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by

Irene Fonseca and Gilles Francfort

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3D-2D ASYMPTOTIC ANALYSIS OF AN OPTIMAL DESIGN PROBLEM FOR THIN FILMS

Irene Fonseca † & Gilles Francfort ‡

Abstract The Gamma-limit of a rescaled version of an optimal material distribution problem for a cylindrical two-phase elastic mixture in a thin three-dimensional domain is explicitly computed. Its limit is a two-dimensional optimal design problem on the cross-section of the thin domain; it involves optimal energy bounds on two-dimensional mixtures of a related two-phase bulk material. Thus, it is shown in essence that 3D-2D asymptotics and optimal design commute from a variational standpoint.

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§1. Introduction

Thin film technology has significantly improved in the past few years. The increasing diversity of available manufacturing methods provides for a better control of the thickness of the thin film and renders more realistic the idealized view of a film as a thin plate. In such a context the transformation of bulk properties as the material domain is scaled thinner and thinner becomes a relevant issue in thin film behavior. In more mathematical terms, a 3D-2D asymptotic analysis of the thermomechanical and electromagnetic bulk properties of a flat domain as the thickness of the domain tends to 0 must be performed.

As far as mechanical properties are concerned this type of analysis has been the object of many investigations in the past twenty years, although mostly in a linear or semilinear context (see [C], [CD]; see also [MS1], [MS2]); only recently has some progress been made in a truly nonlinear setting (see [FRS], [LDR], [BJ]; see also [ABaP], [ABuP], [G]). Fully nonlinear elasticity is the assumed behavior throughout the present study, although in this work, as well as in the previously quoted papers, the energy densities considered are not totally in agreement with the principles of nonlinear elasticity because they violate the physical requirement that the lattice energy density blow up as the determinant of the deformation strain tends to 0^+. To our knowledge the only relevant work that respects such a requirement is to be found in [B]; a similar analysis in our context seems rather hopeless at this time.

Two somewhat related issues provide the inspiration for the present study. On the one hand, the improved technological ability to pattern thin film substrates permits to create regions of specific shape and stiffness within the film. The goal is then the optimization of the mechanical film performance through an optimal distribution of the stiffness of the film. This is a problem of optimal design, which has thus to be coupled with a 3D-2D asymptotic analysis. On the other hand, elastic materials are prone to defects that adversely affect their elastic stiffness; those defects end to grow with the applied loads. The goal is then the prediction of the evolution of the damaged areas, which will also be affected by the scaling in the thickness.

In more specific terms, two settings are being considered. In both settings, $\Omega(\varepsilon)$ is a thin three-dimensional domain of the form

$$\Omega(\varepsilon) = \omega \times (-\varepsilon, \varepsilon),$$

† Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
‡ LPMTM, Institut Galilée, Université Paris Nord, Ave. J.B. Clement, 93430 Villetaneuse, France

with ω a bounded, open domain in $\mathbb{R}^2$, $\varepsilon > 0$, and, for the sake of illustration, $\Omega(\varepsilon)$ is assumed to be clamped on its lateral boundary $\partial\omega \times (-\varepsilon, \varepsilon)$. The thin domain $\Omega(\varepsilon)$ is filled with two elastic materials with respective energy densities W_1 and W_2, where W_i $(i = 1, 2)$ is a continuous real-valued function on $\mathbb{R}^{3\times 3}$. Denote by $\chi(\cdot)$ the characteristic function of the first phase so that at any point $x \in \Omega(\varepsilon)$ the elastic energy is $\chi(x)W_1(\cdot) + (1 - \chi(x))W_2(\cdot)$.

The first setting is that of the *optimal design* of a two-phase mixture of elastic materials in a thin film. Optimal design of two-phase mixtures is a contemporary topic which originates in the works of F. Murat and L. Tartar (see [M], [MT]); similar ideas may be found in [L]. It has since been the subject of numerous papers (e.g. see [GC],[KS], [BK], [AK], [ABFJ]).

Assume that the load f on $\Omega(\varepsilon)$ as well as the volume fraction of each phase,

$$\lambda := \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi(x)\, dx \quad \in [0, 1],$$

are given. Here, and in what follows, $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$. For a fixed thickness ε the *compliance* $c^\varepsilon(\chi)$ is defined as

(1.1)

$$c^\varepsilon(\chi) := -\inf_u \left\{ \frac{1}{\varepsilon} \left[\int_{\Omega(\varepsilon)} (\chi W_1 + (1-\chi)W_2)(Du)\, dx - \int_{\Omega(\varepsilon)} f \cdot u\, dx \right] : u = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon) \right\}.$$

Remark 1.1. The definition (1.1) of the compliance coincides with the classical definition of compliance in the linearized setting, i.e. when W_1 and W_2 are quadratic and the gradient of the transformation Du is replaced by the symmetrized gradient $e(u) := 1/2(Du + Du^T)$. In this case $-2c^\varepsilon(\chi)$ is precisely the work done by the load f (see e.g. [ABFJ]).

The best design will be that which minimizes the compliance, and the problem becomes

$$-\sup_\chi \left\{ -c^\varepsilon(\chi) : \chi \in L^\infty(\Omega(\varepsilon); \{0, 1\}),\ \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi(x)\, dx = \lambda \right\}. \tag{1.2}$$

As a two-field minimization problem (1.1)–(1.2) reduces to

$$\mathrm{I}^\varepsilon_{\text{best}} := -\sup_\chi \inf_u \left\{ \frac{1}{\varepsilon} \left[\int_{\Omega(\varepsilon)} (\chi W_1 + (1-\chi)W_2)(Du)\, dx - \int_{\Omega(\varepsilon)} f \cdot u\, dx \right] : u = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon), \right.$$
$$\left. \chi \in L^\infty(\Omega(\varepsilon); \{0, 1\}),\ \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi(x)\, dx = \lambda \right\}.$$

Our goal is then to compute the limit of $\mathrm{I}^\varepsilon_{\text{best}}$ as ε tends to 0^+. We are not, as of yet, in a position to solve this problem. The results described in Section 2 will only permit the computation of the limit as ε tends to 0^+ of

(1.3)

$$\mathrm{I}^\varepsilon_{\text{worst}} := \inf_{\chi, u} \left\{ \frac{1}{\varepsilon} \left[\int_{\Omega(\varepsilon)} (\chi W_1 + (1-\chi)W_2)(Du)\, dx - \int_{\Omega(\varepsilon)} f \cdot u\, dx \right] : u = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon), \right.$$
$$\left. \chi \in L^\infty(\Omega(\varepsilon); \{0, 1\}),\ \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi(x)\, dx = \lambda \right\},$$

that is of finding the worst possible design! Actually, our result is even more restrictive because the designs may not be arbitrary: only cylindrical domains can be considered, as $\chi = \chi(x_\alpha)$ will not depend upon the transverse variable x_3 in (1.3).

The second setting is that of *brutal damage evolution* of a thin film. We follow there the model proposed in [FMa] (see also [AB]). The material is assumed to brutally lower its elastic energy at a point x from W_2 to W_1 (with $W_1 \le W_2$) whenever the deformation strain $Du(x)$ at x satisfies

$$W_2(Du(x)) - W_1(Du(x)) > \kappa,$$

where κ is a *critical energy release rate*. Then, for a fixed thickness $\varepsilon > 0$, and for a given load $f(x)$ on $\Omega(\varepsilon)$, the problem is shown to be equivalent to determining

$$\kappa^\varepsilon := \inf_{\chi,u} \left\{ \frac{1}{\varepsilon} \left[\int_{\Omega(\varepsilon)} (\chi(x_\alpha)W_1 + (1-\chi(x_\alpha))W_2)(Du)\,dx + \kappa \int_{\Omega(\varepsilon)} \chi(x_\alpha)\,dx - \int_{\Omega(\varepsilon)} f\cdot u\,dx \right] : \right.$$
$$\left. u = 0 \text{ on } \partial\omega \times (-\varepsilon,\varepsilon), \chi \in L^\infty(\Omega(\varepsilon);\{0,1\}) \right\}.$$

Specifically, we assume that the energy densities W_i $(i = 1,2)$ are such that

$$\alpha|F|^p \le W_i(F) \le \beta(1+|F|^p), \quad F \in \mathbb{R}^{3\times 3},$$

where $\alpha, \beta > 0$, $1 < p < +\infty$. Further, we define the two-dimensional energy densities

$$\overline{W}_i(\overline{F}) := \inf_{z\in\mathbb{R}^3} W_i(\overline{F}\,|\,z),\ \overline{F} \in \mathbb{R}^{3\times 2}$$

where $(\overline{F}\,|\,z)$ denotes the 3×3 matrix with first two columns those of $\overline{F}$ and last column the vector z. The results described in detail in Section 2 below (cf. in particular Theorems 2.3, 2.4, Remarks 2.6, 2.8) and a simple rescaling argument – setting $y_3 := x_3/\varepsilon$ which transforms $\Omega(\varepsilon)$ into $\omega \times (-1,1)$ – would immediately imply that

$$\lim_{\varepsilon\to 0^+} \mathrm{I}^\varepsilon_{\text{worst}} = \min_{\theta,u} \left\{ 2\int_\omega \overline{W}^\star(\theta(x_\alpha), Du(x_\alpha))\,dx_\alpha - \int_\omega F(x_\alpha)\cdot u(x_\alpha)\,dx_\alpha \,:\, u \in W_0^{1,p}(\omega;\mathbb{R}^3), \right.$$
$$\left. \theta \in L^\infty(\omega;[0,1]),\ \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \theta(x_\alpha)\,dx_\alpha = \lambda \right\}, \tag{1.5}$$

and

$$\lim_{\varepsilon\to 0^+} \kappa^\varepsilon = \min_{\theta,u} \left\{ 2\int_\omega \overline{W}^\star(\theta(x_\alpha), Du(x_\alpha))\,dx_\alpha + 2\kappa\int_\omega \theta(x_\alpha)\,dx_\alpha - \int_\omega F(x_\alpha)\cdot u(x_\alpha)\,dx_\alpha : \right.$$
$$\left. u \in W_0^{1,p}(\omega;\mathbb{R}^3), \theta \in L^\infty(\omega;[0,1]) \right\}. \tag{1.6}$$

In (1.5), (1.6), $F(x_\alpha) := \int_{-1}^1 f(x_\alpha, x_3)\,dx_3$ while $\overline{W}^\star$ is defined in (2.2) from the $\overline{W}_i$'s.

Roughly speaking, the elastic energy density of the thin film in both settings is that obtained by the lower energy bound on the effective behavior of the compatible two-phase mixtures of the energies $\overline{W}_i$ (cf. Remark 2.2).

We conclude this introduction with a few words regarding notation. Here, and in what follows, x_α designates the pair of variables x_1, x_2, and x'_α denotes the same pair when it varies over the unit square Q' of $\mathbb{R}^2$. D_α will be identified with the pair D_1, D_2, $|x_\alpha| := \sqrt{x_1^2 + x_2^2}$, and dx_α (resp. dx'_α) will stand for $dx_1 dx_2$ (resp. $dx'_1 dx'_2$). Also, we will pay great attention to the order in which limits are being taken. Specifically, $\lim_{m,n\to+\infty}$ means $\lim_{m\to+\infty}\lim_{n\to+\infty}$ while $\lim_{n,m\to+\infty}$ means $\lim_{n\to+\infty}\lim_{m\to+\infty}$, with obvious generalizations to a higher number of limits. Finally,

$\to$ will always denote strong convergence, whereas $\rightharpoonup$ (resp. $\overset{*}{\rightharpoonup}$) will denote weak (resp. weak-*) convergence.

The following section describes the obtained results and gives an overview of the remainder of the paper.

§2. Statement of the Main Results

We assume that $W_i(F)$, $i = 1, 2$, is a continuous real-valued function on $\mathbb{R}^{3\times 3}$ such that

$$\alpha |F|^p \leq W_i(F) \leq \beta(1 + |F|^p), \quad F \in \mathbb{R}^{3\times 3}, \tag{2.1}$$

where $\alpha, \beta > 0$ and $1 < p < +\infty$.

For $i = 1, 2$, we define

$$\overline{W}_i(\overline{F}) := \inf_{z\in\mathbb{R}^3} W_i(\overline{F}\,|\,z), \quad \overline{F} \in \mathbb{R}^{3\times 2}.$$

Remark 2.1. It is proved in Proposition 1 of [LDR] that $\overline{W}_i$ is continuous and satisfies (2.1).

We further define, for any characteristic function χ on the open unit square Q' of $\mathbb{R}^2$ and any $\theta \in [0, 1]$,

$$\overline{W}_\chi(x'_\alpha, \overline{F}) := \chi(x'_\alpha)\overline{W}_1(\overline{F}) + (1 - \chi(x'_\alpha))\overline{W}_2(\overline{F}),$$

$$\overline{W}^\star_\chi(\overline{F}) := \inf_\varphi \left\{ \int_{Q'} \overline{W}_\chi(x'_\alpha, \overline{F} + D\varphi)\, dx'_\alpha : \varphi \in W^{1,p}_0(Q'; \mathbb{R}^3) \right\},$$

$$\overline{W}^\star(\theta, \overline{F}) := \inf_\chi \left\{ \overline{W}^\star_\chi(\overline{F}) : \chi \in L^\infty(Q'; \{0, 1\}), \int_{Q'} \chi(x'_\alpha)\, dx'_\alpha = \theta \right\}. \tag{2.2}$$

Remark 2.2. This remark asserts that $\overline{W}^\star(\theta, \overline{F})$ may be seen as an optimal energy bound on the periodic mixtures at fixed volume fraction of W_1 and W_2. Indeed, in the spirit of e.g. Lemma 2.1 in [K], it can be easily shown that $\overline{W}^\star(\theta, \overline{F})$ has the alternative characterization

$$\overline{W}^\star(\theta, \overline{F}) := \inf_\chi \left\{ \overline{W}^0_\chi(\overline{F}) : \chi \in L^\infty(Q'; \{0, 1\}), \int_{Q'} \chi(x'_\alpha)\, dx'_\alpha = \theta \right\},$$

where

$$\overline{W}^0_\chi(\overline{F}) := \inf_\varphi \left\{ \int_{Q'} \overline{W}_\chi(x'_\alpha, \overline{F} + D\varphi)\, dx'_\alpha : \varphi \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{per}}(Q'; \mathbb{R}^3) \right\}, \tag{2.3}$$

and $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{per}}(Q'; \mathbb{R}^3)$ is the set of all Q'-periodic elements of $\mathrm{W}^{1,\mathrm{p}}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^3)$. If W_1 and W_2 are convex then (2.3) may be identified with the energy density associated to the Γ-limit of the following functional defined on $W^{1,p}(\omega; \mathbb{R}^3)$:

$$I^n_\chi(u) := \int_\omega \overline{W}_\chi(nx_\alpha, Du)\, dx_\alpha,$$

where for every $\overline{F} \in \mathbb{R}^{3\times 2}$ the function $\overline{W}_\chi(\cdot, \overline{F})$ has been Q'-periodically extended to all of $\mathbb{R}^2$ (cf. e.g. [Mar]). If not, the Γ-limit of I^n_χ admits as energy density

$$\overline{W}^\#_\chi(\overline{F}) := \inf_{k\in\mathbb{N}} \inf_\varphi \left\{ \frac{1}{k^2} \int_{kQ'} \overline{W}_\chi(x'_\alpha, \overline{F} + D\varphi)\, dx'_\alpha : \varphi \in \mathrm{W}^{1,\mathrm{p}}_{\mathrm{per}}(kQ'; \mathbb{R}^3) \right\},$$

as demonstrated in [Mu]. Although in general

$$\overline{W}^{\#}_{\chi}(\overline{F}) < \overline{W}^{0}_{\chi}(\overline{F}),$$

as in Remark 3.8 of [FF] it may be seen that

$$\overline{W}^{\star}(\theta, \overline{F}) = \inf_{\chi} \left\{ \overline{W}^{\#}_{\chi}(\overline{F}) : \chi \in L^{\infty}(Q'; \{0,1\}), \int_{Q'} \chi(x'_\alpha)\, dx'_\alpha = \theta \right\}.$$

Let ω be a bounded open set in $\mathbb{R}^2$, and for a subset A of ω, $\theta \in L^{\infty}(\omega; [0,1])$, $v \in W^{1,p}(\omega; \mathbb{R}^3)$, $\theta_0, \lambda \in [0,1]$, define

(2.4)

$$\begin{aligned} J(v;\theta;A) := \inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \Bigg\{ & \lim_{\varepsilon\to 0+} \int_{A\times(-1,1)} (\chi_\varepsilon(x_\alpha)W_1 + (1-\chi_\varepsilon(x_\alpha))W_2) \left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon \right) dx_\alpha dx_3 : \\ & v_\varepsilon \in W^{1,p}(\omega\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \in L^{\infty}(A;\{0,1\}), \\ & v_\varepsilon \to v \quad \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^{\infty}(A;[0,1]) \Bigg\}, \end{aligned}$$

$$\begin{aligned} G_\lambda(v;A) := \inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \Bigg\{ & \lim_{\varepsilon\to 0+} \int_{A\times(-1,1)} (\chi_\varepsilon(x_\alpha)W_1 + (1-\chi_\varepsilon(x_\alpha))W_2) \left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon} D_3 v_\varepsilon \right) dx_\alpha dx_3 : \\ & v_\varepsilon \in W^{1,p}(\omega\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \in L^{\infty}(A;\{0,1\}), \\ & v_\varepsilon \to v \quad \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \frac{1}{\mathcal{L}^2(A)} \int_A \chi_\varepsilon(x_\alpha)\, dx_\alpha = \lambda \Bigg\}. \end{aligned}$$

The following theorems are the main results of the present study:

Theorem 2.3.

$$J(v;\theta;A) = 2 \int_A \overline{W}^{\star}(\theta(x_\alpha), Dv(x_\alpha))\, dx_\alpha.$$

Theorem 2.4.

$$G_\lambda(v;A) = \inf_{\theta} \left\{ J(v;\theta;A) : \frac{1}{\mathcal{L}^2(A)} \int_A \theta(x_\alpha)\, dx_\alpha = \lambda \right\}.$$

Several remarks are timely at this point.

Remark 2.5. It is not a priori obvious that the integral on the right-hand side of the equality in Theorem 2.3 is meaningful. Remark 3.2 will assert the integrability of $x_\alpha \mapsto \overline{W}^{\star}(\theta(x_\alpha), Dv(x_\alpha))$.

Remark 2.6. The statements of both theorems could be rephrased in terms of the original sequence of domains $\Omega(\varepsilon)$ and rescaled energy densities (see Section 1).

Remark 2.7. The coercivity hypothesis (i.e., the first inequality in (2.1)) may be removed in Theorems 2.3, 2.4, as demonstrated below. Of course the significance of any of those theorems is debatable in such an enlarged context because, in the absence of coercivity, sequences $\{\overline{u}_\varepsilon, \overline{\chi}_\varepsilon\}$ of approximate minimizers of

$$(\chi, v) \mapsto \int_{\omega\times(-1,1)} (\chi(x_\alpha)W_1 + (1-\chi(x_\alpha))W_2) \left(D_\alpha v \Big| \frac{1}{\varepsilon} D_3 v \right) dx_\alpha - L(v),$$

where L is any bounded linear map on $W^{1,p}(\omega;\mathbb{R}^3)$, might not be such that $||\overline{u}_\varepsilon||_{W^{1,p}(\omega\times(-1,1))}$ remain bounded, so that Theorems 2.3, 2.4, become irrelevant.

In any case, if $\alpha = 0$ in (2.1), define

$$W_i^\eta(F) := W_i(F) + \eta|F|^p, \ \eta > 0.$$

Then, according to Theorem 2.3, $J^\eta(v;\theta;A)$ defined as in (2.4) with W_i^η in lieu of W_i satisfies

$$J^\eta(v;\theta;A) = 2\int_A \overline{W^\eta}^\star(\theta(x_\alpha), Dv(x_\alpha))\,dx_\alpha,$$

with obvious notation. Note that, by virtue of the coercivity of W_i^η,

$$J^\eta(v;\theta;A) = \tilde{J}^\eta(v;\theta;A), \tag{2.5}$$

where

$$\tilde{J}^\eta(v;\theta;A) := \inf_{\{\chi_\varepsilon\},\{v_\varepsilon\}} \left\{ \lim_{\varepsilon\to 0^+} \int_{A\times(-1,1)} (\chi_\varepsilon(x_\alpha)W_1^\eta + (1-\chi_\varepsilon(x_\alpha))W_2^\eta)\left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon}D_3 v_\varepsilon\right)\,dx_\alpha dx_3 : \right.$$
$$\left. v_\varepsilon \rightharpoonup v \quad \text{in } W^{1,p}(A\times(-1,1);\mathbb{R}^3), \chi_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(A;[0,1]) \right\}.$$

Define $\tilde{J}(v;\theta;A)$ in a similar manner. Then, clearly

$$\tilde{J}^\eta(v;\theta;A) \geq \tilde{J}(v;\theta;A).$$

Conversely, if for $\delta > 0$ there is a sequence $\{\chi_\varepsilon, v_\varepsilon\}$ such that

$$\tilde{J}(v;\theta;A) \geq \lim_{\varepsilon\to 0^+} \int_{A\times(-1,1)} (\chi_\varepsilon(x_\alpha)W_1 + (1-\chi_\varepsilon(x_\alpha))W_2)\left(D_\alpha v_\varepsilon \Big| \frac{1}{\varepsilon}D_3 v_3\right)\,dx_\alpha dx_3 - \delta,$$

with $\{v_\varepsilon\}$ bounded in $W^{1,p}(A\times(-1,1);\mathbb{R}^3)$, then, for any $\eta > 0$,

$$\tilde{J}(v;\theta;A) \geq \tilde{J}^\eta(v;\theta;A) - \eta \limsup_{\varepsilon\to 0^+} \int_{A\times(-1,1)} |Dv_\varepsilon|^p\,dx_\alpha dx_3 - \delta.$$

Thus

$$\limsup_{\eta\to 0^+} \tilde{J}^\eta(v;\theta;A) - \delta \leq \tilde{J}(v;\theta;A),$$

and letting δ tend to 0^+ we conclude that

$$\lim_{\eta\to 0^+} \tilde{J}^\eta(v;\theta;A) = \tilde{J}(v;\theta;A). \tag{2.6}$$

Recalling (2.5) and by Theorem 2.3 we have

$$\tilde{J}(v;\theta;A) = 2\lim_{\eta\to 0^+} \int_{A\times(-1,1)} \overline{W^\eta}^\star(\theta(x_\alpha), Dv(x_\alpha))\,dx_\alpha. \tag{2.7}$$

An argument similar to that which led to (2.6) would entail

$$\lim_{\eta\to 0^+} \overline{W^\eta}^\star(\theta(x_\alpha), \overline{F}) = \overline{W}^\star(\theta(x_\alpha), \overline{F}). \tag{2.8}$$

Since by (2.1), for all $\overline{F} \in \mathbb{R}^{3\times 2}$, $0 < \eta < 1$, and for a.e. $x \in \omega$,

$$\overline{W^\eta}^\star(\theta(x_\alpha), \overline{F}) \leq (\beta+1)(1+|\overline{F}|^p),$$

(2.7), (2.8), and Lebesgue's Dominated Convergence Theorem imply that

$$\tilde{J}(v;\theta;A) = 2\int_{A\times(-1,1)} \overline{W}^\star(\theta(x_\alpha), Dv(x_\alpha))\,dx_\alpha.$$

An analogous argument would apply to Theorems 2.4. Once again, note that the admissible sequences $\{v_\varepsilon\}$ in the definition of $\tilde{J}(v;\theta;A)$ must be such that they remain bounded in $W^{1,p}(A\times(-1,1);\mathbb{R}^3)$.

Remark 2.8. The results of Theorems 2.3, 2.4, can accommodate the imposition of various boundary conditions on $\partial\omega \times (-1,1)$. In particular, the assertions remain valid if J is defined with sequences $\{v_\varepsilon\}$ in $\mathcal{W} := \big\{v \in W^{1,p}(\omega \times (-1,1); \mathbb{R}^3) : v = 0 \text{ on } \partial\omega \times (-1,1)\big\}$ (cf. Section 1). The proofs would be identical or simpler.

The following regularity property of $\overline{W}^\star(\theta, F)$ will be used in the proof of Lemma 4.3 below.

Proposition 2.9. *$\overline{W}^\star$ is an upper semi-continuous function of $(\theta, \overline{F})$ on* $[0,1] \times \mathbb{R}^{3\times 2}$.

Proof. Assume that $(\theta_k, \overline{F}_k)$ converge to $(\theta_\infty, \overline{F}_\infty)$ as k tends to $+\infty$, and, for each $\delta > 0$, let $\varphi \in W_0^{1,p}(Q'; \mathbb{R}^3)$ and $\chi \in L^\infty(Q'; \{0,1\})$ be such that

$$\overline{W}^\star(\theta_\infty, \overline{F}_\infty) \geq \int_{Q'} (\chi(x'_\alpha)\overline{W}_1 + (1-\chi(x'_\alpha))\overline{W}_2)(\overline{F}_\infty + D_\alpha\varphi)\, dx'_\alpha - \delta, \tag{2.9}$$

with

$$\int_{Q'} \chi(x'_\alpha)\, dx'_\alpha = \theta_\infty.$$

By virtue of Remark 2.1, Lebesgue's Dominated Convergence Theorem implies that

$$\begin{aligned}\int_{Q'} &(\chi(x'_\alpha)\overline{W}_1 + (1-\chi(x'_\alpha))\overline{W}_2)(\overline{F}_\infty + D_\alpha\varphi)\, dx'_\alpha \\ &= \lim_{k\to+\infty} \int_{Q'} (\chi(x'_\alpha)\overline{W}_1 + (1-\chi(x'_\alpha))\overline{W}_2)(\overline{F}_k + D_\alpha\varphi)\, dx'_\alpha,\end{aligned}$$

so that, in view of (2.9),

$$\overline{W}^\star(\theta_\infty, \overline{F}_\infty) \geq \lim_{k\to+\infty} \int_{Q'} (\chi(x'_\alpha)\overline{W}_1 + (1-\chi(x'_\alpha))\overline{W}_2)(\overline{F}_k + D_\alpha\varphi)\, dx'_\alpha - \delta. \tag{2.10}$$

If $\theta_k \neq \theta_\infty$, we propose to change χ into χ_k in (2.10) without modifying the value of the limit. This is a straightforward procedure. Assume for example that $\theta_k > \theta_\infty$. Then simply set $\chi_k := \chi + 1_{A_k}$, where A_k is a subset of $\{x'_\alpha \in Q' : \chi(x'_\alpha) = 0\}$ of measure $\theta_k - \theta_\infty$, which is always possible since $\theta_k - \theta_\infty \leq 1 - \theta_\infty = \mathcal{L}^2(\{x'_\alpha \in Q' : \chi(x'_\alpha) = 0\})$. If $\theta_k < \theta_\infty$ then remove a set A_k from $\{x'_\alpha \in Q' : \chi(x'_\alpha) = 1\}$ of measure $\theta_\infty - \theta_k$. In any case, since θ_k converges to θ_∞ as k tends to ∞, $\mathcal{L}^2(A_k)$ tends to 0 and we have

$$\limsup_{k\to+\infty} \int_{A_k} (1 + |\overline{F}_k|^p + |D_\alpha\varphi|^p)\, dx'_\alpha = 0.$$

Consequently, from (2.10) together with the analogue of (2.1) for $\overline{W}_i$ (cf. Remark 2.1),

$$\begin{aligned}\overline{W}^\star(\theta_\infty, \overline{F}_\infty) &\geq \limsup_{k\to+\infty} \int_{Q'} (\chi_k(x'_\alpha)\overline{W}_1 + (1-\chi_k(x'_\alpha))\overline{W}_2)(\overline{F}_k + D_\alpha\varphi)\, dx'_\alpha \\ &\quad - C \limsup_{k\to+\infty} \int_{A_k} (1 + |\overline{F}_k|^p + |D_\alpha\varphi|^p)\, dx'_\alpha - \delta \\ &\geq \limsup_{k\to+\infty} \overline{W}^\star(\theta_k, \overline{F}_k) - \delta.\end{aligned}$$

It suffices to let $\delta \to 0^+$.

□

Remark 2.10. For each fixed $\overline{F} \in \mathbb{R}^{3\times 2}$, the function $\overline{W}^\star(\cdot, \overline{F})$ is a continuous function of $\theta \in [0,1]$. We will not prove this result here; the proof would be very close to that of (3.4) in Lemma 3.1 below.

Remark 2.11. Note that, by virtue of Proposition 2.9, $x_\alpha \mapsto \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))$ is a measurable function whenever $(\theta, v) \in L^\infty(Q'; [0,1]) \times W^{1,p}(Q'; \mathbb{R}^3)$.

We conclude this section with a brief overview of the paper. In Section 3 we will prove that $J(v;\theta;\cdot)$ defined in (2.4) satisfies

$$J(v;\theta;A) \geq 2\int_A \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))\, dx_\alpha, \tag{2.11}$$

for any open subset A of ω (cf. Lemma 3.1). This will establish the integrability of $x_\alpha \mapsto \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))$ (cf. Remarks 2.11 and 3.2).

Section 4 is devoted to the proof of the converse inequality in (2.11). Lemma 4.1 addresses the case of affine v's and constant θ's; then, Lemma 4.3 treats the case of piecewise affine v's and piecewise constant θ's, which, in turn, yields the proof of Theorem 2.3.

The proof of Theorem 2.4 is given in the short Section 5.

§3. Study of $J(v;\theta;\cdot)$

In the sequel (v,θ) is an arbitrary element of $W^{1,p}(\omega;\mathbb{R}^3) \times L^\infty(\omega;[0,1])$, and $\mathcal{M}(\mathbb{R}^N)$ denotes the space of Radon measures on $\mathbb{R}^N$. If $\mu \in \mathcal{M}(\mathbb{R}^N)$ and B is a Borel set of $\mathbb{R}^N$, then the *restriction measure of μ to B* is defined as

$$\mu\lfloor B(X) := \mu(B \cap X) \quad \text{for all Borel set } X \subset \mathbb{R}^N.$$

The present section is devoted to the proof of the following lemma.

Lemma 3.1. *For any open subset A of ω,*

$$J(v;\theta;A) \geq 2\int_A \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))\, dx_\alpha. \tag{3.1}$$

Remark 3.2. In view of Remark 2.11, the integral on the right-hand side of (3.1) is meaningful and finite.

Proof of Lemma 3.1. By the very definition of $J(v;\theta;A)$ (see (2.4)) there exists a sequence $\{\chi_n, v_n, \varepsilon_n\}$ in $L^\infty(A;\{0,1\}) \times W^{1,p}(A\times(-1,1);\mathbb{R}^3) \times \mathbb{R}^+$ such that

$$\begin{cases} \chi_n \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(A;[0,1]), \\ v_n \to v & \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \\ \varepsilon_n \to 0^+, & \end{cases}$$

and

$$J(v;\theta;A) = \lim_{n\to+\infty} \int_{A\times(-1,1)} (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right) dx_\alpha dx_3. \tag{3.2}$$

The finiteness of $J(v;\theta;A)$ implies that

$$\mu_n := (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right)\ \mathcal{L}^3\lfloor A\times(-1,1)$$

is a bounded sequence of nonnegative finite Radon measures on $\mathbb{R}^3$, hence there exists a nonnegative finite Radon measure μ on $\mathbb{R}^3$ such that a subsequence of μ_n – still indexed by n with no loss of generality – satisfies

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{R}^3).$$

Let us denote by $\hat{\mu}$ the finite Radon measure on $\mathbb{R}^2$ defined as

$$\hat{\mu}(B) := \mu(B \times (-1,1)) \quad \text{for all Borel set } B \subset \mathbb{R}^2,$$

so that, in view of (3.2),

$$J(v;\theta;A) \geq \hat{\mu}(A). \tag{3.3}$$

We will show below that the Radon-Nikodym derivative of $\hat{\mu}$ with respect to the Lebesgue measure on $\mathbb{R}^2$ satisfies

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) \geq 2\overline{W}^\star(\theta(x_0), D_\alpha v(x_0)), \tag{3.4}$$

for almost every Lebesgue point $x_0 \in A$ of θ which is also a point of approximate differentiability for v. Then, (3.3)–(3.4) imply that

$$J(v;\theta;A) \geq \hat{\mu}(A) \geq \int_A \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha)\,dx_\alpha \geq 2\int_A \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))\,dx_\alpha,$$

and this proves Lemma 3.1.

We now prove (3.4). Denote by $Q'(x_0,\delta)$ the (open) square of side δ centered at x_0, and consider a sequence $\{\delta_q\}$, with $\delta_q \to 0^+$ such that $\mu(\partial(Q'(x_0,\delta_q) \times (-1,1)))) = 0$. From the definition of $\hat{\mu}$ together with that of the Radon-Nikodym derivative (see e.g. [EG], Section 1.6),

(3.5)

$$\begin{aligned}\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) &= \lim_{q\to+\infty} \frac{1}{\delta_q^2}\,\hat{\mu}(Q'(x_0,\delta_q)) \\ &= \lim_{q\to+\infty} \frac{1}{\delta_q^2}\,\mu(Q'(x_0,\delta_q)\times(-1,1)) \\ &= \lim_{q\to+\infty} \frac{1}{\delta_q^2} \lim_{n\to+\infty} \mu_n(Q'(x_0,\delta)\times(-1,1)) \\ &= \lim_{q\to+\infty} \frac{1}{\delta_q^2} \lim_{n\to+\infty} \int_{Q'(x_0,\delta)\times(-1,1)} (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right) dx_\alpha dx_3.\end{aligned}$$

Setting

$$\begin{cases} \chi_{q,n}(x'_\alpha) := \chi_n((x_0)_\alpha + \delta_q x'_\alpha), \\ v_{q,n}(x'_\alpha, x_3) := \dfrac{v_n((x_0)_\alpha + \delta_q x'_\alpha, x_3) - v(x_0)}{\delta_q}, \end{cases}$$

(3.5) now reads as

(3.6)

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) = \lim_{q,n\to+\infty} \int_{Q'\times(-1,1)} (\chi_{q,n}(x'_\alpha)W_1 + (1-\chi_{q,n}(x'_\alpha))W_2)\left(D_\alpha v_{q,n} \Big| \frac{\delta_q}{\varepsilon_n} D_3 v_{q,n}\right) dx'_\alpha dx_3.$$

The result would then be obvious if

$$\int_{Q'} \chi_{q,n}(x'_\alpha)\,dx'_\alpha = \theta(x_0), \tag{3.7}$$

and if, for some $\psi \in W^{1,p}(Q' \times (-1,1);\mathbb{R}^3)$ with $\psi(x_\alpha, x_3) = 0$ on $\partial Q' \times (-1,1)$,

$$D_\alpha v_{q,n}(x'_\alpha, x_3) = D_\alpha v(x_0) + D_\alpha \psi(x'_\alpha, x_3). \tag{3.8}$$

In such a case, Fubini's theorem would imply that, for a. e. $x_3 \in (-1,1)$, $\psi(\cdot, x_3) \in W_0^{1,p}(Q';\mathbb{R}^3)$ and (2.2) together with (3.6) would yield

$$\begin{aligned}\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) &\geq \int_1^1 \overline{W}^\star(\theta(x_0), D_\alpha v(x_0))\, dx_3 \\ &= 2\overline{W}^\star(\theta(x_0), D_\alpha v(x_0)).\end{aligned}$$

Unfortunately, there is no guarantee that (3.7), (3.8), hold true, and so the sequence $\{\chi_{q,n}, v_{q,n}\}$ must be modified accordingly. We distinguish two cases, the former when $0 < \theta(x_0) < 1$, the latter when $\theta(x_0) = 0$ or 1. In both cases

$$\begin{aligned}\lim_{q,n\to+\infty} \int_{Q'} \chi_{q,n}(x'_\alpha)\, dx'_\alpha &= \lim_{q\to+\infty} \frac{1}{\delta_q^2} \lim_{n\to+\infty} \int_{Q'(x_0,\delta_q)} \chi_n(x_\alpha)\, dx_\alpha \\ &= \lim_{q\to+\infty} \frac{1}{\delta_q^2} \int_{Q'(x_0,\delta_q)} \theta(x_\alpha)\, dx_\alpha \\ &= \theta(x_0),\end{aligned} \tag{3.9}$$

since x_0 is a Lebesgue point for θ. Set

$$A_{q,n} := \{x'_\alpha \in Q' : \chi_n((x_0)_\alpha + \delta_q x'_\alpha) = 1\}.$$

Then (3.9) implies that

$$\lim_{q,n\to+\infty} \mathcal{L}^2(A_{q,n}) = \theta(x_0). \tag{3.10}$$

<u>*Case* $0 < \theta(x_0) < 1$.</u>

By (3.6) and at the expense of extracting a subsequence of $\{q,n\}$, still labeled $\{q,n\}$, we are always at liberty, in view of the coercivity of W_i (cf. (2.1)), to assume that the sequence $\{\lambda_{q,n}\}$ of nonnegative finite Radon measures

$$\lambda_{q,n} := \left(1 + |D_\alpha v_{q,n}|^p + \left|\frac{\delta_q}{\varepsilon_n} D_3 v_{q,n}\right|^p\right) \mathcal{L}^3 \lfloor Q' \times (-1,1),$$

converges weakly-* in $\mathcal{M}(\mathbb{R}^3)$ to a nonnegative finite Radon measure λ, as $q, n \to +\infty$; we then define $\hat{\lambda}(B) := \lambda(B \times [-1,1])$ for all Borel sets $B \subset \mathbb{R}^2$.

For a fixed pair (q,n) we modify $\chi_{q,n}$ as follows. If $\mathcal{L}^2(A_{q,n}) = \theta(x_0)$ then define $\hat{\chi}_{q,n} := \chi_{q,n}$. Assume that such is not the case. If $\mathcal{L}^2(A_{q,n}) < \theta(x_0)$ then set

$$K_{q,n} := \left[\left[\frac{1}{\sqrt{\theta(x_0) - \mathcal{L}^2(A_{q,n})}}\right]\right],$$

where $[[x]]$ stands for the integer part of x. Then, for q,n large enough and since $\theta(x_0) < 1$, by (3.1) we have that

$$K_{q,n}(\theta(x_0) - \mathcal{L}^2(A_{q,n})) \leq \sqrt{\theta(x_0) - \mathcal{L}^2(A_{q,n})} \leq 1 - \theta(x_0), \tag{3.11}$$

so that it is possible to decompose $Q' \setminus A_{q,n}$ (a set of measure at least $1-\theta(x_0)$) as

$$Q' \setminus A_{q,n} = \cup_{i=1}^{K_{q,n}} \hat{A}_i \cup B,$$

where

$$\mathcal{L}^2(\hat{A}_i) = \theta(x_0) - \mathcal{L}^2(A_{q,n}), \quad i = 1, \dots, K_{q,n}.$$

Now, by virtue of the coercivity of W_i (see (2.1)) together with (3.6), there exists an index $i(q,n) \in \{1, \dots, K_{q,n}\}$ such that

$$\int_{\hat{A}_{i(q,n)} \times (-1,1)} \left(1 + |D_\alpha v_{q,n}|^p + \left|\frac{\delta_q}{\varepsilon_n} D_3 v_{q,n}\right|^p\right) dx'_\alpha dx_3 \le \frac{C}{K_{q,n}}. \tag{3.12}$$

With $1_{\hat{A}_{i(q,n)}}$ denoting the characteristic function of $\hat{A}_{i(q,n)}$, define

$$\hat{\chi}_{q,n} := \chi_{q,n} + 1_{\hat{A}_{i(q,n)}},$$

so that

$$\int_{Q'} \hat{\chi}_{q,n}\, dx'_\alpha = \theta(x_0). \tag{3.13}$$

A similar construction may be performed in the case where $\mathcal{L}^2(A_{q,n}) > \theta(x_0) > 0$, but this time removing from $A_{q,n}$ a set $\hat{A}_{i(q,n)}$ satisfying (3.12) with

$$K_{q,n} := \left[\left[\frac{1}{\sqrt{\mathcal{L}^2(A_{q,n}) - \theta(x_0)}}\right]\right],$$

and

$$K_{q,n}(\mathcal{L}^2(A_{q,n}) - \theta(x_0)) \le \theta(x_0). \tag{3.14}$$

A characteristic function $\hat{\chi}_{q,n}$ satisfying (3.13) is thereby constructed.

Note that the argument proposed above fails if $\theta(x_0) \in \{0,1\}$ because we cannot satisfy (3.11) or (3.14).

In any case, in view of (3.12), (3.13), we may find a characteristic function $\hat{\chi}_{q,n}$ such that

$$\int_{Q'} \hat{\chi}_{q,n}(x'_\alpha)\, dx'_\alpha = \theta(x_0), \tag{3.15}$$

and

$$\lim_{q,n\to+\infty} \int_{\{x'_\alpha \in Q' : \chi_{q,n}(x'_\alpha) \neq \hat{\chi}_{q,n}(x'_\alpha)\} \times (-1,1)} \left(1 + |D_\alpha v_{q,n}|^p + \left|\frac{\delta_q}{\varepsilon_n} D_3 v_{q,n}\right|^p\right) dx'_\alpha dx_3 = 0. \tag{3.16}$$

Clearly (3.6), (2.1) and (3.16) imply that

(3.17)
$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) \ge \liminf_{q,n\to+\infty} \int_{Q' \times (-1,1)} (\hat{\chi}_{q,n}(x'_\alpha) W_1 + (1 - \hat{\chi}_{q,n}(x'_\alpha)) W_2) \left(D_\alpha v_{q,n} \middle| \frac{\delta_q}{\varepsilon_n} D_3 v_{q,n}\right) dx'_\alpha dx_3.$$

We now introduce, for $k \ge 2$,

$$w_{k,q,n} := \varphi_k v_{q,n} + (1 - \varphi_k) Dv(x_0) \cdot x'_\alpha,$$

where $\varphi_k \in C_0^\infty(Q')$ is such that

$$\left\{\begin{array}{l} 0 \le \varphi_k \le 1, \quad ||D_\alpha \varphi_k||_{L^\infty} \le k^2, \\ \quad \varphi_k = 1 \quad \text{if } x'_\alpha \in Q'(0, 1-1/k), \\ \quad \varphi_k = 0 \quad \text{if } x \notin Q'(0, 1-1/(1+k)). \end{array}\right. \tag{3.18}$$

Note that $w_{k,q,n} = Dv(x_0) \cdot x'_\alpha$ on $\partial Q' \times (-1,1)$ and that $w_{k,q,n}(\cdot, x_3) \in W^{1,p}(Q'; \mathbb{R}^3)$ for a.e. $x \in (-1,1)$ by application of Fubini's Theorem. Thus, by virtue of (3.15), (3.17), (3.18), and using the bound from above in (2.1),

(3.19)

$$\begin{aligned}
\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) &\ge \liminf_{q,n\to+\infty} \int_{Q'(0,1-1/k)\times(-1,1)} (\hat{\chi}_{q,n}(x'_\alpha) W_1 + (1-\hat{\chi}_{q,n}(x'_\alpha))W_2) \left(D_\alpha w_{k,q,n} \Big| \frac{\delta_q}{\varepsilon_n} D_3 w_{k,q,n} \right) dx'_\alpha dx_3 \\
&\ge \liminf_{q,n\to+\infty} \left\{ \int_{Q'\times(-1,1)} (\hat{\chi}_{q,n}(x'_\alpha) W_1 + (1-\hat{\chi}_{q,n}(x'_\alpha))W_2) \left(D_\alpha w_{k,q,n} \Big| \frac{\delta_q}{\varepsilon_n} D_3 w_{k,q,n} \right) dx'_\alpha dx_3 \right. \\
&\quad - \beta \int_{(Q'\setminus Q'(0,1-1/(1+k)))\times(-1,1)} (1 + |D_\alpha v(x_0)|^p)\, dx'_\alpha dx_3 \\
&\quad - \beta k^{2p} \int_{(Q'(0,1-1/(k+1))\setminus Q'(0,1-1/k))\times(-1,1)} |v_{q,n} - Dv(x_0)\cdot x'_\alpha|^p \, dx'_\alpha dx_3 \\
&\quad \left. -C \int_{(Q'(0,1-1/(k+1))\setminus Q'(0,1-1/k))\times(-1,1)} \left[1 + |D_\alpha v_{q,n}|^p + \left| \frac{\delta_q}{\varepsilon_n} D_3 v_{q,n} \right|^p \right] dx'_\alpha dx_3 \right\} \\
&\ge 2\overline{W}^\star(\theta(x_0), D_\alpha v(x_0)) - \frac{C}{(k+1)^2} \\
&\quad - \limsup_{q,n\to+\infty} \beta k^{2p} \int_{Q'\times(-1,1)} |v_{q,n} - Dv(x_0)\cdot x'_\alpha|^p \, dx'_\alpha dx_3 \\
&\quad - C \limsup_{q,n\to+\infty} \lambda_{q,n}((Q'(0,1-1/(1+k)) \setminus Q'(0,1-1/k)) \times (-1,1)).
\end{aligned}$$

Now

$$\begin{aligned}
&\limsup_{q,n\to+\infty} \lambda_{q,n}((Q'(0,1-1/(1+k)) \setminus Q'(0,1-1/k)) \times (-1,1)) \\
&\qquad \le \lambda(\overline{(Q'(0,1-1/(1+k)) \setminus Q'(0,1-1/k)) \times (-1,1)}) \\
&\qquad \le \hat{\lambda}(Q' \setminus Q'(0,1-1/(k-1)),
\end{aligned} \tag{3.20}$$

while

(3.21)

$$\begin{aligned}
&\limsup_{q,n\to+\infty} \int_{Q'\times(-1,1)} |v_{q,n} - Dv(x_0)\cdot x'_\alpha|^p \, dx'_\alpha dx_3 \\
&\quad = \lim_{q\to+\infty} \frac{1}{\delta_q^{p+2}} \lim_{n\to+\infty} \int_{Q'(x_0,\delta_q)\times(-1,1)} |v_n(x_\alpha, x_3) - v(x_0) - Dv(x_0)\cdot(x - x_0)|^p \, dx_\alpha dx_3 \\
&\quad = \lim_{q\to\infty} \frac{1}{\delta_q^{p+2}} \int_{Q'(x_0,\delta_q)\times(-11)} |v(x_\alpha, x_3) - v(x_0) - Dv(x_0)\cdot(x - x_0)|^p \, dx_\alpha dx_3 \\
&\quad = 0,
\end{aligned}$$

since x_0 is a point of approximate differentiability for v.

In view of (3.20) and (3.21), (3.19) becomes

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0) \ge 2\overline{W}^\star(\theta(x_0), D_\alpha v(x_0)) - \frac{C}{(1+k)^2} - \hat{\lambda}(Q' \setminus Q'(0,1-1/(k-1)),$$

and (3.4) is obtained by letting k tend to $+\infty$ upon observing that $\{Q'(0,1-1/(k-1))\}$ is an increasing sequence of open sets with set limit the open square Q'.

The result is proved in the case where $0<\theta(x_0)<1$.

Case $\theta(x_0)\in\{0,1\}$.

Let us consider the case where $\theta(x_0)=1$; the case $\theta(x_0)=0$ would be handled in a similar manner.

Firstly, note that if $\theta(x_0)=1$ then (3.9) actually implies the strong convergence to 1 of $\chi_{q,n}$ as $q,n\to+\infty$ in $L^r(Q')$, for all $1\le r<+\infty$. A direct application of Egorov's Theorem yields, for any $k>0$, the existence of a $\mathcal{L}^2$-measurable set A_k such that $\mathcal{L}^2(A_k)<1/k$ and

$$\lim_{q,n\to+\infty}\chi_{q,n}=1 \quad \text{uniformly on } \; Q'\setminus A_k.$$

Then, by virtue of (3.6) together with (2.1),

$$\begin{aligned}\frac{d\hat\mu}{d\mathcal{L}^2}(x_0)&\ge\liminf_{q,n\to+\infty}\int_{(Q'\setminus A_k)\times(-1,1)}W_1\left(D_\alpha v_{q,n}\Big|\frac{\delta_q}{\varepsilon_n}D_3v_{q,n}\right)dx'_\alpha dx_3\\&\ge\liminf_{q,n\to+\infty}\int_{Q'\times(-1,1)}1_{(Q'\setminus A_k)\times(-1,1)}(x'_\alpha,x_3)\,\overline{W}_1(D_\alpha v_{q,n})\,dx'_\alpha dx_3,\end{aligned}\tag{3.23}$$

where $1_{(Q'\setminus A_k)\times(-1,1)}$ denotes the characteristic function of $(Q'\setminus A_k)\times(-1,1)$ in $Q'\times(-1,1)$. Now it was earlier noticed in (3.21) that

$$v_{q,n}\to Dv(x_0)\cdot x'_\alpha \quad \text{in } L^p(Q'\times(-1,1);\mathbb{R}^3),$$

and, by virtue of (3.6) and (2.1), the convergence actually holds true in the weak topology of $W^{1,p}(Q;\mathbb{R}^3)$. But then, application of a classical lower semicontinuity result for quasiconvex integrands (see [AF], Statement III.7) yields

$$\begin{aligned}&\liminf_{q,n\to+\infty}\int_{Q'\times(-1,1)}1_{(Q'\setminus A_k)\times(-1,1)}(x'_\alpha,x_3)\,\overline{W}_1(D_\alpha v_{q,n})\,dx'_\alpha dx_3\\&\qquad\ge\int_{Q'\times(-1,1)}1_{(Q'\setminus A_k)\times(-1,1)}(x'_\alpha,x_3)\,Q_3\overline{W}_1(D_\alpha v(x_0))\,dx'_\alpha dx_3\\&\qquad=2\mathcal{L}^2(Q'\setminus A_k)\,Q_3\overline{W}_1(D_\alpha v(x_0)),\end{aligned}$$

where $Q_3\overline{W}_1$ stands for the quasiconvexification of $\overline{W}_1$,

$$Q_3\overline{W}_1(\overline{F}):=\inf_{\varphi}\left\{\frac{1}{2}\int_{Q'\times(-1,1)}\overline{W}_1(\overline{F}+D_\alpha\varphi)dx'_\alpha dx_3:\varphi\in W_0^{1,p}(Q'\times(-1,1);\mathbb{R}^3)\right\},\quad \overline{F}\in\mathbb{R}^{3\times2}.$$

In view of (3.23) and for $\eta>0$, by Remark 2.1 we may find $\varphi\in C_0^\infty(Q'\times(-1,1);\mathbb{R}^3)$ such that for all k

$$\begin{aligned}\frac{d\hat\mu}{d\mathcal{L}^2}(x_0)&\ge2\mathcal{L}^2(Q'\setminus A_k)\,Q_3\overline{W}_1(D_\alpha v(x_0))\\&\ge\mathcal{L}^2(Q'\setminus A_k)\int_{Q'\times(-1,1)}\overline{W}_1(\overline{F}+D_\alpha\varphi)\,dx'_\alpha dx_3-\eta\\&\ge2\mathcal{L}^2(Q'\setminus A_k)\,\overline{W}^\star_{\chi_0}(D_\alpha v(x_0))-\eta\\&=2\mathcal{L}^2(Q'\setminus A_k)\,\overline{W}^\star(1,\overline{F})-\eta,\end{aligned}\tag{3.24}$$

where χ_0 is the characteristic function of Q'. Letting $k,\eta\to+\infty$ in 3.24 yields

$$\frac{d\hat\mu}{d\mathcal{L}^2}(x_0)\ge2\overline{W}^\star(1,D_\alpha v(x_0)),$$

and the result is proved in the case $\theta(x_0) = 1$.

□

§4. Further Study of $J(v;\theta;\cdot)$; Proof of Theorem 2.3

In a first step it is proved in Lemma 4.1 that equality holds in (3.1) whenever $v = v_\infty$ and $\theta = \theta_\infty$ are, respectively, affine and constant. This result yields Theorem 2.3 in the case of affine v's and constant θ's.

Lemma 4.1. *For any open subset A of ω*

$$J(v_\infty;\theta_\infty;A) \le 2\mathcal{L}^2(A)\,\overline{W}^\star(\theta_\infty, D_\alpha v_\infty).$$

Proof. The proof relies on a blow-up argument in the spirit of [FMu].

By the definition (2.5) of $\overline{W}^\star$, for any $\eta > 0$ there exist $\chi^\eta \in L^\infty(Q';\{0,1\})$ with $\int_{Q'}\chi^\eta(x'_\alpha)dx'_\alpha = \theta_\infty$, and $\varphi^\eta \in W_0^{1,p}(Q';\mathbb{R}^3)$ such that

$$\overline{W}^\star(\theta_\infty, D_\alpha v_\infty) \ge \int_{Q'} (\chi^\eta(x'_\alpha)\overline{W}_1 + (1-\chi^\eta(x'_\alpha))\overline{W}_2)(D_\alpha v_\infty + D_\alpha\varphi^\eta)\,dx'_\alpha dx_3 - \eta.$$

A measurability selection criterion (see [ET]) and the upper bound in (2.1) allow us to find $\xi^\eta \in L^p(Q';\mathbb{R}^3)$ such that

$$\overline{W}^\star(\theta_\infty, D_\alpha v_\infty) \ge \int_{Q'} (\chi^\eta(x'_\alpha)W_1 + (1-\chi^\eta(x'_\alpha))W_2)(D_\alpha v_\infty + D_\alpha\varphi^\eta\,|\,\xi^\eta(x'_\alpha))\,dx'_\alpha - 2\eta. \tag{4.1}$$

In addition, continuity properties of $\overline{W}_i$ (cf. Remark 2.1) together with the density of $W_0^{1,\infty}(Q';\mathbb{R}^3)$ in $W_0^{1,p}(Q';\mathbb{R}^3)$ and in $L^p(Q';\mathbb{R}^3)$, allow us to take φ^η, ξ^η in $W_0^{1,\infty}(Q';\mathbb{R}^3)$ in (4.1).

Extend $\chi^\eta, \varphi^\eta, \xi^\eta$ Q'-periodically to $\mathbb{R}^2$ and set

$$\begin{cases} v_n^\eta(x_\alpha, x_3) := v_\infty(x_\alpha) + \dfrac{1}{n}\varphi^\eta(nx_\alpha) + \dfrac{1}{n^2}x_3\xi^\eta(nx_\alpha), \\ \chi_n^\eta(x_\alpha) := \chi^\eta(nx_\alpha). \end{cases}$$

Note that

$$\chi_n^\eta \overset{*}{\rightharpoonup} \theta_\infty \quad \text{in } L^\infty(A),$$

while

$$v_n^\eta \to v_\infty \quad \text{in } L^p(A\times(-1,1);\mathbb{R}^3).$$

Then, according to the definition (2.4) of $J(v_\infty;\theta_\infty;\cdot)$,

$$\begin{aligned} J(v_\infty;\theta_\infty;A) &\le \liminf_{n\to+\infty} \int_{A\times(-1,1)} (\chi_n^\eta(x_\alpha)W_1 + (1-\chi_n^\eta(x_\alpha))W_2)\left(D_\alpha v_n^\eta\middle| n^2 D_3 v_n^\eta\right)\,dx_\alpha dx_3 \\ &= \liminf_{n\to+\infty} \int_{A\times(-1,1)} (\chi^\eta(nx_\alpha)W_1 + (1-\chi^\eta(nx_\alpha))W_2)\Big(D_\alpha v_\infty + D_\alpha\varphi^\eta(nx_\alpha) \\ &\qquad\qquad + \frac{1}{n}x_3 D_\alpha\xi^\eta(nx_\alpha)\Big|\xi^\eta(nx_\alpha)\Big)\,dx_\alpha\,dx_3. \end{aligned} \tag{4.2}$$

Now W_1 and W_2 are uniformly continuous on compact sets, so that, for any $M, \varepsilon > 0$, there exists $\delta > 0$ such that

$$F, G \in B(0,M), |F-G| < \delta \Rightarrow |W_i(F) - W_i(G)| < \varepsilon, \quad i = 1,2. \tag{4.3}$$

Take

$$M := |D_\alpha v_\infty| + ||D_\alpha \varphi^\eta||_{L^\infty} + ||\xi^\eta||_{W^{1,\infty}}.$$

Then, according to (4.3) and for $n > ||D_\alpha \xi^\eta||_{L^\infty}/\delta$,

$$\begin{aligned}&\int_{A\times(-1,1)} (\chi^\eta(nx_\alpha)W_1 + (1-\chi^\eta(nx_\alpha))W_2)\Big(D_\alpha v_\infty + D_\alpha\varphi^\eta(nx_\alpha)\\ &\qquad\qquad + \frac{1}{n}x_3 D_\alpha \xi^\eta(nx_\alpha)\Big|\xi^\eta(nx_\alpha)\Big)\,dx_\alpha dx_3\\ &\quad\le \int_{A\times(-1,1)} (\chi^\eta(nx_\alpha)W_1 + (1-\chi^\eta(nx_\alpha))W_2)\Big(D_\alpha v_\infty + D_\alpha\varphi^\eta(nx_\alpha)\Big|\xi^\eta(nx_\alpha)\Big)\,dx_\alpha dx_3 + 2\varepsilon\mathcal{L}^2(A).\end{aligned}$$

Thus, recalling (4.2),

(4.4)

$$\begin{aligned}&J(v_\infty;\theta_\infty;A)\\ &\quad\le \liminf_{n\to+\infty}\int_{A\times(-1,1)} (\chi^\eta(nx_\alpha)W_1 + (1-\chi^\eta(nx_\alpha))W_2)\Big(D_\alpha v_\infty + D_\alpha\varphi^\eta(nx_\alpha)\Big|\xi^\eta(nx_\alpha)\Big)\,dx_\alpha dx_3\\ &\quad + 2\varepsilon\mathcal{L}^2(A).\end{aligned}$$

Since $(\chi^\eta(\cdot)W_1 + (1-\chi^\eta(\cdot))W_2)(D_\alpha v_\infty + D_\alpha\varphi^\eta(\cdot)|\xi^\eta(\cdot))$ is a periodic function in $L^\infty(\mathbb{R}^2)$, it converges weakly-* to its average, and (4.4) becomes, with the help of (4.1),

(4.5)

$$\begin{aligned}J(v_\infty;\theta_\infty;A) &\le 2\mathcal{L}^2(A)\int_{Q'} (\chi^\eta(x_\alpha)W_1 + (1-\chi^\eta(x_\alpha))W_2)\Big(D_\alpha v_\infty + D_\alpha\varphi^\eta(x_\alpha)\Big|\xi^\eta(x_\alpha)\Big)\,dx_\alpha\\ &\quad + 2\varepsilon\mathcal{L}^2(A)\\ &\le 2\mathcal{L}^2(A)\overline{W}^\star(\theta_\infty, D_\alpha v_\infty) + 4\eta\mathcal{L}^2(A) + 2\varepsilon\mathcal{L}^2(A).\end{aligned}$$

The result is obtained upon letting ε and then η tend to 0 in (4.5).

□

In order to extend the result of Lemma 4.1 to the case where either v is not affine or θ is not constant, we show that we may allow some flexibility in the selection of a sequence $\{\varepsilon_n\}$ such that

$$J(v;\theta;A) = \lim_{n\to+\infty}\int_{A\times(-1,1)} (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n\Big|\frac{1}{\varepsilon_n}D_3 v_n\right)dx_\alpha dx_3,$$

as defined in (2.4). Specifically, we prove the following

Lemma 4.2. *Let v_∞ be affine and θ_∞ be constant. Then for any sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$ as $n\to+\infty$, and for any subset A of ω, there exists a subsequence $\{\varepsilon^A_{k(n)}\}$ of $\{\varepsilon_n\}$ and a sequence $\{\chi^A_n, v^A_n\}$ in $L^\infty(A;\{0,1\})\times W^{1,p}(A\times(-1,1);\mathbb{R}^3)$ such that, as n tends to ∞,*

$$\begin{cases}\chi^A_n \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(A;\{0,1\}),\\ v^A_n \to v_\infty & \text{in } L^p(A\times(-1,1);\mathbb{R}^3),\end{cases}$$

with

$$v^A_n = v_\infty \quad \text{in a neighborhood of } A\times(-1,1),$$

and

$$J(v_\infty;\theta_\infty;A) = \lim_{n\to+\infty}\int_{A\times(-1,1)} (\chi^A_n(x_\alpha)W_1 + (1-\chi^A_n(x_\alpha))W_2)\left(D_\alpha v^A_n\Big|\frac{1}{\varepsilon^A_{k(n)}}D_3 v^A_n\right)dx_\alpha dx_3.$$

Proof. In a first step the lemma is proved for any cube $Q'(a,r)$ of center $a \in \mathbb{R}^2$ and side length $r > 0$. In a second step the result is established for a general A.

Step 1. The result is proved for a cube $Q' = Q'(a,r)$.

Without loss of generality, we take $a := 0, r := 1$. From the definition (2.4) of $J(v_\infty;\theta_\infty;Q')$ there exists a sequence $\{\chi_n, v_n, \alpha_n\}$ in $L^\infty(Q';\{0,1\}) \times W^{1,p}(Q' \times (-1,1);\mathbb{R}^3) \times \mathbb{R}_+$ with

$$\begin{cases} \chi_n \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q';[0,1]), \\ v_n \rightharpoonup v_\infty & \text{strongly in } L^p(Q' \times (-1,1);\mathbb{R}^3), \\ \alpha_n \to 0^+, \end{cases}$$

and

$$J(v_\infty;\theta_\infty;Q') = \lim_{n\to+\infty} \int_{Q'\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n(x'_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\alpha_n} D_3 v_n\right) dx'_\alpha dx_3. \tag{4.6}$$

Consider the sequence of measures

$$\lambda_n := \left(1 + |D_\alpha v_n|^p + \frac{1}{\alpha_n^p}|D_3 v_n|^p\right) \mathcal{L}^3 \lfloor Q' \times (-1,1).$$

By virtue of (4.6) and of the coercivity hypothesis in (2.1), $\{\lambda_n\}$ is a bounded sequence of finite, nonnegative Radon measures on $\mathbb{R}^3$, hence there exists a finite, nonnegative Radon measure λ on $\mathbb{R}^3$ such that a subsequence of $\{\lambda_n\}$ – still indexed by n with no loss of generality – satisfies

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \quad \text{in } \mathcal{M}(\mathbb{R}^3).$$

We also define $\hat{\lambda}(B) := \lambda(B \times [-1,1])$ for any Borel subset B of $\mathbb{R}^2$. Introduce the sequence $\{\varphi_k\}$ already considered in (3.18). Define further

$$w_{k,n} := \varphi_k v_n + (1-\varphi_k)v_\infty,$$

and note that $w_{k,n} = v_\infty$ on $\partial Q' \times (-1,1)$ while

$$w_{k,n} \to v_\infty \quad \text{in } L^p(Q' \times (-1,1);\mathbb{R}^3). \tag{4.7}$$

In addition, and using an argument similar to that in the proof of Lemma 3.1,

$$\begin{aligned} J(v_\infty;\theta_\infty;Q') &\geq \liminf_{n\to+\infty} \int_{Q'(0,1-1/k)\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n(x'_\alpha))W_2)\left(D_\alpha w_{k,n} \Big| \frac{1}{\alpha_n} D_3 w_{k,n}\right) dx'_\alpha dx_3 \\ &\geq \liminf_{n\to+\infty} \int_{Q'\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n(x'_\alpha))W_2)\left(D_\alpha w_{k,n} \Big| \frac{1}{\alpha_n} D_3 w_{k,n}\right) dx'_\alpha dx_3 \\ &\quad - C\mathcal{L}^2(Q' \setminus Q'(0,1-1/k)) - C \limsup_{n\to+\infty} \left\{ k^{2p} \int_{Q'\times(-1,1)} |v_n - v_\infty|^p \, dx'_\alpha dx_3 \right. \\ &\quad \left. + \lambda_n((Q'(0,1-1/(1+k)) \setminus Q'(0,1-1/k)) \times (-1,1)) \right\} \\ &\geq \liminf_{n\to+\infty} \int_{Q'\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n(x'_\alpha))W_2)\left(D_\alpha w_{k,n} \Big| \frac{1}{\alpha_n} D_3 w_{k,n}\right) dx'_\alpha dx_3 \\ &\quad - C\mathcal{L}^2(Q' \setminus Q'(0,1-1/k)) - C\hat{\lambda}((Q' \setminus Q'(0,1-1/(k-1))), \end{aligned}$$

so that, upon letting k tend to $+\infty$, we obtain

$$J(v_\infty;\theta_\infty;Q') \geq \liminf_{k,n\to+\infty} \int_{Q'\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n(x'_\alpha))W_2) \left(D_\alpha w_{k,n} \Big| \frac{1}{\alpha_n} D_3 w_{k,n} \right) dx'_\alpha dx_3.$$

By virtue of (2.4), (4.7), together with a diagonalization process for the sequence $\{w_{k,n}\}$, we conclude that there exist sequences

$$\begin{cases} \hat{v}_k := w_{k,n(k)}, \\ \hat{\chi}_k := \chi_{n(k)}, \\ \hat{\alpha}_k := \alpha_{n(k)}, \end{cases}$$

such that

$$\begin{cases} \hat{v}_k \to v_\infty & \text{in } L^p(Q'\times(-1,1);\mathbb{R}^3), \\ \hat{\chi}_k \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q';\{0,1\}), \\ \hat{\alpha}_k \to 0^+ \end{cases}$$

and

$$J(v_\infty;\theta_\infty;Q') = \lim_{k\to+\infty} \int_{Q'\times(-1,1)} (\hat{\chi}_k(x'_\alpha)W_1 + (1-\hat{\chi}_k(x'_\alpha)W_2) \left(D_\alpha \hat{v}_k \Big| \frac{1}{\hat{\alpha}_k} D_3 \hat{v}_k \right) dx'_\alpha dx_3.$$

In other words, we are at leisure to assume that the sequence $\{v_n\}$ in (4.6) is such that

$$v_n = v_\infty \quad \text{on } \partial Q' \times (-1,1). \tag{4.8}$$

We now extend χ_n and $v_n - v_\infty$ by Q'-periodicity to all of $\mathbb{R}^2 \times (-1,1)$ and define, for any $m \in (0,+\infty)$ and any $(x_\alpha, x_3) \in \mathbb{R}^3$,

$$\begin{cases} \chi_{n,m}(x_\alpha) := \chi_n(m x_\alpha), \\ v_{n,m}(x_\alpha,x_3) := v_\infty + \dfrac{1}{m}(v_n - v_\infty)(m x_\alpha, x_3), \end{cases}$$

and note that, in view of (4.8), $v_{n,m} \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^2 \times (-1,1);\mathbb{R}^3)$. Further, it is easily checked that, as $n,m \to +\infty$,

$$\begin{cases} v_{n,m} \to v_\infty & \text{in } L^p(Q'\times(-1,1);\mathbb{R}^3) \\ \chi_{n,m} \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q'\times(-1,1);\mathbb{R}^3). \end{cases} \tag{4.9}$$

A $1/m$-scaled periodic function converges weakly to its mean; thus, since $D_\alpha v_\infty$ is constant while $D_3 v_\infty = 0$, for a.e. $x_3 \in (-1,1)$

$$\begin{aligned} &\lim_{m\to+\infty} \int_{Q'} (\chi_{n,m}(x'_\alpha)W_1 + (1-\chi_{n,m})(x'_\alpha)W_2) \left(D_\alpha v_{n,m} \Big| \frac{m}{\alpha_n} D_3 v_{n,m} \right) dx'_\alpha \\ &= \int_{Q'} (\chi_n(x'_\alpha)W_1 + (1-\chi_n)(x'_\alpha)W_2) \left(D_\alpha v_n \Big| \frac{1}{\alpha_n} D_3 v_n \right) dx'_\alpha, \end{aligned}$$

so that, by virtue of the bound from above in (2.1) together with (4.6), (4.9),

$$\begin{aligned} &\lim_{n\to+\infty} \lim_{m\to+\infty} \int_{Q'\times(-1,1)} (\chi_{n,m}(x'_\alpha)W_1 + (1-\chi_{n,m})(x'_\alpha)W_2) \left(D_\alpha v_{n,m} \Big| \frac{m}{\alpha_n} D_3 v_{n,m} \right) dx'_\alpha dx_3 \\ &= \lim_{n\to+\infty} \int_{Q'\times(-1,1)} (\chi_n(x'_\alpha)W_1 + (1-\chi_n)(x'_\alpha)W_2) \left(D_\alpha v_n \Big| \frac{1}{\alpha_n} D_3 v_n \right) dx'_\alpha dx_3 \\ &= J(v_\infty;\theta_\infty;Q'). \end{aligned}$$

Thus, for any n there exists $m(n)$ such that if $m \geq m(n)$ then

(4.10)
$$\left| \int_{Q' \times (-1,1)} (\chi_{n,m}(x'_\alpha) W_1 + (1 - \chi_{n,m})(x'_\alpha) W_2) \left(D_\alpha v_{n,m} \Big| \frac{m}{\alpha_n} D_3 v_{n,m} \right) dx'_\alpha dx_3 - J(v_\infty; \theta_\infty; Q') \right| \leq \omega(n)$$

where $\lim_{n\to+\infty} \omega(n) = 0$, and $\lim_{n\to+\infty} m(n) = +\infty$.

If $\{\varepsilon_k\}$ is such that $\varepsilon_k \to 0^+$ as $k \to +\infty$, choose $k(n)$ large enough such that $m_n := \frac{\alpha_n}{\varepsilon_{k(n)}} \geq m(n)$, hence $m_n \to +\infty$ as $n \to +\infty$, and in view of (4.9), (4.10),

$$\begin{cases} \chi_{n,m_n} \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q' \times (-1,1); \mathbb{R}^3) \\ v_{n,m_n} \to v_\infty & \text{in } L^p(Q' \times (-1,1); \mathbb{R}^3), \end{cases}$$

and

$$\int_{Q' \times (-1,1)} (\chi_{n,m_n}(x'_\alpha) W_1 + (1 - \chi_{n,m_n})(x'_\alpha) W_2) \left(D_\alpha v_{n,m_n} \Big| \frac{1}{\varepsilon_{k(n)}} D_3 v_{n,m_n} \right) dx'_\alpha dx_3 \to J(v_\infty; \theta_\infty; Q'),$$

as $n \to +\infty$. Note that m_n is not necessarily an integer number, and so we cannot guarantee a priori that $v_{n,m_n} = v_\infty$ on $\partial Q' \times (-1,1)$. However, the argument used in (4.6)–(4.8) allows us to meet this boundary condition upon extracting a further subsequence and modifying v_{n,m_m} on a thin transition layer in the neighborhood of $\partial Q'$. This proves the result in the case where $Q' = Q'(0,1)$.

Step 2. Let $A_m := \cup_{i=1}^{N(m)} Q'(a_i^m, r_i^m) \subset\subset A$ be a union of disjoint cubes such that $\mathcal{L}^2(A \setminus A_m) \leq 1/m$, $m \in \mathbb{N}$. From the first step and after $N(m)$ extractions, there exists a subsequence $\{\varepsilon_{k(n,m)}\}$ of $\{\varepsilon_n\}$ and $N(m)$ sequences $\{\chi^i_{n,m}, v^i_{n,m}\}$, $i = 1, \dots, N(m)$, with

$$\begin{cases} \chi^i_{n,m} \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(Q'(a_i^m, r_i^m); [0,1]), \\ v^i_{n,m} \to v_\infty & \text{in } L^p(Q'(a_i^m, r_i^m) \times (-1,1); \mathbb{R}^3), \end{cases}$$

(4.11)
$$v^i_{n,m} = v_\infty \quad \text{on } \partial Q'(a_i^m, r_i^m) \times (-1,1),$$

and

(4.12)
$$\int_{Q'(a_i^m, r_i^m) \times (-1,1)} (\chi^i_{n,m}(x'_\alpha) W_1 + (1 - \chi^i_{n,m})(x'_\alpha) W_2) \left(D_\alpha v^i_{n,m} \Big| \frac{1}{\varepsilon_{k(n,m)}} D_3 v^i_{n,m} \right) dx'_\alpha dx_3 \to J(v_\infty; \theta_\infty; Q'(a_i^m, r_i^m)),$$

as $n \to +\infty$. By virtue of (4.11), the n-indexed sequence

$$\begin{cases} \chi_{n,m} := \displaystyle\sum_{i=1}^{N(m)} \chi^i_{n,m} 1_{Q'(a_i^m, r_i^m)} + \chi_{A \setminus A_m} \tau_n, \\ v_{n,m} := \displaystyle\sum_{i=1}^{N(m)} v^i_{n,m} 1_{Q'(a_i^m, r_i^m)} + v_\infty \chi_{A \setminus A_m}, \end{cases}$$

belongs to $L^\infty(A; \{0,1\}) \times W^{1,p}(A \times (-1,1); \mathbb{R}^3)$, where $\chi_{A \setminus A_m}$ is the characteristic function of the set $A \setminus A_m$, $\tau \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2; \{0,1\})$ is a Q'-periodic function with $\int_{Q'} \tau(x')\, dx' = \theta_\infty$, and $\tau_n(x') := \tau(nx'), x' \in Q'$. Then, as $n \to +\infty$ we have

$$\begin{cases} \chi_{n,m} \overset{*}{\rightharpoonup} \theta_\infty & \text{in } L^\infty(A; [0,1]), \\ v_{n,m} \to v_\infty & \text{in } L^p(A \times (-1,1); \mathbb{R}^3), \end{cases}$$

while, by virtue of (4.12) together with the bound from above in (2.1),

(4.13)

$$
\begin{aligned}
J(v_\infty;\theta_\infty;A) &\le \liminf_{m,n\to+\infty} \int_{A\times(-1,1)} (\chi_{n,m}(x_\alpha)W_1 + (1-\chi_{n,m})(x_\alpha)W_2)\left(D_\alpha v_{n,m}\Big|\frac{1}{\varepsilon_{k(n,m)}}D_3 v_{n,m}\right) dx_\alpha dx_3 \\
&\le \liminf_{m\to+\infty} \left\{ \sum_{i=1}^{N(m)} J(v_\infty;\theta_\infty;Q'(a_i^m,r_i^m)) + C\mathcal{L}^2(A\setminus A_m) \right\} \\
&= \liminf_{m\to+\infty} \sum_{i=1}^{N(m)} J(v_\infty;\theta_\infty;Q'(a_i^m,r_i^m)).
\end{aligned}
$$

Now lemmata 3.1 and 4.1 yield

$$
\begin{aligned}
\liminf_{m\to+\infty} \sum_{i=1}^{N(m)} J(v_\infty;\theta_\infty;Q'(a_i^m,r_i^m)) &\le \liminf_{m\to+\infty} 2\sum_{i=1}^{N(m)} \mathcal{L}^2(Q'(a_i^m,r_i^m))\,\overline{W}^\star(\theta_\infty,D_\alpha v_\infty) \\
&= 2\mathcal{L}^2(A)\,\overline{W}^\star(\theta_\infty,D_\alpha v_\infty) \\
&\le J(v_\infty;\theta_\infty;A),
\end{aligned}
$$

so that the first inequality in (4.13) is actually an equality. A diagonalization process provides a subsequence $\{\varepsilon^A_{k(m)}\} := \{\varepsilon_{k(n,m)}\}$ and a sequence $\{\chi^A_m, v^A_m\} := \{\chi_{n(m),m}, v_{n(m),m}\}$ with the desired properties.

□

We now prove the equality in Lemma 3.1 for piecewise affine and continuous v's and piecewise constant θ's; this is the object of

Lemma 4.3. *If v_∞ is piecewise affine and continuous and θ_∞ is piecewise constant then*

$$
J(v_\infty;\theta_\infty;A) \le 2\int_A \overline{W}^\star(\theta, D_\alpha v)\,dx_\alpha.
$$

Furthermore, the analogue of Lemma 4.2 holds for such pairs (v_∞,θ_∞).

Proof. Let

$$
\begin{cases}
v := \sum_{k=1}^{N} v_k 1_{A_k}, & v_k \text{ affine}, \\
\theta := \sum_{k=1}^{N} \theta_k 1_{A_k}, & \theta_k \text{ constant}, \\
\cup_{k=1}^N \overline{A_k}\cap\omega = \omega, & A_k \text{ disjoint, open subsets of } \omega.
\end{cases}
$$

By virtue of Lemma 4.2, and upon N extractions of subsequences, there exists a sequence $\{\varepsilon_n\}$ (a subsequence, say, of $\{1/n\}$) and N sequences $\{\chi_{n,k}, v_{n,k}\}$, $k = 1,\dots,N$, which satisfy the properties of Lemma 4.2 on $A_k\cap A$. The sequence $\{\chi_n, v_n\}$ defined as

$$
\begin{cases}
\chi_n := \sum_{k=1}^{N} \chi_{n,k} 1_{A_k\cap A}, \\
v_n := \sum_{k=1}^{N} v_{n,k} 1_{A_k\cap A},
\end{cases}
$$

is then immediately seen to meet the requirements so as to be admissible in (2.4) (with $v_n = v_\infty$ on a neighborhood of $\partial A \times (-1,1)$). By virtue of lemmata 4.1 and 4.2 we have

$$\begin{aligned} J(v;\theta;A) &\le \liminf_{n\to+\infty} \int_{A\times(-1,1)} (\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right)\, dx_\alpha dx_3 \\ &= \liminf_{n\to+\infty} \left\{ \sum_{k=1}^N \int_{(A_k\cap A)\times(-1,1)} (\chi_{n,k}(x_\alpha)W_1 + (1-\chi_{n,k}(x_\alpha))W_2)\left(D_\alpha v_{n,k} \Big| \frac{1}{\varepsilon_n} D_3 v_{n,k}\right)\, dx_\alpha dx_3 \right\} \\ &= \sum_{k=1}^N J(v_k;\theta_k;A_k\cap A) \\ &\le 2\sum_{k=1}^N \mathcal{L}^2(A_k\cap A)\, \overline{W}^\star(\theta_k, D_\alpha v_k) \\ &= 2\int_A \overline{W}^\star(\theta_\infty, D_\alpha v_\infty)\, dx_\alpha. \end{aligned}$$

□

We are now in a position to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. If v is an arbitrary element of $W^{1,p}(\omega;\mathbb{R}^3)$ and θ an arbitrary element of $L^\infty(\omega;[0,1])$, we let $\{v_k,\theta_k\}$ be a sequence of piecewise affine and continuous v's and piecewise constant θ's such that, as $k\to+\infty$,

$$\begin{cases} v_k \to v & \text{in } W^{1,p}(\omega;\mathbb{R}^3), \\ \theta_k \to \theta & \text{in } L^q(\omega;[0,1]), 1\le q<+\infty. \end{cases} \tag{4.14}$$

According to Lemma 4.3 there exist, for each k, a subsequence $\{\varepsilon_{(n,k)}\}$ of, say, $\{1/n\}$, and a pair $\{\chi_{n,k}, v_{n,k}\}$ satisfying the properties of Lemma 4.2 (with $v_\infty = v_k, \theta_\infty = \theta_k$). A diagonalization process where $\{\varepsilon_k\} := \{\varepsilon_{(n(k),k)}\}$ and $\varepsilon_{(n(k+1),k+1)} < \varepsilon_{(n(k),k)}, k\in\mathbb{N}$, immediately yields, in view of (4.14), a sequence $\{\chi_k,\tilde v_k\}$ in $L^\infty(A;\{0,1\})\times W^{1,p}(A\times(-1,1);\mathbb{R}^3)$ such that

$$\begin{cases} \chi_k \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(A;[0,1]), \\ \tilde v_k \to v & \text{in } L^p(A\times(-1,1);\mathbb{R}^3), \end{cases}$$

and

$$\begin{aligned} J(v;\theta;A) &\le \lim_{k\to+\infty} \int_{A\times(-1,1)} (\chi_k(x_\alpha)W_1 + (1-\chi_k(x_\alpha))W_2)\left(D_\alpha \tilde v_k \Big| \frac{1}{\varepsilon_k} D_\alpha \tilde v_3\right)\, dx_\alpha dx_3 \\ &= \liminf_{k\to+\infty} J(v_k;\theta_k : A) \\ &= 2\liminf_{k\to+\infty} \int_A \overline{W}^\star(\theta_k, D_\alpha v_k)\, dx_\alpha, \end{aligned} \tag{4.15}$$

where we have used lemmata 3.1 and 4.1.

The bound from above for W_i (or rather for $\overline{W}_i$ – see Remark 2.1) in (2.1), the first convergence in (4.14), Fatou's Lemma and the upper semicontinuity property of $\overline{W}_i$ (see Proposition 2.1) imply that

$$\liminf_{k\to+\infty} \int_A \left\{\beta(1+|D_\alpha v_k|^p) - \overline{W}^\star(\theta_k, D_\alpha v_k)\right\} dx_\alpha \ge \int_A \left\{\beta(1+|D_\alpha v|^p)\, dx_\alpha - \int_A \overline{W}^\star(\theta, D_\alpha v)\right\} dx_\alpha,$$

thus

$$\int_A \overline{W}^\star(\theta, D_\alpha v)\,dx_\alpha \geq \limsup_{k\to+\infty}\int_A \overline{W}^\star(\theta_k, D_\alpha v_k)\,dx_\alpha,$$

which, together with (4.15), finally yields

$$J(v;\theta;A) \leq 2\int_A \overline{W}^\star(\theta(x_\alpha), D_\alpha v(x_\alpha))\,dx_\alpha.$$

This inequality and Lemma 3.1 complete the proof of Theorem 2.1.

□

Remark 4.4. We remark that the proof of Theorem 2.3 may be carried out without using Lemma 4.2, as we may always take $\{\varepsilon_n\} = \{1/n\}$. Indeed, it can be seen easily from the proof of Lemma 4.1 that, upon rescaling and up to a translation, if v_∞ is affine, if θ_∞ is constant, and if $Q'(a,r)$ is a square on $\mathbb{R}^2$, then for all $\eta > 0$ there exist $\{v_n^\eta\}, \{\chi_n^\eta\}$ such that as $n \to +\infty$

$$v_n^\eta \to v_\infty \quad \text{in } L^p(Q'(a,r);\mathbb{R}^3), \quad v_n^\eta = v_\infty \quad \text{on } \partial Q'(a,r)\times(-1,1), \tag{4.16}$$

$$\chi_n^\eta \overset{*}{\rightharpoonup} \theta_\infty \quad \text{in } L^\infty(Q'(a,r)), \tag{4.17}$$

and

$$\begin{aligned}
J(v_\infty;\theta_\infty;Q'(a,r)) &\leq \liminf_{\eta\to0^+}\lim_{n\to+\infty}\int_{Q'(a,r)\times(-1,1)}(\chi_n^\eta(x_\alpha)W_1 + (1-\chi_n^\eta(x_\alpha))W_2)\left(D_\alpha v_n^\eta\middle|n^2D_3v_n^\eta\right)\,dx_\alpha dx_3\\
&\leq \limsup_{\eta\to0^+}\lim_{n\to+\infty}\int_{Q'(a,r)\times(-1,1)}(\chi_n^\eta(x_\alpha)W_1 + (1-\chi_n^\eta(x_\alpha))W_2)\left(D_\alpha v_n^\eta\middle|n^2D_3v_n^\eta\right)\,dx_\alpha dx_3\\
&\leq 2\mathcal{L}^2(Q'(a,r))\overline{W}^\star(\theta_\infty, D_\alpha v_\infty),
\end{aligned}$$

where we used the fact that the $\liminf_{n\to+\infty}$ in (4.2) is actually $\lim_{n\to+\infty}$. In view of Lemma 3.1 we have

$$\begin{aligned}
J(v_\infty;\theta_\infty;Q'(a,r)) &= \lim_{\eta\to0^+}\lim_{n\to+\infty}\int_{Q'(a,r)\times(-1,1)}(\chi_n^\eta(x_\alpha)W_1 + (1-\chi_n^\eta(x_\alpha))W_2)\left(D_\alpha v_n^\eta\middle|n^2D_3v_n^\eta\right)\,dx_\alpha dx_3\\
&= 2\mathcal{L}^2(Q'(a,r))\overline{W}^\star(\theta_\infty, D_\alpha v_\infty),
\end{aligned}$$

hence, we may extract subsequences

$$v_n := v_n^{\eta(n)},\ \chi_n := \chi_n^{\eta(n)}$$

satisfying (4.16), (4.17) and

$$\begin{aligned}
J(v_\infty;\theta_\infty;Q'(a,r)) &= \lim_{n\to+\infty}\int_{Q'(a,r)\times(-1,1)}(\chi_n(x_\alpha)W_1 + (1-\chi_n(x_\alpha))W_2)\left(D_\alpha v_n\middle|n^2D_3v_n\right)\,dx_\alpha dx_3\\
&= 2\mathcal{L}^2(Q'(a,r))\overline{W}^\star(\theta_\infty, D_\alpha v_\infty).
\end{aligned} \tag{4.18}$$

If now we consider a triangle T on the plane, given $m \in \mathbb{N}$ we may cover T with squares of the type $Q'(a,r)$, $a \in \mathbb{R}^2, r > 0$, up to a set of measure at most $1/m$, so that using the construction of Step 2 in Lemma 4.2 applied to our sequences constructed in (4.16), (4.17) and (4.18), we obtain a double sequence $\{v_{n,m}, \chi_{n,m}\}$ satisfying (4.16), (4.17) on T and for each $m \in \mathbb{N}$ fixed, and also

$$\begin{aligned}
J(v_\infty;\theta_\infty;T) &= \lim_{m\to\infty}\lim_{n\to+\infty}\int_{T\times(-1,1)}(\chi_{n,m}(x_\alpha)W_1 + (1-\chi_{n,m}(x_\alpha))W_2)\left(D_\alpha v_{n,m}\middle|n^2D_3v_{n,m}\right)\,dx_\alpha dx_3\\
&= 2\mathcal{L}^2(T)\overline{W}^\star(\theta_\infty, D_\alpha v_\infty).
\end{aligned}$$

As before, we extract a subsequence $\{v_{n,m(n)}, \chi_{n,m(n)}\}$ verifying (4.16), (4.17) and (4.18) with T in place of $Q'(a,r)$. Given the matching boundary conditions imposed on these sequences, it is now clear that if v_∞ is piecewise affine on a triangulation of the plane, and if θ_∞ is piecewise constant, then Lemma 4.3 applies with the same proof, and with $\{\varepsilon_n\} = \{1/n\}$. We may proceed with the proof of Theorem 2.3 where $\{\varepsilon_k\} = \{1/k\}$, and where in (4.14) we use the fact that piecewise affine functions on triangulations of the plane are dense in $W^{1,p}(\omega;\mathbb{R}^3)$.

§5. Proof of Theorem 2.5

Throughout this section, v is an arbitrary element of $W^{1,p}(\omega;\mathbb{R}^3)$.

Define

$$I_\lambda(v;A) := \inf_\theta \left\{ J(v;\theta;A) : \theta \in L^\infty(A;[0,1]), \frac{1}{\mathcal{L}^2(A)} \int_A \theta(x_\alpha)\, dx_\alpha = \lambda \right\}.$$

Then the inequality

$$G_\lambda(v;A) \geq I_\lambda(v;A)$$

is immediate because if $\{\chi_\varepsilon\}$ is such that

$$\frac{1}{\mathcal{L}^2(A)} \int_A \chi_\varepsilon(x_\alpha)\, dx_\alpha = \lambda,$$

then a subsequence of $\{\chi_\varepsilon\}$, still indexed by ε, is such that

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(A;[0,1]),$$

with

$$\frac{1}{\mathcal{L}^2(A)} \int_A \theta(x_\alpha)\, dx_\alpha = \lambda.$$

The following lemma proves the converse inequality, hence Theorem 2.4.

Lemma 5.1.

$$G_\lambda(v;A) \leq I_\lambda(v;A).$$

Proof. For a fixed $\eta > 0$, consider $\theta \in L^\infty(A;[0,1])$ such that

$$\begin{cases} \dfrac{1}{\mathcal{L}^2(A)} \displaystyle\int_A \theta(x_\alpha)\, dx_\alpha = \lambda \\ J(v;\theta;A) \leq I_\lambda(v;A) + \eta. \end{cases}$$

Choose a sequence $\{\chi_n, v_n, \varepsilon_n\}$ such that

$$\begin{cases} \chi_n \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(A;[0,1]), \\ v_n \to v & \text{in } L^p(A \times (-1,1);\mathbb{R}^3), \\ \varepsilon_n \to 0^+ \end{cases}$$

and

$$J(v;\theta;A) = \lim_{n\to+\infty} \int_{A\times(-1,1)} (\chi_n(x_\alpha) W_1 + (1-\chi_n(x_\alpha)) W_2) \left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n \right) dx_\alpha dx_3.$$

Then

$$I_\lambda(v,A) + \eta \geq \lim_{n\to+\infty} \int_{A\times(-1,1)} (\chi_n(x_\alpha) W_1 + (1-\chi_n(x_\alpha)) W_2) \left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n \right) dx_\alpha dx_3 \tag{5.1}$$

and the proof is complete if for every n

$$\int_A \chi_n(x_\alpha)\, dx_\alpha = \lambda.$$

In general this is not so, and it is merely true that

$$\lim_{n\to+\infty} \int_A \chi_n(x_\alpha)\, dx_\alpha = \lambda.$$

The proof that χ_n may be modified to yield a $\hat{\chi}_n$ with

$$\int_A \hat{\chi}_n(x_\alpha)\, dx_\alpha = \lambda, \tag{5.2}$$

and without changing the value of the limit in (5.1) is exactly analogue to that in the proof of Lemma 3.1. It will not be repeated at this point. Thus (5.1) implies that

$$\begin{aligned} I_\lambda(v,A) + \eta &\geq \lim_{n\to+\infty} \int_{A\times(-1,1)} (\hat{\chi}_n(x_\alpha)W_1 + (1-\hat{\chi}_n(x_\alpha))W_2) \left(D_\alpha v_n \Big| \frac{1}{\varepsilon_n} D_3 v_n\right) dx_\alpha dx_3 \\ &\geq G_\lambda(v,A), \end{aligned}$$

where the last inequality holds true in view of (5.2) and the definition of $G_\lambda(v,A)$.

The proof of Lemma 5.1 is complete upon letting $\eta \to 0^+$.

□

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