow a \cdot x = y$, that is, $\varphi_a(X_2) = \varphi_{a'}(X_2') \Longleftrightarrow a \cdot x = a' \cdot x'$, where $G = (G, \cdot)$. Therefore,

$$\varphi_{a'}\varphi_a(X_2) = X_2' \quad \Longleftrightarrow \quad \left(a'^{-1} \cdot a\right) \cdot x = x'. \tag{6}$$

From this we get that for any $b \in G$, there exists $b' \in G$ such that

$$\varphi_{a'}\varphi_a(X_2) = \varphi_{b'}\varphi_b(X_2) \tag{7}$$

for every $X_2 \in \Lambda_2$ or, equivalently, for every $x \in G$.

Let Φ be the projectivity group generated by all products $\varphi_{a'}\varphi_a$ where both a, a' range over G. Obviously, Φ leaves both Λ_2 and Λ_3 invariant. In particular, Φ induces a permutation group on Λ_2. We show that if $\mu \in \Phi$ fixes Λ_2 pointwise, then μ is trivial. Since $n > 3$, the projectivity μ has at least four fixed points in $PG(2, \mathbb{K})$. Therefore, μ is either trivial or a homology. Assume that μ is nontrivial, and let C be the center and c the axis of μ. Take a line ℓ through C that contains a point $P \in \Lambda_3$ and assume that C is a point in Λ_2. Then P is the unique common point of ℓ and Λ_3. Since μ preserves Λ_2, μ must fix P. Therefore, μ fixes Λ_3 pointwise, and hence Λ_3 is contained in c. But then μ cannot fix any point in Λ_2 other than C since the definition of a dual 3-net implies that c is disjoint from Λ_2. This contradiction means that μ is trivial, that is, Φ acts faithfully on Λ_2.

Therefore, (7) states that for any $a, a', b \in G$, there exists $b' \in G$ satisfying the equation $\varphi_{a'}\varphi_a = \varphi_{b'}\varphi_b$. This yields that Φ is an Abelian group of order n acting on Λ_2 as a sharply transitive permutation group. Also,

$$\Phi = \{\varphi_a\varphi_e \mid a \in G\},$$

where e is the identity of G. Therefore, $\Phi \cong G$, and G is Abelian.

Let Ψ be the projectivity group generated by Φ together with some φ_a where $a \in G$. Then $|\Psi| = 2n$, and Ψ comprises the elements in Φ and the involutory homologies φ_a with a ranging over G. Obviously, Ψ interchanges Λ_2 and Λ_3, while it leaves Λ_1 invariant acting on Λ_1 as a transitive permutation group.

Two cases are investigated according as Φ contains a homology or does not. Observe that Φ contains no elation since every elation has infinite order when $p = 0$ while its order is at least p when $p > 0$ but $p > n$ is assumed throughout the paper.

In the former case, let $\rho \in \Phi$ be a homology with center $C \in \Lambda_1$ and axis c. Since ρ commutes with every element in Φ, the point C is fixed by Φ, and the line is preserved by Φ. Assume that C is also the center of ϕ_a with some $a \in G$. The group of homologies generated by ϕ_a and ρ preserves every line through C, and it has order greater than 2. But then it cannot interchange Λ_2 with Λ_3. Therefore, the center of every ϕ_a with $a \in G$ lies on c. This shows that Λ_1 is contained in c.

In the case where Φ contains no homology, Φ has odd order, and $\delta \in \Phi$ has three fixed points, which are the vertices of a triangle Δ. Since δ commutes with every element in Φ, the triangle Δ is left invariant by Φ.

If Φ fixes each vertex of δ, then Φ must be cyclic since, otherwise, Ψ would contain a homology. Therefore, Ψ is a dihedral group, and we show that Λ_1 is contained in a line. For this purpose, take a generator $\rho = \varphi_a\varphi_b$ of Φ, and consider the line ℓ through the centers of φ_a and φ_b. Obviously, ρ preserves ℓ, and this holds for every power of ρ. Hence, Ψ also preserves ℓ. Since every φ_c is conjugate to φ_a under Ψ, this shows that the center of φ_c must lie on ℓ as well. Therefore, Λ_1 is contained in ℓ.

We may assume that some $\rho \in \Phi$ acts on the vertices of Δ as a 3-cycle. Let Δ' be the triangle whose vertices are the fixed points of ρ. Then $\rho^3 = 1$ since ρ^3 fixes not only the vertices of Δ' but also those of Δ'. Therefore, $\Phi = \langle\rho\rangle \times \Theta$ where Θ is the cyclic subgroup of Φ fixing each vertex of Δ. A subgroup of Θ of index ≤ 3 fixes each vertex of Δ' and hence is trivial. Therefore, $|\Theta| = 3$ and $\Phi \cong C_3 \times C_3$. This shows that $n = 9$ and if Λ_1 is not contained in a line, then the configuration of their points, that is, the centers of the homologies in Ψ, is isomorphic to $AG(2,3)$, the affine plane of order 3. Such a configuration can also be viewed as the set of the nine common inflection points of the nonsingular plane cubics of a pencil $\mathcal{P}$, each cubic left invariant by Ψ. For a point $P_2 \in \Lambda_2$, take that cubic $\mathcal{F}$ in $\mathcal{P}$ that contains P_2. Since the orbit of P_2 under the action of Ψ consists of the points in $\Lambda_2 \cup \Lambda_3$, it follows that $\mathcal{F}$ contains each point of $(\Lambda_1, \Lambda_2, \Lambda_3)$. □

The following result is a corollary of Proposition 25.

Proposition 26 *Let* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *be a dual* 3*-net of order* $n \ge 4$ *realizing a group* G*. If every point of* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *is the center of an involutory homology that preserves* $(\Lambda_1, \Lambda_2, \Lambda_3)$, *then* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *is triangular.*

Proof By Proposition 11 and Example 1, $(\Lambda_1, \Lambda_2, \Lambda_3)$ is not of conic-line type. For $n = 9$, $(\Lambda_1, \Lambda_2, \Lambda_3)$ does not lie on any nonsingular cubic $\mathcal{F}$ since no nonsingular cubic has twenty-seven inflection points. Therefore, the assertion follows from Proposition 25. □

A useful generalization of Proposition 26 is given in the proposition below.

Proposition 27 *Let* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *be a dual* 3*-net of order* $n \ge 4$ *realizing a group* G*. Let* $\mathcal{U}$ *be the set of all involutory homologies preserving* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *whose centers are points of* $(\Lambda_1, \Lambda_2, \Lambda_3)$*. If* $|\mathcal{U}| \ge 3$ *and* $\mathcal{U}$ *contains two elements whose centers lie in different components, then the following assertions hold*:

(i) *Every component contains the same number of points that are centers of involutory homologies in* $\mathcal{U}$.
(ii) *The points of* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *that are centers of involutory homologies in* $\mathcal{U}$ *form a triangular dual* 3*-subnet* $(\Gamma_1, \Gamma_2, \Gamma_3)$.
(iii) *Let* M *be the cyclic subgroup associated to* $(\Gamma_1, \Gamma_2, \Gamma_3)$*. Then either* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *is also triangular, or*

$$|G| < \begin{cases} |G:M|^2, & \textit{when } \gcd(3, |G|) = 1; \\ 3|G:M|^2, & \textit{when } \gcd(3, |G|) = 3. \end{cases}$$

Proof Let $\mathcal{G}$ be the projectivity group preserving $(\Lambda_1, \Lambda_2, \Lambda_3)$. Let (ijk) denote any permutation of (123). As we have already observed in the proof of Proposition 25, if

$\varphi \in \mathcal{G}$ is an involutory homology with center $P \in \Lambda_i$, then φ preserves Λ_i and interchanges Λ_j with Λ_k. If $\sigma \in \mathcal{G}$ is another involutory homology with center $R \in \Lambda_j$, then $\sigma\varphi\sigma$ is also an involutory homology whose center S is the common point of Λ_k and the line ℓ through P and R. In terms of dual 3-subnets, this yields (i) and (ii). Let m be the order of $(\Gamma_1, \Gamma_2, \Gamma_3)$. For $m = 2$, $(\Gamma_1, \Gamma_2, \Gamma_3)$ is triangular. For $m = 3$, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is the Hesse configuration, and hence $(\Gamma_1, \Gamma_2, \Gamma_3)$ is triangular. This holds for $m \geq 4$ by Proposition 26 applied to $(\Gamma_1, \Gamma_2, \Gamma_3)$.

To prove (iii), assume that $(\Lambda_1, \Lambda_2, \Lambda_3)$ is not triangular and take a point P from some component, say Λ_3, that does not lie on the sides of the triangle associated to $(\Gamma_1, \Gamma_2, \Gamma_3)$. Since $(\Gamma_1, \Gamma_2, \Gamma_3)$ is triangular, it can play the role of $(\Lambda_1, \Lambda_2, \Lambda_3)$ in Sect. 4.2, and we use the notation introduced there. By the second assertion of Proposition 9, the point has as many as $|\Theta|$ distinct images, all lying in Λ_3. Therefore, $|G| = |\Lambda_3| > |\Theta|$. Using Proposition 9, $|\Theta|$ can be written in function of $|M|$ giving the assertion. □

Let $\mathcal{U}_2$ be the set of all involutory homologies with center in Λ_2 which interchanges Λ_1 and Λ_3. There is a natural injective map Ψ from $\mathcal{U}_2$ to G, where $\Psi(\psi) = g$ holds if and only if the point $g_2 \in \Lambda_2$ is the center of ψ.

Proposition 28 *Let $(\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ realizing a group G. If $|\mathcal{U}_2| \geq 2$, then the following hold.*

(i) *$\mathcal{U}_2$ is closed by conjugation, that is, $\psi\omega\psi \in \mathcal{U}_2$ whenever $\psi, \omega \in \mathcal{U}_2$.*
(ii) *If $g, h \in \Psi(\mathcal{U}_2)$, then $gh^{-1}g \in \Psi(\mathcal{U}_2)$.*
(iii) *If G has a cyclic subgroup H of order 6 with $|H \cap \Psi(\mathcal{U}_2)| \geq 3$ and $1 \in H \cap \Psi(\mathcal{U}_2)$, then either $\Psi(\mathcal{U}_2) = H$, or $\Psi(\mathcal{U}_2)$ is the subgroup of H of order 3.*

Proof For $\psi, \omega \in \mathcal{U}_2$, the conjugate $\tau = \psi\omega\psi$ of ω by ψ is also an involutory homology. Let $g = \Psi(\psi)$ and $h = \Psi(\omega)$. Then the center of τ is $\psi(h_2)$. For $x \in G$, the image of x_1 under τ is $y_3 \in \Lambda_3$ with $y = xgh^{-1}g$. This shows that the center of τ is also in Λ_2; more precisely,

$$\Psi(\psi\omega\psi) = \Psi(\psi)\big(\Psi(\omega)\big)^{-1}\Psi(\psi). \tag{8}$$

In the case where G has a cyclic subgroup H of order 6, assume the existence of three distinct elements $\psi, \omega\rho \in \mathcal{U}_2$ such that $g = \Psi(\psi)$, $h = \Psi(\omega)$, and $r = \Psi(\rho)$ with $g, h, r \in H$. Then H contains $gh^{-1}g$, $hg^{-1}h$, g^2, and h^2. From this assertion (iii) follows. □

8 Dual 3-nets containing algebraic 3-subnets of order n with $n \geq 5$

A key result is the following proposition.

Proposition 29 *Let G be a group containing a proper Abelian subgroup H of order $n \geq 5$. Assume that a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes G such that all its dual 3-subnets $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$ realizing H as a subgroup of G are algebraic. Let $\mathcal{F}_j$ be the* *cubic through the points of* $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$. *If* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *is not algebraic, then* Γ_2 *contains three collinear points, and one of the following holds*:

(i) Γ_2 *is contained in a line.*
(ii) $n = 5$, *and there is an involutory homology with center in* Γ_2 *that preserves every* $\mathcal{F}_j$ *and interchanges* Λ_1 *and* Λ_3.
(iii) $n = 6$, *and there are three involutory homologies with center in* Γ_2 *that preserve every* $\mathcal{F}_j$ *and interchange* Λ_1 *and* Λ_3.
(iv) $n = 9$, *and* Γ_2 *consists of the nine common inflection points of* $\mathcal{F}_j$.

We need the following technical lemma.

Lemma 5 *Let* $A = (A, \oplus)$, $B = (B, +)$ *be Abelian groups and consider the injective maps* $\alpha, \beta, \gamma : A \to B$ *such that* $\alpha(x) + \beta(y) + \gamma(z) = 0$ *if and only if* $z = x \oplus y$. *Then*, $\alpha(x) = \varphi(x) + a$, $\beta(x) = \varphi(x) + b$, $\gamma(x) = -\varphi(x) - a - b$ *for some injective homomorphism* $\varphi : A \to B$ *and elements* $a, b \in B$.

Proof Define $a = \alpha(0)$, $b = \beta(0)$, and $\varphi(x) = -\gamma(x) - a - b$. For $x = 0$ and $z = y$, we obtain that $\alpha(0) + \beta(y) + \gamma(y) = 0$, whence $\beta(y) = -\gamma(y) - a = \varphi(y) + b$. Similarly, for $y = 0$ and $z = x$, we obtain that $\alpha(x) + \beta(0) + \gamma(x) = 0$, whence $\alpha(x) = -\gamma(x) - b = \varphi(x) + a$. Finally, for any $x, y \in G$,

$$\begin{aligned} \varphi(x) + \varphi(y) - \varphi(x + y) &= \varphi(x) + a + \varphi(y) + b - \big(\varphi(x + y) + a + b\big) \\ &= \alpha(x) + \beta(y) + \gamma(x + y) = 0. \end{aligned}$$

Therefore, $\varphi : A \to B$ is a group homomorphism. □

Let $A = (A, \oplus)$ be an Abelian group, and α, β, γ injective maps from A to $PG(2, \mathbb{K})$. The triple (α, β, γ) is a *realization* of A if the points $\alpha(x), \beta(y), \gamma(z)$ are collinear if and only if $z = x \oplus y$. Since $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes G, the natural labeling gives rise to a realization (α, β, γ) such that $\alpha(G) = \Lambda_1$, $\beta(G) = \Lambda_2$, $\gamma(G) = \Lambda_3$. Let $u \in G$. Since H is a subgroup of G, the triple

$$\big(\alpha_u(x) = \alpha(ux), \beta(y) = \beta(y), \gamma_u(z) = \alpha(uz)\big)$$

provides a realization of H such that

$$\alpha_u(H) = \Gamma_1^u, \qquad \beta(H) = \Gamma_2, \qquad \gamma_u(H) = \Gamma_3^u.$$

Therefore, Lemma 5 has the following corollary, where $(\mathcal{F}_j, *)$ denotes the additive group of the plane cubic $\mathcal{F}_j$ through the points of $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$, where for $u = 1$, we write $(\mathcal{F}, +), \alpha, \beta, \gamma, \Gamma_1, \Gamma_2, \Gamma_3$.

Lemma 6 *There exist two realizations from* H *into* $PG(2, \mathbb{K})$, *say* (α, β, γ) *and* $(\alpha_j, \beta_j, \gamma_j)$, *with*

$$\alpha(H) = \Gamma_1, \qquad \beta(H) = \Gamma_2, \qquad \gamma(H) = \Gamma_3,$$

$$\alpha_j(H) = \Gamma_1^j, \qquad \beta_j(H) = \Gamma_2, \qquad \gamma_j(H) = \Gamma_3^j$$

such that

$$\alpha(x) = \varphi(x) + a, \qquad \beta(y) = \varphi(y) + b, \qquad \gamma(z) = \varphi(z) + c,$$
$$\alpha_j(x) = \varphi_j(x) * a_j, \qquad \beta_j(y) = \varphi_j(y) * b_j, \qquad \gamma_j(z) = \varphi_j(z) * c_j$$

for every $x, y, z \in H$, *where both* $\varphi : H \to (\mathcal{F}, +)$ *and* $\varphi_j : H \to (\mathcal{F}_j, *)$ *are injective homomorphisms, and* $\varphi(y) + b = \varphi_j(y) * b_j$ *for every* $y \in H$.

To prove Proposition 29, we point out that $3b \in \varphi(H)$ if and only if Γ_2 contains three collinear points. Suppose that $\varphi(x_1)+b, \varphi(x_2)+b, \varphi(x_3)+b$ are three collinear points. Then $\varphi(x_1) + b + \varphi(x_2) + b + \varphi(x_3) + b = 0$, whence $\varphi(x_1 + x_2 + x_3) + 3b = 0$. Therefore, $3b \in \varphi(H)$. Conversely, if $\varphi(t) = 3b$, take three pairwise distinct elements $x_1, x_2, x_3 \in H$ such that $x_1 + x_2 + x_3 + t = 0$. Then $\varphi(x_1) + b + \varphi(x_2) + b + \varphi(x_3) + b = 0$. Therefore, the points $\varphi(x_1) + b, \varphi(x_2) + b$ and $\varphi(x_3) + b$ of Γ_2 are collinear. Notice that the element $t = -x_1 - x_2 - x_3 \in H$ is the same even if we make the computation with φ_j and b_j.

We separately deal with two cases.

8.1 Γ_2 contains no three collinear points

By the preceding observation, $3b \notin \varphi(H)$. For any $z \in H$, take four different elements x_1, y_1, x_2, y_2 in H such that

$$z = x_1 \oplus y_1 = x_2 \oplus y_2. \tag{9}$$

Then $\varphi(x_1) + b + \varphi(y_1) + b = \varphi(z) + 2b = \varphi(x_2) + b + \varphi(y_2) + b$. Let $P_i = \beta(x_i)$ and $Q_i = \beta(y_i)$ for $i = 1, 2$. Then $P_i \neq Q_i$, and the lines P_1Q_1 and P_2Q_2 meet in a point S in $\mathcal{F}$ outside Γ_2. The same holds for $\mathcal{F}_j$. Therefore, each point S is a common point of $\mathcal{F}$ and $\mathcal{F}_i$ other than those in Γ_2. As S only depends on z which can be freely chosen if $|H| \geq 4$, there are at least n such points S. Hence, $\mathcal{F} \cap \mathcal{F}_j$ contains at least $2n \geq 10$ points. By Bézout's theorem, either $\mathcal{F} = \mathcal{F}_j$, or they are reducible. We may assume that the latter case occurs. By Lemma 4, we may assume that both $(\Gamma_1, \Gamma_2, \Gamma_3)$ and $(\Gamma_1^j, \Gamma_2, \Gamma_3^j)$ are of conic-line type. Here Γ_2 is contained in an irreducible conic $\mathcal{C}$ that is a common component of $\mathcal{F}$ and $\mathcal{F}_j$. By Proposition 13, $\mathcal{F} = \mathcal{F}_j$.

8.2 Γ_2 contains three collinear points

This time, $3b \in \varphi(H)$. Let $\varphi(t) = 3b$ with $t \in H$. If either $\mathcal{F}$ or $\mathcal{F}_j$ is reducible, then Γ_2 is contained in a line. Therefore, both $\mathcal{F}$ and $\mathcal{F}_j$ are assumed to be irreducible.

First, suppose in addition that $t \notin 3H$. For any $x \in H$, let $y = 2(\ominus x) \ominus t$. Observe that $y \neq x$. Since

$$2\big(\varphi(x) + b\big) + \varphi(y) + b = \varphi(t) + \varphi(2x) + \varphi(y) = 0,$$

the point $Q=\beta(y)$ is the tangential point of $P=\beta(x)$ on $\mathcal{F}$. Therefore, β determines the tangents of $\mathcal{F}$ at its points in Γ_2. This holds for $\mathcal{F}_j$. By Lemma 6, $\mathcal{F}$ and $\mathcal{F}_j$ share the tangents at each of their common points in Γ_2. Therefore, $|\mathcal{F}\cap\mathcal{F}_j|\geq 2n\geq 10$, and thus $\mathcal{F}=\mathcal{F}_j$.

It remains to investigate the case where $3b=\varphi(3t_0)$ for some $t_0\in H$. Replacing b by $b-\varphi(t_0)$ shows that $3b=0$ may be assumed. Therefore, the point $P=\varphi(y)+b$ with $y\in H$ is an inflection point of $\mathcal{F}$ if and only if $3y=0$. Furthermore, if $3y\neq 0$, then $Q=\varphi(\ominus(2y))+b$ is the tangential point of P on $\mathcal{F}$. Therefore, β determines the tangents of $\mathcal{F}$ at its points in Γ_2. The same holds for $\mathcal{F}_j$. By Lemma 6, $P=\beta(y)$ is an inflection point of both $\mathcal{F}$ and $\mathcal{F}_j$ or none of them. In the latter case, $\mathcal{F}$ and $\mathcal{F}_j$ have the same tangent at P.

Let m be the number of common inflection points of $\mathcal{F}$ and $\mathcal{F}_j$ lying in Γ_2. Obviously, $P=\varphi(0)+b$ is such a point, and hence $m\geq 1$. On the other hand, m may take only three values, namely 1, 3, and 9. If $m=9$, then $\mathcal{F}$ is nonsingular, and Γ_2 consists of all the nine inflection points of $\mathcal{F}$. The same holds for $\mathcal{F}_j$. If $m=3$, then $\mathcal{F}$ and $\mathcal{F}_j$ share their tangents at $n-3$ common points. Therefore, $2n-3\leq 9$, whence $n\leq 6$.

If $n=6$, there are three common inflection points of $\mathcal{F}$ and $\mathcal{F}_j$, and they are collinear. Let H be the additive group of integers modulo 6. Then the inflection points of $\mathcal{F}$ lying on Γ_2 are $P_i=\varphi(i)+b$ with $i=0,2,4$, while the tangential point of $P_i=\varphi(i)+b$ with $i=1,3,5$ is $P_{-2i}=\varphi(-2i)+b$. Now fix a projective frame with homogeneous coordinates (X,Y,Z) in such a way that

$$\begin{array}{lll} P_0=(1,0,1), & P_1=(0,0,1), & P_2=(0,1,1),\\ P_3=(0,1,0), & P_4=(-1,1,0), & P_5=(1,0,0). \end{array}$$

A straightforward computation shows that $\mathcal{F}_j$ is in the pencil $\mathcal{P}$ comprising the cubics $\mathcal{G}_\lambda$ with equation

$$(X-Z)(Y-Z)(X+Y)+\lambda XYZ=0,\quad \lambda\in\mathbb{K},$$

with the cubic $\mathcal{G}_\infty$ with equation $XYZ=0$. The intersection divisor of the plane cubics in $\mathcal{P}$ is $P_0+P_2+P_4+2P_1+2P_3+2P_5$. Moreover, the points P_0, P_2, P_4 are inflection points of all irreducible cubics in $\mathcal{P}$, and

$$\begin{aligned} \psi_0&:(X,Y,Z)\to(Z,-Y,X),\\ \psi_2&:(X,Y,Z)\to(-X,Z,Y),\\ \psi_4&:(X,Y,Z)\to(Y,X,Z) \end{aligned}$$

are the involutory homologies preserving every cubic in $\mathcal{P}$, the center of ψ_i being P_i for $i=0,2,4$.

If $n=5$, the zero of H is the only element y with $3y=0$. This shows that $\mathcal{F}$ (and $\mathcal{F}_j$) has only one inflection point P_0 in Γ_2 and P_0 is not the tangential point of another point in Γ_2. Each of the remaining four points is the tangential point of exactly one point in Γ_2. These four points may be viewed as the vertices of a quadrangle $P_1P_2P_3P_4$ such that the side P_iP_{i+1} is tangent to $\mathcal{F}$ at P_i for every i with $P_5=P_1$.

Therefore, the intersection divisor of $\mathcal{F}$ and $\mathcal{F}_j$ is $P_0+2P_1+2P_2+2P_3+2P_4$, and $\mathcal{F}_j$ is contained in a pencil $\mathcal{P}$.

Fix a projective frame with homogeneous coordinates (X, Y, Z) in such a way that

$$P_1=(0,0,1), \qquad P_2=(1,0,0), \qquad P_3=(1,1,1), \qquad P_4=(0,1,0).$$

Then $P_0=(1,1,0)$. The pencil $\mathcal{P}$ is generated by the cubics $\mathcal{G}$ and $\mathcal{D}$ with equations $Y(X-Z)Z=0$ and $X(Y-X)(Y-Z)=0$, respectively. Therefore, it consists of cubics G_λ with equation

$$Y^2X-X^2Y+(\lambda-1)XYZ+X^2Z-\lambda YZ^2=0,$$

together with $\mathcal{G}=\mathcal{G}_\infty$. Since the line $Z=0$ contains three distinct base points of the pencil, P_0 is a nonsingular point of $\mathcal{G}_\lambda$ for every $\lambda\in\mathbb{K}$, and the tangent ℓ_λ to $\mathcal{G}_\lambda$ at P_0 has equation $-X+Y+\lambda Z=0$. Assume that Q_0 is an inflection point of $\mathcal{G}_\lambda$. Then ℓ_λ contains no point $P=(X,Y,1)$ from $\mathcal{G}_\lambda$, that is, the polynomials $Y^2X-X^2Y+(\lambda-1)XY+X^2-\lambda Y=0$ and $-X+Y+\lambda=0$ have no common solutions. On the other hand, eliminating Y from these polynomials gives λ^2. This shows that Q_0 is an inflection point for every irreducible cubic in $\mathcal{P}$. Hence, $P_0=Q_0$. Therefore, the involutory homology

$$\varphi:(X,Y,Z)\mapsto(-Y+Z,-X+Z,Z)$$

with center P_0 preserves each cubic in $\mathcal{P}$.

This completes the proof of Proposition 29.

In the case where H is an Abelian normal subgroup of G, we have the following result.

Proposition 30 *Let G be a group containing a proper Abelian normal subgroup H of order $n\geq 5$. If a dual* 3*-net $(\Lambda_1,\Lambda_2,\Lambda_3)$ realizes G such that all its dual* 3*-subnets realizing H as a subgroup of G are algebraic, then either* (I) *or* (II) *of Theorem* 1 *holds.*

Proof The essential tool in the proof is Proposition 29. Assume on the contrary that neither (I) nor (II) occurs.

If every H-member is contained in a line, then every dual 3-net realizing H as a subgroup of G is triangular. By Proposition 23, either (I) of (II) follows.

Take an H-member not contained in a line. Since H is a normal subgroup, that H-member can play the role of Γ_2 in Proposition 29. Therefore, one of the three sporadic cases in Proposition 29 holds. Furthermore, from the proof of that proposition it follows that every $\mathcal{F}_j$ is irreducible, and hence neither Γ_1^j nor Γ_3^j is contained in a line. Therefore, no H-member is contained in a line. Since H is a normal subgroup, every 3-subnet $(\Gamma_1^i,\Gamma_2^j,\Gamma_3^s)$ realizing H as a subgroup of G lies in an irreducible plane cubic $\mathcal{F}(i,j)$.

Therefore, we can assume that all H-members have the exceptional configurations described in (ii), (iii), or (ivc) of Proposition 29. We separately deal with the cases $n=5, 6$, and 9.

$n = 9$ From (iv) of Proposition 29 it follows that the cubics $\mathcal{F}_j$ share their nine inflection points, which form Γ_2. So it is possible to avoid this case by replacing Γ_2 with Γ_1 so that Γ_2 will not have any inflection point of $\mathcal{F}$.

$n = 6$ Every H-member Γ_2 contains three collinear points, say Q_1, Q_2, Q_3, so that Q_r is the center of an involutory homology ψ_r interchanging Λ_1 and Λ_3. Relabeling the points of the dual 3-net permits us to assume that $Q_1 = 1_2$. Then for all $x \in G$, ψ_1 interchanges the points x_1 and x_3. The point $a_2 \in \Lambda_2$ is the intersection of the lines $y_1(ya)_3$, with $y \in G$. These lines are mapped to the lines $(ya)_1 y_3$, which all contain the point $(a^{-1})_2$ of Λ_2. Therefore, the involutory homology ψ_1 leaves Λ_2 invariant. This holds for all involutory homologies with center in $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Since the H-members partition each component of $(\Lambda_1, \Lambda_2, \Lambda_3)$ and every H-member comprises six points, it turns out that half of the points of $(\Lambda_1, \Lambda_2, \Lambda_3)$ are the centers of involutory homologies preserving $(\Lambda_1, \Lambda_2, \Lambda_3)$. Therefore, Proposition 27(iii) applies. As in Proposition 27, let M denote the subgroup of G such that the dual 3-subnet consisting of the centers of involutory homologies realizes M. As $|G : M| = 2$, Proposition 27(iii) implies $|G| < 6$, a contradiction.

$n = 5$ The arguments in discussing case $n = 6$ can be adapted for case $n = 5$. This time, Proposition 29 gives $|G : M| = 5$. By Proposition 27(iii), if G contains an element of order 3, then $|G| < 75$; otherwise, $|G| < 25$. In the former case, the element of order 3 of G is in $C_G(H)$, and hence G contains a cyclic normal subgroup of order 15. Then, $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic by Proposition 29. If G has no element of order 3, then $|G| < 25$, and G contains a normal subgroup of order 10 that is either cyclic or dihedral. By Propositions 24 and 29 either (I) or (II) of Theorem 1 holds. □

The following result is a corollary of Proposition 30.

Theorem 4 *Every dual* 3*-net* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *realizing an Abelian group* G *is algebraic.*

Proof By absurd, let n be the smallest integer for which a counterexample $(\Lambda_1, \Lambda_2, \Lambda_3)$ to Theorem 4 exists. Since any dual 3-net of order ≤ 8 is algebraic by Propositions 15 and 18, we have that $n \ge 9$. Furthermore, again by Proposition 18, G has composite order. Since n is chosen to be as small as possible, by Proposition 30, $|G|$ has only one prime divisor, namely either 2 or 3. Since $|G| \ge 9$, either $|G| = 2^r$ with $r \ge 4$, or $|G| = 3^r$ with $r \ge 2$. In the former case, G has a subgroup M of order 8, and every dual 3-subnet realizing M is algebraic by Proposition 15. But this, together with Proposition 30, shows that $(\Lambda_1, \Lambda_2, \Lambda_3)$ is not a counterexample. In the latter case, G contains no element of order 9, and hence it is an elementary Abelian group. But then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic by Proposition 16. □

9 Dual 3-nets realizing 2-groups

Proposition 31 *Let* G *be a group of order* $n = 2^h$ *with* $h \ge 2$. *If* G *can be realized by a dual* 3*-net* $(\Lambda_1, \Lambda_2, \Lambda_3)$, *then one of the following holds.*

(i) *G is cyclic.*
(ii) *$G \cong C_m \times C_k$ with $n = mk$.*
(iii) *G is a dihedral.*
(iv) *G is the quaternion group of order* 8.

Proof For $n = 4, 8$, the classification follows from Propositions 15 and 22 and from [17, Theorem 4.2]. Up to isomorphisms, there exist fourteen groups of order 16; each has a subgroup H of index 2 that is either an Abelian or a dihedral group. In the latter case, G is itself dihedral, by Proposition 24. So, Proposition 30 applies to G and H, yielding that G is Abelian. This completes the proof for $n = 16$. By induction on h we assume that Proposition 31 holds for $n = 2^h \geq 16$, and we are going to show that this remains true for 2^{h+1}. Let H be a subgroup of G of index 2. Then $|H| = 2^h$, and one of the cases (i), (ii), and (iii) hold for H. Therefore, the assertion follows from Propositions 30 and 24. □

10 Dual 3-nets containing algebraic 3-subnets of order n with $2 \leq n \leq 4$

It is useful to investigate separately two cases according as $n = 3, 4$, or 2. An essential tool in the investigation is $M = C_G(H)$, the centralizer of H in G.

Proposition 32 *Let G be a finite group containing a normal subgroup H of order n with $n = 3$ or $n = 4$. Then every dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizing G is either algebraic or of tetrahedron type, or G is isomorphic either to the quaternion group of order* 8, *or to* Alt_4, *or to* Sym_4.

Proof First, we investigate the case where $M > H$. Take an element $m \in M$ outside H. Then the subgroup T of G generated by m and H is Abelian and larger than H. Since $|H| \geq 3$, we have $|T| \geq 6$. If all H-members of $(\Lambda_1, \Lambda_2, \Lambda_3)$ are contained in a line, then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or of tetrahedron type by Proposition 23. Assume that Γ_2 is an H-member that is not contained in a line. Let Γ_2' be the T-member containing Γ_2. We claim that $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic. If not, then one of the exceptional cases (iii) or (iv) of Proposition 29 must hold. Clearly, in these cases, $|H| = 3$. However, the centers of the involutory homologies mentioned in Proposition 29 correspond to the points in the H-member Γ_2. As these centers must be collinear, we obtain that Γ_2 is contained in a line, a contradiction.

Assume that $M = H$. Then G/H is an automorphism group H. If H is C_3 or C_4, then $|\mathrm{Aut}(H)| = 2$, and G is either a dihedral group or the quaternion group of order 8. If $H \cong C_2 \times C_2$, then G is a subgroup of Sym_4. The possibilities for G other than H and the dihedral group of order 8 are Alt_4 and Sym_4. Since all these groups are allowed in the proposition, the proof is finished. □

Proposition 33 *Let G be a finite group with a central involution that contains no normal subgroup H of order* 4. *Then a dual* 3*-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizing G is either algebraic or of tetrahedron type.*

Proof Let H be the normal subgroup generated by the (unique) central involution of G. Two cases are separately investigated according as a minimal normal subgroup $\bar{N}$ of the factor group $\bar{G} = G/H$ is solvable or not. Let σ be the natural homomorphism $G \to \bar{G}$. Let $N = \sigma^{-1}(\bar{N})$.

If $\bar{N}$ is solvable, then $\bar{N}$ is an elementary Abelian group of order d^h for a prime d. Furthermore, N is a normal subgroup of G and $\bar{N} = N/H$. If N is Abelian, then $|N| \geq 6$, and the assertion follows from Proposition 30 and Theorem 4.

Bearing this in mind, the case where $d = 2$ is investigated first. Then N has order 2^{h+1} and is a normal subgroup of G. By Proposition 31, N is either Abelian, or it is the quaternion group Q_8 of order 8. We may assume that $N \cong Q_8$. By Proposition 31, N is not contained in a larger 2-subgroup of G. Therefore, N is a (normal) Sylow 2-subgroup of G. We may assume that G is larger than N. If $M = C_G(N)$ is also larger than N, take an element $t \in M$ of outside N. Then t has odd order ≥ 3. The group T generated by N and t has order $8m$, and its subgroup D generated by t together with an element of N of order 4 is a (normal) cyclic subgroup of M of order $4m$. But this contradicts Proposition 30 as T is neither Abelian nor dihedral. Therefore, $M = N$, and hence G/N is isomorphic to a subgroup L of the automorphism group $\mathrm{Aut}(Q_8)$. Hence, $|G|/|N|$ divides 24. On the other hand, since N is a Sylow 2-subgroup of G, $|G/N|$ must be odd. Therefore, $|G| = 24$. Two possibilities arise according as either $G \cong SL(2,3)$ or G is the dicyclic group of order 24. The latter case cannot actually occur by Proposition 30 as the dicyclic group of order 24 has a (normal) cyclic subgroup of order 12.

To rule the case $G \cong SL(2,3)$ out, we rely on Propositions 29 and 28 since $SL(2,3)$ has four cyclic groups of order 6. For this purpose, we show that every point in Λ_2 is the center of an involutory homology preserving $(\Lambda_1, \Lambda_2, \Lambda_3)$ whence the assertion will follow from Proposition 25 applied to Λ_2. With the notation in Sect. 7, (iii) Proposition 29 yields that $|\mathcal{U}_2| \geq 3$. With the notation introduced in the proof of Proposition 18, we may assume that the point 1_2 is the center of an involutory homology ϵ in $\mathcal{U}_2$. From (iii) of Proposition 28 it follows that every (cyclic) subgroup of G of order 6 provides (at least) two involutory homologies other than ϵ. Therefore, $|\mathcal{U}_2| \geq 9$, and every point $u_2 \in \Lambda_2$ such that $u^3 = 1$ is the center of an involutory homology in $\mathcal{U}$. A straightforward computation shows that every element in G other than the unique involution e can be written as $gh^{-1}g$ with $g^3 = h^3 = 1$. Thus, $|\mathcal{U}| \geq 23$. The involutory homology with center 1_2 cannot actually be an exception. To show this, take an element $g \in G$ of order 4. Then g_2 is the center of an element in $\mathcal{U}$. Since $1 = g^2 = g \cdot 1 \cdot g$, this holds for 1_2. Therefore, $|\mathcal{U}| = 24$. By (i) of Proposition 28, $\mathcal{U}$ also preserves Λ_2. This completes the proof.

Now, the case of odd d is investigated. Since $|H| = 2$ and d are coprime, Zassenhaus' theorem [9, 10.1 Hauptsatz] ensures a complement $W \cong \bar{N}$ such that $N = W \ltimes H = W \times H$. Obviously, W is an Abelian normal subgroup of G of order at least 3. The assertion follows from Propositions 30 and 32.

If $\bar{N}$ is not solvable, then it has a non-Abelian simple group $\bar{T}$. Let $\bar{S}_2$ be a Sylow 2-subgroup of $\bar{T}$. By Proposition 31, the realizable 2-group S_2 is either cyclic, or product of two cyclic groups, or dihedral, or quaternion of order 8. Thus, $\bar{S}_2$ is either cyclic, or product of two cyclic groups, or dihedral. As $\bar{T}$ is simple, $\bar{S}_2$ cannot be cyclic. In the remaining cases we can use the classification of finite simple groups of

2-rank 2 to deduce that either $\bar{T} \cong PSL(2, q^h)$ with an odd prime q and $q^h \geq 5$, or $\bar{T} \cong \mathrm{Alt}_7$; cf. the Gorenstein–Walter theorem [6].

If $H \not\leq T'$, then $T = H \times T'$. As $T' \cong \bar{T}$, T' contains an elementary Abelian subgroup of order 4, and G contains an elementary Abelian group of order 8, a contradiction. Therefore, T is a central extension of either $PSL(2, q^h)$ with q^h as before, or Alt_7 with a cyclic group of order 2. From a classical result of Schur [1, Chap. 33], either $T \cong SL(2, q)$, or T is the unique central extension of Alt_7 with a cyclic group of order 2. In the latter case, no dual 3-net can actually realize T since Proposition 31 applies, a Sylow 2-subgroup of T being isomorphic to a generalized quaternion group of order 16. To finish the proof, it suffices to observe that $SL(2, q^h)$, with q^h as before, contains $SL(2, 3)$, whereas no dual 3-net can realize $SL(2, 3)$ as we have already pointed out. □

11 3-Nets and non-Abelian simple groups

Proposition 34 *If a dual* 3*-net realizes a non-Abelian simple group* G, *then* $G \cong \mathrm{Alt}_5$.

Proof Let G be a non-Abelian simple group, and consider a Sylow 2-subgroup S_2 of G. By Proposition 31, S_2 is dihedral since no Sylow 2-subgroup of a non-Abelian simple group is either cyclic or the direct product of cyclic groups, see [5, Theorem 2.168], or the quaternion group of order 8, see [3]. From the Gorenstein–Walter theorem [6], either $G \cong PSL(2, q^h)$ with an odd prime q and $q^h \geq 5$, or $G \cong \mathrm{Alt}_7$. In the former case, G has a subgroup T of order $q^h(q^h - 1)/2$ containing a normal subgroup of order q^h. Here T is not Abelian and is dihedral only for $q^h = 5$. Therefore, Theorem 4 and Proposition 30 leave only one case, namely $q = 5$. This also shows that Alt_7 cannot occur since Alt_7 contains $PSL(2, 7)$. □

Notice that, by Proposition 17, computer results show that if $p = 0$, then Alt_4 cannot be realized in $PG(2, \mathbb{K})$. This implies that no dual 3-net can realize Alt_5.

12 The proof of Theorem 1

Take a minimal normal subgroup H of G. If H is not solvable, then H is either a simple group or the product of isomorphic simple groups. By Proposition 20, the latter case cannot actually occur as every simple group contains an elementary Abelian subgroup of order 4. Therefore, if H is not solvable, $H \cong \mathrm{Alt}_5$ may be assumed by Proposition 34. Two cases are considered separately according as the centralizer $C_G(H)$ of H in G is trivial or not. If $|C_G(H)| > 1$, take a nontrivial element $u \in C_G(H)$ and define U to be the subgroup of G generated by u together with a dihedral subgroup D_5 of H of order 10. Since u centralizes D_5, the latter subgroup is a normal subgroup of U. Hence, D_5 is a normal dihedral subgroup of U. By Proposition 24, M itself must be dihedral. Since the center of a dihedral group has order 2, this implies that u is an involution. Now, the subgroup generated by u, together with an elementary Abelian subgroup of H of order 4, generates an elementary Abelian subgroup of order 8. But this contradicts Proposition 20. Therefore, $C_G(H)$ is trivial, and, equivalently, G is contained in the automorphism group of H. From this it follows that either $G = H$ or $G \cong PGL(2,5)$. In the latter case, G contains a subgroup isomorphic to the semidirect product of C_5 by C_4. But this contradicts Proposition 24. Hence, if H is not solvable, then $H \cong \mathrm{Alt}_5$.

If H is solvable, then it is an elementary Abelian group of order $d \geq 2$. If d is a power of 2, then $d = 2$ or $d = 4$, and Theorem 1 follows from Propositions 31 and 33. If d is a power of an odd prime, Theorem 1 is obtained by Propositions 30 and 32.

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