

## 3-Quasi-Sasakian manifolds

Beniamino Cappelletti Montano · Antonio De Nicola ·  
Giulia Dileo

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**Abstract** We correct the results in section 6 of [B. Cappelletti Montano, A. De Nicola, G. Dileo, 3-Quasi-Sasakian manifolds, Ann. Global Anal. Geom. 33 (2008), 397–409], concerning the corrected energy of the Reeb distribution of a compact 3-quasi-Sasakian manifold. The results are slightly different than what was originally claimed and they are obtained by using results in [B. Cappelletti Montano, A. De Nicola, G. Dileo, The geometry of a 3-quasi-Sasakian manifold, Int. J. Math., to appear, arXiv:0801.1818], where the geometry of these manifolds is more deeply investigated.

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### Introduction

In [2] the authors study the geometry of 3-quasi-Sasakian manifolds, which include as special cases 3-Sasakian and 3-cosymplectic manifolds. A 3-quasi-Sasakian manifold is an almost 3-contact metric manifold  $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$  such that each almost contact metric structure  $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$  is quasi-Sasakian. It is proven in [2] that the distribution generated by the Reeb vector fields  $\xi_1, \xi_2, \xi_3$  is integrable, defining a canonical totally geodesic and Rie-

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B. Cappelletti Montano · G. Dileo  
Dipartimento di Matematica, Università degli Studi di Bari, Via E. Orabona 4, 70125 Bari, Italy  
e-mail: cappelletti@dm.uniba.it

G. Dileo  
e-mail: dileo@dm.uniba.it

A. De Nicola (✉)  
Departamento de Matemática Fundamental, Universidad de La Laguna, Av. Astrofísico F.co Sánchez,  
s/n, 38206 La Laguna, Tenerife, Islas Canarias, Spain  
e-mail: antondenicola@gmail.com

mannian foliation of  $M^{4n+3}$ . The characteristic vector fields obey the commutation relations  $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$  for any even permutation  $(\alpha, \beta, \gamma)$  of  $\{1, 2, 3\}$  and some  $c \in \mathbb{R}$ . Furthermore, the ranks of the 1-forms  $\eta_1, \eta_2, \eta_3$  coincide, so that 3-quasi-Sasakian manifolds are classified according to their well-defined rank, which is of the form  $4l + 1$  in the Abelian case ( $c = 0$ ), and  $4l + 3$  in the non-Abelian one,  $0 \leq l \leq n$ . As a single application, we compute in [2] the corrected energy of the canonical foliation of a compact 3-quasi-Sasakian manifold, in the attempt to generalize a result of Blair and Turgut Vanli concerning 3-Sasakian manifolds ([1]).

The corrected energy  $\mathcal{D}(\mathcal{V})$  of a  $p$ -dimensional distribution of a compact Riemannian manifold was defined by Chacón and Naveira in [4]. They also proved that the Reeb distribution of the natural 3-Sasakian structure on the sphere  $S^{4n+3}$  is a minimum of the corrected energy in the set of all integrable 3-dimensional distributions. In [1] Blair and Turgut Vanli tried to extend this result to the Reeb distribution of an arbitrary compact 3-Sasakian manifold. Unfortunately, as it is remarked by Perrone in [5], their demonstration does not prove the minimality of the corrected energy. As for the corrected energy of the Reeb distribution in a 3-quasi-Sasakian manifold, our demonstration of minimality in [2] contains the same gap as in [1].

In this erratum, we distinguish between 3-quasi-Sasakian manifolds of rank  $4l + 1$  and those of rank  $4l + 3$ . We use the results contained in [3], where the geometry of these manifolds is more deeply investigated. Indeed, a 3-quasi-Sasakian manifold of rank  $4l + 1$  turns out to be a 3-cosymplectic manifold and in this case, supposing the manifold to be compact, the corrected energy of the Reeb distribution vanishes. As regards compact 3-quasi-Sasakian manifolds of rank  $4l + 3$ , we prove that the Reeb distribution represents a minimum for the corrected energy among a suitable subset of all integrable 3-dimensional distributions.

### Corrected energy of 3-quasi Sasakian manifolds

The corrected energy  $\mathcal{D}(\mathcal{V})$  of a  $p$ -dimensional distribution  $\mathcal{V}$  on a compact oriented Riemannian manifold  $(M^m, g)$  is defined as (cf. [4])

$$\mathcal{D}(\mathcal{V}) = \int_M \left( \sum_{a=1}^m \|\nabla_{e_a} \xi\|^2 + q(q-2) \|\vec{H}_{\mathcal{H}}\|^2 + p^2 \|\vec{H}_{\mathcal{V}}\|^2 \right) d\text{vol},$$

where  $\{e_1, \dots, e_m\}$  is a local orthonormal adapted frame with  $e_1, \dots, e_p \in \mathcal{V}_x$  and  $e_{p+1}, \dots, e_{m-p+q} \in \mathcal{H}_x = \mathcal{V}_x^\perp$ , and  $\xi = e_1 \wedge \dots \wedge e_p$  is a  $p$ -vector which determines the distribution  $\mathcal{V}$  regarded as a section of the Grassmann bundle  $G(p, M^m)$  of oriented  $p$ -planes in the tangent spaces of  $M^m$ . Finally  $\vec{H}_{\mathcal{H}}$  and  $\vec{H}_{\mathcal{V}}$  are the mean curvatures of the distributions  $\mathcal{H}$  and  $\mathcal{V}$  (see [4] and [2] for the details). It is proven in [4] that if  $\mathcal{V}$  is integrable then

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha} d\text{vol}, \quad (1)$$

where  $c_{i\alpha} = K(e_i, e_\alpha)$  is the sectional curvature of the plane spanned by  $e_i \in \mathcal{H}$  and  $e_\alpha \in \mathcal{V}$ . Moreover, the equality in (1) holds if and only if  $\mathcal{V}$  is totally geodesic and  $e_1, \dots, e_p$  are  $\mathcal{H}$ -conformal, that is  $(\mathcal{L}_{e_i} g)(X, Y) = f_i g(X, Y)$ , for any  $X, Y \in \mathcal{H}$  and  $i \in \{1, \dots, p\}$ , where  $\mathcal{L}_{e_i}$  denotes the Lie derivative and  $f_i$  is a function on  $M$ .

Now, let  $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$  be a compact 3-quasi Sasakian manifold and let  $\xi$  denote the Reeb distribution determined by the 3-vector  $\xi_1 \wedge \xi_2 \wedge \xi_3$ . It is proven in [2] that the corrected energy of  $\xi$  is given by

$$\mathcal{D}(\xi) = \int_M \left( \sum_{\alpha=1}^3 \|\nabla \xi_\alpha\|^2 - \frac{3}{2}c^2 \right) d\text{vol}. \quad (2)$$

If  $M^{4n+3}$  is a 3-quasi-Sasakian manifold of rank  $4l + 1$ , then it is necessarily a 3-cosymplectic manifold (see [3]). Therefore,  $\nabla \xi_\alpha = 0$  and  $c = 0$ . Using (2), it follows that the corrected energy  $\mathcal{D}(\xi)$  vanishes.

Now, let us consider a 3-quasi-Sasakian manifold  $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$  of rank  $4l + 3$ , with  $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ ,  $c \neq 0$ . It is proven in [3] that  $M^{4n+3}$  is locally the Riemannian product of a 3- $\alpha$ -Sasakian manifold  $M^{4l+3}$ , where  $\alpha = \frac{c}{2}$ , and a hyper-Kähler manifold  $M^{4m}$ , with  $m = n - l$ . In particular,  $M^{4m}$  is a leaf of the distribution  $\mathcal{E}^{4m} := \{X \in TM \mid \text{for any } \alpha \in \{1, 2, 3\} \ i_X \eta_\alpha = 0 \text{ and } i_X d\eta_\alpha = 0\}$ , while  $M^{4l+3}$  is a leaf of the orthogonal distribution  $\mathcal{E}^{4l+3}$ . Moreover, the Ricci tensor of  $M^{4n+3}$  is given by

$$\text{Ric}(X, Y) = \begin{cases} \frac{c^2}{2}(2l+1)g(X, Y), & \text{if } X, Y \in \Gamma(\mathcal{E}^{4l+3}); \\ 0, & \text{elsewhere.} \end{cases} \quad (3)$$

We can prove the following.

**Theorem 1** *Let  $M^{4n+3}$  be a compact 3-quasi-Sasakian manifold of rank  $4l+3$ . Then, among the integrable 3-dimensional distributions  $\mathcal{V}$  of  $M^{4n+3}$  such that  $\mathcal{V} \subset \mathcal{E}^{4l+3}$  and  $K(\mathcal{V}) \leq \frac{3}{4}c^2$ , the Reeb distribution  $\xi$  minimizes the corrected energy  $\mathcal{D}(\mathcal{V})$ , where  $K(\mathcal{V}) := K(e_1, e_2) + K(e_1, e_3) + K(e_2, e_3)$  is the curvature of the distribution  $\mathcal{V}$ . Moreover  $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\xi)$  if and only if  $K(\mathcal{V}) = \frac{3}{4}c^2$ ,  $\mathcal{V}$  is totally geodesic and  $e_1, e_2, e_3$  are  $\mathcal{H}$ -conformal.*

*Proof* We compute the corrected energy  $\mathcal{D}(\xi)$  of the canonical distribution given by (2). Since for a quasi-Sasakian structure  $\|\nabla \xi_\alpha\|^2 = \text{Ric}(\xi_\alpha, \xi_\alpha)$ , applying (3), we have  $\mathcal{D}(\xi) = 3c^2 l \text{vol}(M^{4n+3})$ . Now, let  $\mathcal{V}$  be a 3-dimensional integrable distribution such that  $\mathcal{V} \subset \mathcal{E}^{4l+3}$  and  $K(\mathcal{V}) \leq \frac{3}{4}c^2$ . We prove that  $\mathcal{D}(\mathcal{V}) \geq \mathcal{D}(\xi)$ . Let  $\{e_1, \dots, e_{4n+3}\}$  be a local orthonormal adapted frame with  $e_1, e_2, e_3 \in \mathcal{V}$  and  $e_4, \dots, e_{4n+3} \in \mathcal{H} = \mathcal{V}^\perp$ . Using (3) again, we get

$$\begin{aligned} \sum_{\alpha=1}^3 \sum_{i=1}^{4n} K(e_i, e_\alpha) &= \sum_{\alpha=1}^3 \sum_{i=1}^{4n+3} K(e_i, e_\alpha) - \sum_{\alpha, \beta=1}^3 K(e_\alpha, e_\beta) \\ &= \sum_{\alpha=1}^3 \text{Ric}(e_\alpha, e_\alpha) - 2K(\mathcal{V}) \\ &= \frac{3}{2}c^2(2l+1) - 2K(\mathcal{V}). \end{aligned} \quad (4)$$

Arguing as in [5],  $K(\mathcal{V})$  depends only on the distribution, in the sense that it is invariant under adapted orthonormal frame changes. Moreover, supposing  $K(\mathcal{V}) \leq \frac{3}{4}c^2$  and applying (1), we have

$$\mathcal{D}(\mathcal{V}) \geq 3c^2 l \text{vol}(M^{4n+3}) = \mathcal{D}(\xi),$$

and the equality holds if and only if  $K(\mathcal{V}) = \frac{3}{4}c^2$ ,  $\mathcal{V}$  is totally geodesic and  $e_1, e_2, e_3$  are  $\mathcal{H}$ -conformal.  $\square$

In the above theorem, if  $l < n$ , since the distribution  $\mathcal{E}^{4l+3}$  defines a Riemannian foliation, then  $(\mathcal{L}_{e_i} g)(X, Y) = 0$  for any  $i \in \{1, 2, 3\}$  and  $X, Y \in \mathcal{E}^{4m}$ . Therefore,  $e_1, e_2, e_3$

are  $\mathcal{H}$ -conformal if and only if the distribution  $\mathcal{V}$  defines a Riemannian foliation. As for 3-quasi-Sasakian manifolds of maximal rank  $4n+3$ , they are necessarily 3- $\alpha$ -Sasakian manifolds, with  $\alpha = \frac{c}{2}$  (see [3, Corollary 4.4]). Hence, we obtain the following.

**Corollary 2** *Let  $M^{4n+3}$  be a compact 3- $\alpha$ -Sasakian manifold. Then, among the integrable 3-dimensional distributions  $\mathcal{V}$  of  $M^{4n+3}$  with curvature  $K(\mathcal{V}) \leq 3\alpha^2$ , the Reeb distribution  $\xi$  minimizes the corrected energy  $\mathcal{D}(\mathcal{V})$ . Moreover  $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\xi)$  if and only if  $\mathcal{V}$  is totally geodesic,  $e_1, e_2, e_3$  are  $\mathcal{H}$ -conformal and  $K(\mathcal{V}) = 3\alpha^2$ .*

The sphere  $S^{4n+3}(r)$  of radius  $r$  can be canonically endowed with a 3- $\alpha$ -Sasakian structure  $(\phi_\delta, \xi_\delta, \eta_\delta, g)$  with  $\alpha = \frac{1}{r}$  ([3]). Since for any 3-dimensional distribution  $\mathcal{V}$ ,  $K(\mathcal{V}) = 3\alpha^2$ , then the Reeb distribution  $\xi$  minimizes the corrected energy among the integrable 3-dimensional distributions of  $S^{4n+3}(r)$ .

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