3D COPRIME ARRAYS IN SPARSE SENSING

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ABSTRACT

Coprime arrays are a class of sensor arrays that play a crucial role in various signal processing tasks because of their desirable properties such as sparsity and increased degrees of freedom (DOF) of coarrays. In this contribution, a new class of three-dimensional (3D) arrays is constructed from pure cubic fields. By studying the properties of cubic integers, we convert the problem of finding two coprime 3-by-3 integer matrices to that of two coprime integers in the ring of integers of a cubic field, which significantly reduces the design complexity and expands the design space of these matrices. The proposed construction offers naturally commutative matrices and includes generalized circulant matrices as a special case (under certain restriction of a parameter). The surged DOF is guaranteed by the generalized Chinese Remainder Theorem (CRT) for rings and ideals.

Index Terms— Coprime sensing, sparse arrays, lattices, cubic fields, Chinese Reminder Theorem.

1. INTRODUCTION

Coprime arrays are a family of sparse arrays where the subarrays are allocated on coprime lattices. Thanks to the Chinese Remainder Theorem (CRT), such arrays enjoy a gain in the degrees of freedom (DOF) and thus find applications in multidimensional array processing based on difference or sum coarrays, such as DFT filter banks [1, 2] and direction of arrival estimations in passive and active scenarios [3, 4]. Herein, DOF is defined as the cardinality of the coarray. Other advantages of the sparse array based on CRT include its symmetric geometry which simplifies the computational complexity [5, 6], and provides a gain of DOF with the fixed array aperture by using the optimum lattice for sphere packing [7].

The design and applications of one-dimensional (1D) coprime arrays can be found in [1,3], which offer a surged gain of DOF compared to uniform linear arrays (ULA) [8]. Recently, the study of coprime arrays has drawn attention in multidimensional signal processing [2], which requires the use of pairwise coprime matrices. The essentialness of such matrix pairs can also be seen in the context of multirate systems [9–11]. One family of adjugate pairs of triangular matrices were reported in [12] where the coprimality conditions were proved by using Bezout's identity with matrix representations. Another approach of finding coprime integer matrices with specific structures was introduced in [13] based on minors, which allows systematic but laborious verification of coprime pairs of integer matrices in high dimensions. The problem of generating all commuting coprime matrices of a given size was left open.

In this paper, we propose a novel design of coprime arrays in the 3D space by exploiting the rings of integers of cubic fields, which can offer $O(p^M)$ DOF with a number Mp of sensors allocated on M subarrays. The concept of CRT arrays based on quadratic fields was introduced in [14] and was further investigated in [4] along with their applications in sparse sensing. Motivated by this, this paper explores the field extension of dimension three, which provides 3-by-3 matrices. Compared to existing works on coprime 3-by-3 matrices [12, 13], the construction given in this paper is simpler and more general. In particular, the generalized circulant matrices given in [13] coincide with a particular case of matrices obtained here (under certain restriction of a parameter; see the end of Section 3.2 for details). A salient feature of integer matrices constructed from number fields is that they are commutative by nature, which is required in most aforementioned applications. Potential applications of this work include multidimensional DFT [15], massive MIMO [16], and 3D microphone arrays [17].

The rest of the paper is organized as follows. The concept of CRT arrays is briefly reviewed in Section 2. Section 3 proposes a general theorem for coprime matrices with dimension three, based on which 3D CRT array configurations are presented. Section 4 concludes the paper.

Notations: Bold font lowercase letters (e.g., $\mathbf{z}_1$), bold font uppercase letters (e.g., $\mathbf{G}$) and calligraphy font alphabets (e.g., $\mathcal{D}$) denote vectors, matrices, principal ideals and sets respectively. $\mathbb{Z}$ and $\mathbb{Q}$ denote rational integers $\{\cdots -1, 0, 1\cdots\}$ and rational numbers $\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\}$ respectively.

2. REVIEW OF CRT ARRAYS IN ONE AND TWO DIMENSIONS

Given n linearly independent column vectors $\mathbf{g}_1, \cdots \mathbf{g}_n$ of dimension n, an nD lattice Λ is defined as all linear combinations of these column vectors, i.e.,

$$\Lambda = \left\{ \sum_{k=1}^{n} x_k \mathbf{g}_k : x_k \in \mathbb{Z} \right\}. \tag{1}$$

The set $\{\mathbf{g}_1, \cdots \mathbf{g}_n\}$ is called the *basis* and the matrix that consists of this basis is the *generator matrix* of Λ which can be written straightforwardly as $\mathbf{G} = [\mathbf{g}_1| \cdots |\mathbf{g}_n]$.

Given a number field K, its ring of integers denoted by $\mathcal{O}_K$ can form a lattice Λ under canonical embedding σ. More generally, any ideal $\mathcal{I}$ in $\mathcal{O}_K$ forms a *sublattice* of $\Lambda = \sigma(\mathcal{O}_\mathcal{K})$.

Definition 1 (CRT array) *Let $\Lambda_1 = \sigma(\mathcal{I})$ and $\Lambda_2 = \sigma(\mathcal{J})$ denote two sublattices of Λ, which are generated by two coprime ideals $\mathcal{I}$ and $\mathcal{J}$ respectively. A CRT array comprises two subarrays which are allocated on [4]:*

$$\mathcal{S}_1 = \{\mathbf{z}_m : \mathbf{z}_m \in \Lambda/\Lambda_1\}, \mathcal{S}_2 = \{\mathbf{z}_n : \mathbf{z}_n \in \Lambda/\Lambda_2\}.$$

Definition 2 (Coarrays) *The* coarray *of $\mathcal{S}_1$ and $\mathcal{S}_2$ is composed of all the sum vectors between them:*

$$\mathcal{S} = \{\mathbf{z}_m + \mathbf{z}_n : \mathbf{z}_m \in \mathcal{S}_1, \mathbf{z}_n \in \mathcal{S}_2\}. \tag{2}$$

If $\mathcal{S}_1$ is symmetric with respect to the origin and same with $\mathcal{S}_2$, $\mathcal{S}$ is identical to the difference coarray set $\{\mathbf{z}_m - \mathbf{z}_n : \mathbf{z}_m \in \mathcal{S}_1, \mathbf{z}_n \in \mathcal{S}_2\}$.

According to the generalized Chinese Remainder Theorem rephrased by a ring $\mathcal{R}$ and its ideals, if $\mathcal{I}$ and $\mathcal{J}$ are coprime, there exists a ring isomorphism:

$$R/\mathcal{I}\mathcal{J} \simeq R/\mathcal{I} \times R/\mathcal{J}, \tag{3}$$

which asserts the quadratic gain of DOF [18]. In the 1D case, $\Lambda = \mathbb{Z}$ which is the set of all rational integers such as $\cdots - 1, 0, 1, 2, 3, \cdots$. For instance, given two prime ideals $\langle 3 \rangle$ and $\langle 5 \rangle$, a CRT array is allocated on $\Lambda_1 = 3k$ and $\Lambda_2 = 5k$, $k \in \mathbb{Z}$, which coincides with the coprime array introduced in [1]. As an example in the 2D case, consider the ring $\mathbb{Z}[\omega]$ of Eisenstein integers. $\mathcal{I}$ and $\mathcal{J}$ can be $\langle 1 + 2\sqrt{3}i \rangle$ and $\langle 1 - 2\sqrt{3}i \rangle$ respectively, where $1 + 2\sqrt{3}i$ and $1 - 2\sqrt{3}i$ are coprime Eisenstein integers. In this case Λ is the hexagonal lattice A_2, which is the optimum lattice for sphere packing in 2D and has a gain of 15.5% in DOF compared to uniformly rectangular arrays [7].

3. DESIGN OF 3D CRT ARRAYS

3.1. Algebraic Construction of 3D Lattices

For convenience, we restrict ourselves to those *pure cubic fields* whose ring of integers $\mathcal{O}_K$ is moreover a principal ideal domain[1]. A pure cubic field K is a field extension of $\mathbb{Q}$ in the form of $\mathbb{Q}(\sqrt[3]{r})$ where r is a non-unit cubic-free integer. Expressing r as $r = ab^2$ where a and b are square-free and coprime, an *integral basis* of $\mathbb{Q}(\theta)$ where $\theta = \sqrt[3]{r}$ is [19, Theorem 6.4.13]

(a) if $r \not\equiv \pm 1 \pmod 9$

$$\left\{1,\, \theta,\, \frac{\theta^2}{b}\right\}, \text{ or} \tag{4}$$

(b) if $r \equiv \pm 1 \pmod 9$

$$\left\{1,\, \theta,\, \frac{\theta^2 + r\theta + b^2}{3b}\right\}. \tag{5}$$

[1]The rings of integers of cubic fields with discriminant between -268 and 1944 (inclusively) are principal ideal domains.

If $r \not\equiv \pm 1 \pmod 9$, $m = m_1 + m_2\theta + m_3\frac{\theta^2}{b}$ is an *algebraic integer* in $\mathcal{O}_K$ where $m_1, m_2, m_3 \in \mathbb{Z}$. The integral basis of the principal ideal generated by $m \in \mathcal{O}_K$ is

$$\begin{aligned} &\left(m_1 + m_2\theta + m_3\frac{\theta^2}{b}\right)\left\{1, \theta, \frac{\theta^2}{b}\right\} \\ =&\Big\{m_1 + m_2\theta + m_3\frac{\theta^2}{b}, m_3\frac{r}{b} + m_1\theta + m_2\theta^2, \\ &m_2\frac{r}{b} + m_3\frac{r}{b^2}\theta + m_1\frac{\theta^2}{b}\Big\}. \end{aligned} \tag{6}$$

By stacking the coefficients corresponding to basis (4), the matrix representation of m is

$$\mathbf{B}_m = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_3ab & m_1 & m_2b \\ m_2ab & m_3a & m_1 \end{pmatrix}, \text{ for } r \not\equiv \pm 1 \pmod 9. \tag{7}$$

Likewise, the matrix of m corresponding to basis (5) is

$$\mathbf{B}_m = \begin{pmatrix} m_1 & m_2 & m_3 \\ T_1 & T_2 & T_3 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \end{pmatrix}, \text{ for } r \equiv \pm 1 \pmod 9 \tag{8}$$

where $T_1 = m_3\frac{b(a-r)}{3} - m_2b^2$, $T_2 = m_1 - m_2r - m_3\frac{(ar-1)b}{3}$, $T_3 = 3m_2b + m_3r$, $\Gamma_1 = m_2\frac{b(a-r)}{3} + m_3\frac{2ar-r^2-b^2}{9}$, $\Gamma_2 = m_2\frac{b(1-ar)}{3} + m_3\frac{a(1-r^2)}{9}$, and $\Gamma_3 = m_1 + m_2r + m_3\frac{b(ar+2)}{3}$. Obviously, all entries in (7) are integers. For basis (5), because $r \equiv \pm 1 \pmod 9$ is equivalent to $a^2 \equiv b^2 \mod 9$ [19], it can be shown that all entries in (8) are in $\mathbb{Z}$ as well. For example, in T_1, it can be verified that $b(a-r) = ab - ab^3 = ab(1-b^2)$ where $b(1-b^2)$ can always be divided by 3 (if b cannot be divided by 3, $b = 3k \pm 1$ where $k \in \mathbb{Z}$). We omit the details due to the space limit. In fact, the matrix representation of any algebraic integer in $\mathcal{O}_K$ is an integer matrix because algebraic integers are eigenvalues of integer matrices from the matrix point of view [20].

Similar to complex conjugates, the algebraic conjugates of an algebraic number is calculated from canonical embeddings [19, Section 4.2.4]. For illustrative purposes, let us rewrite m in terms of $\{1, \theta, \theta^2\}$, i.e.,

$$m = u_1 + u_2\theta + u_3\theta^2. \tag{9}$$

If $r \not\equiv \pm 1 \pmod 9$, $u_1 = m_1$, $u_2 = m_2$, $u_3 = m_3/b$, and $u_1 = m_1 + m_3\frac{b}{3}$, $u_2 = m_2 + m_3\frac{ab}{3}$ and $u_3 = \frac{m_3}{3b}$ otherwise. With $n = 3$, there are three embeddings that map $m \in \mathcal{O}_K$ into the set of complex numbers $\mathbb{C}$:

$$\begin{aligned} m &\to u_1 + u_2\theta + u_3\theta^2, \\ m &\to m' = u_1 + u_2\omega\theta + u_3\omega^2\theta^2, \\ m &\to m'' = u_1 + u_2\omega^2\theta + u_3\omega\theta^2, \end{aligned} \tag{10}$$

where $\omega = e^{j2\pi/3}$ is the root of $\omega^2 + \omega + 1 = 0$ (cubic root of unity), and m' and m'' are referred as *algebraic conjugates* of m. It is notable that the relation between m along with its conjugates and FFT matrix is:

$$\begin{pmatrix} m \\ m' \\ m'' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} u_1 \\ u_2\theta \\ u_3\theta^2 \end{pmatrix}. \quad (11)$$

Note that m' and m'' are not necessarily in $\mathbb{Q}(\theta)$ unless the extension is Galois [19]. Hence, let us define $\hat{m} = m'm'' \in \mathcal{O}_K$. For instance, with basis (4),

$$\hat{m} = \hat{m}_1 + \hat{m}_2\theta + \hat{m}_3\frac{\theta^2}{b}, \quad (12)$$

where $\hat{m}_1 = m_1^2 - m_2m_3ab$, $\hat{m}_2 = m_3^2a - m_1m_2$, and $\hat{m}_3 = m_2^2b - m_1m_3$. Thus the matrix representation of $\hat{m}$ with basis $\left\{1,\, \theta,\, \frac{\theta^2}{b}\right\}$ is

$$\mathbf{B}_{\hat{m}} = \begin{pmatrix} \hat{m}_1 & \hat{m}_2 & \hat{m}_3 \\ \hat{m}_3ab & \hat{m}_1 & \hat{m}_2b \\ \hat{m}_2ab & \hat{m}_3a & \hat{m}_1 \end{pmatrix}. \quad (13)$$

Note that $\mathbf{B}_{\hat{m}}$ is the adjugate matrix of $\mathbf{B}_m$ and all elements in $\mathbf{B}_{\hat{m}}$ are in $\mathbb{Z}$. Similarly, with basis (5), the matrix corresponding to $\hat{m}$ can be calculated where all entries are in $\mathbb{Z}$. Therefore, it is feasible to exploit the matrices of algebraic integers for any application that requires pairs of coprime integer matrices. In the following sections, a 3D lattice for assigning sensors will be generated as the set given in (1) where the generator matrix is a matrix representation of an element in $\mathcal{O}_K$.

3.2. Coprime Algebraic Integers and Their Matrix Representations

Due to the requirement of coprimality in the generalized CRT (3), we shall search for two coprime ideals in $\mathcal{O}_K$. One way of finding these ideals is via prime decomposition, which gives prime ideals that are coprime by nature. The criteria of this prime decomposition can be found in [19, Proposition 6.4.14].

In this section, we provide a computationally tractable approach to the construction of 3D coprime matrices. First we shall define the coprimality of algebraic integers and of integer matrices: two algebraic integers k and v are *coprime*, if and only if there exist $\alpha, \beta \in \mathcal{O}_K$ such that $k\alpha + v\beta = 1$ [21]; similarly, two integer matrices $\mathbf{B}_k$ and $\mathbf{B}_v$ are *coprime*, if and only if there exist integer matrcies $\mathbf{B}_\alpha$ and $\mathbf{B}_\beta$ such that $\mathbf{B}_\alpha\mathbf{B}_k + \mathbf{B}_\beta\mathbf{B}_v = \mathbf{I}$ [13, 22]. The following lemma relates the coprimality of algebraic integers and that of their matrices.

Lemma 1 *Two cubic integers are coprime if and only if their corresponding matrices are coprime.*

Proof: Let k and v be two coprime cubic integers with $\mathbf{B}_k$ and $\mathbf{B}_v$ being their corresponding matrices. From the Bezout's identity, $k\alpha + v\beta = 1$ holds. Taking the conjugations of both sides of this equation and stacking them accordingly yield

$$\mathbf{P}_\alpha\mathbf{P}_k + \mathbf{P}_\beta\mathbf{P}_v = \mathbf{I}, \quad (14)$$

where $\mathbf{P}_k$ is a diagonal matrix with k, k', k'' being its diagonal entries and k', k'' are conjugates of k by embeddings (10), and same with $\mathbf{P}_\alpha, \mathbf{P}_\beta$ and $\mathbf{P}_v$. Since algebraic integers are eigenvalues of their matrices with the integral bases being their eigenvectors [20], $\mathbf{B}_k$ can be decomposed as $\mathbf{B}_k = \mathbf{Q}^{-1}\mathbf{P}_k\mathbf{Q}$ where $\mathbf{Q}$ is the eigenvector matrix and same with other algebraic integers. Right multiplying $\mathbf{Q}$ and left multiplying $\mathbf{Q}^{-1}$ to (14) results $\mathbf{B}_\alpha\mathbf{B}_k + \mathbf{B}_\beta\mathbf{B}_v = \mathbf{I}$. The sufficiency of the condition can be proved by checking the first row of $\mathbf{P}_\alpha\mathbf{P}_k + \mathbf{P}_\beta\mathbf{P}_v = \mathbf{I}$. ■

Next, let us review the definition of the norm of an algebraic integer for the simplification of the coprime conditions. The norm of an arbitrary integer of degree n is given in [19, Section 3.6.2]. In a pure cubic field ($n = 3$), the norm of m is

$$\mathrm{N}(m) = mm'm'' = m\hat{m}, \quad (15)$$

which is always in $\mathbb{Z}$ and $\mathrm{N}(m) = |\det(\mathbf{B}_m)|$ for all $m \in \mathcal{O}_K$. Recall the generalized greatest common divisor in rings of integers [21, Definition 6.1.3]. The following facts hold for any three cubic integers $k, v, p \in \mathcal{O}_K$:

1. $\mathrm{GCD}(k, v) = \mathrm{GCD}(k + \alpha v, v), \forall \alpha \in \mathcal{O}_K$;
2. $\mathrm{GCD}(k, vp) = 1$ if and only if $\mathrm{GCD}(k, v) = 1$ and $\mathrm{GCD}(k, p) = 1$.

From the Bezout's identity, two algebraic integers k and v are coprime, if and only if $\mathrm{GCD}(k, v) = 1$. Using the notations above, before deriving the sufficient and necessary condition for two algebraic integers in the forms of m and $\hat{m}$, we propose the following lemma:

Lemma 2 *$m \in \mathcal{O}_K$ and $s \in \mathbb{Z}$ are coprime if and only if $\mathrm{N}(m)$ and s are coprime.*

Proof: For the sufficiency of the condition, let us assume $\mathrm{GCD}(m, s) = 1$ where $m \in \mathcal{O}_K$ and $s \in \mathbb{Z}$. From the generalized Bezout's identity, there exist $k, v \in \mathcal{O}_K$, such that

$$mk + sv = 1. \quad (16)$$

By taking the conjugates of all elements in both right and left hand sides of (16), the following two equations hold:

$$m'k' + su' = 1, \quad (17)$$

$$m''k'' + su'' = 1, \quad (18)$$

where k' and k'' are algebraic conjugates of k as defined in (10) and same with u' and u''. Because of the definition of the conjugation, (16), (17) and (18) are equivalent.

Multiplying both sides of (17) and (18) yields

$$(m'k' + su')(m''k'' + su'') = m'm''(k'k'')+ \\ s(su'u'' + u'm''k'' + m'k'u'') = m'm''\mu + s\delta = 1, \quad (19)$$

where $\mu = k'k''$ and $\delta = su'u'' + u'm''k'' + m'k'u''$. From Section 3.1, it is easy to observe that $k'k''$ and $u'u''$ are also in $\mathcal{O}_K$. Using the definition of embeddings (10) and noticing that $\omega^2 + \omega = -1$, it can be verified that $u'm''k'' + m'k'u''$ is also in $\mathcal{O}_k$. In short, $m'm'', \mu, \delta$ are all in $\mathcal{O}_K$ where the generalized Bezout's identity applies. By fact 2, $\mathrm{GCD}(\mathrm{N}(m), s) = 1$ holds. The necessity of the condition can be proved by noticing that $\mathrm{GCD}(\mathrm{N}(m), s) = 1$ is equivalent to $\mathrm{GCD}(m, s) = 1$ and $\mathrm{GCD}(m'm'', s) = 1$. ■

Using the above notations and Lemma 2, the following theorem provides the condition of the coprimality:

Theorem 1 *In a pure cubic field, m and $\hat{m}$ are coprime if and only if*

$$\mathrm{GCD}(\mathrm{N}(m)\,,\,3\hat{u}_1) = 1, \tag{20}$$

where $3\hat{u}_1 = 3m_2m_3ab - 3m_1^2$, *if* $r \not\equiv \pm 1 \pmod 9$, *and* $3\hat{u}_1 = 3m_1^2 + (2m_1 - m_2a)m_3b + m_3^2\frac{b^2(1-a^2)}{3}$ *otherwise*.

Proof: According to Bezout's identity, m and $\hat{m}$ are coprime if and only if

$$\mathrm{GCD}(\hat{m}\,,\,m) = 1. \tag{21}$$

For $m = u_1 + u_2\theta + u_3\theta^2$, $\hat{m}$ can be expressed as:

$$\begin{aligned}\hat{m} = m'm'' = (u_1 + u_2\omega\theta + u_3\omega^2\theta^2)(u_1 + u_2\omega^2\theta \\ + u_3\omega\theta^2) = \hat{u}_1 + \hat{u}_2\theta + \hat{u}_3\theta^2,\end{aligned} \tag{22}$$

where $\hat{u}_1 = u_1^2 - u_2u_3r$, $\hat{u}_2 = u_3^2r - u_1u_2$, $\hat{u}_3 = u_2^2 - u_1u_3$. u_1, u_2 and u_3 are expressed in (9). Therefore, by fact 1, (21) can be rewritten as:

$$\mathrm{GCD}(\hat{m}, m) \;=\; \mathrm{GCD}(-\hat{m} + m^2 - 3u_1m\,,\,m) \;= 1. \tag{23}$$

Since $m^2 - \hat{m} - 3u_1m = 3ru_2u_3 - 3u_1 = -3\hat{u}_1$, the coprimality condition becomes:

$$\mathrm{GCD}(m\,,\,3\hat{u}_1) = 1, \tag{24}$$

Notice that $3\hat{u}_1$ is always in $\mathbb{Z}$. If $r \equiv \pm 1 \pmod 9$, i.e., $a^2 \equiv b^2 \mod 9$, $b^2(1-a^2) = b^2(1-b^2) - 9k'b^2$ where $k' \in \mathbb{Z}$; Because $b^2(1-b^2)$ can always be divided by 3 (Section 3.1), $b^2(1-a^2)/3$ is in $\mathbb{Z}$. The other case is obvious. By Lemma 2, (24) holds if and only if (20) holds. ■

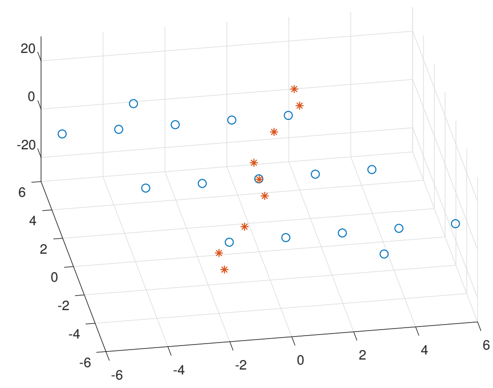
According to Lemma 1, the coprimality of algebraic integers implies the coprimality of their matrices, and vice versa. Therefore, the generalized circulant matrices that were discussed in [13] can be interpreted as a special case of matrices of algebraic integers by substituting $b = 1$ to (7) and (13) for cubefree r, and in this case, the coprime condition of Theorem 1 is equivalent to that in [13]. It is remarkable that by considering the cases $r \not\equiv \pm 1 \pmod 9$ with $b \neq 1$ and $r \equiv \pm 1 \pmod 9$, Theorem 1 offers significantly more options of coprime matrix pairs.

3.3. 3D CRT Arrays

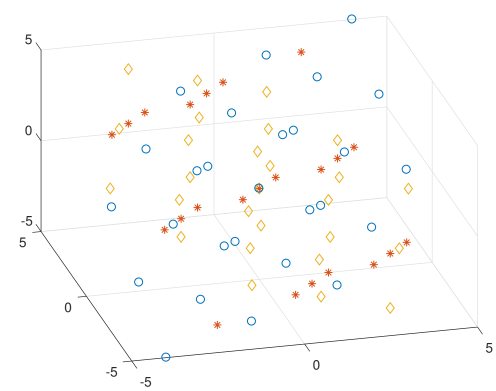
Examples of novel 3D CRT arrays are illustrated in Fig. 1. By Theorem 1, it is easy to check two coprime integers $\gamma = 5 + 2\sqrt[3]{12} + \sqrt[3]{12^2}$ and $\hat{\gamma} = 1 + 2\sqrt[3]{12} - \sqrt[3]{12^2}$ in the ring of integers of $\mathbb{Q}(\sqrt[3]{12})$. Because $r \not\equiv \pm 1 \mod 9$, the basis of the ring of integers is $\left\{1,\ \sqrt[3]{12},\ \sqrt[3]{12^2}/2\right\}$. By substituting $a = 3$ and $b = 2$ to (7) and (13), we obtain the matrices corresponding to γ and $\hat{\gamma}$

$$\mathbf{B}_\gamma = \begin{pmatrix} 5 & 2 & 2 \\ 12 & 5 & 4 \\ 12 & 6 & 5 \end{pmatrix}, \mathbf{B}_{\hat{\gamma}} = \begin{pmatrix} 1 & 2 & -2 \\ -12 & 1 & 4 \\ 12 & -6 & 1 \end{pmatrix} \tag{25}$$

respectively. Another example is $v = 3 + \sqrt[3]{4}$, $\eta = -1 + 2\sqrt[3]{2} + \sqrt[3]{4}$, and $u = 1 - 2\sqrt[3]{4}$ in $\mathbb{Z}[\sqrt[3]{2}]$, and their corresponding matrices $\mathbf{B}_v$, $\mathbf{B}_\eta$ and $\mathbf{B}_u$ calculated by (8) are



(a)



(b)

Fig. 1. Examples of 3D CRT arrays: (a) lattices generated by $\sigma(\langle\,\gamma\,\rangle)$ in red stars and $\sigma(\langle\,\hat{\gamma}\,\rangle)$ in blue dots. (b) lattices generated by $\sigma(\langle\,u\,\rangle)$ in red stars $\sigma(\langle\,v\,\rangle)$ in blue dots, and $\sigma(\langle\,\eta\,\rangle)$ in yellow diamonds.

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & 2 \\ 4 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -2 \\ -4 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \tag{26}$$

respectively. In this case $\mathbf{B}_v$ is not the adjugate matrix of $\mathbf{B}_u$ or $\mathbf{B}_\eta$, i.e., the three integers do not have to satisfy the conjugation relationship.

4. CONCLUDING REMARKS

A new class of 3D coprime arrays is designed by mapping algebraic integers of cubic fields into the 3D space. Based on generalized Bezout's identity, the coprimality condition of two 3-by-3 matrices is reduced to that for $\mathrm{N}(m)$ (or $|\det(\mathbf{B}_m)|$) and $\hat{u}_1$ in $\mathbb{Z}$, which is significantly easier to compute. Future work will focus on applications of the presented methods, as well as general forms of coprime matrices of dimension n and the generalization of the coprimality condition of two or more arbitrary algebraic integers.

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