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4-CRITICAL 4-VALENT PLANAR GRAPHS CONSTRUCTED WITH CROWNS

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Abstract.

A construction of arbitrarily large 4-critical 4-valent planar graphs with aid of socalled "crowns" is given which also proves a conjecture of B. Grünbaum. Furthermore some coloring properties of the aid graphs are derived

Introduction.

All graphs considered are finite, connected, planar, undirected, and have no loops. A vertex is r-valent if it is incident with r edges. A graph is r-valent if each of its vertices is r-valent. A (vertex) coloring is an integer valued function on the vertices of a graph. It is a proper coloring if adjacent vertices have different colors (values). A graph G has the chromatic number $\chi(G) = k$ if there is a proper coloring of G with k different colors (a k-coloring) but none with fewer colors. G is k-critical if $\chi(G) = k$ and $\chi(G') < k$ for every proper subgraph G' of G.

4-critical 4-valent graphs were known in the past only for the nonplanar case [1]. G. A. Dirac and T. Gallai even conjectured [1] that every 4-critical planar graph contains 3-valent vertices. But the author found in 1984 [3] a 4-critical 4-valent planar graph G^* (see Fig. 1). In this paper a construction is given which generates infinite families of 4-critical 4-valent planar graphs and which is based on G^* and on some symmetric planar graphs here called "crowns". This also proves a recent conjecture of B. Grünbaum [4] that there exist arbitrarily large 4-critical 4-valent planar graphs. Some lemmas concerning crown colorings support the proofs of the results.

In the sequel, vertex indices shall be taken modulo s (s integer, $s > 1$; $x_{s+1} = x_1$, etc.). An edge joining two vertices x_1, x_2 is denoted (x_1, x_2), $c(x)$ denotes the color of x. In a colored vertex sequence an r-*block* is a maximal subsequence of r consecutive vertices of the same color ($r > 1$). Each graph G is assumed to be properly embedded in an infinite plane with a given outside region (G is said to be *plane*). G^0 arises from G by removal of the edges which bound the

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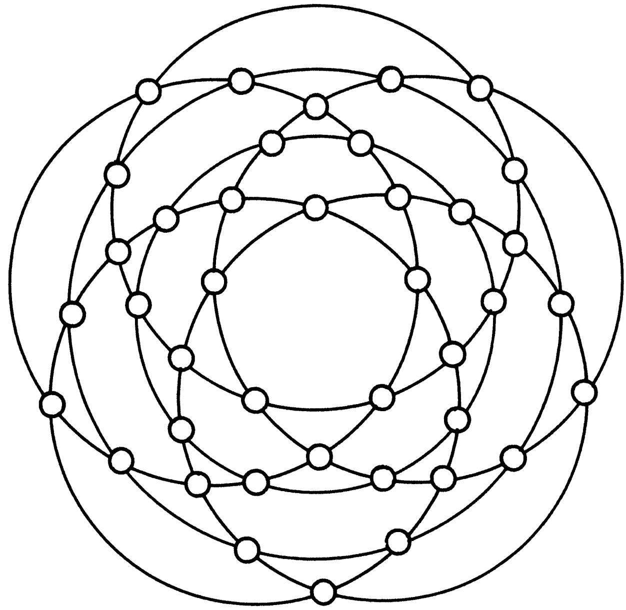

Figure 1. The 4-critical graph G^*.

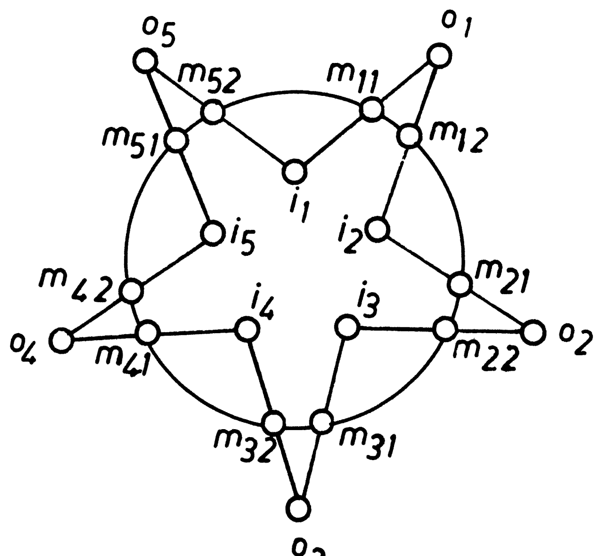

Figure 2. The 5-crown C_5.

outside region of G. Let G have an outside region such that its boundary contains vertices, some of which are labelled. Then $[G]$ denotes the graph which arises from G by joining consecutive labelled vertices by new edges, drawn along the boundary of the outside region of G (if we label all vertices which bound the outside region of G then $G = [G^0]$).

The operation of "crowning".

DEFINITION 1. An *s-crown* C_s is a plane graph, the vertex-set $V(C_s)$ of which is the disjoint union of the following subsets: $O = \{o_j\}$ (outside vertices), $I = \{i_j\}$ (inside vertices), $M = \{m_{jk}\}$ (midside vertices). o_j is adjacent to m_{j1}, m_{j2}; i_j to $m_{j-1,2}$, m_{j1}; m_{j1} to $m_{j-1,2}$, m_{j2}, i_j, o_j; m_{j2} to m_{j1}, $m_{j+1,1}$, i_{j+1}, o_j ($j = 1, \ldots, s$; $k = 1, 2$; Fig. 2 shows C_5 as an example).

DEFINITION 2. LET A_G be an s-gon (with vertices $a_1, \ldots, a_s$ in cyclic order) that bounds the outside region of plane graph G. The plane graph given by

$$F = G \circ C_s, \tag{1}$$

arises from G^0 and $[C_s]$ (where $o_1, \ldots, o_s$ in C_s are labelled) by identifying the boundaries of

- a closed disc containing G^0, such that its boundary contains only the vertices $a_1, \ldots, a_s$ from G^0
- $[C_s]$ minus an open disc from its inside region (the one having $i_1, \ldots, i_s$ on its boundary), such that the boundary of what remains contains only vertices $i_1, \ldots, i_s$ from $[C_s]$

in such a way that a_j is identified with i_j ($j = 1, \ldots, s$). F shall be called an *s-crowning* of G. If G is 4-valent then so is F. An n-fold s-crowning

$$F = (\ldots((G \circ C_s) \circ C_s) \ldots) \circ C_s$$

shall be abbreviated by

$$F = G \circ C_s^n. \tag{2}$$

REMARK 1. Let L_s be a 4-valent plane graph, the vertex-set $V(L_s)$ of which is the disjoint union of the following subsets: $U = \{u_j\}$ (inside vertices) and $A = \{a_j\}$ (outside vertices). u_j is adjacent to u_{j-1}, u_{j+1}, a_{j-1}, a_j; a_j to a_{j-1}, a_{j+1}, u_j, u_{j+1} ($j = 1, \ldots, s$; for $s = 5$ see Fig. 3). Then G^* (from Fig. 1) is a 2-fold 5-crowning of L_5:

$$G^* = L_5 \circ C_5^2. \tag{3}$$

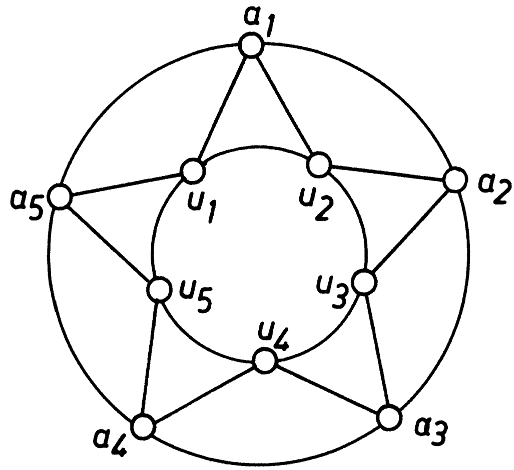

Figure 3. The graph L_5.

The results.

PROPOSITION.

(a) *Let G be a* 4*-critical plane graph bounded on the outside by a triangle. Then*

$$F = G \circ C_3 \tag{4}$$

is also 4*-critical.*

(*b*) *Let*

$$F_s^{(k)} = L_s \circ C_s^k. \tag{5}$$

Then $F_5^{(k)}$ *is* 4*-critical for* $k > 1$ (for L_s see Remark 1 and Fig. 3).

REMARK 2. With a suitable embedding in the plane, repeated 3-crowning of the graph G^* (Fig. 1) generates an infinite family of 4-critical 4-valent plane graphs. Similarly, $F_5^{(k)}$ with $k > 1$ is another such family, and its smallest member is also G^*.

Lemmas and proofs.

Let $c(I)$ be a given k-coloring ($k < 4$) of the inside vertices of C_s. Then $c(C, I)$ denotes a 3-coloring of C_s which extends $c(I)$. The existence of some $c(C, I)$ which extends any $c(I)$ is established in Lemma 2. Let $c(A)$, $c(B)$ be k-colorings of vertex sequences $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ resp. $c(A)$, $c(B)$ are said to be *isochromal* if they can be made to coincide by an index translation (reflexion) or a color permutation or a combination of both transitions. A coloring of an edge sequence $A = \{a_1, \dots, a_n\}$ is denoted $(c(a_1), \dots, c(a_n))$.

REMARK 3. For any 3-coloring of C_s and for any $j \in \{1,\dots,s\}$:
if $c(i_j) = c(i_{j+1}) = c$ then

$$c(o_j) = c. \tag{6}$$

LEMMA 1. (color invariance).

(a) *At every 3-coloring of C_s using colors 1, 2 & 3, the numbers of inside and outside vertices colored with j ($j = 1, 2, 3$) are equal.*

(b) *If $c(I)$ is an inside coloring of C_s using at most two colors then for each $c(C, I)$ one of the following holds*

$$c(o_j) = c(i_j) \quad \text{for} \quad j = 1,\dots,s \quad \text{or} \tag{7'}$$

$$c(o_j) = c(i_{j+1}) \quad \text{for} \quad j = 1,\dots,s. \tag{7''}$$

PROOF. Let I_j, M_j, O_j be the subsets of inside, midside, and outside vertices resp. which are colored by j ($j = 1, 2, 3$). Then $|I_j| + |M_j| = |O_j| + |M_j| = s$, and hence

$$|I_j| = |O_j|, \tag{8}$$

which proves (a).

In case (b) let $c(I)$ have colors 1, 2 (1). Then all m_{j1} or all m_{j2} ($j = 1,\dots,s$) must have color 3, and from this follows (7″) or (7′) resp.

If $c(A)$ is a k-coloring of a vertex sequence A so that the number of vertices colored by j ($j \in 1,\dots,k\}$) is greater than the number of the remaining vertices of A then $c(A)$ shall be called a *dominant coloring*. Then A contains at least one block. From Lemma 1 follows immediately:

COROLLARY. *If $c(I)$ is a dominant k-coloring of the inside vertices of C_s ($k < 4$) then each $c(C, I)$ restricts to a dominant outside k-coloring $c(O)$.*

LEMMA 2. *Let $c(I)$ be a k-coloring ($k < 4$) of the inside vertices of C_s.*

(a) *For every $c(I)$ there exists some $c(C, I)$.*

(b) *If I has no block then there is a $c(C, I)$ which restricts to an outside k-coloring $c(O)$, and $c(O)$ is isochromal with $c(I)$.*

PROOF. We introduce a "diagonal coloring" of C_s (which forces a $c(C, I)$) by the following (r integer > 0; $j = 1,\dots,s$): If

$$c(i_j) \neq c(i_{j+1}) = \dots = c(i_{j+r}) \neq c(i_{j+r+1}) \quad \text{then} \quad c(o_{j+r}) = c(m_{j2}) = c(i_j). \tag{9}$$

(6) and (9) cause a unique outside coloring $c(O)$. It remains to determine the midside coloring $c(M)$. In the antiblock case ($r = 1$) the 4 neighbours of $m_{j+1,1}$ are colored with exactly 2 different colors and therefore $c(m_{j+1,1})$ is uniquely determined. In the block case ($r > 1$) $c(o_{j+r})$ determines the colors of $m_{j+1,1}$,

$m_{j+1,2}, \ldots, m_{j+r,1}$ which must be different from the block color. In the blockfree case (b) clearly (9) gives on outside coloring $c(O)$ which is isochromal with $c(I)$.

Lemma 3. *Let G be a plane graph and $F = G \circ C_s$. If $G(G^0)$ has chromatic number 3 then so has F (F^0).*

Proof. If G^0 has chromatic number 3 then so has F^0 because of the existence of a $c(C, I)$ for every 3-coloring $c(I)$ of the inside vertices of C_s (Lemma 2 (a)). From Lemma 2 (b) follows that F has chromatic number at most 3 if G has. Clearly F has chromatic number at lest 3, since it contains triangles.

Proof of the Proposition. For the k-criticality of a graph H it is sufficient that

(10) $$\chi(H) = k \quad (H \text{ is } k\text{-chromatic}), \text{ and}$$

(11) $$\chi(H - e) = k - 1$$

holds for each edge e of H. A path P in a k-chromatic graph H shall be called a *critical path*, it for some coloring of the graph $H - P$ using $k - 1$ colors, there are two colors c_1, c_2, so that $c_1(c_2)$ does not occur among the neighbours outside P of all (all interior) vertices of P. For each edge e of a critical path P of H holds (11). In the following let $k = 4$.

(a) Let $H = F$ (F from (4)). Since G in (4) is 4-critical, any 3-coloring of G^0 restricts to a dominant coloring of the outside vertices a_1, a_2, a_3 of G (isochromal with (1, 1, 2)). Let $c(a_1) = 1, c(a_2) = 1$, and $c(a_3) = 2$. From Lemma 1 (b) follow for o_j, m_{jk} in C_3 ($j = 1, 2, 3$; $k = 1, 2$):

(12) $$c(o_1) = c(o_2) = 1,\ c(o_3) = 2 \quad \text{or}$$

(13) $$c(o_1) = c(o_3) = 1,\ c(o_2) = 2,$$

which gives (10). If (12) holds then we have $c(M) = (2,3; 2,3; 1,3)$ as midside coloring ((13) forces an analogous midside coloring). From Lemma 2 (b) follows (11) if e is an edge of G^0. From the above follows (11) for e (o_1, o_2). Moreover, the vertex sequence $\{o_1, o_2, m_{22}, m_{31}, m_{32}, i_1\}$ forms a critical path P of F. From the symmetry of C_3 now follows (11) for each edge e of F and hence the 4-criticality of F in (4).

(b) From Lemma 3 follows that each $(F_s^{(k)})^0$ $(s > 1, k \geqq 0)$ is 3-colorable. Each 3-coloring of L_5^0 restricts to a dominant coloring of A isochromal with (1, 1, 1, 2, 3). For $H = F_5^{(k)}$ follows (10) from the Corollary. The proof of the 4-criticality proceeds by induction. We start with $G^* = F_5^{(2)}$ which is shown to be 4-critical by [3]. Assume $F_5^{(k-1)}$ is 4-critical for some $k > 2$. From the 4-criticality and from Lemma 1 (color invariance) follows that there are 3-colorings of $(F_5^{(k-1)})^0$ which restrict to outside colorings isochromal with (1, 1, 2, 1, 3). Such a 3-coloring we extend to a 3-coloring of $(F_5^{(k)})^0$:

(1, 1, 2, 1, 3)	inside of C_5,
(3,2; 3,1; 3,2; 3,2; 1,2)	midside of C_5,
(1, 2, 1, 1, 3)	outside of C_5.

The vertices $o_4, o_3, m_{31}, m_{22}, m_{21}, i_2$, form a critical path P' of $F_5^{(k)}$. From the symmetry of $F_5^{(k)}$ and from the 4-criticality of $F_5^{(k-1)}$ follows (11) for each edge e of $F_5^{(k)}$.

Concluding remarks.

1. Without detailed proofs we state that:

(14) $$\chi(F_s^{(k)}) = 4 \quad \text{for} \quad s = 2, 4, 5 \quad \text{and} \quad k >= 0,$$

(15) $$\chi(F_s^{(k)}) = 3 \quad \text{for} \quad s = 3, s > 5 \quad \text{and} \quad k > 0.$$

(14) follows from the above Lemmas and the Corollary. To show (15) it is sufficient to establish 3-colorings of $F_s^{(1)}$ for $s = 3$ and $s > 5$. From Lemma 3 then follows (15) for $k > 1$.

2. Neither $F_2^{(k)}$ nor $F_4^{(k)}$ is 4-critical for any k. The first contains double-edges and the second cannot be 4-critical because every 3-coloring of $(F_4^{(k)})^0$ restricts to an outside 2-coloring with 2 2-blocks or to a 1-coloring (Lemma 2).

3. An old conjecture of H. Grötzsch ws stated by H. Sachs as follows: "Let G be a finite 4-regular plane graph generated by a set of simple closed curves (= Jordan curves) (i.e., every vertex of G is an intersection point of exactly two of the generating curves of G (which are not allowed to touch)). H. Grötzsch conjectured that $\chi(G) = 3$." The author found two counterexamples, namely the graph $F_4^{(1)}$ which was published in 1984 [2] and the graph $F_5^{(2)} = G^*$ [3]. It is easy to verify that $F_s^{(s-3)}$ for $s > 2$ is of the Grötzschian type, but there are no more 4-chromatic graphs among them because of statement (15).

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