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CHAPTER 4

Monotone Dynamical Systems

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0. Introduction

This chapter surveys a restricted but useful class of dynamical systems, namely, those enjoying a comparison principle with respect to a closed order relation on the state space. Such systems, variously called monotone, order-preserving or increasing, occur in many biological, chemical, physical and economic models.

The following notation will be used. $\mathbb{Z}$ denotes the set of integers; $\mathbb{N} = \{0, 1, \dots\}$, the set of natural numbers; $\mathbb{N}_+$ is the set of positive integers, and $\mathbb{R}$ is the set of real numbers. For $u, v \in \mathbb{R}^n$ (= Euclidean n-space), we write

$$\begin{aligned} u \leqslant v &\iff u_i \leqslant v_i, \\ u < v &\iff u_i \leqslant v_i, \quad u \neq v, \\ u \ll v &\iff u_i < v_i, \end{aligned}$$

where $i = 1, \dots, n$. This relation $\leqslant$ is called the *vector order* in $\mathbb{R}^n$.

The prototypical example of monotone dynamics is a Kolmogorov model of cooperating species,

$$\dot{x}_i = x_i G_i(x), \quad x_i \geqslant 0, \quad i = 1, \dots, n \tag{0.1}$$

in the positive orthant $\mathbb{R}^n_+ = [0, \infty)^n$, where $G : \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuously differentiable. x_i denotes the population and G_i the *per capita* growth rate of species i. Cooperation means that an increase in any population causes an increase of the growth rates of all the other populations, modeled by the assumption that $\partial G_i / \partial x_j \geqslant 0$ for $i \neq j$. The right-hand side $F_i = x_i G_i$ of (0.1) then defines a *cooperative vector field* $F : \mathbb{R}^n \to \mathbb{R}^n$, meaning that $\partial F_i / \partial x_j \geqslant 0$ for $i \neq j$.

Assume for simplicity that solutions to Eq. (0.1) are defined for all $t \geqslant 0$. Let $\Phi = \{\Phi_t : \mathbb{R}^n_+ \to \mathbb{R}^n_+\}_{t \geqslant 0}$ denote the resulting semiflow in $\mathbb{R}^n_+$ that describes the evolution of states in positive time: the solution with initial value u is given by $x(t) = \Phi_t(u)$. The key to the long-term dynamics of cooperative vector fields is an important differential inequality due to Müller [148] and Kamke [91].

$$u \leqslant v \text{ and } t \geqslant 0 \implies \Phi_t(u) \leqslant \Phi_t(v).$$

In other words: *The maps Φ_t preserve the vector order.* A semiflow Φ with this property is called *monotone*. Monotone semiflows and their discrete-time counterparts, order-preserving maps, form the subject of Monotone Dynamics.

Returning to the biological setting, we may make the assumption that each species directly or indirectly affect all the others. This is modeled by the condition that the Jacobian matrices $G'(x)$ are irreducible. An extension of the Müller–Kamke theorem shows that in the *open orthant* $\operatorname{Int} \mathbb{R}^n$, the restriction of Φ is *strongly monotone*: If $u, v \in \operatorname{Int} \mathbb{R}^n$, then

$$u < v \text{ and } t > 0 \implies \Phi_t(u) \ll \Phi_t(v).$$

A semiflow with this property is *strongly monotone*.

Similar order-preserving properties are found in other dynamical settings, including delay differential equations and quasilinear parabolic partial differential equations. Typically the state space is a subset of a (real) Banach space Y with a distinguished closed cone $Y_+ \subset Y$. An order relation is introduced by $x \geqslant y \Leftrightarrow x - y \in Y_+$. When Y is a space of real valued functions on some domain, Y_+ is usually (but not always) the cone of functions with values in $\mathbb{R}_+ := [0, \infty)$. When $Y = \mathbb{R}^n$, the cooperative systems defined above use the cone $\mathbb{R}^n_+$.

Equations (0.1) model an ecology of competing species if $\partial G_i / \partial x_j \leqslant 0$ for $i \neq j$. The resulting vector field K with components $K_i = x_i G_i$ is not generally cooperative, but its negative $F = -K$ *is* cooperative. Many dynamical properties of the semiflow of K can be deduced from that of F, which is monotone.

We will see that the long-term behavior of monotone systems is severely limited. Typical conclusions, valid under mild restrictions, include the following:

- If all forward trajectories are bounded, the forward trajectory of almost every initial state converges to an equilibrium.
- There are no attracting periodic orbits other than equilibria, because every attractor contains a stable equilibrium.
- In $\mathbb{R}^3$, every compact limit set that contains no equilibrium is a periodic orbit that bounds an invariant disk containing an equilibrium.
- In $\mathbb{R}^2$, each component of any solution is eventually increasing or decreasing.

Other cones in $\mathbb{R}^n$ are also used, especially the orthants defined by restricting the sign of each coordinate. For example, a system of two competing species can be modeled by ODEs

$$\dot{y}_i = y_i H_i(y); \quad y_i \geqslant 0, \quad i = 1, 2$$

with $\partial H_i / \partial y_j < 0$ for $i \neq j$. The coordinate change $x_1 = y_1$, $x_2 = -y_2$ converts this into a cooperative system in the second orthant K defined by $x_1 \geqslant 0 \geqslant x_2$. This system is thus both competitive and cooperative, albeit for different cones. Not surprisingly, the dynamics are very simple.

In view of such powerful properties of cooperative vector fields, it would be useful to know when a given field F in an open set $D \subset \mathbb{R}^n$ can be made cooperative or competitive by changing coordinates. The following sufficient condition appears to be due to DeAngelis et al. [39]; see also Smith [193], Hirsch [74]. Assume the Jacobian matrices $[a_{ij}(x)] = F'(x)$ have the following two properties:

(1) (Sign stability) If $i \neq j$ then a_{ij} does not change sign in D;

(2) (Sign symmetry) $a_{ij} a_{ji} \geqslant 0$ in D.

Let Γ be the combinatorial labeled graph with nodes $1, \dots, n$ and an edge e_{ij} joining i and j labeled $\sigma_{ij} \in \{+, -\}$ if and only if $i \neq j$ and there exists $p \in D$ such that $\operatorname{sgn} a_{ij}(p) = \sigma_{ij} \neq 0$. Then F is cooperative (respectively, competitive) relative to some orthant if and only if in every closed loop in Γ the number of negative labels is even (respectively, odd).

Order-preserving dynamics also occur in discrete time systems. Consider a nonautonomous Kolmogorov system $\dot{x}_i = x_i H_i(t, x)$, where the map $H := (H_1, \dots, H_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ has period $\tau > 0$ in t. Denote by $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ the Poincaré map, which to $x \in \mathbb{R}^n_+$ assigns $y(\tau)$ where $y(t)$ denotes the solution with initial value x. Then T is monotone provided the $\partial H_i/\partial x_j \geqslant 0$ for $i \neq j$, and strongly monotone in the open orthant when these matrices are also irreducible. Most of the results stated above have analogs for T.

Convergence and stability properties of several kinds of order-preserving semiflows are developed in Sections 1 and 2, in the setting of general ordered metric spaces. Section 3 treats ODEs whose flows preserve the order defined by a cone in $\mathbb{R}^n$. Delay differential equations are studied in Section 4. In Section 5 we present results on order-preserving maps. The final section applies the preceding results to second order quasilinear parabolic equations.

1. Strongly order-preserving semiflows

This section introduces the basic definitions and develops the main tools of monotone dynamics. Several results on density of quasiconvergent points are proved, and used to establish existence of stable equilibria.

1.1. *Definitions and basic results*

The setting is a semiflow $\Phi = \{\Phi_t\}_{0 \leqslant t < \infty}$ in a (partially) ordered metric space that preserves the weak order relation: $x \leqslant y$ implies $\Phi_t(x) \leqslant \Phi_t(y)$. Such semiflows, called *monotone*, have severely restricted dynamics; for example, in $\mathbb{R}^n$ with the vector ordering there cannot be stable periodic orbits other than equilibria. But for generic convergence theorems we need semiflows with the stronger property of being "strongly order preserving," together with mild compactness assumptions. In later sections we will see that these conditions are frequently encountered in applications. The centerpiece of this section is the Limit Set Dichotomy, a fundamental tool for the later theory.

1.1.1. *Ordered spaces* Let Z be a metric space and $A, B \subset Z$ subsets. The closure of A is denoted by $\overline{A}$ and its interior by $\operatorname{Int} A$. The distance from A to B is defined as $\operatorname{dist}(A, B) := \inf_{a \in A, b \in B} d(a, b)$. When B is a singleton $\{b\}$ we may write this as $\operatorname{dist}(A, b) = \operatorname{dist}(b, A)$.

X always denotes an *ordered space*. This means X is endowed with a metric d and an order relation $\mathsf{R} \subset X \times X$. As usual we write $x \leqslant y$ to mean $(x, y) \in \mathsf{R}$, and the order relation is:

(i) reflexive: $x \leqslant x$ for all $x \in X$,
(ii) transitive: $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$,
(iii) antisymmetric: $x \leqslant y$ and $y \leqslant x$ implies $x = y$.

In addition, the ordering is compatible with the topology in the following sense:

(iv) if $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leqslant y_n$, then $x \leqslant y$.

This is just to say that R is a closed subset of $X \times X$.

We write $x < y$ if $x \leqslant y$ and $x \neq y$. Given two subsets A and B of X, we write $A \leqslant B$ ($A < B$) when $x \leqslant y$ ($x < y$) holds for each choice of $x \in A$ and $y \in B$. The relation $A \leqslant B$ *does not imply* "$A < B$ or $A = B$"!

The notation $x \ll y$ means that there are open neighborhoods U, V of x, y respectively such that $U \leqslant V$. Equivalently, (x, y) belongs to the interior of R. The relation $\ll$, sometimes referred to as the *strong ordering*, is transitive; in many cases it is empty. We write $x \geqslant y$ to mean $y \leqslant x$, and similarly for $>$ and $\gg$.

We call X an *ordered subspace* of an ordered space X' if $X \subset X'$, and the order and topology on X are inherited from X'. When this is this case, the relation $u < v$ for points $u, v \in X$ means the same thing whether u and v are considered as points of X, or points of X'. But there are simple examples for which $u \ll v$ is true in X', yet false in X.

Let X be an ordered space. The *lower boundary* of a set $U \subset X$ is the set of points x in the boundary of U such that every neighborhood of x contains a point $y \in U$ with $y > x$. The *upper boundary* of U is defined dually.

Two points $x, y \in X$ are *order related* if $x < y$ or $y < x$; otherwise they are *unrelated*. A subset of X is *unordered* if it does not contain order related points. The empty set and singletons are unordered.

The (closed) *order interval* determined by $u, v \in X$ is the closed set

$$[u, v] = [u, v]_X := \{x \in X\colon\ u \leqslant x \leqslant v\}$$

which may be empty. The *open order interval* is the open set

$$[[u, v]] = \{x \in X\colon\ u \ll x \ll v\}.$$

A subset of X is *order bounded* if it lies in an order interval, and *order convex* if it contains $[u, v]$ whenever it contains u and v.

A point $x \in X$ is *accessible from below* if there is a sequence $x_n \to x$ with $x_n < x$; such a sequence is said to approximate x from below. We define *accessible from above* dually, that is, by replacing $<$ with $>$. In most applications there is a dense open subset of points that are accessible from both above and below.

The *supremum* $\sup S$ of a subset $S \subset X$, if it exists, is the unique point a such that $a \geqslant S$ and $x \geqslant S \Rightarrow x \geqslant a$. The *infimum* $\inf S$ is defined dually, i.e., substituting $\leqslant$ for $\geqslant$. A *maximal element* of S is a point $a \in S$ such that $x \in S$ and $x \geqslant a$ implies $x = a$. A *minimal element* is defined dually.

The following basic facts are well known:

LEMMA 1.1. *Assume the ordered space X is compact.*

(i) *Every sequence in X that is increasing or decreasing converges.*

(ii) *If X is totally ordered, it contains a supremum and an infimum.*

(iii) *X contains a maximal element and a minimal element.*

PROOF. (i) If p and q denote subsequential limits, then $p \leqslant q$ and $q \leqslant p$, hence $p = q$.

(ii) For each $x \in X$, the set $B_x := \{y \in X\colon\ y \geqslant x\}$ is compact, and every finite family of such sets has nonempty intersection because X is totally ordered. Therefore there exists $a \in \bigcap_x B_x$, and clearly $a = \sup X$. Similarly, $\inf X$ exists.

(iii) Apply (ii) to a maximal totally ordered subset (using Zorn's lemma). □

An *ordered Banach space* is an ordered space whose underlying metric space is a Banach space Y, and such that the set $Y_+ = \{y \in Y\colon\ y \geqslant 0\}$ is a cone, necessarily closed and convex. Thus Y_+ is a closed subset of Y with the properties:

$$\mathbb{R}_+ \cdot Y_+ \subset Y_+, \qquad Y_+ + Y_+ \subset Y_+, \qquad Y_+ \cap (-Y_+) = \{0\}.$$

We always assume $Y_+ \neq \{0\}$.

When $\operatorname{Int} Y_+$ is nonempty we call Y a *strongly ordered* Banach space. In this case $x \ll y \Leftrightarrow y - x \in \operatorname{Int} Y_+$.

The most important examples of ordered Banach spaces are completions of normed vector spaces of real-valued functions on some set Ω, with the positive cone corresponding to nonnegative functions. This cone defines the *functional ordering*. The simplest case is obtained from $\Omega = \{1, 2, \ldots, n\}$: here $Y = \mathbb{R}^n$ and $Y_+ = \mathbb{R}^n_+$, the *standard cone* comprising vectors with all components nonnegative. For the corresponding *vector ordering*, $x \leqslant y$ means that $x_i \leqslant y_i$ for all i. Other function spaces are used in Sections 4 and 6.

When Y is an ordered Banach space, the notation $X \subset Y$ tacitly assumes that X is an ordered subspace of Y (but not necessarily a linear subspace).

A subset S of an ordered Banach space is *p-convex* if it contains the line segment spanned by u, v whenever $u, v \in S$ and $u < v$.

1.1.2. *Semiflows* All maps are assumed to be continuous unless the contrary is indicated.

A *semiflow* on X is a map $\Psi : \mathbb{R}_+ \times X \to X$, $(t, x) \mapsto \Psi_t(x)$ such that:

$$\Psi_0(x) = x, \qquad \Psi_t\big(\Psi_s(x)\big) = \Psi_{t+s}(x) \quad (t, s \geqslant 0,\ x \in X).$$

Thus Ψ can be viewed as a collection of maps $\{\Psi_t\}_{t \in \mathbb{R}_+}$ such that Ψ_0 is the identity map of X and $\Psi_t \circ \Psi_s = \Psi_{t+s}$, and such that $\Psi_t(x)$ is continuous in (t, x).

A *flow* in a space M is a continuous map $\Psi : \mathbb{R} \times M \to M$, written $\Psi(t, x) = \Psi_t(x)$, such that

$$\Psi_0(x) = x, \qquad \Psi_t\big(\Psi_s(x)\big) = \Psi_{t+s}(x) \quad (t, s \in \mathbb{R}, x \in X).$$

Restricting a flow to $\mathbb{R}_+ \times M$ gives a semiflow. A C^1 vector field F on a compact manifold M, tangent to the boundary, generates a *solution flow*, for which the trajectory of x is the solution $u(t)$ to the initial value problem $\mathrm{d}u/\mathrm{d}t = F(u)$, $u(0) = x$.

The *trajectory* of x is the map $[0, \infty) \to X$, $t \mapsto \Psi_t(x)$; the image of the trajectory is the *orbit* $O(x, \Psi)$, denoted by $O(x)$ when Ψ is understood. When $O(x) = \{x\}$ then x is an *equilibrium*. The set of equilibria is denoted by E.

x and its orbit are called *T-periodic* if $T > 0$ and $\Psi_T(x) = x$; such a T is a *period* of x. In this case $\Psi_{t+T}(x) = \Psi_t(x)$ for all $t \geqslant 0$, so $O(x) = \Psi([0, T] \times \{x\})$. A periodic point is *nontrivial* if it is not an equilibrium.

A set $A \subset X$ is *positively invariant* if $\Psi_t A \subset A$ for all $t \geqslant 0$. It is *invariant* if $\Psi_t A = A$ for all $t \geqslant 0$. Orbits are positively invariant and periodic orbits are invariant.

A set K is said to *attract* a set S if for every neighborhood U of K there exists $t_0 \geqslant 0$ such that $t > t_0 \Rightarrow \Psi_t(S) \subset U$; when $S = \{x\}$ we say K attracts x. An *attractor* is a nonempty invariant set L that attracts a neighborhood of itself. The union of all such neighborhoods is the *basin* of L. If the basin of an attractor L is all of X then L is a *global attractor*.

The *omega limit set* of $x \in X$ is

$$\omega(x) = \omega(x, \Psi) := \bigcap_{t \geqslant 0} \overline{\bigcup_{s \geqslant t} \Psi_s(x)}.$$

This set is closed and positively invariant. When $\overline{O(x)}$ is compact, $\omega(x)$ is nonempty, compact, invariant and connected and it attracts $\overline{O(x)}$ (see, e.g., Saperstone [175]).

A point $x \in X$ is *quasiconvergent* if $\omega(x) \subset E$; the set of quasiconvergent points is denoted by Q. We call x *convergent* when $\omega(x)$ is singleton $\{p\}$; in this case $\Phi_t(x) \to p \in E$. We sometimes signal this by the abuse of notation $\omega(x) \in E$. The set of convergent points is denoted by C.

When all orbit closures are compact and E is totally disconnected (e.g., countable), then $Q = C$; because in this case every omega limit set, being a connected subset of E, is a singleton. For systems of ordinary differential equations generated by smooth vector fields, the Kupka–Smale theorem gives generic conditions implying that E is discrete (see Peixoto [157]); but in concrete cases it is often difficult to verify these conditions.

1.1.3. *Monotone semiflows* A map $f : X_1 \to X_2$ between ordered spaces is *monotone* if

$$x \leqslant y \implies f(x) \leqslant f(y),$$

strictly monotone if

$$x < y \implies f(x) < f(y),$$

and *strongly monotone* if

$$x < y \implies f(x) \ll f(y).$$

Let Φ denote a semiflow in the ordered space X. We call Φ monotone or strictly monotone according as each map Φ_t has the corresponding property.

We call Φ *strongly order-preserving*, SOP for short, if it is monotone and whenever $x < y$ there exist open subsets U, V of x, y respectively, and $t_0 \geqslant 0$, such that

$$\Phi_{t_0}(U) \leqslant \Phi_{t_0}(V).$$

Monotonicity of Φ then implies that $\Phi_t(U) \leqslant \Phi_t(V)$ for all $t \geqslant t_0$.

We call Φ *strongly monotone* if

$$x < y,\ 0 < t \implies \Phi_t(x) \ll \Phi_t(y)$$

and *eventually strongly monotone* if it is monotone and whenever $x < y$ there exists $t_0 > 0$ such that

$$t \geqslant t_0 \implies \Phi_t(x) \ll \Phi_t(y).$$

This property obviously holds when Φ is strongly monotone. We shall see in Section 6 that many parabolic equations generate SOP semiflows in function spaces that are not strongly ordered and therefore do not support strongly monotone semiflows.

Strong monotonicity was introduced in Hirsch [68,69], while SOP was proposed later by Matano [133,134] and modified slightly by Smith and Thieme [197,199]. We briefly explore the relation between these two concepts.

PROPOSITION 1.2. *If Φ is eventually strongly monotone, it is SOP. If X is an open subset of a Banach space Y ordered by a cone Y_+, Φ is SOP and the maps $\Phi_t : X \to X$ are open, then Φ is eventually strongly monotone. In particular, Φ is eventually strongly monotone provided Y is finite-dimensional, Φ is SOP and the maps Φ_t are injective.*

PROOF. If $x < y$ and Φ is eventually strongly monotone, then there exists $t_0 > 0$ such that $\Phi_{t_0}(x) \ll \Phi_{t_0}(y)$. Take neighborhoods $\widetilde{U}$ of $\Phi_{t_0}(x)$ and $\widetilde{V}$ of $\Phi_{t_0}(y)$ such that $\widetilde{U} < \widetilde{V}$. By continuity of Φ_{t_0}, there are neighborhoods U of x and V of y such that $\Phi_{t_0}(U) \subset \widetilde{U}$ and $\Phi_{t_0}(V) \subset \widetilde{V}$. Therefore, $\Phi_{t_0}(U) < \Phi_{t_0}(V)$ so Φ is SOP.

Suppose that $X \subset Y$ is open and ordered by Y_+ and Φ is SOP. If $x < y$ and U, V are open neighborhoods as in the definition of SOP, the inequality $\Phi_t(U) \leqslant \Phi_t(V)$ together with the fact that $\Phi_t(U)$ and $\Phi_t(V)$ are open in Y imply that $\Phi_t(x) \ll \Phi_t(y)$. □

The following very useful result shows that the defining property of SOP semiflows, concerning points $x < y$, extends to a similar property for compact sets $K < L$:

LEMMA 1.3. *Assume Φ is SOP and K, L are compact subsets of X satisfying $K < L$. Then there exists real numbers $t_1 \geqslant 0$, $\epsilon > 0$ and neighborhoods U, V of K, L respectively such that*

$$t \geqslant t_1 \quad \text{and} \quad 0 \leqslant s \leqslant \epsilon \implies \Phi_{t+s}(U) \leqslant \Phi_t(V).$$

PROOF. Let $x \in K$. For each $y \in L$ there exist $t_y \geqslant 0$, a neighborhood U_y of x, and a neighborhood V_y of y such that $\Phi_t(U_y) \leqslant \Phi_t(V_y)$ for $t \geqslant t_y$ since Φ is strongly order preserving. $\{V_y\}_{y \in L}$ is an open cover of L, so we may choose a finite subcover: $L \subset \bigcup_{i=1}^n V_{y_i} := \widetilde{V}$ where $y_i \in L$, $1 \leqslant i \leqslant n$. Let $\widetilde{U}_x = \bigcap_{i=1}^n U_{y_i}$, which is a neighborhood of x, and let $\tilde{t}_x = \max_{1 \leqslant i \leqslant n} t_{y_i}$. Then $\Phi_{\tilde{t}}(\widetilde{U}_x) \subset \Phi_{\tilde{t}}(U_{y_i}) \leqslant \Phi_{\tilde{t}}(V_{y_i})$, so $\Phi_t(\widetilde{U}_x) \leqslant \Phi_t(V_{y_i})$ for $t \geqslant \tilde{t}_x$. It follows that

$$t \geqslant \tilde{t}_x \implies \Phi_t(\widetilde{U}_x) \leqslant \Phi_t(\widetilde{V}).$$

Extract a finite subcover $\{\widetilde{U}_{x_j}\}$ of K from the family $\{\widetilde{U}_x\}$. Setting $U := \bigcup_j \widetilde{U}_{x_j} \supset K$ and $t_1 := \max_{1 \leqslant j \leqslant m} \tilde{t}_{x_j}$, we have

$$t \geqslant t_1 \implies \Phi_t(U) = \bigcup_j \Phi_t(\widetilde{U}_j) \leqslant \Phi_t(V).$$

In order to obtain the stronger conclusion of the lemma, note that for each $z \in K$ there exists $\epsilon_z > 0$ and a neighborhood U'_z of z such that $\Phi([0, \epsilon_z) \times W_z) \subset U$. Choose $z_1, \ldots, z_m$ in K so that $K \subset \bigcup_j U'_{z_j}$. Define $U' = \bigcup_j U'_{z_j}$ and $\epsilon = \min_j\{\epsilon_{z_j}\}$. If $x \in U'$ and $0 \leqslant s < \epsilon$ then $x \in U'_{z_j}$ for some j so $\Phi_s(x) \in U$. Thus $\Phi([0, \epsilon) \times U') \subset U$ so $\Phi_s(U') \subset U$. It follows that $\Phi_{t+s}(U') \subset \Phi_t(U) \leqslant \Phi_t(V)$ for $t \geqslant t_1$, $0 \leqslant s < \epsilon$. □

Several fundamental results in the theory of monotone dynamical systems are based on the following sufficient conditions for a solution to converge to equilibrium.

THEOREM 1.4 (Convergence Criterion). *Assume Φ is monotone, $x \in X$ has compact orbit closure, and $T > 0$ is such that $\Phi_T(x) \geqslant x$. Then $\omega(x)$ is an orbit of period T. Moreover, x is convergent if the set of such T is open and nonempty or Φ is SOP and $\Phi_T(x) > x$.*

PROOF. Monotonicity implies that $\Phi_{(n+1)T}(x) \geqslant \Phi_{nT}(x)$ for $n = 1, 2, \ldots$ and therefore $\Phi_{nT}(x) \to p$ as $n \to \infty$ for some p by the compactness of the orbit closure. By continuity,

$$\begin{aligned}
\Phi_{t+T}(p) &= \Phi_{t+T}\Bigl(\lim_{n\to\infty} \Phi_{nT}(x)\Bigr) \\
&= \lim_{n\to\infty} \Phi_{(n+1)T+t}(x) \\
&= \lim_{n\to\infty} \Phi_t\bigl(\Phi_{(n+1)T}(x)\bigr) \\
&= \Phi_t(p)
\end{aligned}$$

for all $t \geqslant 0$. Hence p is T-periodic.

To prove $\omega(x) = O(p)$, suppose $t_j \to \infty$ and $\Phi_{t_j}(x) \to q \in \omega(x)$ as $j \to \infty$, and write $t_j = n_j T + r_j$ where n_j is a natural number and $0 \leqslant r_j < T$. By passing to a subsequence if necessary, we may assume that $r_j \to r \in [0, T]$. Taking limits as $j \to \infty$ and noting that $n_j \to \infty$, we have by continuity:

$$\lim \Phi_{t_j}(x) = \lim \Phi_{r_j}\bigl(\lim \Phi_{n_j T}(x)\bigr) = \lim \Phi_{r_j}(p) = \Phi_r(p) = q.$$

Therefore $\omega(x) \subset O(p)$, and the opposite inclusion holds because $p \in \omega(x)$. This proves the first assertion of the theorem.

Suppose $\Phi_t(x) \geqslant x$ for all t in a nonempty open interval $(T - \epsilon, T + \epsilon)$. The first assertion shows that $\omega(x)$ is an orbit $O(p)$ of period τ for every $\tau \in (T - \epsilon, T + \epsilon)$. All elements of $O(p)$ have the same set G of periods; G is closed under addition and contains $(T - \epsilon, T + \epsilon)$. If $0 \leqslant s < \epsilon$ and $t \geqslant 0$ then

$$\Phi_{t+s}(p) = \Phi_t\bigl(\Phi_s(p)\bigr) = \Phi_t\bigl(\Phi_{s+T}(p)\bigr) = \Phi_t(p).$$

Hence $[0, \epsilon) \subset G$ and therefore $G = \mathbb{R}_+$, which implies $p \in E$. This proves the second assertion.

If $\Phi_T(x) > x$ and Φ is SOP then there exist neighborhoods U of x and V of $\Phi_T(x)$ and $t_0 > 0$ such that $\Phi_{t_0}(U) \leqslant \Phi_{t_0}(V)$. It follows that $\Phi_{t_0}(x) \leqslant \Phi_{t_0+T+\epsilon}(x)$ for all ϵ sufficiently small. The previous assertion implies $\omega(x) = p \in E$. □

1.2. *Nonordering of omega limit sets*

The next result is the first of several describing the order geometry of limit sets.

PROPOSITION 1.5 (Nonordering of Periodic Orbits). *A periodic orbit of a monotone semiflow is unordered.*

PROOF. Let x have minimal period $s > 0$ under a monotone semiflow Φ. Suppose $x \leqslant z \in O(x)$. By compactness of $O(x)$ there is a maximal $y \in O(x)$ such that $y \geqslant z \geqslant x$. By periodicity and monotonicity $y = \Phi_t(x) \leqslant \Phi_t(y)$, $t > 0$, hence $y = \Phi_t(y)$ by maximality. Therefore t is an integer multiple of s, so $x = \Phi_t(x) = y$, implying $x = z$. □

The following result, which implies (1.5), is a broad generalization of the obvious fact that for ODEs in $\mathbb{R}$, nonconstant solutions are everywhere increasing or everywhere decreasing. Let $J \subset \mathbb{R}$ be an interval and $f : J \to X$ a map. A compact subinterval $[a, b] \subset J$ is *rising* for f provided $f(a) < f(b)$, and *falling* if $f(b) < f(a)$.

THEOREM 1.6. *A trajectory of a monotone semiflow cannot have both a rising interval and a falling interval.*

This originated in Hirsch [67], with improvements in Smith [194], Smith and Waltman [203]. An analog for maps is given in Theorem 5.4.

PROOF. Let Φ be a monotone semiflow in X and fix a trajectory $f : [0, \infty) \to X$, $f(t) := \Phi_t(x)$. Call an interval $[d, d']$ *weakly falling* if $f(d) \geqslant f(d')$. Monotonicity shows that when this holds, the *right translates* of $[d, d']$—the intervals $[d + u, d' + u]$ with $u \geqslant 0$—are also weakly falling.

Proceeding by contradiction, we assume f has a falling interval $[a, a + r]$ and a rising interval $[c, c + q]$. To fix ideas we assume $a \leqslant c$, the case $c \leqslant a$ being similar. Define

$$b := \sup\{t \in [c, c + q]\colon\ f(t) \leqslant f(c),\ s := c + q - b\}.$$

Then $[b, b + s]$ is a rising interval in $[c, c + q]$, and

$$b < t \leqslant b + s \quad \Longrightarrow \quad f(t) \not\leqslant f(b). \tag{1.1}$$

Claim 1: *No interval* $[b - l, b]$ *is weakly falling*. Assume the contrary. Then (i) $l > s$, and (ii) $[b - (l - s), b]$ is weakly falling. To see (i), observe that $f(b + l) \leqslant f(b)$ because

$[b, b+l]$ is a right translate of $[b-l, b]$; hence $l \leqslant s$ would entail $b < b+l \leqslant b+s$, contradicting (1.1) with $t = b+l$. To prove (ii), note that right translation of $[b-l, b]$ shows that $[b-l+s, b+s]$ is weakly falling, implying $f(b-(l-s)) \geqslant f(b+s) > f(b)$; hence $[b-(l-s), b]$ is falling. Repetition of this argument with l replaced by $l-s, l-2s, \ldots$ leads by induction on n to the absurdity that $l - ns > s$ for all $n \in \mathbb{N}$.

Claim 2: $r > s$. For $f(b+r) \leqslant f(b)$ because $[b, b+r]$ is falling, as it is a right translate of $[a, a+r]$. Therefore $r > s$, for otherwise $b < b+r \leqslant b+s$ and (1.1) leads to a contradiction.

As $b+s \geqslant a+r$, we can translate $[a, a+r]$ to the right by $(b+s)-(a+r)$, obtaining the weakly falling interval $[b+s-r, b+s]$. Note that $b+s-r < b$ by Claim 2. From $f(b+s-r) \geqslant f(b+s) > f(b)]$ we conclude that $[b-(r-s), b]$ is falling. But this contradicts Claim 1 with $l = r-s$. □

LEMMA 1.7. *An omega limit set for a monotone semiflow Φ cannot contain distinct points x, y having respective neighborhoods U, V such that $\Phi_r U \leqslant \Phi_r V$ for some $r \geqslant 0$.*

PROOF. We proceed by contradiction. Suppose there exist distinct points $x, y \in \omega(z)$ having respective neighborhoods U, V such that $\Phi_r U \leqslant \Phi_r V$ for some $r \geqslant 0$. Then $\omega(z)$ is not a periodic orbit, for otherwise from $\Phi_r(x) \leqslant \Phi_r(y)$ we infer $x \leqslant y$ and hence $x < y$, violating Nonordering of Periodic Orbits.

There exist real numbers $a < b < c$ be such that $\Phi_a(z) \in U$, $\Phi_b(z) \in V$, $\Phi_c(z) \in U$. Therefore the properties of r, U and V imply

$$\Phi_{a+r}(z) \leqslant \Phi_{b+r}(z), \qquad \Phi_{b+r}(z) \geqslant \Phi_{c+r}(z).$$

As $\omega(z)$ is not periodic, the semiflow is injective on the orbit of z; hence the order relations above are strict. But this contradicts Theorem 1.6. □

It seems to be unknown whether omega limit sets of monotone semiflows must be unordered. This holds for SOP semiflows by the following theorem due to Smith and Thieme [197, Proposition 2.2]; the strongly monotone case goes back to Hirsch [66]. This result is fundamental to the theory of monotone semiflows:

THEOREM 1.8 (Nonordering of Omega Limit Sets). *Let $\omega(z)$ be an omega limit set for a monotone semiflow Φ.*

(i) *No points of $\omega(z)$ are related by $\ll$.*
(ii) *If $\omega(z)$ is a periodic orbit or Φ is SOP, no points of $\omega(z)$ are related by $<$.*

PROOF. Assume $x, y \in \omega(z)$. If $\omega(z)$ is a periodic orbit then x, y are unrelated (Proposition 1.5). If $x \ll y$ or $x < y$ and Φ is SOP, there are respective neighborhoods U, V of x, y such that $\Phi_r(U) \leqslant \Phi_r(V)$ for some $r \geqslant 0$; but this violates Lemma 1.7. □

COROLLARY 1.9. *Assume Φ is SOP.*

(i) *If an omega limit set has a supremum or infimum, it reduces to a single equilibrium.*

(ii) *If the equilibrium set is totally ordered, every quasiconvergent point with compact orbit closure is convergent.*

PROOF. Part (i) follows from Theorem 1.8(ii), since the supremum or infimum, if it exists, belongs to the limit set. Part (ii) is a consequence of (i). □

1.3. *Local semiflows*

For simplicity we have assumed trajectories are defined for all $t \geqslant 0$, but there are occasions when we need the more general concept of a *local semiflow* in X. This means a map $\Psi : \Omega \to X$, with $\Omega \subset [0,\infty) \times X$ an open neighborhood of $\{0\} \times X$, such that the maps

$$\Psi_t : D_t \to X, \ x \mapsto \Psi(t,x) \quad (0 \leqslant t < \infty)$$

satisfy the following conditions: D_t is an open, possibly empty set in X, Ψ_0 is the identity map of X, and $\Psi_{s+t} = \Psi_s \circ \Psi_t$ in the sense that $D_{s+t} = D_t \cap \Psi_t^{-1}(D_s)$ and $\Psi_{s+t}(x) = \Psi_s(\Psi_t(x))$ for $x \in D_{s+t}$.

The trajectory of x is defined as the map

$$I_x \to X, \ t \mapsto \Psi_t(x), \quad \text{where } I_x = \{t \in \mathbb{R}_+ \colon x \in D_t\}.$$

The composition law implies I_x is a half open interval $[0, \tau_x)$; we call $\tau_x \in (0,\infty]$ the *escape time* of x. It is easy to see that every point with compact orbit closure has infinite escape time. Thus a local semiflow with compact orbit closures is a semiflow. In dealing with local semiflows we adopt the convention that the notations $\Psi_t(x)$ and $\Psi_t(U)$ carry the assumptions that $t \in I_x$ and $U \subset D_t$. The image of I_x under the trajectory of x is the orbit $O(x)$. The omega limit set $\omega(x)$ is defined as $\omega(x) = \bigcap_{t \in I_x} \overline{O(\Psi_t(x))}$.

A *local flow* is a map $\Theta : \Lambda \to X$ where $\Lambda \subset \mathbb{R} \times X$ is an open neighborhood of $\{0\} \times X$, and the (possibly empty) maps

$$\Theta_t : D_t \to X, \ x \mapsto \Theta(t,x) \quad (-\infty \leqslant t < \infty)$$

satisfy the following conditions: Θ_0 is the identity map of $D_0 := X$, Θ_t is a homeomorphism of D_t onto D_{-t} with inverse Θ_{-t}, and

$$x \in (\Theta_s)^{-1} D_r \implies \Theta_r \circ \Theta_s(x) = \Theta_{r+s}(x).$$

Θ is a *flow* provided $D_t = X$ for all t.

The set $J_x := \{t \in \mathbb{R} \colon x \in D_t\}$ is an open interval around 0. The positive and negative *semiorbits* of x are the respective sets

$$\gamma^+(x) = \gamma^+(x,\Theta) := \{\Theta_t(x) \colon t \in J_x, t \geqslant 0\},$$

$$\gamma^-(x) = \gamma^-(x,\Theta) := \{\Theta_t(x) \colon t \in J_x, t \leqslant 0\}.$$

The *time-reversal* of Θ is the local flow $\widetilde{\Theta}$ defined by $\widetilde{\Theta}(t,x) = \Theta(-t,x)$.

The omega limit set $\omega(x)$ (for Θ) is defined to be $\omega(x) = \bigcap_{t\in I_x, t\geqslant 0} \overline{O(\Psi_t(x))}$. The *alpha limit set* $\alpha(x) = \alpha(x,\Theta)$ of x is defined as the omega limit set of x under the time-reversal of Θ^+.

Let F be a locally Lipschitz vector field F on a manifold M tangent along the boundary. Denote by $t \mapsto u(t;x)$ the maximally defined solution to $\dot{u} = F(u)$, $u(0,x) = x$. There is a local flow Θ^F on M such that $\Theta_t(x) = u(t;x)$. The time-reversal of Θ^F is Θ^{-F}. When M is compact, Θ^F is a flow. If we assume that F, rather than being tangent to the boundary, is transverse inward, we obtain a local semiflow.

Our earlier results are readily adapted to monotone local semiflows. In particular, omega limit sets are unordered. Theorems 1.8 and 1.6 have the following extension:

THEOREM 1.10. *Let Φ be a monotone local semiflow.*

(a) *No trajectory has both a rising and a falling interval.*

(b) *No points of an omega limit set are related by $\ll$, or by $<$ if Φ is SOP.*

(c) *The same holds for alpha limit sets provided Φ is a local flow.*

PROOF. The proofs of Theorems 1.6 and 1.8 also prove (a) and (b), and (c) follows by time reversal. □

1.4. *The limit set dichotomy*

Throughout the remainder of Section 1 we adopt the following assumptions:

(H) Φ is a strongly order preserving semiflow in an ordered space X, with every orbit closure compact.

Our goal now is to prove the important Limit Set Dichotomy:

If $x < y$ then either $\omega(x) < \omega(y)$, or $\omega(x) = \omega(y) \subset E$.

LEMMA 1.11 (Colimiting Principle). *Assume $x < y$, $t_k \to \infty$, $\Phi_{t_k}(x) \to p$ and $\Phi_{t_k}(y) \to p$ as $k \to \infty$. Then $p \in E$.*

PROOF. Choose neighborhoods U of x and V of y and $t_0 > 0$ such that $\Phi_{t_0}(U) \leqslant \Phi_{t_0}(V)$. Let $\delta > 0$ be so small that $\{\Phi_s(x)\colon 0 \leqslant s \leqslant \delta\} \subset U$ and $\{\Phi_s(y)\colon 0 \leqslant s \leqslant \delta\} \subset V$. Then $\Phi_s(x) \leqslant \Phi_r(y)$ whenever $t_0 \leqslant r, s \leqslant t_0 + \delta$. Therefore,

$$\Phi_{t_k - t_0}\big(\Phi_s(x)\big) \leqslant \Phi_{t_k - t_0}\big(\Phi_{t_0}(y)\big) = \Phi_{t_k}(y) \tag{1.2}$$

for all $s \in [t_0, t_0 + \delta]$ and all large k. As

$$\Phi_{t_k - t_0}\big(\Phi_s(x)\big) = \Phi_{s - t_0}\big(\Phi_{t_k}(x)\big) = \Phi_r\big(\Phi_{t_k}(x)\big),$$

where $r = s - t_0 \in [0, \delta]$ if $s \in [t_0, t_0 + \delta]$, we have

$$\Phi_r\big(\Phi_{t_k}(x)\big) \leqslant \Phi_{t_k}(y)$$

for large k and $r \in [0, \delta]$. Passing to the limit as $k \to \infty$ we find that $\Phi_r(p) \leqslant p$ for $0 \leqslant r \leqslant \delta$. If, in (1.2), we replace $\Phi_s(x)$ by $\Phi_{t_0}(x)$ and replace $\Phi_{t_0}(y)$ by $\Phi_s(y)$, and argue as above then we find that $p \leqslant \Phi_r(p)$ for $0 \leqslant r \leqslant \delta$. Evidently, $\Phi_r(p) = p$, $0 \leqslant r \leqslant \delta$ and therefore for all $r \geqslant 0$, so $p \in E$. □

THEOREM 1.12 (Intersection Principle). *If $x < y$ then $\omega(x) \cap \omega(y) \subset E$. If $p \in \omega(x) \cap \omega(y)$ and $t_k \to \infty$, then $\Phi_{t_k}(x) \to p$ if and only if $\Phi_{t_k}(y) \to p$.*

PROOF. If $p \in \omega(x) \cap \omega(y)$ then there exists a sequence $t_k \to \infty$ such that $\Phi_{t_k}(x) \to p$ and $\Phi_{t_k}(y) \to q \in \omega(y)$, and $p \leqslant q$ by monotonicity. If $p < q$ then we contradict the Nonordering of Limit Sets since $p, q \in \omega(y)$. Hence $p = q$. The Colimiting Principle then implies $p \in E$. □

The proof of the next result has been substantially simplified over previous versions.

LEMMA 1.13. *Assume $x < y$, $t_k \to \infty$, $\Phi_{t_k}(x) \to a$, and $\Phi_{t_k}(y) \to b$ as $k \to \infty$. If $a < b$ then $O(a) < b$ and $O(b) > a$.*

PROOF. The set $W := \{t \geqslant 0\colon\ \Phi_t(a) \leqslant b\}$ contains 0 and is closed. We prove $W = [0, \infty)$ by showing that W is also open. Observe first that if $t \in W$, then $\Phi_t(a) < b$. For equality implies $b \in \omega(x) \cap \omega(y) \subset E$, and then the Intersection Principle entails $\Phi_{t_k}(x) \to b$, giving the contradiction $a = b$.

Suppose $\bar{t} \in W$ is positive. By SOP there are open sets U, V with $\Phi_{\bar{t}}(a) \in U$, $b \in V$ and $t_1 \geqslant 0$ such that $\Phi_t(U) \leqslant \Phi_t(V)$ for $t \geqslant t_1$. There exists $\delta \in (0, \bar{t}/2)$ such that $\Phi_s(a) \in U$ for $|s - \bar{t}| \leqslant \delta$, so we can find an integer $\kappa > 0$ such that $\Phi_s(\Phi_{t_k}(x)) \in U$ for $k \geqslant \kappa$. Choose $k_0 \geqslant \kappa$ such that $\Phi_{t_{k_0}}(y) \in V$. Then we have $\Phi_{t+s+t_{k_0}}(x) \leqslant \Phi_{t+t_{k_0}}(y)$ for $t \geqslant t_1$. Setting $t = t_k - t_{k_0}$ for large k in this last inequality yields $\Phi_{t_k+s}(x) \leqslant \Phi_{t_k}(y)$ for large k. Taking the limit as $k \to \infty$ we get $\Phi_s(a) \leqslant b$ for $|s - \bar{t}| \leqslant \delta$. A similar argument in the case $\bar{t} = 0$ considering only $s \in [0, \delta]$ gives the previous inequality for such s. Therefore, W is both open and closed so $W = [0, \infty)$. This proves $O(a) < b$, and $O(b) > a$ is proved dually. □

LEMMA 1.14 (Absorption Principle). *Let $u, v \in X$. If there exists $x \in \omega(u)$ such that $x < \omega(v)$, then $\omega(u) < \omega(v)$. Similarly, if there exists $x \in \omega(u)$ such that $\omega(v) < x$, then $\omega(v) < \omega(u)$.*

PROOF. Apply Lemma 1.3 to obtain open neighborhoods U of x and V of $\omega(v)$ and $t_0 > 0$ such that

$$r \geqslant t_0 \quad \Longrightarrow \quad \Phi_r(U) \leqslant \Phi_r(V),$$

hence $\Phi_r(U) \leqslant \omega(v)$ since $\omega(v)$ is invariant. As $x \in \omega(u)$, there exists $t_1 > 0$ such that $\Phi_{t_1}(u) \in U$. Hence for $\Phi_{t_0+t_1}(u) \leqslant \omega(v)$, and monotonicity implies that $\Phi_{s+t_0+t_1}(u) \leqslant$

$\omega(v)$ for all $s \geqslant 0$. This implies that $\omega(u) \leqslant \omega(v)$. If $z \in \omega(u) \cap \omega(v)$ then $z = \sup \omega(u) = \inf \omega(v)$, whence $\{z\} = \omega(u) = \omega(v)$ by Corollary 1.9(ii). But this is impossible since $x < \omega(v)$ and $x \in \omega(u)$, so we conclude that $\omega(u) < \omega(v)$. □

LEMMA 1.15 (Limit Set Separation Principle). *Assume $x < y$, $a < b$ and there is a sequence $t_k \to \infty$ such that $\Phi_{t_k}(x) \to a$, $\Phi_{t_k}(y) \to b$. Then $\omega(x) < \omega(y)$.*

PROOF. By Lemma 1.13, $O(a) < b$, and therefore $\omega(a) \leqslant b$. If $b \in \omega(a)$ then Corollary 1.9 implies that $\omega(a) = b \in E$. Applying the Absorption Principle with $u = x$, $v = a$, $x = a$, we have $a \in \omega(x)$, $a < \omega(a) = b$ which implies that $\omega(x) < \omega(a)$. This is impossible as $\omega(a) \subset \omega(x)$. Consequently, $\omega(a) < b$. By the Absorption Principle again (with $u = a$, $v = y$), we have $\omega(a) < \omega(y)$. Since $\omega(a) \subset \omega(x)$, the Absorption Principle gives $\omega(x) < \omega(y)$. □

We now prove the fundamental tool in the theory of monotone dynamics, stated for strongly monotone semiflows in Hirsch [66,68].

THEOREM 1.16 (Limit Set Dichotomy). *If $x < y$ then either*

(a) $\omega(x) < \omega(y)$, *or*

(b) $\omega(x) = \omega(y) \subset E$.

If case (b) *holds and $t_k \to \infty$ then $\Phi_{t_k}(x) \to p$ if and only if $\Phi_{t_k}(y) \to p$.*

PROOF. If $\omega(x) = \omega(y)$ then $\omega(x) \subset E$ by the Intersection Principle, Theorem 1.12, which also establishes the final assertion. If $\omega(x) \neq \omega(y)$ then we may assume that there exists $q \in \omega(y) \setminus \omega(x)$, the other case being similar. There exists $t_k \to \infty$ such that $\Phi_{t_k}(y) \to q$. By passing to a subsequence if necessary, we can assume that $\Phi_{t_k}(x) \to p \in \omega(x)$. Monotonicity implies $p \leqslant q$ and, in fact, $p < q$ since $q \notin \omega(x)$. By the Limit Set Separation Principle, $\omega(x) < \omega(y)$. □

Among the many consequences of the Convergence Criterion is that a monotone semiflow in a strongly ordered Banach space cannot have a periodic orbit γ that is *attracting*, meaning that γ attracts all points in some neighborhood of itself (Hadeler [55], Hirsch [69]). The following consequence of the Limit Set Dichotomy implies the same conclusion for periodic orbits of SOP semiflows:

THEOREM 1.17. *Let γ be a nontrivial periodic orbit, some point of which is accessible from above or below. Then γ is not attracting.*

The accessibility hypothesis is used to ensure that there are points near p that are order-related to p but different from p. Some such hypothesis is required, as otherwise we could simply take $X = \gamma$, and then γ is attracting!

PROOF. Suppose $\gamma \subset W$ attracts an open set W. By hypothesis there exists $p \in \gamma$ and $x \in W$ such that $x > p$ or $x < p$ and $\omega(x) = \gamma$. To fix ideas we assume $x > p$. Then

$p \in \omega(x)$, so the Limit Set Dichotomy implies $p \in E$. Hence the contradiction that γ contains an equilibrium. □

It turns out that the periodic orbits γ considered above are not only not attracting; they enjoy the strong form of instability expressed in the next theorem.

A set $K \subset X$ is *minimal* if it is nonempty, invariant, and every orbit it contains is dense in K.

THEOREM 1.18. *Let K be a compact minimal set that is not an equilibrium, some point of which is accessible from below or above. Then there exists $\delta > 0$ with the following property: Every neighborhood of K contains a point x comparable to some point of K, such that* $\mathrm{dist}(\Phi_t(x), K) > \delta$ *for all sufficiently large t.*

PROOF. We may assume there exists a sequence $\tilde{x}_n \to p \in K$ with $\tilde{x}_n > p$. Suppose there is no such δ. Then there exist a subsequence $\{x_n\}$ and points $y_n \in \omega(x_n)$ such that $y_n \to q \in K$. Minimality of K implies $\omega(p) = \omega(q) = K$. Since $x_n > p$, the Limit Set Dichotomy implies $\omega(x_n) \geqslant \omega(p)$; therefore $y_n \geqslant K$, so $q \geqslant K$. It follows that $q = \sup K$, and Corollary 1.9 implies the contradiction that K is a singleton. □

A stronger form of instability for periodic orbits is given in Theorem 2.6.

1.5. *Q is plentiful*

One of our main goals is to find conditions that make quasiconvergent points generic in various senses. The first such results are due to Hirsch [66,73]; the result below is an adaptation of Smith and Thieme [199, Theorem 3.5].

We continue to assume Φ is an SOP semiflow with compact orbit closures.

A *totally ordered arc* is the homeomorphic image of a nontrivial interval $I \subset \mathbb{R}$ under a map $f : I \to X$ satisfying $f(s) < f(t)$ whenever $s, t \in I$ and $s < t$.

THEOREM 1.19. *If $J \subset X$ is a totally ordered arc, $J \setminus Q$ is at most countable.*

Stronger conclusions are obtained in Theorems 2.8 and 2.24.

The following global convergence theorem is adapted from Hirsch [73, Theorem 10.3].

COROLLARY 1.20. *Let Y be an ordered Banach space. Assume $X \subset Y$ is an open set, a closed order interval, or a subcone of Y_+. If $E = \{p\}$, every trajectory converges to p.*

PROOF. If X is open in Y, there exists a totally ordered line segment $J \subset X$ and quasi-convergent points $u, v \in J$ with $u < x < v$, by Theorem 1.19. Therefore $\Phi_t(u) \to p$ and $\Phi_t(v) \to p$, so monotonicity and closedness of the order relation imply $\Phi_t(x) \to p$.

If $X = [a, b]$, the trajectories of a and b converge to p by the Convergence Criterion 1.4, and the previous argument shows all trajectories converge to p. Similarly if X is a subcone of Y_+. □

PROOF OF THEOREM 1.19. Let $W = \overline{\Phi([0,\infty) \times J)}$. Continuity of Φ implies that W is a separable metric space which is positively invariant under Φ. Therefore we may as well assume that X is a separable metric space.

We show that if $x \in J$ and

$$\inf\{\mathrm{dist}(\omega(x), \omega(y)) \colon y \in J, y \neq x\} = 0,$$

then $x \in Q$. Choose a sequence $x_n \in J$, $x_n \neq x$ such that $\mathrm{dist}(\omega(x), \omega(x_n)) \to 0$. We may assume that $x_n < x$ for all n. Taking a subsequence, we conclude from the Limit Set Dichotomy: Either some $\omega(x_n) = \omega(x)$, or every $\omega(x_n) < \omega(x)$.

In the first case, $x \in Q$. In the second case, choose $y_n \in \omega(x_n)$, $z_n \in \omega(x)$ such that $d(y_n, z_n) \to 0$. After passing to subsequences, we assume $y_n, z_n \to z \in \omega(x)$. Because $y_n \leqslant \omega(x)$, we conclude that $z \leqslant \omega(x)$. As $z \in \omega(x)$, Corollary 1.9 implies $\omega(x) = \{z\}$. Hence $x \in Q$ in this case as well.

It follows that for every $x \in J \setminus Q$, there exists an open set U_x containing $\omega(x)$ such that $U_x \cap \omega(y) = \emptyset$ for every $y \in J \setminus \{x\}$. By the axiom of choice we get an injective mapping

$$J \setminus Q \to X, \quad x \mapsto p_x \in \omega(x) \subset U_x.$$

The separable metric space X has a countable base $\mathcal{B}$. A second application of the axiom of choice gives a map

$$J \setminus Q \to \mathcal{B}, \quad x \mapsto V_x \subset U_x, \quad p_x \in V_x.$$

This map is injective. For if x, y are distinct points of $J \setminus Q$, then $V_x \neq V_y$ because V_x, being contained in U_x, does not meet $\omega(y)$; but $p_y \in V_y \cap \omega(y)$. This proves $J \setminus Q$ is countable. □

Let Y be an ordered Banach space and assume $X \subset Y$ is an ordered subspace (not necessarily linear). When Y is finite-dimensional, Theorem 1.19 implies $X \setminus Q$ has Lebesgue measure zero, hence almost every point is quasiconvergent. For infinite-dimensional Y there is an analogous result for Gaussian measures (Hirsch [73, Lemma 7.7]). The next result shows that in this case Q is also plentiful in the sense of category.

A subset of a topological space S is *residual* if it contains the intersection of countably many dense open subsets of S. When S is a complete metric space every residual set is dense by the Baire category theorem.

The assumption on X in the following result holds for many subsets of an ordered Banach space, including all convex sets and all sets with dense interior.

THEOREM 1.21. *Assume X is a subset of an ordered Banach space Y, and a dense open subset $X_0 \subset X$ is covered by totally ordered line segments. Then Q is residual in X.*

PROOF. It suffices to show that the set $Q_1 := Q \cup (Y \setminus X_0)$ is residual in Y. Note that $Y \setminus Q_1 = X_0 \setminus Q$. Let $L \subset Y$ be the 1-dimensional space spanned by some positive vector. Every translate $y + L$ meets $Y \setminus Q_1$ in a finite or countably infinite set by Theorem 1.19, hence $(y + L) \cap Q_1$ is residual in the line $y + L$. By the Hahn–Banach theorem there is a closed linear subspace $M \subset Y$ and a continuous linear isomorphism $F\colon Y \approx M \times L$ such that $F(x + L) = \{x\} \times L$ for each $x \in M$. Therefore $F(Q_1) \cap (\{x\} \times L)$ is residual in $\{x\} \times L$ for all $x \in X_0$, whence $F(Q_1)$ is residual in $M \times L$ by the Kuratowski–Ulam Theorem (Oxtoby [154]). This implies Q_1 is residual in Y. □

Additional hypotheses seem to be necessary in order to prove density of Q in general ordered spaces. The next theorem obtains the stronger conclusion that Q has dense interior. A different approach will be explored in Section 2.

A point x is *doubly accessible from below* (respectively, above) if in every neighborhood of x there exist f, g with $f < g < x$ (respectively, $x < f < g$).

Consider the following condition on a semiflow satisfying (H):

(L) Either every omega limit set has an infimum in X and the set of points that are doubly accessible from below has dense interior, or every omega limit set has a supremum in X and the set of points that are doubly accessible from above has dense interior.

This holds when X is the Banach space of continuous functions on a compact set with the usual ordering, for then every compact set has a supremum and infimum, and every point is doubly accessible from above and below.

THEOREM 1.22. *If* (L) *holds, then* $X \setminus Q \subset \overline{\operatorname{Int} C}$, *and* $\operatorname{Int} Q$ *is dense.*

The proof is based on the following result. For $p \in E$ define $C(p) := \{z \in X\colon \omega(z) = \{p\}\}$. Note that $C = \bigcup_{p \in E} C(p)$.

LEMMA 1.23. *Suppose $x \in X \setminus Q$ and $a = \inf \omega(x)$. Then $\omega(a) = \{p\}$ with $p < \omega(x)$, and $x \in \overline{\operatorname{Int} C(p)}$ provided x is doubly accessible from below.*

PROOF. Fix an arbitrary neighborhood M of x. Note that $a < \omega(x)$ because $\omega(x)$ is unordered (Theorem 1.8). By invariance of $\omega(x)$ we have $\Phi_t a \leqslant \omega(x)$, hence $\Phi_t a \leqslant a$. Therefore the Convergence Criterion Theorem 1.4 implies $\omega(a)$ is an equilibrium $p \leqslant a$. Because $p < \omega(x)$, SOP yields a neighborhood N of $\omega(x)$ and $s \geqslant 0$ such that $p \leqslant \Phi_t N$ for all $t \geqslant s$. Choose $r \geqslant 0$ with $\Phi_t x \in N$ for $t \geqslant r$. Then $p \leqslant \Phi_t x$ if $t \geqslant r + s$. The set $V := (\Phi_{r+s})^{-1}(N) \cap M$ is a neighborhood of x in M with the property that $p \leqslant \Phi_t V$ for all $t \geqslant r + 2s$. Hence:

$$u \in V \implies p \leqslant \omega(u). \tag{1.3}$$

Now assume x doubly accessible from below and fix $y_1, y \in V$ with $y_1 < y < x$. By the Limit Set Dichotomy $\omega(y) < \omega(x)$, because $\omega(x) \not\subset E$. By SOP we fix a neighborhood $U \subset V$ of y_1 and $t_0 > 0$ such that $\Phi_{t_0} u \leqslant \Phi_{t_0} y$ for all $u \in U$. The Limit Set Dichotomy implies $\omega(u) = \omega(y)$ or $\omega(u) < \omega(y)$; as $\omega(y) < \omega(x)$, we therefore have:

$$u \in U \implies \omega(u) < \omega(x). \tag{1.4}$$

For all $u \in U$, (1.4) implies $\omega(u) \leqslant \omega(a) = \{p\}$, while (1.3) entails $p \leqslant \omega(u)$. Hence $U \subset C(p) \cap M$, and the conclusion follows. □

PROOF OF THEOREM 1.22. To fix ideas we assume the first alternative in (L), the other case being similar. Let X_0 denote a dense open set of points doubly accessible from below. Lemma 1.23 implies $X_0 \subset Q \cup \overline{\operatorname{Int} C} \subset Q \cup \overline{\operatorname{Int} Q}$, hence the open set $X_0 \setminus \overline{\operatorname{Int} Q}$ lies in Q. This prove $X_0 \setminus \overline{\operatorname{Int} Q} \subset \operatorname{Int} Q$, so $X_0 \setminus \overline{\operatorname{Int} Q} = \emptyset$. Therefore $\overline{\operatorname{Int} Q} \supset X_0$, hence $\overline{\operatorname{Int} Q} \supset \overline{X_0} = X$. □

EXAMPLE 1.24. An example in Hirsch [73] shows that generic quasiconvergence and the Limit Set Dichotomy need not hold for a monotone semiflow that does not satisfy SOP. Let X denote the ordered Banach space $\mathbb{R}^3$ whose ordering is defined by the "ice-cream" cone $X_+ = \{x \in \mathbb{R}^3\colon x_3 \geqslant \sqrt{x_1^2 + x_2^2}\}$. The linear system $x_1' = -x_2$, $x_2' = x_1$, $x_3' = 0$ generates a flow Φ with global period 2π which merely rotates points about the x_3-axis. Evidently X_+ is invariant, so linearity of Φ implies monotonicity. On the other hand, Φ is not strongly order preserving: If $a = (1, 0, 1)$ (or any other point on ∂Y_+ except the origin 0), SOP would require $\Phi_t(a) \gg 0$ for $t > 0$ because Φ_t is a homeomorphism, but this fails for all $t > 0$. The Limit Set Dichotomy fails to hold: For $a = (1, 0, 1)$ and $b = (2, 0, 2)$ it is easy to see that $a < b$ (for the ordering defined by X_+) and $\omega(a) \cap \omega(b) = \emptyset$, but $\omega(a) \not< \omega(b)$. As $E = C = Q = \{x\colon x_1 = x_2 = 0\}$ and most points belonging to periodic orbits of minimal period 2π, quasiconvergence is rare. In fact, the set of nonquasiconvergent points—the complement of the x_3-axis—is open and dense. It is not known whether there is a similar example with a polyhedral cone.

1.6. *Stability in normally ordered spaces*

We continue to assume the semiflow Φ is SOP with compact orbit closures.

The *diameter* of a set Z is $\operatorname{diam} Z := \sup_{x,y \in Z} d(x, y)$.

We now introduce some familiar stability notions. A point $x \in X$ is *stable* (relative to $R \subset X$) if for every $\epsilon > 0$ there exists a neighborhood U of x such that $\operatorname{diam} \Phi_t(U \cap R) < \epsilon$ for all $t \geqslant 0$. The set of stable points is denoted by S.

Suppose x_0 is stable. Then omega limit sets of nearby points are close to $\omega(x_0)$, and if all orbit closures are compact, the map $x \mapsto \omega(x)$ is continuous at x_0 for the Hausdorff metric on the space of compact sets.

x is *stable from above* (respectively, *from below*) if x is stable relative to the set of points $\geqslant x$ (resp., $\leqslant x$). The set of points stable from above (resp., below is denoted by S_+ (resp., S_-).

The *basin of* x in R is the union of all subsets of R of the form $V \cap R$ where $V \subset X$ is an open neighborhood of x such that

$$\lim_{t \to \infty} \operatorname{diam} \Phi_t(V \cap R) = 0.$$

Notice that $\omega(x) = \omega(y)$ for all y in the basin.

If the basin of x in R is nonempty, we say x is *asymptotically stable* relative to R. This implies x is stable relative to R. If x is asymptotically stable relative to X we say x is *asymptotically stable*. The set of asymptotically stable points is an open set denoted by A.

x is *asymptotically stable from above* (respectively, *below*) if it is asymptotically stable relative to the set of points $\geqslant x$ (resp., $\leqslant x$). The basin of x relative to this set is called the *upper* (resp., *lower*) basin of x. The set of such x is denoted by A_+ (resp., A_-).

Note that continuity of Φ shows that asymptotic stability relative to R implies stability relative to R. In particular, $A \subset S$, $A_+ \subset S_+$ and $A_- \subset S_-$.

These stability notions for x depend only on the topology of X, and not on the metric, provided the orbit of x has compact closure.

The metric space X is *normally ordered* if there exists a *normality constant* $\kappa > 0$ such that $d(x, y) \leqslant \kappa d(u, v)$ whenever $u, v \in X$ and $x, y \in [u, v]$. In a normally ordered space order intervals are bounded and the diameter of $[u, v]$ goes to zero with $d(u, v)$. Many common function spaces, including L^p spaces and the Banach space of continuous functions with the uniform norm, are normally ordered by the cone of nonnegative functions. But spaces whose norms involve derivatives are not normally ordered. Normality is required in order to wring the most out of the Sequential Limit Set Trichotomy. The propositions that follow record useful stability properties of SOP dynamics in normally ordered spaces.

PROPOSITION 1.25. *Assume X is normally ordered.*

(a) *$x \in S_+$ (respectively, S_-) provided there exists a sequence $y_n \to x$ such that $y_n > x$ (resp., $y_n < x$) and $\lim_{n\to\infty} \sup_{t>0} d(\Phi_t(x), \Phi_t(y_n)) = 0$.*

(b) *$x \in S$ provided $x \in S_+ \cap S_-$ and x is accessible from above and below.*

(c) *$x \in A$ provided $x \in A_+ \cap A_-$ and x is accessible from above and below.*

(d) *Suppose $a < b$ and $\omega(a) = \omega(b)$. Then $a \in A_+$ and $b \in A_-$. If $a < x < b$ then $x \in A$ and the basin of x includes $[a, b] \setminus \{a, b\}$.*

In particular, (d) shows that an equilibrium e is in A_+ if $x > e$ and $\Phi_t(x) \to e$ (provided X is normally ordered); and dually for A_-.

PROOF. We prove (a) for the case $y_n > x$. Given $\epsilon > 0$, choose m and t_0 so that

$$t > t_0 \implies d\big(\Phi_t(x), \Phi_t(y_m)\big) < \epsilon.$$

By SOP there exists a neighborhood W of x and $t_1 > t_0$ such that

$$t > t_1,\ v \in W \implies \Phi_t(v) < \Phi_t(y_m).$$

Fixing t_1, we shrink W to a neighborhood W_ϵ of x so that

$$0 < t \leqslant t_1,\ v \in W_\epsilon \implies d\big(\Phi_t(x), \Phi_t(v)\big) < \kappa\epsilon,$$

where $\kappa > 0$ is the normality constant. If $x < v \in W_\epsilon$ and $t > t_1$ then $\Phi_t(x) \lneq \Phi_t(v) \leqslant \Phi_t(y_m)$, and therefore

$$t > t_1,\ x < v \in W_\epsilon \implies d\big(\Phi_t(x), \Phi_t(v)\big) \leqslant \kappa d\big(\Phi_t(x), \Phi_t(y_m)\big) \leqslant \kappa\epsilon.$$

Hence we have proved

$$0 < t < \infty,\ v \in W_\epsilon \implies d\big(\Phi_t(x), \Phi_t(v)\big) < \kappa\epsilon.$$

As ϵ is arbitrary, this proves $x \in S_+$.

To prove (b), let $u_n, v_n \to x$ with $u_n < x < v_n$. Because $x \in S_+ \cap S_-$, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d(y, x) < \delta$ and $y < x$ or $y > x$, then $\sup_{t>0} d(\Phi_t(y), \Phi_t(x)) < \epsilon$. Choose k such that $d(u_k, x) < \delta$ and $d(v_k, x) < \delta$. By SOP there is a neighborhood W_ϵ of x such that $\Phi_t(u_k) \leqslant \Phi_t(W_\epsilon) \leqslant \Phi_t(v_k)$ for sufficiently large t. Normality implies that for such t,

$$\begin{aligned}\kappa^{-1} \operatorname{diam} \Phi_t(W_e) &\leqslant d\big(\Phi_t(u_k), \Phi_t(v_k)\big)\\ &\leqslant d\big(\Phi_t(u_k), \Phi_t(x)\big) + d\big(\Phi_t(x), \Phi_t(v_k)\big) < 2\epsilon.\end{aligned}$$

As κ is constant and ϵ is arbitrary, this proves x is stable.

The proofs of (c) and (d) are similar. □

PROPOSITION 1.26. *Assume X is normally ordered, $p \in E$, and $\{K_n\}$ is a sequence of nonempty compact invariant sets such that $K_n < p$ and* $\operatorname{dist}(K_n, p) \to 0$. *Then*:

(a) *p is stable from below.*

(b) *If z is such that $\omega(z) = p$, then z is stable from below.*

In particular, if p is the limit of a sequence of equilibria $< p$ then p is stable from below.

PROOF. (a) Given $\epsilon > 0$, fix m such that $\operatorname{dist}(K_m, p) < \epsilon$. By Lemma 1.3 there is a neighborhood W of p and $t_0 > 0$ such that $t > t_0 \implies \Phi_t(W) \geqslant K_m$, and therefore

$$t > t_0,\ v \in W,\ p > v \implies d\big(\Phi_t(p), \Phi_t(v)\big) \leqslant \kappa d\big(\Phi_t(p), \Phi_t(K_m)\big) \leqslant \kappa\epsilon.$$

Pick a neighborhood $W_\epsilon \subset W$ of p so small that

$$0 \leqslant t \leqslant t_0,\ v \in W_\epsilon \implies d\big(\Phi_t(p), \Phi_t(v)\big) < \kappa\epsilon.$$

Then

$$0 \leqslant t < \infty,\ v \in W_\epsilon,\ v < p \implies d\big(\Phi_t(p), \Phi_t(v)\big) < \kappa\epsilon.$$

This proves $p \in S_-$, because ϵ is arbitrary.

(b) Choose a neighborhood U of z and $t_1 \geqslant 0$ such that $\Phi_{t_1}(U) \subset W$. Assume $y \in U$, $y < z$. If $t \geqslant t_1 + t_0$, then $K_m \leqslant \Phi_t(y) \leqslant \Phi_t(z)$, and therefore by normality, $d(\Phi_t(y), \Phi_t(z)) \leqslant \kappa \operatorname{dist}(K_m, \Phi_t(z))$. As $\Phi_t(z) \to p$, there exists $t_2 \geqslant t_1 + t_0$ such that

$$t \geqslant t_2 \implies d\big(\Phi_t(y), \Phi_t(z)\big) \leqslant \kappa \operatorname{dist}(K_m, p) < \kappa\epsilon.$$

Fix this t_2. By continuity of Φ there is a neighborhood $U_1 \subset U$ of z so small that

$$0 \leqslant t \leqslant t_2 \implies d\big(\Phi_t(y), \Phi_t(z)\big) \leqslant \kappa\epsilon.$$

As ϵ is arbitrary, this implies $z \in S_-$. □

1.7. *Stable equilibria in strongly ordered Banach spaces*

In spaces that are not normally ordered we cannot directly use the results of the previous subsection to characterize stable equilibria. For strongly monotone semiflows in strongly ordered Banach spaces we work around this by introducing a weaker norm that makes the order normal, and for which the semiflows are continuous and SOP. This permits use of the earlier results.

Let Y be a strongly ordered Banach space. The *order topology* on Y is the topology generated by open order intervals. An *order norm* on the topological vector space $\widehat{Y}$ is defined by fixing $u \gg 0$ and assigning to x the smallest ϵ such that $x \in [-\epsilon u, \epsilon u]$. It is easy to see that $\widehat{Y}$ is normally ordered by the order norm, with normality constant 1. Every order neighborhood of p in $\widehat{Y}$ contains $[p - \epsilon u, p + \epsilon u]$ for all sufficiently small numbers $\epsilon > 0$. For example, $Y = C^1([0, 1])$ with the usual C^1-norm and with Y_+ the cone of nonnegative functions is strongly ordered but not normally ordered; putting $u := 1$, the order norm becomes the usual supremum.

The induced topology on any subset $Z \subset Y$ is also referred to as the order topology, and the resulting topological space is denoted by $\widehat{Z}$. A neighborhood in $\widehat{Z}$ is an *order neighborhood*.

Every open subset of $\widehat{Z}$ is open in Z, i.e., the identity map of Z is continuous from Z to $\widehat{Z}$. Therefore $\widehat{Z} = Z$ as topological spaces when Z is compact. As shown below, if Ψ is a monotone local semiflow in Z, it is also a local semiflow in $\widehat{Z}$, denoted by $\widehat{\Psi}$. Evidently Ψ and $\widehat{\Psi}$ have the same orbits and the same invariant sets.

LEMMA 1.27. *Let Ψ be a monotone local semiflow in a subset X of a strongly ordered Banach space Y, that extends to a monotone local semiflow in an open subset of Y. Then*:

(a) *$\widehat{\Psi}$ is a monotone local semiflow.*

(b) *If Ψ is a strongly monotone, then $\widehat{\Psi}$ is SOP.*

PROOF. It suffices to prove (a) and (b) when X is open in Y, which condition is henceforth assumed.

$\widehat{\Psi}$ is monotone because Ψ is monotone. To prove continuity of $\widehat{\Psi}$, let $N = [[a, b]]_Y \cap X$ and $(t_0, x_0) \in \widehat{\Psi}^{-1}(N)$. As the latter is open in $\mathbb{R}_+ \times X$, there exists $\epsilon > 0$ and U, an open neighborhood of x_0 in X, such that

$$\big[(t_0 - \epsilon, t_0 + \epsilon) \cap \mathbb{R}_+\big] \times U \subset \widehat{\Psi}^{-1}(N).$$

We may choose $u, v \in U$ such that $x \in [[u, v]]_Y$. If $z \in [[u, v]]_Y \cap X$ and $|t - t_0| < \epsilon$ then by monotonicity and $u, v \in U$ we have $a \ll \widehat{\Psi}_t(u) \leqslant \widehat{\Psi}_t(z) \leqslant \widehat{\Psi}_t(v) \ll b$. Thus,

$$\big[(t_0 - \epsilon, t_0 + \epsilon) \cap \mathbb{R}_+\big] \times \big([[u, v]]_Y \cap X\big) \subset \widehat{\Psi}^{-1}(N),$$

proving the continuity of $\widehat{\Psi}$.

Assume $x, y \in X$, $x < y$ and let $t_0 > 0$ be given. By strong monotonicity of Ψ there are respective open neighborhoods $U, V \subset X$ of x, y such that $\Psi_{t_0}(U) \leqslant \Psi_{t_0}(V)$ (see Proposition 1.2). Choose $w, u, v, z \in X$ such that $u, w \in U$, $v, z \in V$ and

$$w \ll x \ll u, \qquad v \ll y \ll z$$

so that $[[w, u]]_Y \cap X$ and $[[v, z]]_Y \cap X$ are order neighborhoods in X of x, y respectively. Monotonicity of Ψ implies

$$\Psi_{t_0}\big([[w, u]]_Y \cap X\big) \leqslant \Psi_{t_0}\big([[v, z]]_Y \cap X\big). \qquad \square$$

An equilibrium p for $\Psi : \mathbb{R}^+ \times X \to X$ is *order stable* (respectively, *asymptotically order stable* if p is stable (respectively, asymptotically stable) for $\widehat{\Psi}$.

PROPOSITION 1.28. *Let Ψ be a monotone local semiflow in a subset X of a strongly ordered Banach space Y, that extends to a monotone local semiflow in some open subset of Y. Assume p is an equilibrium having a neighborhood W that is attracted to a compact set $K \subset X$. If p is order stable (respectively, asymptotically order stable), it is stable (respectively, asymptotically stable).*

PROOF. Suppose p is order stable and let U be a neighborhood of p. As $\widehat{K} = K$, there is a closed order neighborhood N_0 of p such that $N_0 \cap K \subset U \cap K$. By order stability there exists an order neighborhood N_1 of p such that $O(N_1) \subset N_0$. Compactness of $N_0 \cap K$ implies there is an open set $V \supset K$ there is an open set $V \supset K$ such that $N_0 \cap V \subset U$. Because K attracts W, there is a neighborhood $U_2 \subset W$ of p and $r \geqslant 0$ such that

$$t \geqslant r \implies \Psi_t(U_2) \subset V.$$

By continuity of Ψ_r at $p = \Psi_r(p)$ there is a neighborhood $U_3 \subset U_2$ of p such that

$$0 \leqslant t \leqslant r \implies \Psi_t(U_3) \subset V.$$

and thus $O(U_3) \subset V$. Therefore $N_1 \cap U_3$ is a neighborhood of p such that

$$O(N_1 \cap U_3) \subset O(N_1) \cap O(U_3) \subset N_0 \cap V \subset U.$$

This shows p is stable.

Assume p is asymptotically order stable and choose an order neighborhood $M \subset X$ of p that is attracted to p by $\widehat{\Psi}$. We show that $M \cap W$ is in the basin of p for Ψ. Consider arbitrary sequences $\{x_k\}$ in $M \cap W$ and $t_k \to \infty$ in $[0, \infty)$. Fix $u \gg 0$. By the choice of M there are positive numbers $\epsilon_k \to 0$ such that

$$p - \epsilon_k u \ll \Psi_{t_k}(x_k) \ll p + \epsilon_k u.$$

This implies $\Psi_{t_k}(x_k) \to p$ in X, because the order relation on X is closed and $\{\Psi_{t_k}(x_k)\}$ is precompact in X by the choice of W and compactness of K. $\square$

1.8. *The search for stable equilibria*

The following results illustrate the usefulness of a dense set of quasiconvergent points. Φ denotes a strongly order preserving semiflow in X; Hypothesis (H) of Section 1.4 is still in force.

PROPOSITION 1.29. *Assume Q is dense. Let $p, q \in E$ be such that $p < q$, p is accessible from above, and q is accessible from below. Then there exists $z \in X$ satisfying one of the following conditions*:

(a) *$p < z < q$, and $\Phi_t(z) \to p$ or $\Phi_t(z) \to q$*;

(b) *$p < z < q$ and $z \in E$*;

(c) *$z > p$ and $p \in O(z)$, or $z < q$ and $q \in O(z)$.*

PROOF. By SOP there are open neighborhoods U, V of p, q respectively and $t_0 \geqslant 0$ such that $\Phi_t U \leqslant \Phi_t V$ for $t \geqslant t_0$. Choose sequences $x_n \to p$ in U and $y_n \to q$ in V with $p < x_n$, $y_n < q$. We assume $p \notin O(x_n)$ and $q \notin O(y_n)$), as otherwise (c) is satisfied. Then

$$t \geqslant t_0 \quad \Longrightarrow \quad p < \Phi_t(x_n) \leqslant \Phi_t(y_n) < q.$$

Choose open neighborhoods U_1, W, V_1 of $p, \Phi_{t_0}(y_1), q$ respectively such that for some $t_1 \geqslant t_0$:

$$t \geqslant t_1 \quad \Longrightarrow \quad \Phi_t(U_1) \leqslant \Phi_t(W) \leqslant \Phi_t(V_1).$$

Choose $w \in Q \cap W$ and a sequence $s_k \to \infty$, $s_k \geqslant t_1$ such that $\Phi_{s_k}(w) \to e \in E$. Fix m so large that $x_m \in U_1$, $y_m \in V_1$. Then for sufficiently large k,

$$p < \Phi_{s_k}(x_m) \leqslant \Phi_{s_k}(w) \leqslant \Phi_{s_k}(y_m) < q.$$

It follows that $p \leqslant e \leqslant q$. If $e = p$ or q then $\omega(\Phi_{s_k}(w)) = p$ or q by the Convergence Criterion 1.4, giving (a) with $z = \Phi_{s_k}(w)$. Therefore if (a) does not hold, (b) holds with $z = e$. □

The assumption in Proposition 1.29 that Q is dense can be considerably weakened, for example, to p (or q) being interior to $\overline{Q}$: Assume $y_1 \in \overline{\operatorname{Int} Q}$ and set $w = \Phi_{s_k}(w_0)$, $w_0 \in (\operatorname{Int} Q) \cap {\Phi_{s_k}}^{-1}(W)$, etc. In fact, density of Q can be replaced with the assumption that p or q lies in the interior of the set $Q_\#$ of points x such that there is a sequence $x_i \to x$ with $\lim_{i\to\infty} \operatorname{dist}(\omega(x_i), E) = 0$. Clearly $Q_\#$ is closed and contains Q, so density of Q implies $Q_\# = X$.

THEOREM 1.30. *Suppose X is normally ordered and the following three conditions hold*:

(a) *Q is dense*;

(b) *if $e \in E$ and e is not accessible from above (below) then $e = \sup X$ ($e = \inf X$)*;

(c) *there is a maximal totally ordered subset $R \subset E$ that is nonempty and compact.*

Then R contains a stable equilibrium, an asymptotically stable equilibrium if R is finite.

PROOF. By Lemma 1.1, $\sup R$ ($\inf R$) exists and is a maximal (minimal) element of E. We first prove that every maximal equilibrium q is in A_+. This holds vacuously when $q = \sup X$. Suppose $q \neq \sup X$. If q is in the orbit of some point $> q$ then $q \in A_+$ by Proposition 1.25(d). Hence we can assume:

$$t \geqslant 0,\ y > q \implies \Phi_t(y) > q.$$

By hypothesis we can choose $y > q$. By SOP there is an open neighborhood U of q and $s > 0$ such that $\Phi_s(y) \geqslant \Phi_s(U)$. By hypothesis we can choose $z \in U$ such that $\Phi_s(y) \neq \Phi_s(z)$ and $z > q$. Set $x_2 = \Phi_s(y)$, $x_1 = \Phi_s z$. Then $x_2 > x_1 > q$, By SOP and the assumption above there is a neighborhood V_2 of x_2 and $t_0 \geqslant 0$ such that

$$t > t_0 \implies q < \Phi_t(x_1) \leqslant \Phi_t(V_2).$$

Choose $v \in V_2 \cap Q$. Then $q < \Phi_t(v)$ for $t \geqslant t_0$, hence $q \leqslant \omega(v) = \omega(\Phi_{t_0}(v)) \subset E$. Therefore $\Phi_t(v) \to q$ by maximality of q, so and Proposition 1.25(d) implies $q \in A_+$, as required. The dual argument shows that every minimal equilibria is in A_-.

Assumption (c) and previous arguments establish that $q = \sup R$ and $p = \inf R$ satisfy $p \leqslant q$ and $q \in A_+$, $p \in A_-$.

Suppose $p = q$; in this case we prove $q \in A$. As q is both maximal and minimal in E, we have $q \in A_+ \cap A_-$. If q is accessible from above and below then $q \in A$ by Proposition 1.25(b). If q is not accessible from above then by hypothesis $q = \sup X$, in which case the fact that $q \in A_-$ implies $q \in A$. Similarly, $q \in A$ if q is not accessible from below.

Henceforth we assume $p < q$. As R is compact and $R \cap S_- \neq \emptyset$ because $p \in R$, it follows that R contains the equilibrium $r := \sup(R \cap S_-)$. Note that $r \in S_-$, because this holds by definition of r if r is isolated in $\{r' \in R\colon r' \leqslant r\}$, and otherwise $r \in S_-$ by Proposition 1.26(a). If $r = q$ a modification of the preceding paragraph proves $q \in S$.

Henceforth we assume $r < q$; therefore r is accessible from above.

If r is not accessible from below then $r = p = \inf X$ so $r \in S$ and we are done; so we may as well assume r is accessible from below as well as from above. If r is the limit of a sequence of equilibria $> r$ then $r \in S_+$ by the dual of Proposition 1.26, hence $r \in S$ by Proposition 1.25(b). Therefore we can assume R contains a smallest equilibrium $r_1 > r$. Note that $r_1 \notin S_-$ by maximality of r. We apply Proposition 1.29 to r, r_1: among its conclusions, the only one possible here is that $z > r$ and $\Phi_t(z) \to r$ (and perhaps $r \in O(z)$). Therefore $r \in S_+$ by Proposition 1.25(a), whence $r \in S$ by 1.25(b). When R is finite, a modification of the preceding arguments proves $\max(R \cap A_-) \subset A$. □

Assumption (b) in the Theorem 1.30 holds for many subsets X of an ordered Banach space Y, including open sets, subcones of Y_+, closed order intervals, and so forth. This result is similar to Theorem 10.2 of Hirsch [73], which establishes equilibria that are merely order stable, but does not require normality.

Assumption (c) holds when E is compact, and also in the following situation: $X \subset Y$ where Y is an L^p space, $1 \leqslant p < \infty$, and E is a nonempty, closed, and order bounded subset of X; then every order bounded increasing or decreasing sequence converges. If (c) holds and some Φ_t is real analytic with spatial derivatives that are compact and strongly positive operators, then R is finite. This follows from the statements and proofs of Lemma 3.3 and Theorem 2 in Jiang and Yu [90].

For related results on stable equilibria see Jiang [86], Mierczyński [138,139], and Hirsch [69].

THEOREM 1.31. *Let Φ be a semiflow in a subset X of a strongly ordered Banach space Y, that extends to a strongly monotone local semiflow in some open subset of Y. Assume hypotheses* (a), (b), (c) *of Theorem* 1.30 *hold, and every equilibrium has a neighborhood attracted to a compact set. Let $R \subset E$ be as in* 1.30(c). *Then R contains a stable equilibrium, and an asymptotically stable equilibrium when R is finite.*

PROOF. Our strategy is to apply Theorem 1.30 to the semiflow $\widehat{\Phi}$ in $\widehat{X}$ (see Section 1.7). Give $\widehat{X}$ the metric coming from an order norm on $\widehat{Y}$; this makes $\widehat{X}$ is normally ordered. Lemma 1.27 shows that $\widehat{\Phi}$ is SOP. Therefore R contains an equilibrium p that is stable for $\widehat{\Phi}$, by Theorem 1.30. This means p is order stable for Φ, whence Proposition 1.28 shows that p is stable for Φ. The final assertion follows similarly. □

Stable equilibria are found under various assumptions in Theorems 2.9, 2.10, 2.11, 2.26, 3.14, 4.12.

2. Generic convergence and stability

2.1. *The sequential limit set trichotomy*

Throughout Section 2 we assume Hypothesis (H) of Section 1.4:

> *Φ is a strongly order preserving semiflow in an ordered space X, with all orbit closures compact.*

The main result is that the typical orbit of an SOP semiflow is stable and approaches the set E of equilibria. Existence of stable equilibria is established under additional compactness assumptions.

The index n runs through the positive integers.

A point x is *strongly accessible from below* (*respectively*, above) if there exists a sequence $\{y_n\}$ converging to x such that $y_n < y_{n+1} < x$ (resp., $y_n > y_{n+1} > x$). In this case we say $\{y_n\}$ *strongly approximates x from below* (resp., *from above*).

The sequence $\{x_n\}$ is *omega compact* if $\bigcup_n \omega(x_n)$ is compact.

Define sets $BC, AC \subset X$ as follows:

$x \in BC \iff x$ is strongly accessible from below by an omega compact sequence,

$x \in AC \iff x$ is strongly accessible from above by an omega compact sequence.

In this notation "B" stands for "below," "A" for above, and "C" for "compact."

We will also use the following condition on a set $W \subset X$:

(C) Every sequence $\{w_n\}$ in W that strongly approximates a point of W from below or above is omega compact.

This does not assert that any point is strongly accessible from below or above. But if every point of W is accessible from above and W satisfies (C), then $W \subset AC$; and similarly for BC.

The next two propositions imply properties stronger than (C). Recall that a map $f : X \to X$ is *completely continuous* provided $\overline{f(B)}$ is compact for every bounded set $B \subset X$; and f *conditionally completely continuous* provided $\overline{f(B)}$ is compact whenever B and $f(B)$ are bounded subsets of X.

The *orbit* of any set $B \subset X$ is $O(B) = \bigcup_t \Phi_t(B)$.

PROPOSITION 2.1. *Assume the following two conditions:*
(a) *every compact set has a bounded orbit, and*
(b) *Φ_s is conditionally completely continuous for some $s > 0$.*
If $L \subset X$ is compact, then $\overline{\bigcup_{x \in L} \omega(x)}$ is compact and this implies X has property (C).

PROOF. $O(L)$ is a bounded set by (a), and positively invariant, so (b) implies compactness of $\overline{\Phi_s(O(L))}$. As the latter set contains $\omega(x)$ for all $x \in L$, the first assertion is proved. The second assertion follows from precompactness of $\{x_n\}$. □

PROPOSITION 2.2. *Assume $W \subset X$ has the following property: For every $x \in W$ there is a neighborhood $U_x \subset X$ and a compact set M_x that attracts every point in U_x. Then $\overline{O(x)}$ is compact for every $x \in W$, and $\overline{\bigcup_{y \in U_x} \omega(y)}$ is compact. If $z_n \to x \in W$ then $\overline{\bigcup_n \omega(z_n)}$ is compact, therefore W has property* (C).

PROOF. It is easy to see that $\overline{O(x)}$ is compact and $\overline{\bigcup_{y \in U_x} \omega(y)}$ is compact because it lies in M_x. Fix $k \geqslant 0$ such that $z_n \in U_x$ for all $n \geqslant k$. Then

$$\overline{\bigcup_n \omega(z_n)} = \bigcup_{1 \leqslant n \leqslant k} \omega(z_n) \cup M_x,$$

which is the union of finitely many compact sets, hence compact. Condition (C) follows trivially. □

The key to stronger results on generic quasiconvergence and stability is the following result of Smith and Thieme [197]:

THEOREM 2.3 (Sequential Limit Set Trichotomy). *Let $\{\tilde{x}_n\}$ be an omega compact sequence strongly approximating $z \in BC$ from below. Then there is a subsequence $\{x_n\}$ such that exactly one of the following three conditions holds:*

(a) *There exists* $u_0 \in E$ *such that*

$$\omega(x_n) < \omega(x_{n+1}) < \omega(z) = \{u_0\}$$

and

$$\lim_{n\to\infty} \operatorname{dist}(\omega(x_n), u_0) = 0.$$

In this case $z \in C$.

(b) *There exists* $u_1 = \sup\{u \in E\colon\ u < \omega(z)\}$ *and*

$$\omega(x_n) = \{u_1\} < \omega(z).$$

In this case $z \in \overline{\operatorname{Int} C}$. *Moreover* z *has a neighborhood* W *such that if* $w \in W$, $w < z$ *then* $\Phi_t(w) \to u_1$ *and* $\Phi_t(w) > u_1$ *for sufficiently large* t.

(c) $\omega(x_n) = \omega(z) \subset E$.
In this case $z \in \overline{\operatorname{Int} Q}$. *Moreover* $\omega(w) = \omega(z) \subset E$ *for every* $w < z$ *sufficiently near* z.

Note that z is convergent in (a), and strongly accessible from below by convergent points in (b). In (c), z is quasiconvergent and strongly accessible from below by quasiconvergent points.

If $z \in AC$ there is an analogous dual result, obtained by reversing the order relation in X. Although we do not state it formally, we will use it below. If $z \in AC \cap BC$ then both results apply. See Proposition 3.6 in Smith and Thieme [197].

PROOF OF THEOREM 2.3. By the Limit Set Dichotomy 1.16, either there exists a positive integer j such that $\omega(\tilde{x}_n) = \omega(\tilde{x}_m)$ for all $m, n \geqslant j$, or else there exists a subsequence $\{\tilde{x}_{n_i}\}$ such that $\omega(\tilde{x}_{n_i}) < \omega(\tilde{x}_{n_{i+1}})$ for all i. Therefore there is a subsequence $\{x_n\}$ such that $\omega(x_n) < \omega(x_{n+1})$ for all n, or $\omega(x_n) = \omega(x_{n+1})$ for all n.

Case I: $\omega(x_n) < \omega(x_{n+1})$. We will see that (a) holds. The Limit Set Dichotomy 1.16 implies $\omega(x_n) \leqslant \omega(z)$. In fact, that $\omega(x_n) < \omega(z)$. Otherwise $\omega(x_k) \cap \omega(z) \neq \emptyset$ for some k, and the Limit Set Dichotomy implies the contradiction $\omega(x_k) = \omega(z) \geqslant \omega(x_{k+1}) > \omega(x_k)$.

Define $K = \overline{\bigcup \omega(x_n)}$, a nonempty compact invariant set. Consider the set

$$\Lambda = \Big\{y\colon\ y = \lim_{n\to\infty} y_n,\ y_n \in \omega(x_n)\Big\} \subset K.$$

Clearly Λ is invariant and closed, and compactness of K implies Λ is compact and nonempty. We show that Λ is a single equilibrium. Suppose $y, v \in \Lambda$, so that $y_n \to y$, $v_n \to v$ with $y_n, v_n \in \omega(x_n)$. Since $y_n < v_{n+1}$ and $v_n < y_{n+1}$, we have $y \leqslant v$ and $v \leqslant y$, so $v = y$. Thus we can set $\Lambda = \{u_0\}$, and invariance implies $u_0 \in E$.

The definition of Λ and compactness of K imply $\lim_{n\to\infty} \operatorname{dist}(\omega(x_n), u_0) = 0$. From $\omega(x_n) < \omega(x_{n+1}) < \omega(z)$ we infer

$$\omega(x_n) < u_0 \leqslant \omega(z).$$

If $u_0 \in \omega(z)$ then $\omega(z) = \{u_0\}$ by Corollary 1.9, yielding (a).

We show that $u_0 < \omega(z)$ gives a contradiction. Choose a neighborhood W of $\omega(z)$ and $t_0 \geqslant 0$ such that $u_0 \leqslant \Phi_t(W)$ for all $t \geqslant t_0$ (by Lemma 1.3). There exists $t_1 > 0$ such that $\Phi_{t_1}(z) \in W$, and by continuity of Φ_{t_1} there exists m such that $\Phi_{t_1}(x_m) \in W$. It follows that $u_0 \leqslant \Phi_t(x_m)$ for $t \geqslant t_0 + t_1$. As $u_0 \in E$, we have $u_0 \leqslant \omega(x_m)$. But this contradicts $\omega(x_m) <_! u_0$. Thus (a) holds in Case I.

Case II: $\omega(x_n) = \omega(x_{n+1}) \subset E$. Since $x_n < z$, the Limit Set Dichotomy implies that either $\omega(x_n) = \omega(z)$, which gives (c), or else $\omega(x_n) < \omega(z)$, which we now assume. Choose an equilibrium $u_1 \in \omega(x_1)$. By Lemma 1.3 there exists an open set W containing $\omega(z)$ and $t_0 \geqslant 0$ such that $u_1 \leqslant \Phi_t(W)$ for all $t \geqslant t_0$. Arguing as in Case I, we obtain $u_1 \leqslant \Phi_t(x_m)$ for some m and all large t. Since $u_1 \in \omega(x_m)$, it follows that $\omega(x_m) = u_1$ by Corollary 1.9, and therefore $\omega(x_n) = \{u_1\}$ as asserted in case (b). Finally, if $u \in E$ and $u < \omega(z)$, we argue as above that $\omega(x_m) \geqslant u$ for some m, which implies $u_1 \geqslant u$.

To prove $z \in \overline{\operatorname{Int} Q}$, use SOP to obtain a neighborhood U_n of x_n such that $\Phi_t(x_{n-1}) \leqslant \Phi_t(U_n) \leqslant \Phi_t(x_{n+1})$ for all large t, implying $U_n \subset Q$. A similar argument proves the analogous assertion in (b). □

The following addendum to the Sequential Limit Set Trichotomy provides important stability information. In essence, it associates various kinds of stable points to arbitrary elements $z \in BC$:

PROPOSITION 2.4. *Assume X is normally ordered. In cases* (a), (b) *and* (c) *of the Sequential Limit Set Trichotomy, the following statements are valid respectively*:

(a) *z and u_0 are stable from below*;

(b) *z is not stable from below, $\omega(z)$ is unstable from below, and u_1 is asymptotically stable from above*;

(c) *z is asymptotically stable from below, and $z \in \overline{A}$.*

PROOF. (a) follows from Proposition 1.26(a) and (b).

(b) The first two assertions are trivial. To prove $u_1 \in A_+$, take $w = x_n$ for some large n in the last assertion of (b) and apply 1.25(d) with $a = u_1$.

(c) follows from 1.25(d), taking $b = z$. □

We expect in real world systems that observable motions are stable trajectories. Our next result implies stable trajectories approach equilibria.

PROPOSITION 2.5. $S \cap (BC \cup AC) \subset Q$.

PROOF. When $z \in S \cap (BC \cup AC)$, only (a) and (c) of the Sequential Limit Set Trichotomy are possible, owing to continuity at z of the function $x \mapsto \omega(x)$. In both cases $z \in Q$. □

The inclusion $S \subset Q$ suggests trajectories issuing from nonquasiconvergent points are unlikely to be observed; the next result implies that their limit sets are, not surprisingly, unstable. There are as many concepts of instability as there are of stability, but for our purposes the following very strong property suffices: A set $M \subset X$ is *unstable from above* provided there is an equilibrium $u > M$ such that $\omega(x) = \{u\}$ if $u > x > y$, $y \in M$. Such an equilibrium u is unique, and SOP implies it attracts all points $< u$ in some neighborhood of u. *Unstable from below* is defined dually.

THEOREM 2.6. *Assume $z \in BC \setminus Q$ (respectively, $z \in AC \setminus Q$). Then $\omega(z)$ is unstable from below (resp., above).*

PROOF. To fix ideas we assume $z \in BC \setminus Q$. Then there exists a sequence $x_n \to z$ and an equilibrium u_1 as in conclusion (b) of the Sequential Limit Set Trichotomy. Suppose $u_1 < x < y$, $y \in \omega(z)$. SOP implies there exist open sets W_x and W_y containing x and y, respectively, and $t_0 \geqslant 0$, such that $\Phi_t(W_x) \leqslant \Phi_t(W_y)$ for all $t \geqslant t_0$. As $\Phi_s(z) \in W_y$ for some large s, by continuity $\Phi_s(x_n) \in W_y$ for some large n. Thus $u_1 \leqslant \Phi_t(x) \leqslant \Phi_{t+s}(x_n)$ for all $t \geqslant t_0$. Letting $t \to \infty$ and using the fact that $\omega(x_n) = \{u_1\}$, we find that $\omega(x) = \{u_1\}$. □

A set is *minimal* if it is nonempty, closed and invariant, and no proper subset has these three properties. Every positively invariant nonempty compact set contains a minimal set (by Zorn's Lemma). A minimal set containing more than one point is called *nontrivial*.

COROLLARY 2.7. *A compact, nontrivial minimal set M that meets BC (respectively, AC) is unstable from below (resp., above).*

PROOF. Suppose $z \in M \cap BC$. The assumptions on M imply $M = \omega(z)$ and $M \cap E = \emptyset$. Therefore $z \in BC \setminus Q$, and instability follows from Theorem 2.6. □

When X is a convex subset of a vector space, an alternative formulation of Theorem 2.6 is that $\omega(z)$ belongs to the upper boundary of the basin of attraction of the equilibrium u_1. Corollary 2.7 implies that periodic orbits are unstable. Theorem 2.6 is motivated by Theorem 1.6 in Hirsch [79].

The following sharpening of Theorem 1.19, due to Smith and Thieme [199], is an immediate corollary of the Sequential Limit Set Trichotomy.

THEOREM 2.8. *If $J \subset X$ is a totally ordered arc having property* (C), *then $J \setminus Q$ is a discrete, relatively closed subset of J; hence it is countable, and finite when J is compact.*

PROOF. Every limit point z of $J \setminus Q$ is strongly accessible from above or below by a sequence $\{\tilde{x}_n\}$ in $J \setminus Q$. As Property (C) implies $J \subset BC \cup AC$, there is a sequence $\{x_n\}$ satisfying (a), (b) or (c) of Theorem 2.3 (or its dual result), all of which imply $x_n \in Q$. Thus $J \setminus Q$ contains none of its limit points, which implies the conclusion. □

The following result sharpens Theorems 1.30 and 2.8:

PROPOSITION 2.9. *Assume X is normally ordered and every point is accessible from above and below. Let $J \subset X$ be a totally ordered compact arc having property* (C), *with* *endpoints $a < b$ such that $\omega(a)$ is an equilibrium stable from below and $\omega(b)$ is an equilibrium stable from above. Then J contains a point whose trajectory converges to a stable equilibrium.*

PROOF. Denote by C_s (respectively: C_+, C_-) the set of convergent points whose omega limits belong to S (resp.: to S_+, S_-). Then $C_+ \cap C_- = C_s$ by Proposition 1.25(b).

Set $\sup(J \cap C_-) = z \in J$.

Case 1: $z \notin C_-$. Then $z > a$. Choose a sequence $x_1 < x_2 < \cdots < z$ in $J \cap C_-$ such that $x_n \to z$. By the Sequential Limit Set Trichotomy 2.3 it suffices to consider the following three cases:

(a) There exists $u_0 \in E$ such that

$$\omega(x_n) < \omega(x_{n+1}) < \omega(z) = \{u_0\}.$$

This is not possible, because $u_0 \in S_-$ by Proposition 2.4(a), yielding the contradiction $z \in C_-$.

(b) There exists $u_1 = \sup\{u \in E\colon\ u < \omega(z)\}$, and for all n we have

$$\omega(x_n) = \{u_1\} < \omega(z).$$

Now 2.4(b) has $u_1 \in S_+$, hence $x_n \in C_+$. Therefore $x_n \in C_+ \cap C_- = C_s$, as required.

(c) $\omega(x_n) = \omega(z)$. This is not possible because $x_n \in C_-$ and $\omega(z) = \omega(x_n)$ implies the contradiction $z \in C_-$.

Thus (b) holds, validating the conclusion when $z \notin C_-$.

Case 2: $z \in C_-$. If $z = b$ then $z \in C_+ \cap C_- = C_s$ and there is nothing more to prove. Henceforth we assume $z < b$.

The closed subinterval $K \subset J$ with endpoints z, b satisfies the hypotheses of the theorem. Set $\inf(K \cap C_+) = w \in K$. The dual of the reasoning above shows that if $w \notin C_+$ then the conclusion of the theorem is true.

From now on we assume $w \in C_+$. If $w = z$ there is nothing more to prove, so we also assume $w > z$. Let $L \subset K$ be the closed subinterval with endpoints w and z. Let $\{\bar{x}_n\}$ be a sequence in L converging to w from below.

One of the conclusions (a), (b) or (c) of 2.3 holds. Referring to the corresponding parts of 2.4, we see in case (a) that $\omega(w)$ is an equilibrium $\bar{u}_0$ that is stable from below; but $w > z$, so this contradicts the definition of z. If (b) holds, $\omega(\bar{x}_n)$ is an equilibrium $\bar{u}_1$ stable from above. But $\bar{x}_n < w$, so this contradicts the definition of w. In case (c) we have for all n that $\omega(\bar{x}_n) = \omega(w)$, which is an equilibrium stable from above. But $\bar{x}_n < w$ for $n > 1$, again contradicting the definition of w. □

In the following result the assumption on equilibria holds when Φ has a global compact attractor.

PROPOSITION 2.10. *Assume X is an open subset of a strongly ordered Banach space, Φ is strongly monotone, and every equilibrium has a neighborhood attracted to a compact set. Let $J \subset X$ be a totally ordered compact arc, with endpoints $a < b$ such that $\omega(a)$ is* *an equilibrium stable from below and $\omega(b)$ is an equilibrium stable from above. Then J contains a point whose trajectory converges to a stable equilibrium.*

PROOF. Apply Proposition 2.9 to the to the SOP semiflow $\widehat{\Phi}$ in the normally ordered space $\widehat{X}$ (see Section 1.7), to obtain an equilibrium p that is stable for $\widehat{\Phi}$. This means p is order stable for Φ, hence stable for Φ by Proposition 1.28. □

COROLLARY 2.11. *Let X be a p-convex open set in an ordered Banach space Y. Assume Φ has a compact global attractor. Suppose that either Y is normally ordered, or Y is strongly ordered and Φ is strongly monotone. Then:*

(i) *There is a stable equilibrium.*

(ii) *Let $u, v \in X$ be such that $u < v$ and there exist real numbers $r, s > 0$ such that $u < \Phi_r(u)$, $\Phi_s(v) < v$. Then there is a stable equilibrium in $[u, v]$.*

In case (ii) *with Y normally ordered, the hypothesis of a global attractor can be replaced the assumption that the line segment joining u to v from satisfies condition* (C).

PROOF. We first prove (ii). Monotonicity shows that $\omega(x) \subset [u, v]$ for all $x \in [u, v]$. The Convergence Criterion implies

$$\Phi_t(u) \to a \in E \cap [u, v], \qquad \Phi_t(v) \to b \in E \cap [u, v].$$

We claim that $a \in S_-$ and $b \in S_+$, and $a \in S$ is stable if $a = b$. When Y is normal this follows from Propositions 1.25(b) and (d), and it is easy to prove directly when Φ is strongly monotone. Suppose $a < b$. By p-convexity and Theorems 2.9 and 2.10, the line segment from a to b lies in $[u, v] \cap X$ and contains a point whose x such that $\omega(x)$ is a stable equilibrium z. As noted above, $z \in [u, v]$.

We prove (i) by finding u and v as in (ii). By Theorem 2.8 and compactness of the global attractor, there is a minimal equilibrium p and a maximal equilibrium $q > p$. As X is open, it contains a totally ordered line segment $J < p$. By Theorem 1.19 J contains a quasiconvergent point $u < p$. As $\omega(u) \leqslant p$, minimality of p implies $\Phi_t(u) \to p$. Similarly there exist $v > q$ with $\Phi_t(v) \to q$. It follows from SOP that $u < \Phi_r(u)$, $\Phi_s(v) < v$ for some $r, s > 0$. □

For strongly monotone semiflows, the existence of order stable equilibria in attractors was treated in Hirsch [68,69,73].

2.2. *Generic quasiconvergence and stability*

The following result adapted from Smith and Thieme [197] refines Theorems 1.22 and 1.21:

THEOREM 2.12. (i) $AC \cup BC \subset \overline{\operatorname{Int} Q} \cup C$. *Therefore if $AC \cup BC$ is dense, so is Q.*

(ii) $(\operatorname{Int} AC) \cup (\operatorname{Int} BC) \subset \overline{\operatorname{Int} Q}$. *Therefore if* $(\operatorname{Int} AC) \cup (\operatorname{Int} BC)$ *is dense, so is* $\operatorname{Int} Q$.

PROOF. Every $z \in BC$ is the limit of an omega compact sequence $x_1 < x_2 < \cdots$ such that (a), (b) or (c) of the Sequential Limit Set Trichotomy Theorem 2.3 holds, and $z \in \overline{\operatorname{Int} Q} \cup C$ in each case; the proof for AC is similar.

To prove (ii), assume $z \in \operatorname{Int} BC$. If (a) holds for every point of a neighborhood W of z, then $W \subset C$, whence $z \in \operatorname{Int} Q$. If there is no such W, every neighborhood of z contains a point for which (b) or (c) holds, hence $z \in \overline{\operatorname{Int} Q}$. Similarly for $z \in \operatorname{Int} AC$. $\square$

The next result extends Theorems 8.10 and 9.6 of Hirsch [73] and Theorem 3.9 of Smith and Thieme [197]:

THEOREM 2.13. *Assume X is normally ordered and* $\operatorname{Int}(BC \cup AC)$ *is dense. Then* $A \cup \operatorname{Int} C$ *is dense.*

PROOF. We argue by contradiction. If $A \cup \operatorname{Int} C$ is not dense, there exists an open set U such that

$$U \cap \overline{A} = \emptyset = U \cap \overline{\operatorname{Int} C}.$$

Suppose $z \in U \cap BC$, and let $\{x_n\}$ be a sequence in U strongly approximating z from below. Conclusion (b) of the Sequential Limit Set Trichotomy 2.3 is not possible because $z \notin \overline{\operatorname{Int} C}$, and conclusion (c) is ruled out because $z \notin \overline{A}$ (see Proposition 2.4(c)). Therefore conclusion (a) holds, which makes z convergent; likewise when $z \in U \cap AC$. Thus we have $C \supset U \cap (BC \cup AC)$, so $\operatorname{Int} C \supset U \cap \operatorname{Int}(BC \cup AC)$. But the latter set is nonempty by the density hypothesis, yielding the contradiction $U \cap \operatorname{Int} C \neq \emptyset$. $\square$

The following theorem concludes that generic trajectories are not only quasiconvergent, but also stable. Its full force will come into play in the next subsection, under assumptions entailing a dense open set of convergent points.

THEOREM 2.14. *If X is normally ordered and* $\operatorname{Int}(BC \cap AC)$ *is dense, then* $\operatorname{Int}(Q \cap S)$ *is dense.*

PROOF. $\operatorname{Int} Q$ is dense by Theorem 2.12. To prove density of $\operatorname{Int} S$, it suffices to prove that if $z \in \operatorname{Int}(BC \cap AC)$, then every open neighborhood U of z meets $\operatorname{Int} S$. We can assume $z \notin \overline{A}$ because $A \subset \operatorname{Int} S$. Let $\{x_n\}$ be an omega compact sequence strongly approximating z from below. Suppose (b) or (c) of the Sequential Limit Set Trichotomy 2.3 holds. Then $x_m \in U$ for $m \geqslant m_0$. Fix $m \geqslant m_0$. It follows from Proposition 1.25(d) (with $a = x_m$, $x = x_{m+1}$, $b = x_{m+2}$) that $x_{m+1} \in A$, hence $z \in \overline{A}$; this is proved similarly when $\{x_n\}$ strongly approximates z from above.

Henceforth we can assume z belongs to the open set $W = \operatorname{Int}(BC \cap AC) \setminus \overline{A}$, and consequently that there are omega compact sequences $\{x_n\}$, $\{y_n\}$ strongly approximating z from below and above respectively, for which Theorem 2.3(a) and its dual hold respectively. Then Proposition 1.26 implies $z \in S_+ \cap S_-$, whence $z \in S$ by Proposition 1.25(b). Thus the open set W is contained in $\operatorname{Int} S$, and we have proved $\operatorname{Int} S$ is dense. It follows that $\operatorname{Int} S \cap \operatorname{Int} Q$ is dense. $\square$

2.3. *Improving the limit set dichotomy for smooth systems*

The aim now is to strengthen the Limit Set Dichotomy with additional hypotheses, especially smoothness, in order to obtain the following property:

(ILSD) A semiflow satisfies the *Improved Limit Set Dichotomy* if $x_1 < x_2$ implies that either
(a) $\omega(x_1) < \omega(x_2)$, or
(b) $\omega(x_1) = \omega(x_2) = e \in E$.

We begin with some definitions.

Let X be a subset of the Banach space Y. A map $f : X \to Y$ is said to be *locally C^1 at* $p \in X$ if there exists a neighborhood U of p in X and a continuous *quasiderivative* map $f' : U \to L(Y)$, where $L(Y)$ is the Banach space of bounded operators on Y, such that

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \phi(x, x_0)|x - x_0|, \quad x, x_0 \in U$$

with $\phi(x, x_0) \to 0$ as $x \to x_0$. The following result gives a setting where the quasiderivative is uniquely determined by f. We denote the open ball in Y of center p and radius r by $B_Y(p, r) := \{y \in Y\colon\ |y - p| < r\}$.

LEMMA 2.15. *Let $p \in X \subset Y$ where Y is a strongly ordered Banach space. Assume $f : X \to Y$ is locally C^1 at p, and suppose that either $B_Y(p, r) \cap Y_+ \subset X$ or $B_Y(p, r) \cap (-Y_+) \subset X$ for some $r > 0$. Then $f'(p)$ is uniquely defined.*

PROOF. Suppose $B_Y(p, r) \cap Y_+ \subset X$, the other case being similar. Fix $w \gg 0$ and let $y \in Y$. As $w + y/n := k_n \geqslant 0$ for large n, $y = n(k_n - w)$ so $Y = Y_+ - Y_+$.

Assume

$$f(x) - f(p) = A(x - p) + \phi(x, p)|x - p| = B(x - p) + \psi(x, p)|x - p|,$$

where $A, B \in L(Y)$ and $\phi, \psi \to 0$ as $x \to p$ in X. It suffices to show that $Av = Bv$ for all $v \geqslant 0$. The segment $x = p + sv \in X$ for all small $s \geqslant 0$. Inserting it in the formula above, dividing by s, and letting $s \to 0$ yields the desired result. □

Let Φ be a monotone semiflow on the subset X of the strongly ordered Banach space Y. Concerning X and the set of equilibria E, we assume the following condition on the pair (Y, X):

(OC) Either X is an order convex subset of Y or $E \subset \operatorname{Int} X$. For each $e \in E$ there exists $r > 0$ such that either $B_Y(e, r) \cap Y_+ \subset X$ or $B_Y(e, r) \cap (-Y_+) \subset X$.

This relatively minor restriction is automatically satisfied if X is an open set, an order interval, or the cone Y_+. The second assertion of (OC) trivially holds if $E \subset \operatorname{Int} X$.

We will also assume the following two conditions hold for some $\tau > 0$. A compact, strongly positive linear operator is called a *Krein–Rutman* operator.

(M) $x_1 < x_2 \Longrightarrow \Phi_\tau(x_1) \ll \Phi_\tau(x_2)$
(D*) Φ_τ is locally C^1 at each $e \in E$, with $\Phi_\tau'(e)$ a Krein–Rutman operator.

As motivation for (D*), consider the case that X is an open set in Y and Φ_τ is C^1. If $x \in X$, $y \in Y_+$, $h > 0$, and $x + hy \in X$, then $(\Phi_\tau(x+hy) - \Phi_\tau(x))/h \geqslant 0$ by monotonicity; on taking the limit as $h \to 0$, we get $\Phi_\tau'(x)y \geqslant 0$. Consequently, $\Phi_\tau'(x)Y_+ \subset Y_+$, and hence the assumption that $\Phi_\tau'(x)$ is strongly positive is not such a severe one. Typically, one usually must verify it anyway to prove that Φ_τ is strongly monotone.

Observe that (M) implies that Φ is strongly order preserving on X.

THEOREM 2.16 (Improved Limit Set Dichotomy). *Let Φ be a monotone semiflow on a subset X of the strongly ordered Banach space Y for which* (OC), (M), *and* (D*) *are satisfied. Then* (ILSD) *holds.*

In particular, (ILSD) holds if X is open, the semiflow Φ continuously differentiable and strongly monotone, and the derivative $\Phi_t'(e)$ is a Krein–Rutman operator at each $e \in E$.

Before giving the proof, we explore the spectral and dynamical implications of (D*).

2.3.1. *The Krein–Rutman theorem* The spectrum of a linear operator $A : Y \to Y$ is denoted by $\mathrm{Spec}(A)$. When A is compact (i.e., completely continuous), $\mathrm{Spec}(A)$ consists of a countable set of eigenvalues and perhaps 0, and the eigenvalues have no accumulation point except possibly 0.

Let $\rho(A)$ be the *spectral radius* of A, that is, $\rho(A) = \max\{|\lambda|\colon \lambda \in \mathrm{Spec}(A)\}$. Denote the null space of A by $N(A)$ and the range by $\mathrm{Im}(A)$.

The set $\mathrm{KR}(Y)$ of Krein–Rutman operators on Y is given the metric induced by the uniform norm.

THEOREM 2.17 (Krein–Rutman). *Let $A \in \mathrm{KR}(Y)$ and set $r = \rho(A)$. Then Y decomposes into a direct sum of two closed invariant subspaces Y_1 and Y_2 such that $Y_1 = N(A - rI)$ is spanned by $z \gg 0$ and $Y_2 \cap Y_+ = \{0\}$. Moreover, the spectrum of $A|Y_2$ is contained in the closed ball of radius $\nu < r$ in the complex plane.*

See Krein and Rutman [104], Takáč [214] or Zeidler [244] for proofs.

It follows that each $A \in \mathrm{KR}(Y)$ has a unique unit eigenvector $z(A) \in Y_+$, and $z(A) \in \mathrm{Int}\, Y_+$, $Az(A) = \rho(A)z(A)$.

LEMMA 2.18. *$\rho(A)$ and $z(A)$ are continuous functions of $A \in \mathrm{KR}(Y)$.*

PROOF. The upper semicontinuity of the spectral radius follows from the upper semicontinuity of the spectrum as a function of the operator (Kato [92]). The lower semicontinuity follows from the lower semicontinuity of isolated parts of the spectrum (Kato [92, Chapter IV, Theorem 3.1, Remark 3.3, Theorem 3.16]). Let P_A be the projection onto $N(A) - \rho(A)I)$ along $\mathrm{Im}(A - \rho(A)I)$. Continuity of $A \mapsto P_A$ is proved in [92, Chapter IV, Theorem 3.16]. Let $A_n \to A$ in $\mathrm{KR}(Y)$ and set $z_n = z(A_n)$, $z = z(A)$. Then $(I - P_A))z_n = (P_{A_n} - P_A)z_n \to 0$ as $n \to \infty$, and $\{P_A z_n\}$ is precompact, so $\{z_n\}$ is precompact. If $z_{n_i} \to u$ for some subsequence, then

$$P_A u = \lim A_{n_i} z_{n_i} = \lim \rho(A_{n_i}) z_{n_i} = \rho(A)u.$$

Uniqueness of the positive eigenvector for A (Theorem 2.17) implies $u = z$ and $z_n \to z$. □

For technical reasons it is useful to employ a norm that is more compatible with the partial order. If $w \in \mathrm{Int}\, Y_+$ is fixed, then the set $U = \{y \in Y\colon -w \ll y \ll w\}$ is an open neighborhood of the origin. Consequently, if $y \in Y$, then there exists $t_0 > 0$ such that $t_0^{-1} y \in U$, hence, $-t_0 w \ll y \ll t_0 w$. Define the w-norm by

$$\|y\|_w = \inf\{t > 0\colon -tw \leqslant y \leqslant tw\}.$$

Since $w \in \mathrm{Int}\, Y_+$, there exists $\delta > 0$ such that for all $y \in Y \setminus \{0\}$ we have $w \pm \delta \frac{y}{|y|} \in Y_+$. Thus

$$\|y\|_w \leqslant \delta^{-1} |y|$$

holds for all $y \in Y$, implying that the w-norm is weaker than the original norm. In fact, the two norms are equivalent if Y_+ is normal, but we will have no need for this result. See Amann [6] and Hirsch [73] for more results in this direction. It will be useful to renormalize the positive eigenvector $z(A)$ for $A \in \mathrm{KR}(Y)$. The next result says this can be done continuously. Continuity always refers to the original norm topology on Y unless the contrary is explicitly stated.

LEMMA 2.19. *Let $Z(A) = z(A)/\|z(A)\|_w$ and $\beta(A) = \sup\{\beta > 0\colon Z(A) \geqslant \beta w\}$. Then $\beta(A) > 0$, $Z(A) \geqslant \beta(A)w$, and the maps $A \to Z(A)$ and $A \to \beta(A)$ are continuous on* $\mathrm{KR}(Y)$.

PROOF. Since the w-norm is weaker than the original norm, the map $A \mapsto \|z(A)\|_w$ is continuous. This implies that $Z(A)$ is continuous in A. It is easy to see that $\beta(A) > 0$. Let $\epsilon > 0$ satisfy $2\epsilon < \beta(A)$ and let $A_n \to A$ in $\mathrm{KR}(Y)$. Then $-\epsilon w \leqslant Z(A) - Z(A_n) \leqslant \epsilon w$ for all large n by continuity of Z and because the w-norm is weaker than the original norm. Therefore, $Z(A_n) = Z(A_n) - Z(A) + Z(A) \geqslant (\beta(A) - \epsilon)w$, so $\beta(A_n) \geqslant \beta(A) - \epsilon$ for all large n. Similarly, $Z(A) = Z(A) - Z(A_n) + Z(A_n) \geqslant (\beta(A_n) - \epsilon)w$ for all large n, so $\beta(A) \geqslant \beta(A_n) - \epsilon$ for all large n. Thus, $\beta(A) - \epsilon \leqslant \beta(A_n) \leqslant \beta(A) + \epsilon$ holds for all large n, completing the proof. □

The key to improving the Limit Set Dichotomy is to show that the omega limit set of a point x that is quasiconvergent but not convergent, is uniformly unstable in the linear approximation. The direction of greatest instability at $e \in \omega(x)$ is the positive direction $z(e) := z(\Phi'_\tau(e))$. The number $\rho(e) := \rho(\Phi'_\tau(e))$ gives a measure of the instability.

Nonordering of Limit Sets means that positive directions are, in some rough sense, "transverse" to the limit set. Thus our next result means that the limit set is uniformly unstable in a transverse direction.

LEMMA 2.20. *Assume* (D*). *Let x be quasiconvergent but not convergent. Then $\rho(e) > 1$ for all $e \in \omega(x)$.*

PROOF. Fix $e \in \omega(x)$. Since $\omega(x)$ is connected, e is the limit of a sequence $\{e_n\}$ in $\omega(x) \cap U \setminus \{e_0\}$, where U is the neighborhood of e in the definition of Φ_τ is locally C^1 at e. Then

$$e_0 - e_n = \Phi_\tau(e) - \Phi_\tau(e_n) = \Phi'_\tau(e)(e - e_n) + \mathrm{o}(|e - e_n|),$$

where $\mathrm{o}(|e - e_n|)/|e - e_n| \to 0$ as $n \to \infty$. Put $v_n = (e - e_n)/|e - e_n|$. Then

$$v_n = \Phi'_\tau(e)v_n + r_n, \quad r_n \to 0, \quad n \to \infty.$$

The compactness of $\Phi'_\tau(e)$ implies that v_n has a convergent subsequence v_{n_i}; passing to the limit along this subsequence leads to $v = \Phi'_\tau(e)v$ for some unit vector v. Thus $\rho(e) \geqslant 1$. If $\rho(e) = 1$, then the Krein–Rutman Theorem implies $v = rz(e)$ where $r = \pm 1$. Consequently,

$$(e - e_{n_i})/|e - e_{n_i}| \to rz(e)$$

as $i \to \infty$. It follows that $e \ll e_{n_i}$ or $e \gg e_{n_i}$ for all large i, contradicting the Nonordering of Limit Sets. □

PROOF OF THEOREM 2.16. By the Limit Set Dichotomy (Theorem 1.16), it suffices to prove: If $x_1 < x_2$ and $\omega(x_1) = \omega(x_2) = K \subset E$, then K is a singleton. K is compact and connected, unordered by the Nonordering of Limit Sets, and consists of fixed points of Φ_τ. Arguing by contradiction, we assume K is not a singleton.

Set $v_n = \Phi_{n\tau}(x_1)$, $u_n = \Phi_{n\tau}(x_2)$. Then $\mathrm{dist}(K, u_n) \to 0$ and $\mathrm{dist}(K, v_n) \to 0$ as $n \to \infty$. Moreover (M) and the final assertion of the Limit Set Dichotomy imply

$$u_n - v_n \gg 0, \qquad u_n - v_n \to 0.$$

Fix $w \gg 0$ and define real numbers

$$\alpha_n = \sup\{\alpha \in \mathbb{R}\colon \alpha \geqslant 0,\ \alpha w \leqslant u_n - v_n\}.$$

Then $\alpha_n > 0$ and $\alpha_n \to 0$.

To simplify notation, define $S: X \to X$ by $S(x) := \Phi_\tau(x)$. Choose $e_n \in K$ such that $v_n - e_n \to 0$ as $n \to \infty$. By Lemma 2.20, local smoothness of Φ_τ and compactness of K, there exists $r > 1$ such that $\rho(e) > r$ for all $e \in K$. Let $z_n = Z(e_n)$ be the normalized positive eigenvector for $S'(e_n) = \Phi'_\tau(e_n)$ so $\|z_n\|_w = 1$ and $z_n \leqslant w$. By Lemma 2.19, there exists $\epsilon > 0$ such that $\beta(e_n) \geqslant \epsilon$ for all n. In particular, $w \geqslant z_n \geqslant \epsilon w$ for all n.

Fix a positive integer l such that $r^l\epsilon > 1$.

For each $e \in K$, by (D*) we can choose an open neighborhood W_e of e in X and a continuous map $S' : W_e \to L(Y)$ such that for $x, x_0 \in W_e$ we have

$$Sx - Sx_0 = S'(x_0)(x - x_0) + \phi(x, x_0)|x - x_0|, \quad \lim_{x \to x_0} \phi(x, x_0) = 0.$$

Putting $x_0 = e$ and estimating norms, one easily sees that there exists a convex open neighborhood $U_e \subset W_e$ of e such that $S^i(U_e) \subset W_e$ for $1 \leqslant i \leqslant l$. Furthermore, a simple induction argument implies that S^l is locally C^1 at e with quasiderivative

$$(S^l)' : U_e \to L(Y), \qquad (S^l)'(x) = S'(S^{l-1}x) \circ S'(S^{l-2}x) \circ \cdots \circ S'(x).$$

By compactness of K there is a finite subset $\{e_1, \ldots, e_\nu\} \subset K$ such that the sets U_{e_j} cover K. Set $U_j = U_{e_j}$, $W_j = W_{e_j}$. Then

$$K \subset \bigcup_{j=1}^{\nu} U_j, \qquad S^i(U_j) \subset W_j \quad (1 \leqslant i \leqslant l),$$

and for $z, z_0 \in U_j$

$$S^l(z) - S^l(z_0) = (S^l)'(z_0)(z - z_0) + \phi_{l,j}(z, z_0)|z - z_0|, \quad \lim_{z \to z_0} \phi_{l,j}(z, z_0) = 0,$$

and the usual chain rule expresses $(S^l)'$ in terms of S'.

By (OC), either X is order convex in Y or $E \subset \operatorname{Int} X$. In the order convex case, from $v_n \ll v_n + \alpha_n w \leqslant u_n$ we infer that $v_n + s\alpha_n w \in X$ for all $s \in [0, 1]$. Since $v_n - e_n \to 0$, $v_n - u_n \to 0$, and $\alpha_n \to 0$, for sufficiently large n there exists $j(n) \in \{1, \ldots, \nu\}$ such that $U_{j(n)}$ contains the points v_n, u_n, e_n, and $v_n + s\alpha_n w$ for all $s \in [0, 1]$. When $E \subset \operatorname{Int} X$ the same conclusion holds, and we can take U_j, V_j to be open in Y.

Lemma 2.15 justifies the application of the fundamental theorem of calculus to the map $[0, 1] \to X$, $s \mapsto S^l(v_n + s\alpha_n w)$, leading to

$$S^l(v_n + \alpha_n w) - S^l(v_n) = (S^l)'(e_n)(\alpha_n w) + \alpha_n \delta_n$$

and

$$\delta_n = \int_0^1 \left[(S^l)'(v_n + \eta\alpha_n w) - (S^l)'(e_n)\right] w \, d\eta.$$

Using that $v_n + \alpha_n w - e_n \to 0$, K is compact, and $(S^l)'$ is continuous, it is easy to show that

$$\lim_{n \to \infty} \max_{0 \leqslant \eta \leqslant 1} \left|\left[(S^l)'(v_n + \eta\alpha_n w) - (S^l)'(e_n)\right] w\right| = 0.$$

It follows that $d_n := \|\delta_n\|_w \to 0$ as $n \to \infty$. Because $w \geqslant z_n \geqslant \epsilon w \gg 0$ and $\delta_n \geqslant -d_n w$, for sufficiently large n we have:

$$\begin{aligned} S^l(v_n + \alpha_n w) - S^l(v_n) &\geqslant \left[(S^l)'(e_n)\right]\alpha_n w - \alpha_n d_n w \\ &\geqslant \left[S'(e_n)\right]^l \alpha_n w - \alpha_n d_n w \\ &\geqslant r^l \alpha_n z_n - \alpha_n d_n w \\ &\geqslant (r^l \epsilon - d_n)\alpha_n w \\ &\geqslant \alpha_n w, \end{aligned}$$

and therefore

$$u_{n+l} = S^l(u_n) \geqslant S^l(v_n + \alpha_n w) \geqslant S^l v_n + \alpha_n w = v_{n+l} + \alpha_n w.$$

Thus $\alpha_n w \leqslant u_{n+l} - v_{n+l}$, so the definition of α_{n+l} implies $\alpha_{n+l} \geqslant \alpha_n > 0$ for all sufficiently large n. Therefore the sequence $\{\alpha_i\}_{i\in\mathbb{N}_+}$, which converges to 0, contains a nondecreasing positive subsequence $\{\alpha_{n+kl}\}_{k\in\mathbb{N}_+}$. This contradiction implies K is a singleton. □

A drawback of the Improved Limit Set Dichotomy, Theorem 2.16, is that the topology on X comes from a strongly ordered Banach space $Y \supset X$, severely limiting its application to infinite-dimensional systems. The following extension permits use of (ILSD) in more general spaces:

PROPOSITION 2.21. *Let X^1, X^0 be ordered spaces such that $X^1 \subset X^0$ and the inclusion map $j : X^1 \hookrightarrow X^0$ is continuous and order preserving. For $k = 0, 1$ let Φ^k be a monotone semiflow on X^k with compact orbit closures. Assume for all $t > 0$ that Φ_t^0 maps X^0 continuously into X^1, and $\Phi_t^0|X^1 = \Phi_t^1$. If* (ILSD) *holds for Φ^1, it also holds for Φ^0.*

PROOF. Denote the closure in X^k of any $S \subset X^k$ by $\mathrm{C}_k S$. For $k \in \{0, 1\}$ and $x \in X^k$, let $O_k(x)$ and $\omega_k(x)$ respectively denote the orbit and omega limit set of x.

The hypotheses imply that the compact set $\mathrm{C}_0 O_0(x)$, which is positively invariant for Φ^0, is mapped homeomorphically by Φ_1^0 onto $\mathrm{C}_1 O_1(y) \subset X^1$, which is positively invariant for Φ^1. As Φ^0 and Φ^1 coincide in X^1, we see that $\omega_0(x) = \omega_1(y)$ as compact sets. Hence Φ^0 and Φ^1 have the same collection of omega limit sets, which implies the conclusion. □

THEOREM 2.22 (Improved Sequential Limit Set Trichotomy). *Assume* (ILSD). *Let $\{\tilde{x}_n\}$ be a sequence approximating $z \in BC$ from below, with $\overline{\bigcup_n \omega(\tilde{x}_n)}$ compact. Then there is a subsequence $\{x_n\}$ such that exactly one of the following three conditions holds for all n:*

(a) *There exists $u_0 \in E$ such that*

$$\omega(x_n) < \omega(x_{n+1}) < \omega(z) = \{u_0\}$$

and

$$\lim_{n\to\infty} \mathrm{dist}\big(\omega(x_n), u_0\big) = 0.$$

(b) *There exists* $u_1 = \sup\{u \in E\colon\ u < \omega(z)\}$, *and*

$$\omega(x_n) = \{u_1\} < \omega(z).$$

In this case $z \in \overline{\operatorname{Int} C}$. *Moreover* z *has a neighborhood* W *such that if* $w \in W$, $w < z$ *then* $\Phi_t(w) \to u_1$ *and* $\Phi_t(w) > u_1$ *for sufficiently large* t.

(c$'$) *There exists* $u_2 \in E$ *such that* $\omega(x_n) = \omega(x_0) = u_2$.

Note that z is convergent in (a), strongly accessible from below by convergent points in (b), and convergent in (c$'$).

PROOF. Conclusions (a) and (b) are the same as in the Sequential Limit Set Trichotomy, Theorem 2.3. If 2.3(c) holds, then (c$'$) follows from (ILSD). □

PROPOSITION 2.23. *Assume* (ILSD). *If* $x \in BC \setminus C$ *then* $\omega(x)$ *is unstable from below. If* $x \in AC \setminus C$ *then* $\omega(x)$ *is unstable from above.*

PROOF. This is just Theorem 2.6 if $x \notin Q$. If $x \in BC \cap (Q \setminus C)$, we must have conclusion (b) of Theorem 2.22. This provides $u_1 \in E$ such that $\omega(x_n) = \{u_1\}$ for all n, and the remainder of the proof mimics that of Theorem 2.6. □

A consequence of Proposition 2.23 is that if $x \in BC \cap AC$ is nonconvergent, then $\omega(x)$ lies in both the upper boundary of the basin of attraction of an equilibrium u_0 and the lower boundary of the basin of attraction of an equilibrium v_0, where $u_0 < L < v_0$. Thus $\omega(x)$ forms part of a separatrix separating the basins of attraction of u_0 and v_0.

2.4. *Generic convergence and stability*

The following result concludes that the set C of convergent points is dense and open in totally ordered arcs:

THEOREM 2.24. *Assume* (ILSD) *and let* $J \subset X$ *be a totally ordered arc having property* (C). *Then* $J \setminus C$ *is a discrete, relatively closed subset of* J; *hence it is countable, and finite when* J *is compact.*

PROOF. The proof is like that of Theorem 2.8, using the Improved Limit Set Trichotomy 2.22 instead of the Sequential Limit Set Trichotomy 2.3. □

We can now prove the following generic convergence and stability results:

THEOREM 2.25. *Assume* (ILSD).

(a) $AC \cup BC \subset \overline{\operatorname{Int} C} \cup C$. *In particular, if* $AC \cup BC$ *is dense, so is* $\operatorname{Int} C$ *is dense.*

(b) *If* $\operatorname{Int}(BC \cap AC)$ *is dense and* X *is normally ordered, then* $\operatorname{Int}(C \cap S)$ *is dense.*

PROOF. The proof of (a) is similar to that of Theorem 2.12: take $p \in X \setminus \operatorname{Int} C$ and use the Improved Limit Set Trichotomy (Theorem 2.22), instead of the Limit Set Trichotomy, to show that $p \in \overline{\operatorname{Int} C} \cup C$. Conclusion (b) follows from (a) and Theorem 2.14. □

THEOREM 2.26. *Assume X is a subset of a strongly ordered Banach space Y, and a dense open subset of X is covered by totally ordered line segments. Let* (M) *and* (D*) *hold. Then:*

(a) *The set of convergent points has dense interior.*
(b) *Suppose Y is normally ordered. Then the set of stable points has dense interior.*
(c) *Assume Y is normally ordered; X is open or order convex or a subcone of Y_+; and every closed totally ordered subset of E is compact. Then there is a stable equilibrium, and an asymptotically stable equilibrium when E is finite.*

PROOF. The assumption in (a) implies $BC \cap AC$ has dense interior and condition (OC) holds. Therefore the Improved Limit Set Dichotomy (ILSD) holds by Theorem 2.16, so (a) and (b) follow from Theorem 2.25(a). Conclusion (c) is a consequence of (a) and Theorem 1.30. □

As most orbits with compact closure converge to an equilibrium, it is natural to investigate the nature of the convergence. It might be expected that most trajectories converging to a stable equilibrium are eventually increasing or decreasing. We quote a theorem of Mierczyński that demonstrates this under quite general conditions for smooth strongly monotone dynamical systems, including cases when the equilibrium is not asymptotically stable in the linear approximation. Mierczyński assumes the following hypothesis:

(M_1) X is an open set in a strongly ordered Banach space Y. Φ is C^1 on $(0,\infty) \times X$ and strongly monotone, $\Phi_t'(x)$ is strongly positive for all $t > 0$, $x \in X$, and $\Phi_1'(x)$ is compact.

The following local trichotomy due to Mierczyński [138] builds on earlier work of Poláčik [161]:

THEOREM 2.27. *Assume* (M_1). *Then each equilibrium e satisfying $\rho(\Phi_1'(e)) \leqslant 1$ belongs to a locally invariant submanifold Σ_e of codimension one that is smooth and unordered and has the following property. If $\lim_{t\to\infty} \Phi_t(x) = e$, there exists $t_0 \geqslant 0$ such that one of the following holds as $t \to \infty$, $t \geqslant t_0$:*

(i) *$\Phi_t(x)$ decreases monotonically to e;*
(ii) *$\Phi_t(x)$ increases monotonically to e;*
(iii) *$\Phi_t(x) \in \Sigma_e$.*

Mierczyński also provides further important information: The trajectories in cases (i) and (ii) lie in curves tangent at e to the one-dimensional principle eigenspace Y_1 of $\Phi_1'(e)$ described in the Krein–Rutman Theorem 2.17. The hypersurface Σ_e is locally unique in a neighborhood of e. Its tangent space is the closed complementary subspace Y_2, hence Σ_e is transverse to $z = z(\Phi_1'(e)) \gg 0$ at e. Strong monotonicity implies that when (i) or (ii) holds, e is asymptotically stable for the induced local flow in Σ_e, even when e is not stable.

2.4.1. *Background and related results* Smith and Thieme [197,199] introduced the compactness hypothesis (C) and obtained the Sequential Limit Set Trichotomy. This tool streamlines many of the arguments and leads to stronger conclusions so the presentation here follows [197,199]. Takáč [210] extends the compactness hypothesis, which leads to additional stability concepts.

The results of Smith and Thieme [199] on generic convergence for SOP semiflows were motivated by earlier work of Poláčik [160], who obtained such results for abstract semilinear parabolic evolution systems assuming less compactness but more smoothness than Smith and Thieme.

The set A of asymptotically stable points can be shown to be dense under suitable hypotheses. See, e.g., Hirsch [73, Theorem 9.6]; Smith and Thieme [197, Theorems 3.13 and 4.1].

Hirsch [69] shows that if K is a nonempty compact, invariant set that attracts all points in some neighborhood of itself, then K contains an order-stable equilibrium.

It is not necessary to assume, as we have done here, that the semiflow is globally defined, that is, that trajectories are defined for all $t \geqslant 0$; many of the results adapt to local semiflows. See Hirsch [73], Smith and Thieme [199].

3. Ordinary differential equations

Throughout this section $\mathbb{R}^n$ is ordered by a cone K with nonempty interior. Our first objective is to explore conditions on a vector field that make the corresponding local semiflow monotone with respect to the order defined by K. It is convenient to work with time-dependent vector fields. We then investigate the long-term dynamics of autonomous vector fields f that are K-cooperative, meaning that K is invariant under the forward flow of the linearized system. These results are applied to competitive vector fields by the trick of time-reversal. In fairly general circumstances, limit sets of cooperative or competitive systems in $\mathbb{R}^n$ are invariant sets for systems in $\mathbb{R}^{n-1}$. This leads to particularly sharp theorems for $n = 2$ and 3.

A cone is *polyhedral* if it is the intersection of a finite family of closed half spaces. For example, the standard cone $\mathbb{R}^n_+$ is polyhedral, while the ice-cream cone is not.

The *dual cone* to K is the closed cone K^* in the dual space $(\mathbb{R}^n)^*$ of linear functions on $\mathbb{R}^n$, defined by

$$K^* = \{\lambda \in (\mathbb{R}^n)^*\colon \lambda(K) \geqslant 0\}.$$

To $\lambda \in K^*$ we associate the vector $a \in \mathbb{R}^n$ such that $\lambda(x) = \langle a, x\rangle$ where $\langle a, x\rangle$ denotes the standard inner product on $\mathbb{R}^n$. Under this association K^* is canonically identified with a cone in $\mathbb{R}^n$, namely, the set of vectors a such that a is normal to a supporting hyperplane H of K, and a and K lie in a common halfspace bounded by H.

We use the following simple consequence of general results on the separation of two closed convex sets:

$$x \in K \iff \lambda(x) \geqslant 0 \quad (\lambda \in K^*).$$

See, e.g., Theorem 1.2.8 of Berman et al. [18].

PROPOSITION 3.1. *If* $x \in K$, *then* $x \in \operatorname{Int} K$ *if and only if* $\lambda(x) > 0$ *for all* $\lambda \in K^* \setminus \{0\}$.

PROOF. Suppose $x \in \operatorname{Int} K$, $\lambda \in K^* \setminus \{0\}$, and $v \in X$ satisfies $\lambda(v) \neq 0$. Then $x \pm \epsilon v \in K$ for sufficiently small $\epsilon > 0$, so

$$\lambda(x \pm \epsilon v) = \lambda(x) \pm \epsilon \lambda(v) \geqslant 0,$$

implying that $\lambda(x) > 0$.

To prove the converse, assume $\mu(x) > 0$ for all functionals μ in the compact set $\Gamma = \{\lambda \in K^*\colon \|\lambda\| = 1\}$. As $\inf\{\mu(x)\colon \mu \in \Gamma\} > 0$, continuity of the map $(x, \lambda) \mapsto \lambda(x)$ implies $\mu(y) > 0$ for all y in some neighborhood U of x and all $\mu \in \Gamma$. If $\lambda \in K^*$ then $\|\lambda\|^{-1}\lambda \in \Gamma$ and therefore $\lambda(y) > 0$ for all $y \in U$. This proves $U \subset K$. $\square$

An immediate consequence of Proposition 3.1 is that if $x \in \partial K$, then there exists a nontrivial $\lambda \in K^*$ such that $\lambda(x) = 0$.

3.1. *The quasimonotone condition*

Let $J \subset \mathbb{R}$ be a nontrivial open interval, $D \subset \mathbb{R}^n$ an open set and $f : J \times D \to \mathbb{R}^n$ a locally Lipschitz function. We consider the ordinary differential equation

$$x' = f(t, x). \tag{3.1}$$

For every $(t_0, x_0) \in J \times D$, the initial value problem $x(t_0) = x_0$ has a unique noncontinuable solution defined on an open interval $J(t_0, x_0) \subset \mathbb{R}$. We denote this solution by $t \mapsto x(t, t_0, x_0)$. The notation $x(t, t_0, x_0)$ will carry the tacit assumption that $(t_0, x_0) \in J \times D$ and $t \in J(t_0, x_0)$. For fixed s_0, t_0 the map $x_0 \mapsto x(s_0, t_0, x_0)$ is a homeomorphism between open subsets of $\mathbb{R}^n$, the inverse being $x_0 \mapsto x(t_0, s_0, x_0)$.

System (3.1) is called *monotone* if $x_0 \leqslant x_1 \Longrightarrow x(t, t_0, x_0) \leqslant x(t, t_0, x_1)$.

The time-dependent vector field $f : J \times D \to \mathbb{R}^n$ satisfies the *quasimonotone condition* in D if for all $(t, x), (t, y) \in J \times D$ and $\phi \in K^*$ we have:

(QM) $x \leqslant y$ and $\phi(x) = \phi(y)$ implies $\phi(f(t, x)) \leqslant \phi(f(t, y))$.

The quasimonotone condition was introduced by Schneider and Vidyasagar [177] for finite-dimensional, autonomous linear systems and used later by Volkmann [224] for nonlinear infinite-dimensional systems. The following result is inspired by a result of Volkmann [224] and work of W. Walter [227]. See also Uhl [221], Walcher [226].

THEOREM 3.2. *Assume* f *satisfies* (QM) *in* D, $t_0 \in J$, *and* $x_0, x_1 \in D$. *Let* $\prec$ *denote any one of the relations* $\leqslant, <, \ll$. *If* $x_0 \prec x_1$ *then* $x(t, t_0, x_0) \prec x(t, t_0, x_1)$, *hence* (3.1) *is monotone. Conversely, if* (3.1) *is monotone then* f *satisfies* (QM).

PROOF. Assume that $x(t, t_0, x_i)$, $i = 0, 1$ are defined for $t \in [t_0, t_1]$ and $x_0 \leqslant x_1$. Let $v \gg 0$ be fixed and define $x_\epsilon := x_1 + \epsilon v$ and $f_\epsilon(t, x) := f(t, x) + \epsilon v$ for $\epsilon > 0$. Denote by $x(t) := x(t, t_0, x_0)$ and let $y_\epsilon(t) := x(t, t_0, x_\epsilon, \epsilon)$ denote the solution of the initial value problem $x'(t) = f_\epsilon(t, x)$, $x(t_0) = x_\epsilon$. It is well known that $y_\epsilon(t)$ is defined on $[t_0, t_1]$ for all sufficiently small ϵ. We show that $x(t) \ll y_\epsilon(t)$ for $t_0 \leqslant t \leqslant t_1$ and all sufficiently small $\epsilon > 0$. If not, then as $x(t_0) \ll y_\epsilon(t_0)$, there would exist $\epsilon > 0$ and $s \in (t_0, t_1]$ such that $x(t) \ll y_\epsilon(t)$ for $t_0 \leqslant t < s$ and $y_\epsilon(s) - x(s) \in \partial K$. By Proposition 3.1, there exists a non-trivial $\phi \in K^*$ such that $\phi(y_\epsilon(s) - x(s)) = 0$ but $\phi(y_\epsilon(t) - x(t)) > 0$ for $t_0 \leqslant t < s$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi\big(y_\epsilon(t)\big) - \phi\big(x(t)\big)\right]\Big|_{t=s} \leqslant 0,$$

hence

$$\phi\big(f\big(s, y_\epsilon(s)\big)\big) < \phi\big(f\big(s, y_\epsilon(s)\big)\big) + \epsilon\phi(v) = \phi\big(f_\epsilon\big(s, y_\epsilon(s)\big)\big) \leqslant \phi\big(f\big(s, x(s)\big)\big),$$

where the last inequality follows from the one above. On the other hand, by (QM) we have

$$\phi\big(f\big(s, y_\epsilon(s)\big)\big) \geqslant \phi\big(f\big(s, x(s)\big)\big).$$

This contradiction proves that $x(t) \ll y_\epsilon(t)$ for $t_0 \leqslant t \leqslant t_1$ and all small $\epsilon > 0$. Since $y_\epsilon(t) = x(t, t_0, x_\epsilon, \epsilon) \to x(t, t_0, x_1)$ as $\epsilon \to 0$, by taking the limit we conclude that $x(t, t_0, x_0) \leqslant x(t, t_0, x_1)$ for $t_0 \leqslant t \leqslant t_1$.

Fix t_0 and $t \in J(t_0, x_0)$. As the map $h : x_0 \mapsto x(t, t_0, x_0)$ is injective, from $x_0 < x_1$ we infer $x(t, t_0, x_0) < x(t, t_0, x_1)$. Note that $h(D \cap [x_0, x_1]) \subset [x(t, t_0, x_0), x(t, t_0, x_1)]$. Therefore the relation $x_0 \ll x_1$ implies $\operatorname{Int} D \cap [x_0, x_1] \neq \emptyset$. Injectivity of h and invariance of domain implies $\operatorname{Int}[x(t, t_0, x_0), x(t, t_0, x_1)] \neq \emptyset$, which holds if and only if $x(t, t_0, x_0) \ll x(t, t_0, x_1)$.

Conversely, suppose that (3.1) is monotone, $t_0 \in J$, $x_0, x_1 \in D$ with $x_0 \leqslant x_1$ and $\phi(x_0) = \phi(x_1)$ for some $\phi \in K^*$. Since $x(t, t_0, x_0) \leqslant x(t, t_0, x_1)$ for $t \geqslant t_0$ we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\phi[x(t, t_0, x_1) - x(t, t_0, x_0)]|_{t=t_0} \geqslant 0$, or $\phi(f(t_0, x_1)) \geqslant \phi(f(t_0, x_0))$. Thus (QM) holds. $\square$

Theorem 3.2 has been stated so as to minimize technical details concerning the domain $J \times D$ by assuming that J and D are open. In many applications, D is a closed set, for example, $D = K$ or $D = [a, b]$ where $a \ll b$. The proof can be modified to handle these (and other) cases. If $D = K$ and K is positively invariant for (3.1), the proof is unchanged because whenever $x \in D$ then $x + \epsilon v \in D$ for small positive ϵ, and because K is also positively invariant for the modified equation. If $D = [a, b]$, then the result follows by applying Theorem 3.2 to $f|J \times [[a, b]]$ and using continuity.

A set S is called *positively invariant* under (3.1) if $S \subset D$ and solutions starting in S stay in S, or more precisely:

$$(t_0, x_0) \in J \times S \quad \text{and} \quad t \in J(t_0, x_0),\ t \geqslant t_0 \quad \Longrightarrow \quad x(t, t_0, x_0) \in S.$$

It will be useful to have the following necessary and sufficient condition for invariance of K:

PROPOSITION 3.3. *The cone K is positively invariant under* (3.1) *if and only if $K \subset D$ and for each $t \in J$*

(P) $\lambda \in K^*, x \in \partial K, \lambda(x) = 0 \Longrightarrow \lambda(f(t,x)) \geqslant 0.$

PROOF. The proof that (P) implies positive invariance of K is similar to that of Theorem 3.2. Given $x_1 \in K$, we pass immediately to $x_\epsilon \gg x_1$ and the solution $y_\epsilon(t)$ of the perturbed equation defined in the proof of Theorem 3.2 and show that $y_\epsilon(t) \gg 0$ for $t_0 \leqslant t \leqslant t_1$ by an argument similar to the one used in the aforementioned proof. The result $x(t, t_0, x_1) \geqslant 0$ for $t \geqslant t_0$ is obtained by passage to the limit as $\epsilon \to 0$. The converse is also an easy modification of the converse argument given in the proof of Theorem 3.2. □

Since we will have occasion to apply (P) to systems other than (3.1), it will be convenient to refer to (P) by saying that (P) holds for $f : J \times D \to \mathbb{R}^n$ where $K \subset D$. Hypothesis (P) says that the time-dependent vector field $f(t,x)$ points into K at points $x \in \partial K$.

Let $A(t)$ be a continuous $n \times n$ matrix-valued function defined on the interval J containing t_0 and consider the linear initial value problem for the matrix solution X:

$$X' = A(t)X, \quad X(t_0) = I. \tag{3.2}$$

Observe that (P) and (QM) are equivalent for linear systems; therefore we have:

COROLLARY 3.4. *The matrix solution $X(t)$ satisfies $X(t)K \subset K$ for $t \geqslant t_0$ if and only if for all $t \in J$,* (P) *holds for the function $x \to A(t)x$. In fact,* (P) *implies that $X(t)$ maps $K \setminus \{0\}$ and* Int K *into themselves for all $t > t_0$.*

A matrix A is *K-positive* if $A(K) \subset K$. Corollary 3.4 implies that $X(t)$ is K-positive for $t \geqslant t_0$ if (P) holds.

If for every $t \in J$, there exists $\alpha \in \mathbb{R}$ such that $A + \alpha I$ is K-positive, then (P) holds for A. Indeed, if $\lambda \in K^*$ satisfies $\lambda(x) = 0$ then application of λ to $(A + \alpha I)x \geqslant 0$ yields that $\lambda(A(t)x) \geqslant 0$. The converse is false for general cones but true for polyhedral cones by Theorem 8 of Schneider and Vidyasagar [177]. See also Theorem 4.3.40 of Berman and Neumann [18]. Lemmert and Volkmann [118] give the following example of a matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which satisfies (P) for the ice-cream cone above but $A + \alpha I$ is not K-positive for any α.

Recall that the domain D is *p-convex* if for every $x, y \in D$ satisfying $x \leqslant y$ the line segment joining them also belongs to D. Let $\frac{\partial f}{\partial x}(t,x)$ be continuous on $J \times D$. We say that f (or system (3.1)) is K*-cooperative* if for all $t \in J$, $y \in D$, (P) holds for the function $x \to \frac{\partial f}{\partial x}(t, y)x$. By Corollary 3.4 applied to the variational equation

$$X'(t) = \frac{\partial f}{\partial x}\big(t, x(t, t_0, x_0)\big)X, \quad X(t_0) = I$$

we conclude that if f is K-cooperative then $X(t) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ is K-positive.

THEOREM 3.5. *Let* $\frac{\partial f}{\partial x}(t, x)$ *be continuous on* $J \times D$. *Then* (QM) *implies that* f *is* K*-cooperative. Conversely, if* D *is p-convex and* f *is* K*-cooperative, then* (QM) *holds.*

PROOF. Suppose that (QM) holds, $x \in D$, $h \in \partial K$, and $\phi \in K^*$ satisfies $\phi(h) = 0$. Since $x \leqslant x + \epsilon h$ and $\phi(x) = \phi(x + \epsilon h)$ for small $\epsilon > 0$, (QM) implies that $\phi(f(t, x)) \leqslant \phi(f(t, x + \epsilon h))$. Hence,

$$0 \leqslant \phi\left(\frac{f(t, x + \epsilon h) - f(t, x)}{\epsilon}\right)$$

and the desired result holds on taking the limit $\epsilon \to 0$.

Conversely, suppose that f is K-cooperative and D is p-convex. If $x, y \in D$ satisfy $x \leqslant y$ and $\phi(x) = \phi(y)$ for some $\phi \in K^*$, then either $\phi = 0$ or $y - x \in \partial K$. Consequently

$$\phi\big(f(t, y) - f(t, x)\big) = \int_0^1 \phi\left(\frac{\partial f}{\partial x}\big(t, sy + (1 - s)x\big)(y - x)\right)\mathrm{d}s \geqslant 0$$

because the integrand is nonnegative. □

If for each $(t, x) \in J \times D$ there exists α such that $(\frac{\partial f}{\partial x}(t, x) + \alpha I)$ is K-positive, then f is K-cooperative. This is implied by the remark following Corollary 3.4.

In the special case that $K = \mathbb{R}^n_+$, the cone of nonnegative vectors, it is easy to see by using the standard inner product that we may identify K^* with K. The quasimonotone hypothesis reduces to the Kamke–Müller condition [91,148]: $x \leqslant y$ and $x_i = y_i$ for some i implies $f_i(t, x) \leqslant f_i(t, y)$. This holds by taking $\phi(x) = \langle e_i, x\rangle$ (e_i is the unit vector in the x_i-direction) and noting that every $\phi \in K^*$ can be represented as a positive linear combination of these functionals. If f is differentiable, the Kamke–Müller condition implies

$$\frac{\partial f_i}{\partial x_j}(t, x) \geqslant 0, \quad i \neq j. \tag{3.3}$$

Conversely, if $\frac{\partial f}{\partial x}(t, x)$ is continuous on $J \times D$, (3.3) holds and D is p-convex, then the Kamke–Müller condition holds by an argument similar to the one used in the proof of the converse in Theorem 3.5.

Stern and Wolkowicz [206] give necessary and sufficient conditions for (P) to hold for matrix A relative to the ice-cream cone $K = \{x \in \mathbb{R}^n\colon x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leqslant x_n^2,\ x_n \geqslant 0\}$. Let Q denote the $n \times n$ diagonal matrix with first $n - 1$ entries 1 and last entry -1. Then

(QM) holds for A if and only if $QA + A^T Q + \alpha Q$ is negative semidefinite for some $\alpha \in \mathbb{R}$. Their characterization extends to other ellipsoidal cones.

3.2. *Strong monotonicity with linear systems*

In this section, all matrices are assumed to be square. Recall that the matrix A is strongly positive if $A(K \setminus \{0\}) \subset \operatorname{Int} K$. We introduce the following milder hypothesis on the matrix A, following Schneider and Vidyasagar [177]:

(ST) For all $x \in \partial K \setminus \{0\}$ there exists $\nu \in K^*$ such that $\nu(x) = 0$ and $\nu(Ax) > 0$.

The following result for the case of constant matrices was proved by Elsner [44], answering a question in [177]. Our proof follows that of Theorem 4.3.26 of Berman et al. [18].

THEOREM 3.6. *Let the linear system* (3.2) *satisfy* (P). *Then the fundamental matrix* $X(t_1)$ *is strongly positive for* $t_1 > t_0$ *if there exists* s *satisfying* $t_0 \leqslant s \leqslant t_1$ *such that* (ST) *holds for* $A(s)$.

PROOF. Observe that the set of all s such that (ST) holds for $A(s)$ is open. If the result is false, there exists $x > 0$ such that the solution of (3.2) given by $y(t) = X(t)x$ satisfies $y(t_1) \in \partial K \setminus \{0\}$. By Corollary 3.4, $y(t) > 0$ for $t \geqslant t_0$ and $y(t) \in \partial K$ for $t_0 \leqslant t \leqslant t_1$. Let $s \in (t_0, t_1]$ be such that (ST) holds for $A(s)$. Then there exists $\nu \in K^*$ such that $\nu(y(s)) = 0$ and $\nu(A(s)y(s)) > 0$. As $\nu \in K^*$ and $y(t) \in K$, $h(t) := \nu(y(t)) \geqslant 0$ for $t_0 \leqslant t \leqslant t_1$. But $h(s) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=s} h(t) = \nu(A(s)y(s)) > 0$ which, taken together, imply that $h(s - \delta) < 0$ for small positive δ, giving the desired contradiction. $\square$

If (3.2) satisfies (P) and if $x \in \partial K$ then for all $\phi \in K^*$ such that $\phi(x) = 0$ we have $\phi(A(t)x) \geqslant 0$. Hypothesis (ST) asserts that if $x \neq 0$ then $\phi(A(t)x) > 0$ for at least one such ϕ. Berman et al. [18] refer to (ST) (they include (P) in their definition) by saying that A is strongly K-subtangential; while we do not use this terminology, our notation is motivated by it.

An example in [18] shows that (P) and (ST) are not necessary for strong positivity. Let K be the ice-cream cone $K = \{x \in \mathbb{R}^3\colon x_1^2 + x_2^2 \leqslant x_3^2, x_3 \geqslant 0\}$ and consider the constant coefficient system (3.2) with matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

An easy calculation shows that $(x_1^2 + x_2^2)' = -2x_2^2$ so it follows easily that K is positively invariant, hence (P) holds by Corollary 3.4. The solution satisfying $x(0) = (\cos(\theta), \sin(\theta), 1)^T \in \partial K$ satisfies $x(t) \in \operatorname{Int} K$ for $t > 0$ since the calculation above and the fact that $x_2(t)$ can have only simple zeros implies that $x_1^2 + x_2^2$ is strictly decreasing while x_3 remains unchanged. The linear functional ν, defined $\nu(x) :=$

$(-\cos(\theta), -\sin(\theta), 1)x$ belongs to K^* by an easy calculation and satisfies $\nu(x(0)) = 0$. It is unique, up to positive scalar multiple, with these properties because K is smooth so its positive normal at a point is essentially unique. But $\nu(Ax(0)) = \sin^2(\theta)$ vanishes if $\theta = 0, \pi$. Therefore (ST) fails although $X(t)$ is strongly positive for $t > 0$.

Theorem 3.6 leads to the following result on strong monotonicity for the nonlinear system (3.1).

LEMMA 3.7. *Assume D is p-convex, $\frac{\partial f}{\partial x}(t, x)$ is continuous on $J \times D$ and f is K-cooperative. Let $x_0, x_1 \in D$ satisfy $x_0 < x_1$ and $t > t_0$ with $t \in J(t_0, x_0) \cap J(t_0, x_1)$. If there exists y_0 on the line segment joining x_0 to x_1 and $r \in [t_0, t]$ such that* (ST) *holds for $\frac{\partial f}{\partial x}(r, x(r, t_0, y_0))$ then*

$$x(t, t_0, x_0) \ll x(t, t_0, x_1).$$

PROOF. First, observe that for y_0 on the segment it follows that $t \in J(t_0, y_0)$. We apply the formula

$$x(t, t_0, x_1) - x(t, t_0, x_0) = \int_0^1 \frac{\partial x}{\partial x_0}\big(t, t_0, sx_1 + (1-s)x_0\big)(x_1 - x_0)\, \mathrm{d}s,$$

where $X(t) = \frac{\partial x}{\partial x_0}(t, t_0, y_0)$ is the fundamental matrix for (3.2) corresponding to the matrix $A(t) = \frac{\partial f}{\partial x}(t, x(t, t_0, y_0))$. The left-hand side belongs to $K \setminus \{0\}$ if $x_0 < x_1$ by Theorems 3.5 and 3.2 but we must show it belongs to $\operatorname{Int} K$. For this to be true, it suffices that for each $t > t_0$ there exists $s \in [0, 1]$ such that the matrix derivative in the integrand is strongly positive. In fact, this derivative is K-positive by Corollary 3.4 for all values of the arguments with $t \geqslant t_0$, so application of any nontrivial $\phi \in K^*$ to the integral gives a nonnegative numerical result. If there exists s as above, then the application of ϕ to the integrand gives a positive numerical result for all s' near s by continuity and Proposition 3.1 and hence the integral belongs to $\operatorname{Int} K$ by Proposition 3.1. By Theorem 3.6, $\frac{\partial x}{\partial x_0}(t, t_0, y_0)$ is strongly positive for $t > t_0$ if (ST) holds for $A(r) = \frac{\partial f}{\partial x}(r, x(r, t_0, y_0))$ for some $r \in [t_0, t]$. But this is guaranteed by our hypothesis. □

THEOREM 3.8. *D is p-convex, $\frac{\partial f}{\partial x}(t, x)$ is continuous on $J \times D$, and f is K-cooperative. Suppose for every $x_0, x_1 \in D$ with $x_0 < x_1$ and $t_0 \in J$, there exists y_0 on the line segment joining the x_i such that* (ST) *holds for $\frac{\partial f}{\partial x}(t_0, y_0)$. If $x_0, x_1 \in D$, $x_0 < x_1$, and $t > t_0$ then*

$$t \in J(t_0, x_0) \cap J(t_0, x_1) \implies x(t, t_0, x_0) \ll x(t, t_0, x_1).$$

PROOF. This is an immediate corollary of Lemma 3.7. □

As the main hypothesis of Theorem 3.8 will be difficult to verify in applications, the somewhat stronger condition of irreducibility may be more useful because there is a large body of theory related to it [18,19]. We now introduce the necessary background. A closed subset F of K that is itself a cone is called *a face* of K if $x \in F$ and $0 \leqslant y \leqslant x$ (inequalities induced by K) implies that $y \in F$. For example, the faces of $K = \mathbb{R}^n_+$ are of the form $\{x \in \mathbb{R}^n_+ : x_i = 0,\ i \in I\}$ where $I \subset \{1, 2, \ldots, n\}$. For the ice-cream cone $K = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leqslant x_n^2, x_n \geqslant 0\}$, the faces are the rays issuing from the origin and passing through its boundary vectors. A K-positive matrix A is *K-irreducible* if the only faces F of K for which $A(F) \subset F$ are $\{0\}$ and K. The following is a special case of Theorem 2.3.9 in Berman and Neumann [18]; see Berman and Plemmons [19] for proofs. These references contain additional related results.

THEOREM 3.9. *Let A be an $n \times n$ K-positive matrix. Then the following are equivalent:*

(i) *A is K-irreducible*;
(ii) *No eigenvector of A belongs to ∂K*;
(iii) *A has exactly one unit eigenvector in K and it belongs to* $\operatorname{Int} K$;
(iv) $(I + A)^{n-1}(K \setminus \{0\}) \subset \operatorname{Int} K$.

The famous Perron–Frobenius Theory is developed for K-positive and K-irreducible matrices in the references above. In particular, the spectral radius of A is a simple eigenvalue of A with corresponding eigenvector described in (iii) above.

Below we require the simple observation that if A is K-positive, then the adjoint A^* is K^*-positive. Indeed, if $\nu \in K^*$ then $(A^*\nu)(x) = \nu(Ax) \geqslant 0$ for all $x \in K$ so $A^*\nu \in K^*$. The next result is adapted from Theorem 4.3.17 of Berman et al. [18].

PROPOSITION 3.10. *Let A be an $n \times n$ matrix and suppose that there exists $\alpha \in \mathbb{R}$ such that $B := A + \alpha I$ is K-positive. Then B is K-irreducible if and only if* (ST) *holds for A.*

PROOF. Suppose that $B = A + \alpha I$ is K-positive and (ST) holds. If $Ax = \lambda x$ for some $\lambda \in \mathbb{R}$ and nonzero vector $x \in \partial K$ then there exists $\nu \in K^*$ such that $\nu(x) = 0$ and $\nu(Ax) > 0$. But $\nu(Ax) = \lambda\nu(x) = 0$. Consequently, no eigenvector of B belongs to ∂K so by Theorem 3.9, B is K-irreducible.

Conversely, suppose that B is K-positive and K-irreducible. Let $x \in \partial K$, $x \neq 0$ and let $\nu \in K^*$ satisfy $\nu \neq 0$ and $\nu(x) = 0$. By Theorem 3.9, $C := B + I$ has the property that C^{n-1} is strongly positive so $\nu(C^{n-1}x) > 0$. As C is K-positive, $\nu(C^r x) \geqslant 0$ for $r = 1, 2, \ldots, n-1$. Because $\nu(x) = 0$, we may choose $p \in \{1, 2, \ldots, n-1\}$ such that $\nu(C^p x) > 0$ but $\nu(C^{p-1}x) = 0$. Let $\tilde{\nu} = (C^*)^{p-1}\nu$. Then $\tilde{\nu} \in K^*$, $\tilde{\nu}(x) = 0$ and $\tilde{\nu}(Cx) > 0$. But then A satisfies (ST) because $\tilde{\nu}(Ax) = \tilde{\nu}(Cx) > 0$. □

Motivated by Proposition 3.10, we introduce the following hypothesis for matrix A.

(CI) There exists $\alpha \in \mathbb{R}$ such that $A + \alpha I$ is K-positive and K-irreducible.

In the special case that $K = \mathbb{R}^n_+$, $n \geqslant 2$, matrix A satisfies (CI) if and only if $a_{ij} \geqslant 0$ for $i \neq j$ and there is no permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square. This is equivalent to the assertion that the incidence graph of A is strongly connected. See Berman and Plemmons [19].

The following is an immediate consequence of Theorem 3.8.

COROLLARY 3.11. *D is p-convex, $\frac{\partial f}{\partial x}(t,x)$ is continuous on $J \times D$ and f is K-cooperative. Suppose that for every $x_0, x_1 \in D$ with $x_0 < x_1$ and $t_0 \in J$, there exists y_0 on the line segment joining the x_i such that* (CI) *holds for $\frac{\partial f}{\partial x}(t_0, y_0)$. If $x_0, x_1 \in D$, $x_0 < x_1$, and $t > t_0$ then*

$$t \in J(t_0, x_0) \cap J(t_0, x_1) \implies x(t, t_0, x_0) \ll x(t, t_0, x_1).$$

PROOF. If (CI) holds then, by Proposition 3.10, (ST) holds for $\frac{\partial f}{\partial x}(t,x)$, so the conclusion follows from Theorem 3.8. □

Corollary 3.11 is an improvement of the restriction of Theorem 10 of Kunze and Siegel [111] to the case that K has nonempty interior; their results also treat the case that K has empty interior in $\mathbb{R}^n$ but nonempty interior in some subspace of $\mathbb{R}^n$. Walter [228] gives a sufficient condition for strong monotonicity relative to $K = \mathbb{R}^n_+$ which does not require f to be differentiable.

For polyhedral cones it can be shown that matrix A satisfies (P) and (ST) if and only if there exists $\alpha \in \mathbb{R}$ such that $A + \alpha I$ is K-positive and K-irreducible. See Theorem 4.3.40 of Berman et al. [18]. For the case of polyhedral cones, therefore, Corollary 3.11 and Theorem 3.8 are equivalent.

3.3. *Autonomous K-competitive and K-cooperative systems*

Our focus now is on the autonomous system of ordinary differential equations

$$x' = f(x), \tag{3.4}$$

where f is a vector field on an open subset $D \subset \mathbb{R}^n$; all vector fields are assumed to be continuously differentiable. We change our notation slightly to conform to more dynamical notation, denoting $x(t, 0, x_0)$ by $\Phi_t(x)$, where Φ denotes the dynamical system (= local flow) in D generated by f discussed in Section 1. The notation $\Phi_t(x)$ carries the tacit assumption that $t \in I_x$, the open interval in $\mathbb{R}$ containing the origin on which the trajectory of x under Φ is defined. The *positive semiorbit* (respectively, (*negative semiorbit*) of x is $\gamma^+(x) := \{\Phi_t(x)\colon t \in t \geqslant 0\}$ (respectively, $\gamma^-(x) := \{\Phi_t(x)\colon t \leqslant 0\}$). The limit sets of x can be defined as

$$\omega(x) = \bigcap_{t \geqslant 0} \overline{\bigcup_{\tau \geqslant t} \Phi_\tau(x)}, \qquad \alpha(x) = \bigcap_{t \leqslant 0} \overline{\bigcup_{\tau \leqslant t} \Phi_\tau(x)}.$$

We call f and Eq. (3.4) *K-competitive* in D if the time-reversed system

$$x' = -f(x)$$

is K-cooperative. When K is the standard cone $\mathbb{R}^n_+$, f is competitive if and only if $\partial f_i/\partial x_j \leqslant 0$ for $i \neq j$. Therefore if f is K-competitive with local flow Φ, then $-f$ is K-cooperative with local flow $\widetilde{\Phi}$, where $\widetilde{\Phi}_t(x) = \Phi_{-t}(x)$; and conversely. Thus time-reversal changes K-competitive systems into K-cooperative ones, and vice-versa. This fact will be exploited repeatedly below.

In the remainder of Section 3 we assume $\mathbb{R}^n$ is ordered by a cone $K \subset \mathbb{R}^n$ with nonempty interior.

A map is *locally monotone* if every point in its domain has a neighborhood on which the map is monotone. A local flow or local semiflow Φ is locally monotone if Φ_t is a locally monotone map for all $t > 0$. *Locally strongly monotone* is defined similarly.

THEOREM 3.12. *Let f be a K-cooperative vector field in an open set $D \subset \mathbb{R}^n$, generating the local flow Φ. Then Φ is locally monotone, and monotone when D is p-convex.*

PROOF. If D is p-convex, monotonicity follows from Theorem 3.2 (with $f(t,x) := f(x)$). Suppose D is not p-convex. Denote the domain of Φ_t by D_t.

We first claim: For every $p \in D$ there exists $\tau > 0$ and a neighborhood $N \subset D_\tau$ such that $\Phi_t|N$ is monotone if $t \in [0, \tau]$. But this is obvious since by restricting f to a p-convex neighborhood of p, we can use Theorem 3.2.

Now fix $p \in D$ and let $J(0,p) \cap [0,\infty) = [0,r)$, $0 < r \leqslant \infty$. Let I_p be the set of all nonnegative $s \in [0,r)$ such that there is a neighborhood U_s of p, contained in D_s, such that $\Phi_t|U_s$ is monotone for each $t \in [0,s]$. The previous claim implies that $[0,\tau] \subset I_p$ and, by its definition, I_p is an interval. Furthermore, straightforward applications of the previous claim establish that I_p is both an open and a closed subset of $[0,r)$. It follows that $I_p = [0,r)$. □

The next theorem gives a sufficient condition for strong monotonicity. Define $G(f)$ to be the set of $x \in D$ such that (ST) holds for $A = f'(x)$. Note that $x \in G(f)$ provided (CI) holds for $A = f'(x)$, by Proposition 3.10. If $K = \mathbb{R}^n_+$, a sufficient condition for $x \in G(f)$ is that $f'(x)$ is an irreducible matrix with nonnegative off-diagonal entries.

THEOREM 3.13. *Let f be a K-cooperative vector field in an open set $D \subset \mathbb{R}^n$, generating the local flow Φ. Assume $D \setminus G(f)$ does not contain any totally ordered line segment (which holds when $D \setminus G(f)$ is zero dimensional). Then Φ is locally strongly monotone, and strongly monotone when D is p-convex.*

PROOF. Suppose D is p-convex, in which case Φ is monotone by from Theorem 3.2. By Theorem 3.8, Φ is strongly monotone.

When D is not p-convex, Φ is locally monotone by Theorem 3.12, and the previous paragraph implies Φ is locally strongly monotone. □

The proof of Theorem 3.13 can be adapted to cover certain nonopen domains D, such as an order interval, a closed halfspace, and the cone K; see the discussion following the proof of Theorem 3.2.

Theorem 3.8 implies that Φ is strongly monotone provided D is p-convex and f satisfies the autonomous version of condition (ST) of Section 3.2, namely:

(ST*) For all $u \in D, x \in \partial K \setminus \{0\}$ there exists $\nu \in K^*$ such that $\nu(x) = 0$ and $\nu(f'(u)x) > 0$.

Without p-convexity of D, condition (ST*) yields local strong monotonicity.

3.4. *Dynamics of cooperative and competitive systems*

We continue to assume $\mathbb{R}^n$ is ordered by a cone K having nonempty interior; all notions involving order refer to that defined by K. For this section, the terms "competitive" and "cooperative" are tacitly understood to mean "K-competitive" and "K-cooperative," and monotonicity refers to the ordering defined by K.

We first apply results from Section 2 to obtain a generic stable convergence theorem for cooperative vector fields.

Let Φ denote the local flow generated by a vector field f on $D \subset \mathbb{R}^n$. We assume D is p-convex throughout this section without further mention. When $\Phi_t(x)$ is defined for all $(t, x) \in [0, \infty) \times D$, as when all positive semiorbits have compact closure in D, the corresponding positive local semiflow Φ^+ is a semiflow. To Φ we associate C, S and E, denoting respectively the sets of convergent, stable and equilibrium points for Φ^+.

THEOREM 3.14. *Let f be a cooperative vector field on an open set $D \subset \mathbb{R}^n$, generating a local flow Φ such that*:

(a) *Every positive semiorbit of Φ has compact closure in D*;

(b) *Condition* (ST*) *above is satisfied, and $D = AC \cup BC$.*

Then Φ has the following properties:

(i) *$C \cap S$ contains a dense open subset of D, consisting of points whose trajectories converge to equilibria*;

(ii) *If E is compact there is a stable equilibrium, and an asymptotically stable equilibrium when E is finite.*

PROOF. Assumption (ST*) makes Φ strongly monotone. The hypothesis of Theorem 2.26, with $X = D$, is fulfilled: D is normally ordered and $D = BC \cup AC$. Therefore Theorem 2.26 implies the conclusion. □

Theorem 3.14, like Theorem 3.13, holds for some more general domains D, including relatively open subsets of V where V denotes a closed halfspace, a closed order interval, or the cone K.

One of the main results of this subsection is that n-dimensional competitive and cooperative systems behave like general systems of one less dimension. Theorems 3.21 and 3.22 illustrate this principle for $n = 2$ in a very strong form. In higher dimensions the principle holds for compact limit sets. The key tool in proving this is the following result due to Hirsch [67]:

THEOREM 3.15. *A limit set of a competitive or cooperative system cannot contain two points related by* $\ll$.

PROOF. By time reversal, if necessary, we assume the system is cooperative, hence the local flow is monotone. Now apply Proposition 1.10. □

A periodic orbit of a competitive or cooperative system is a limit set and consequently it cannot contain two points related by $\ll$. The following sharper result will be useful later:

PROPOSITION 3.16. *Nontrivial periodic orbit of a competitive or cooperative system cannot contain two points related by* $<$.

PROOF. By time-reversal we assume the system is cooperative, and in this case the conclusion follows from Proposition 1.10. □

Let Φ, Ψ be flows in respective spaces A, B. We say Φ and Ψ are *topologically equivalent* if there is a homeomorphism $Q: A \to B$ that is a conjugacy between them, i.e., $Q \circ \Phi_t = \Psi_t \circ Q$ for all $t \in \mathbb{R}$. The relationship of topological equivalence is an equivalence relation on the class of flow; it formalizes the notion of "having the same qualitative dynamics."

A system of differential equations $y' = F(y)$, defined on $\mathbb{R}^k$, is called *Lipschitz* if F is Lipschitz. That is, there exists $K > 0$ such that $|F(y_1) - F(y_2)| \leqslant K|y_1 - y_2|$ for all $y_1, y_2 \in \mathbb{R}^k$. With these definitions, we can state a result of Hirsch [67] that follows directly from Theorem 3.15.

THEOREM 3.17. *The flow on a compact limit set of a competitive or cooperative system in* $\mathbb{R}^n$ *is topologically equivalent to a flow on a compact invariant set of a Lipschitz system of differential equations in* $\mathbb{R}^{n-1}$.

PROOF. Let L be the limit set, $v \gg 0$ be a unit vector and let H_v be the hyperplane orthogonal to v, i.e, $H_v := \{x\colon \langle x, v\rangle = 0\}$. The orthogonal projection Q onto H_v is given by $Qx = x - \langle x, v\rangle v$. By Theorem 3.15, Q is one-to-one on L (this could fail only if L contains two points that are related by $\ll$). Therefore, Q_L, the restriction of Q to L, is a Lipschitz homeomorphism of L onto a compact subset of H_v. We argue by contradiction to establish the existence of $m > 0$ such that $|Q_L x_1 - Q_L x_2| \geqslant m|x_1 - x_2|$ whenever $x_1 \neq x_2$ are points of L. If this were false, then there exists sequences $x_n, y_n \in L$, $x_n \neq y_n$ such that

$$\frac{|Q(x_n) - Q(y_n)|}{|x_n - y_n|} = \frac{|(x_n - y_n) - v\langle v, x_n - y_n\rangle|}{|x_n - y_n|} \to 0$$

as $n \to \infty$. Equivalently, $|w_n - v\langle v, w_n\rangle| \to 0$ as $n \to \infty$ where $w_n = x_n - y_n/|x_n - y_n|$. We can assume that $w_n \to w$ as $n \to \infty$ where $|w| = 1$. Then, $w = v\langle v, w\rangle$ and therefore, $\langle v, w\rangle^2 = 1$ so $w = \pm v$. But then $x_n - y_n/|x_n - y_n| \to \pm v$ as $n \to \infty$ and this implies that $x_n \ll y_n$ or $y_n \ll x_n$ for all large n, contradicting Theorem 3.15. Therefore, Q_L^{-1} is Lipschitz on $Q(L)$. Since L is a limit set, it is an invariant set for (3.4). It follows that the dynamical system restricted to L can be modeled on a dynamical system in H_v. In fact, if $y \in Q(L)$ then $y = Q_L(x)$ for a unique $x \in L$ and $\Psi_t(y) \equiv Q_L(\Phi_t(x))$ is a dynamical system on $Q(L)$ generated by the vector field

$$F(y) = Q_L\big(f\big(Q_L^{-1}(y)\big)\big)$$

on $Q(L)$. According to McShane [137], a Lipschitz vector field on an arbitrary subset of H_v can be extended to a Lipschitz vector field on all of H_v, preserving the Lipschitz constant. It follows that F can be extended to all of H_v as a Lipschitz vector field. It is easy to see that $Q(L)$ is an invariant set for the latter vector field. We have established the topological equivalence of the flow Φ on L with the flow Ψ on $Q(L)$. $Q(L)$ is a compact invariant set for the $(n-1)$-dimensional dynamical system on H_v generated by the extended vector field. □

A consequence of Theorem 3.17 is that the flow on a compact limit set, L, of a competitive or cooperative system shares common dynamical properties with the flow of a system of differential equations in one less dimension, restricted to the compact, connected invariant set $Q(L)$. Notice, however, that L may be the limit set of a trajectory not in L, and therefore $Q(L)$ need not be a limit set.

On the other hand, the flow Ψ in a compact limit sets enjoys the topological property of *chain recurrence*, due to Conley [31,30], which will be important in the next subsection. The definition is as follows. Let A be a compact invariant set for the flow Φ. Given two points z and y in A and positive numbers ϵ and t, an (ϵ, t)-chain from z to y in A is an ordered set

$$\{z = x_1, x_2, \dots, x_{m+1} = y; t_1, t_2, \dots, t_m\}$$

of points $x_i \in A$ and times $t_i \geqslant t$ such that

$$\big|\Phi_{t_i}(x_i) - x_{i+1}\big| < \epsilon, \quad i = 1, 2, \dots, m. \tag{3.5}$$

A is *chain recurrent* for Φ if for every $z \in A$ and for every $\epsilon > 0$ and $t > 0$, there is an (ϵ, t)-chain from z to z in A.

Conley proved that when A is compact and connected, a flow Φ in A is chain recurrent if and only if there are no attractors. This useful condition can be stated as follows: For every proper nonempty compact set $S \subset A$ and all $t > 0$, there exists $s > t$ such that $\Phi_s(S) \not\subset \operatorname{Int} S$.

Compactness of A implies that chain recurrence of the flow in A is independent of the metric, and thus holds for any topologically equivalent flow.

It is intuitively clear that, as Conley proved, flows in compact alpha and omega limit sets are chain recurrent. Indeed, orbit segments of arbitrarily long lengths through point x repeatedly pass near any point of $\omega(x) \cup \alpha(x)$. Of course these segments do not necessarily belong to $\omega(x)$; but by taking suitable limits of points in these segments, one can find enough (ϵ, t)-chains in $\omega(x)$ and $\alpha(x)$ to prove the flows in these sets chain recurrent. For a rigorous proof, see Smith [194].

3.5. *Smale's construction*

Smale [182] showed that it is possible to embed essentially arbitrary dynamics in a competitive or cooperative irreducible system. His aim was to warn population modelers that systems designed to model competition could have complicated dynamics. His result is also very useful for providing counterexamples to conjectures in the theory of monotone dynamics, since by time reversal his systems are cooperative. In this section, competitive and cooperative are with respect to the usual cone.

Smale constructed special systems of Kolmogorov type

$$x_i' = x_i M_i(x), \quad 1 \leqslant i \leqslant n, \tag{3.6}$$

in $\mathbb{R}^n_+$ where the M_i are smooth functions satisfying

$$\frac{\partial M_i}{\partial x_j} < 0 \tag{3.7}$$

for all i, j; all sums are understood to be from 1 to n. We refer to such systems as *totally competitive*. They are simple models of competition between n species, where M_i is interpreted as the *per capita* growth rate of species i.

Smale's object was to choose the M_i so that the standard $(n-1)$-simplex $\Sigma_n = \{x \in \mathbb{R}^n_+ \colon \sum x_i = 1\}$ is an attractor in which arbitrary dynamics may be specified.

In order to generate a dynamical system on Σ_n, let H denote the tangent space to Σ_n, that is, $H = \{x \in \mathbb{R}^n \colon \sum x_i = 0\}$, and let $h : \Sigma_n \to H$ be a smooth vector field on Σ_n, meaning that all partial derivatives of h exist and are continuous on Σ_n. We also assume that $h = (h_1, h_2, \ldots, h_n)$ has the form $h_i = x_i g_i(x)$ where the g_i are smooth functions on Σ_n. Then the differential equation

$$x_i' = h_i(x), \quad 1 \leqslant i \leqslant n \tag{3.8}$$

generates a flow in $\mathbb{R}^n_+$ that leaves Σ_n invariant. The form of the h_i ensures that if $x_i(0) = 0$, then $x_i(t) \equiv 0$ so each lower dimensional simplex forming part of the boundary of Σ_n is invariant.

The goal is to construct a competitive system of the form (3.6) satisfying (3.7) such that its restriction to Σ_n is equivalent to (3.8). Let $p : [0, \infty) \to \mathbb{R}_+$ have continuous derivatives of all orders, be identically 1 in a neighborhood of $s = 1$, and vanish outside the interval $[1/2, 3/2]$. As g is a smooth vector field on Σ_n, it has a smooth extension to $\mathbb{R}^n_+$ which we denote by g in order to conserve notation. An example of such an extension is the map $x \mapsto P(\sum x_j) g(x / \sum x_j) / P(1)$, where $P(u) = \int_0^u p(s)\,\mathrm{d}s$.

For $\eta > 0$, define

$$M_i(x) = 1 - S(x) + \eta p\Big(\sum x_j\Big) g_i(x), \quad 1 \leqslant i \leqslant n.$$

Then (3.7) holds for sufficiently small η since $p(\sum x_j)$ vanishes identically outside a compact subset of $\mathbb{R}^n_+$. Consider the system (3.6) with M as above. $\mathbb{R}^n_+$ is positively invariant; and the function $S(x) = \sum_i x_i$, evaluated along a solution $x(t)$ of (3.6), satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} S\big(x(t)\big) = S\big(x(t)\big)\big[1 - S\big(x(t)\big)\big]$$

since $\sum x_i g_i(x) = \sum h_i(x) = 0$. Consequently Σ_n, which is $S^{-1}(1) \cap \mathbb{R}^n_+$, is positively invariant. Moreover if $x(0) \in \mathbb{R}^n_+$ then $S(x(0)) \geqslant 0$. This implies $S(x(t)) \to 1$ as $t \to \infty$, unless $x(t) \equiv 0$, and Σ_n attracts all nontrivial solutions of (3.6) in $\mathbb{R}^n_+$. Restricted to Σ_n, (3.6) becomes

$$x_i' = \eta h_i(x), \quad 1 \leqslant i \leqslant n.$$

Therefore the dynamics of (3.6) restricted to Σ_n is equivalent, up to a change in time scale, to the dynamics generated by (3.8).

As noted above, Smale's construction has implications for cooperative and irreducible systems since the time-reversed system corresponding to (3.6) is cooperative and irreducible in $\operatorname{Int} \mathbb{R}^n_+$. Time-reversal makes the simplex a repellor for a cooperative system Φ in $\mathbb{R}^n_+$. Therefore every invariant set in the simplex is unstable for Φ. Each trajectory of Φ that is not in the simplex is attracted to the equilibrium at the origin or to the virtual equilibrium at ∞. The simplex is the common boundary between the basins of attraction of these two equilibria.

3.6. *Invariant surfaces and the carrying simplex*

It turns out that the essential features of Smale's seemingly very special construction are found in a large class of totally competitive Kolmogorov systems

$$x_i' = x_i M_i(x), \quad x \in \mathbb{R}^n_+. \tag{3.9}$$

Here and below i and j run from 1 to n. Let Φ denote the corresponding local flow. The *unit $(n-1)$ simplex* is $\Delta^{n-1} := \{x \in \mathbb{R}^n_+ \colon \sum x_i = 1\}$.

THEOREM 3.18. *Assume* (3.9) *satisfies the following conditions*:

(a) $\frac{\partial M_i}{\partial x_j} < 0$;

(b) $M_i(0) > 0$;

(c) $M_i(x) < 0$ *for* $|x|$ *sufficiently large.*

Then there exists an invariant compact hypersurface $\Sigma \subset \mathbb{R}^n_+$ *such that*

(i) Σ *attracts every point in* $\mathbb{R}^n_+ \setminus \{0\}$;

(ii) $\Sigma \cap \operatorname{Int} \mathbb{R}^n_+$ *is a locally Lipschitz submanifold*;

(iii) $\Sigma \cap \operatorname{Int} \mathbb{R}^n_+$ *is transverse to every line that is parallel to a nonnegative vector and meets* $\Sigma \cap \operatorname{Int} \mathbb{R}^n_+$;

(iv) Σ *is unordered*;

(v) *Radial projection defines a homeomorphism* $h : \Sigma \to \Delta^{n-1}$ *whose inverse is locally Lipschitz on the open* $(n-1)$*-cell* $\Delta^{n-1} \cap \operatorname{Int} \mathbb{R}^n_+$. *There is a flow* Ψ *on* Δ^{n-1} *such that* $\Phi_t | \Sigma = h \circ \Phi_t \circ h^{-1}$.

COROLLARY 3.19. *If* $n = 3$, *every periodic orbit in* $\mathbb{R}^3_+$ *bounds an unordered invariant disk.*

Assumption (a) is the condition of total competition; (b) and (c) have plausible biological interpretations. The attracting hypersurface Σ, named the *carrying simplex* by M. Zeeman, is analogous to the carrying capacity K in the one-dimensional logistic equation $\mathrm{d}x/\mathrm{d}t = rx(K - x)$. One can define Σ either as the boundary of the set of points whose alpha limit set is the origin, or as the boundary of the compact global attractor. These sets coincide if and only if Σ is unique, in which case it uniformly attracts every compact set in $\mathbb{R}^n_+ \setminus \{0\}$. Uniqueness holds under mild additional assumptions on the maps M_i (Wang and Jiang [230]). The geometry, smoothness and dynamics of carrying simplices have been investigated by Benaïm [14], Brunovsky [21], Mierczyński [140,143,141], Tineo [220], van den Driessche and M. Zeeman [223], Wang and Jiang [230], E. Zeeman [239], E. Zeeman and M. Zeeman [240–242], M. Zeeman [243].

Theorem 3.18 is proved in Hirsch [72] using a general existence theorem for invariant hypersurfaces, of which the following is a generalization:

THEOREM 3.20. *Let* Φ *be a strongly monotone local flow in a p-convex open set* $D \subset \mathbb{R}^n$. *If* $L \subset D$ *is a nonempty compact unordered invariant set,* L *lies in an unordered invariant hypersurface* M *that is a locally Lipschitz submanifold.*

IDEA OF PROOF. Define U to be the set of $x \in D$ such that $\Phi_t(x) \gg y$ for some $t > 0$, and some $y \in L$. Continuity implies U is open, and it is nonempty since it contains $z \in D$ where $z > y \in L$. It can be shown that the lower boundary of U in D (Section 1.1) is a hypersurface with the required properties, by arguments analogous to the proof of Theorem 3.17. □

3.7. *Systems in* $\mathbb{R}^2$

Cooperative and competitive systems in $\mathbb{R}^2$ have particularly simple dynamics. Versions of the following result were proved in Hirsch [67], Theorem 2.7 and Smith [194], Theorem 3.2.2. It is noteworthy that in the next two theorems Φ does not need to be monotone, only locally monotone; hence p-convexity of D is not needed.

THEOREM 3.21. *Let* $D \subset \mathbb{R}^2$ *be an open set and* $g : D \to \mathbb{R}^2$ *a vector field that is cooperative or competitive for the standard cone. Let* $y(t)$ *a nonconstant trajectory defined on an open interval* $I \subset \mathbb{R}$ *containing* 0. *Then there exists* $t_* \in I$ *such that each coordinate* $y_i(t)$ *is nonincreasing or nondecreasing on each connected component of* $I \setminus \{t_*\}$.

PROOF. It suffices to prove that $y_i'(t)$ can change sign at most once. We assume g is cooperative, otherwise reversing time. Let Φ be local flow of g and set $X(t,x) = \frac{\partial \Phi}{\partial x}(t,x)$. The matrix-valued function $X(t,x)$ satisfies the variational equation

$$\frac{\partial}{\partial t} X(t,x) = \frac{\partial g}{\partial x}\big(\Phi(t,x)\big) \cdot X(t,x), \quad X(0,x) = I.$$

Cooperativity and Corollary 3.4 show that $X(t,x)$ has nonnegative entries for $t \geqslant 0$, i.e., matrix multiplication by $X(t,x)$ preserves the standard cone. The tangent vector $y'(t)$ to the curve $y(t)$, being a solution of the variational equation, satisfies $y'(t) = X(t, y(0))y'(0)$. Nonnegativity of $X(t,x)$ implies that if $y'(t_0)$ lies in the first or third quadrants, then $y'(t)$ stays in the same quadrant, and hence its coordinates have constant sign, for $t > t_0$. On the other hand if $y'(t)$ for $t \geqslant t_0$ is never in the first or third quadrants, its coordinates again have constant sign. (Note that $y'(t)$ cannot transit directly between quadrants 1 and 3, or 2 and 4, since it cannot pass through the origin.) We have shown that there is at most one $t_0 \in I$ at which $y'(t)$ changes quadrants. If such a t_0 exists, set $t_* = t_0$; otherwise let $t_* \in I$ be arbitrary. □

Variants of the next result have been proved many times for Kolmogorov type population models (Albrecht et al. [1], Grossberg [53], Hirsch and Smale [80], Kolmogorov [97], Rescigno and Richardson [168], Selgrade [178]).

THEOREM 3.22. *Let g be a K-cooperative or K-competitive vector field in a domain $D \subset \mathbb{R}^2$. If $\gamma^+(x)$ (respectively, $\gamma^-(x)$) has compact closure in D, then $\omega(x)$ (respectively, $\alpha(x)$) is a single equilibrium.*

PROOF. For the standard cone, denoted here by P, this follows from Theorem 3.21. The general case follows by making a linear coordinate change $y = Tx$ mapping K onto the standard cone. Here T is any linear transformation that takes a basis for $\mathbb{R}^2$ contained in ∂P into the standard basis, which lies in ∂K. Then we have $u \leqslant_K v$ if and only if $Tu \leqslant_P Tv$; in other words, T is an order isomorphism. It follows that the system $x' = g(x)$ is K-cooperative (respectively, K-competitive) if and only if the system $y' = h(y) := Tg(T^{-1}y)$ is P-cooperative (respectively, P-competitive). Therefore T is a conjugacy between the local flows Φ, Ψ of the two dynamical systems, that is, $T \circ \Phi_t = \Psi_t \circ T$. Consequently the conclusion for P, proved above, implies the conclusion for K. □

3.8. *Systems in $\mathbb{R}^3$*

The following Poincaré–Bendixson theorem for three-dimensional cooperative and competitive systems is the most notable consequence of Theorem 3.17. It was proved by Hirsch [76] who improved earlier partial results [67,187]. The following result from Smith [194] holds for arbitrary cones $K \subset \mathbb{R}^3$ with nonempty interior:

THEOREM 3.23. *Let g be a K-cooperative or K-competitive vector field in a p-convex domain $D \subset \mathbb{R}^3$. Then a compact limit set of g that contains no equilibrium points is a periodic orbit.*

PROOF. Let Φ denote the flow of the system, and L the limit set. By Theorem 3.17, the restriction of Φ to L is topologically equivalent to a flow Ψ, generated by a Lipschitz planar vector field, restricted to the compact, connected, chain recurrent invariant set $Q(L)$. Since L contains no equilibria neither does $Q(L)$. The Poincaré–Bendixson theorem implies that $Q(L)$ consists of periodic orbits and, possibly, entire orbits whose omega and alpha limit sets are periodic orbits contained in $Q(L)$. The chain recurrence of Ψ on $Q(L)$ will be exploited to show that $Q(L)$ consists entirely of periodic orbits.

Let $z \in Q(L)$ and suppose that z does not belong to a periodic orbit. Then $\omega(z)$ and $\alpha(z)$ are distinct periodic orbits in $Q(L)$. Let $\omega(z) = \gamma$ and suppose for definiteness that z belongs to the interior component, V, of $\mathbb{R}^2 \setminus \gamma$ so that $\Psi_t(z)$ spirals toward γ in V. The other case is treated similarly. Then γ is asymptotically stable relative to V. Standard arguments using transversals imply the existence of compact, positively invariant neighborhoods U_1 and U_2 of γ in V such that $U_2 \subset \mathrm{Int}_V U_1$, $z \notin U_1$ and there exists $t_0 > 0$ for which $\Psi_t(U_1) \subset U_2$ for $t \geqslant t_0$. Let $\epsilon > 0$ be such that the 2ϵ-neighborhood of U_2 in D is contained in U_1. Choose t_0 larger if necessary such that $\Psi_t(z) \in U_2$ for $t \geqslant t_0$. This can be done since $\omega(z) = \gamma$. Then any (ϵ, t_0)-chain in $Q(L)$ beginning at $x_1 = z$ satisfies $\Psi_{t_1}(x_1) \in U_2$ and, by (3.5) and the fact that the 2ϵ-neighborhood of U_2 is contained in U_1, it follows that $x_2 \in U_1$. As $t_2 > t_0$, it then follows that $\Psi_{t_2}(x_2) \in U_2$ and (3.5) again implies that $x_3 \in U_1$. Continuing this argument, it is evident that the (ϵ, t_0)-chain cannot return to z. There can be no (ϵ, t_0)-chain in $Q(L)$ from z to z and therefore we have contradicted that $Q(L)$ is chain recurrent. Consequently, every orbit of $Q(L)$ is periodic. Since $Q(L)$ is connected, it is either a single periodic orbit or an annulus consisting of periodic orbits. It follows that L is either a single periodic orbit or a cylinder of periodic orbits.

To complete the proof we must rule out the possibility that $Q(L)$ consists of an annulus of periodic orbits. We can assume that the system is cooperative. The argument will be separated into two cases: $L = \omega(x)$ or $L = \alpha(x)$.

If $L = \omega(x)$ consists of more than one periodic orbit then $Q(L)$ is an annulus of periodic orbits in the plane containing an open subset O. Then there exists $t_0 > 0$ such that $Q(\Phi_{t_0}(x)) \in O$. Let y be the unique point of L such that $Q(y) = Q(\Phi_{t_0}(x))$. $y = \Phi_{t_0}(x)$ cannot hold since this would imply that L is a single periodic orbit so it follows that either $y \ll \Phi_{t_0}(x)$ or $\Phi_{t_0}(x) \ll y$. Suppose that the latter holds, the argument is similar in the other case. Then there exists $t_1 > t_0$ such that $\Phi_{t_1}(x)$ is so near y that $\Phi_{t_0}(x) \ll \Phi_{t_1}(x)$. But then the Convergence Criterion from Chapter 1 implies that $\Phi_t(x)$ converges to equilibrium, a contradiction to our assumption that L contains no equilibria. This proves the theorem in this case.

If $L = \alpha(x)$ and $Q(L)$ consists of an annulus of periodic orbits, let $C \subset L$ be a periodic orbit such that $Q(L)$ contains C in its interior. $Q(C)$ separates $Q(L)$ into two components. Fix a and b in $L \setminus C$ such that $Q(a)$ and $Q(b)$ belong to different components of $Q(L) \setminus Q(C)$. Since $\Phi_t(x)$ repeatedly visits every neighborhood of a and b as $t \to -\infty$, $Q(\Phi_t(x))$ must cross $Q(C)$ at a sequence of times $t_k \to -\infty$. Therefore, there exist $z_k \in C$ such that $Q(z_k) = Q(\Phi_{t_k}(x))$ and consequently, as in the previous case, either $z_k \ll \Phi_{t_k}(x)$

or $\Phi_{t_k}(x) \ll z_k$ holds for each k. Passing to a subsequence, we can assume that either $z_k \ll \Phi_{t_k}(x)$ holds for all k or $\Phi_{t_k}(x) \ll z_k$ holds for all k. Assume the latter, the argument is essentially the same in the other case. We claim that for every $s < 0$ there is a point $w \in C$ such that $\Phi_s(x) > w$. For if $t_k < s$ then

$$\Phi_s(x) = \Phi_{s-t_k} \circ \Phi_{t_k}(x) < \Phi_{s-t_k}(z_k) \in C.$$

If $y \in L$ then $\Phi_{s_n}(x) \to y$ for some sequence $s_n \to -\infty$. By the claim, there exists $w_n \in C$ such that $\Phi_{s_n}(x) > w_n$. Passing to a subsequence if necessary, we can assume that $w_n \to w \in C$ and $y \geqslant w$. Therefore, every point of L is related by $\leqslant$ to some point of C.

The same reasoning applies to every periodic orbit $C' \subset L$ for which $Q(C')$ belongs to the interior of $Q(L)$: either every point of L is $\leqslant$ some point of C' or every point of L is $\geqslant$ some point of C'. Since there are three different periodic orbits in L whose projections are contained in the interior of $Q(L)$, there will be two of them for which the same inequality holds between points of L and points of the orbit. Consider the case that there are two periodic orbits C_1 and C_2 such that every point of L is $\leqslant$ some point of C_1 and $\leqslant$ some point of C_2. The case that the opposite relations hold is treated similarly. If $u \in C_1$ then it belongs to L so we can find $w \in C_2$ such that $u < w$ (equality can't hold since the points belong to different periodic orbits). But $w \in L$ so we can find $z \in C_1$ such that $w < z$. Consequently, $u, z \in C_1$ satisfy $u < z$, a contradiction to Proposition 3.16. This completes the proof. □

A remarkable fact about three-dimensional competitive or cooperative systems on suitable domains is that the existence of a periodic orbit implies the existence of an equilibrium point inside a certain semi-invariant closed ball having the periodic orbit on its boundary. Its primary use is to locate equilibria, or conversely, to exclude periodic orbits. The construction below is adapted from Smith [187,194] where the case $K = \mathbb{R}^3_+$ was treated; here we treat the general case that K has nonempty interior. The terms "competitive" and "cooperative" will be used to mean K-competitive and K-cooperative for brevity. A related result appears in Hirsch [75]. Throughout the remainder of this section, the system is assumed to be defined on a p-convex subset D of $\mathbb{R}^3_+$.

We can assume the system is competitive. Let γ denote the periodic orbit and assume that there exist p, q with $p \ll q$ such that

$$\gamma \subset [p, q] \subset D. \tag{3.10}$$

Define

$$B = \{x \in \mathbb{R}^3\colon\ x \text{ is not related to any point } y \in \gamma\} = (\gamma + K)^{\mathrm{c}} \cap (\gamma - K)^{\mathrm{c}}.$$

Here we use the notation A^{c} for the complement of the subset A in $\mathbb{R}^3$. Observe that in defining B we ignored the domain D of (3.4), viewing γ as a subset of $\mathbb{R}^3$. Another way to define B is to express its complement as $B^{\mathrm{c}} = (\gamma + K) \cup (\gamma - K)$.

A 3-*cell* is a subset of $\mathbb{R}^3$ that is homeomorphic to the open unit ball.

THEOREM 3.24. *Let γ be a nontrivial periodic orbit of a competitive system in $D \subset \mathbb{R}^3$ and suppose that* (3.10) *holds. Then B is an open subset of $\mathbb{R}^3$ consisting of two connected components, one bounded and one unbounded. The bounded component, $B(\gamma)$, is a 3-cell contained in $[p, q]$. Furthermore, $B(\gamma)$ is positively invariant and its closure contains an equilibrium.*

Combining this result with Theorem 3.23 leads to the following dichotomy from Hirsch [75].

COROLLARY 3.25. *Assume the domain $D \subset \mathbb{R}^3$ of a cooperative or competitive system contains $[p, q]$ with $p \ll q$. Then one of the following holds*:

(i) *$[p, q]$ contains an equilibrium*;

(ii) *the forward and backward semi-orbits of every point of $[p, q]$ meet $D \setminus [p, q]$.*

PROOF. We take the system to be competitive, otherwise reversing time. Assume (ii) is false. Then $[a, b]$ contains a compact limit set L. If L is not a cycle, it contains an equilibrium by Theorem 3.23. If L is a cycle, (i) follows from Theorem 3.24. □

PROOF SKETCH OF THEOREM 3.24. That B is open is a consequence of the fact that $\gamma + K$ and $\gamma - K$ are closed. We show that $B \cap D$ is positively invariant. If $x \in B \cap D$, $y \in \gamma$ and $t > 0$ then $\Phi_{-t}(y) \in \gamma$ so x is not related to it. Since the forward flow of a competitive system preserves the property of being unrelated, $\Phi_t(x)$ is unrelated to y. Therefore, $\Phi_t(x) \in B \cap D$.

As in the proof of Theorem 3.17, for $v > 0$, H_v denotes the hyperplane orthogonal to v and Q the orthogonal projection onto H_v along v. Q is one-to-one on γ so $Q(\gamma)$ is a Jordan curve in H_v. Let H_{i} and H_{e} denote the interior and exterior components of $H_v \setminus Q(\gamma)$. If $x \in Q^{-1}(Q(\gamma))$ then $Q(x) = Q(y)$ for some $y \in \gamma$ and therefore either $x = y$, $x \ll y$ or $y \ll x$. In any case, $x \notin B$. Hence,

$$B = \big(B \cap Q^{-1}(H_{\mathrm{i}})\big) \cup \big(B \cap Q^{-1}(H_{\mathrm{e}})\big).$$

Set $B(\gamma) = B \cap Q^{-1}(H_{\mathrm{i}})$.

Given $z \in H_{\mathrm{i}}$, let $A_z^+ := \{s \in \mathbb{R}\colon z + sv \in \gamma + K\}$ and $A_z^- := \{s \in \mathbb{R}\colon z + sv \in \gamma - K\}$. A_z^+ clearly contains all large s by compactness of γ and it is closed because $\gamma + K$ is closed. If $s \in A_z^+$, there exists $y \in \gamma$ and $k \in K$ such that $z + sv = y + k$ so $z + (s + r)v = y + k + rv$, implying that $s + r \in A_z^+$ for all $r \geqslant 0$. It follows that $A_z^+ = [s_+(z), \infty)$, and similarly, $A_z^- = (-\infty, s_-(z)]$. If $s_-(z) \geqslant z_+(z)$ so $A_z^+ \cap A_z^-$ is nonempty, then there exists $s \in \mathbb{R}$, $k_i \in K$, and $y_i \in \gamma$ such that $z + sv = y_1 + k_1 = y_2 - k_2$. We must have $k_1 = k_2 = 0$ or else $y_2 > y_1$, a contradiction to Proposition 3.16, but then $z + sv = y_1$ so $z = Qy_1$ contradicting that $z \in H_{\mathrm{i}}$. We conclude that $s_-(z) < z_+(z)$ and that $z + sv \in B(\gamma)$ if and only if $s_-(z) < s < s_+(z)$. It follows that

$$B(\gamma) = \big\{z + sv\colon z \in H_{\mathrm{i}},\ s \in \big(s_-(z), s_+(z)\big)\big\}.$$

It is easy to show that the maps $z \mapsto s_{\pm}(z)$ are continuous and satisfy $s_+(z) - s_-(z) \to 0$ as $z \to y \in \gamma$ and this implies that $B(\gamma)$ is a 3-cell. See the argument given in Smith [187, 194].

To prove $B(\gamma) \subset [p, q]$, we identify K^* as the set of x such that $\langle x, k\rangle \geqslant 0$ for all $k \in K$ (where $\langle x, k\rangle$ denotes inner product). Schneider and Vidyasagar [177] proved the elegant result that every vector x has a unique representation

$$x = k - w, \quad k \in K, \ w \in K^*, \ \langle w, k\rangle = 0.$$

Choose any $z \in B \cap (\mathbb{R}^3 \setminus [p, q])$ and write

$$z - p = k - w, \quad k \in K, w \in K^*, \ \langle w, k\rangle = 0,$$

$$q - z = k' - w', \quad k' \in K, w' \in K^*, \ \langle w', k'\rangle = 0.$$

Observe that $w > 0$, $w' > 0$ because $z \in B$.

Either $k > 0$ or $k' > 0$. For if $k = k' = 0$ then $q - p = -(w + w')$, so

$$0 \leqslant \langle w + w', q - p\rangle = -\|w + w'\|^2 \leqslant 0.$$

This entails $w + w' = 0$ and thus $p = q$, a contradiction.

We assume $k > 0$, as the case $k' > 0$ is similar, and even follows formally by replacing K with $-K$. Then $w > 0$. Consider the ray $R = \{z + tk \colon t \geqslant 0\}$. If $y \in \gamma$, then

$$\langle w, z + tk - y\rangle = \langle w, z - p\rangle + \langle w, p - y\rangle \leqslant \langle w, z - p\rangle = -\|w\|^2 < 0.$$

Because z and u are unrelated, there exists $u \in K^*$ such that $\langle u, z - y\rangle > 0$. So

$$\langle u, z + tk - y\rangle = \langle u, z - y\rangle + t\langle z, k\rangle \geqslant \langle u, z - y\rangle > 0.$$

This shows that no point of R is related to any point of γ. Therefore R and hence z are in the unbounded component of B.

As $B(\gamma)$ is a connected component of the positively invariant set B, it is positively invariant. Consequently its closure is a positively invariant set homeomorphic to the closed unit ball in $\mathbb{R}^3$. It therefore contains an equilibrium by a standard argument using the Brouwer Fixed Point Theorem (see, e.g., Hale [57, Theorem I.8.2]). $\square$

If $B(\gamma)$ contains only nondegenerate equilibria $x_1, x_2, \ldots, x_m$, then standard topological degree arguments imply that m is odd and that $1 = \sum_{i=1}^{m} (-1)^{s_i}$ where $s_i \in \{0, 1, 2, 3\}$ is the number of positive eigenvalues of $Df(x_i)$. See Smith [187] for the proof and further information on equilibria in $B(\gamma)$.

There are many papers devoted to competitive Lotka–Volterra systems in $\mathbb{R}^3$, largely stimulated by the work of M. Zeeman. See for example [82,223,237,239,243,240,242] and references therein. The paper of Li and Muldowney [115] contains an especially nice application to epidemiology. Additional results for three-dimensional competitive and cooperative systems can be found in references [66,75–77,41,194,196,247,248].

The recent paper of Ortega and Sánchez [153] is noteworthy for employing a cone related to the ice-cream cone and observing that results for competitive systems are valid for general cones with nonempty interior. They show that if P is a symmetric matrix of dimension n having one positive eigenvalue λ_+ with corresponding unit eigenvector e_+, and $n-1$ negative eigenvalues, then (3.4) is monotone with respect to the order generated by the cone $K := \{x \in \mathbb{R}^n\colon\ \langle Px, x\rangle \geqslant 0,\ \langle x, e_+\rangle \geqslant 0\}$ if and only if there exists a function $\mu : \mathbb{R}^n \to \mathbb{R}$ such that the matrix $P \cdot Df_x + (Df_x)^T \cdot P + \mu(x)P$ is positive semidefinite for all x. They use this result to show that one of the results of R.A. Smith [204] on the existence of an orbitally stable periodic orbit, in the special case $n = 3$, follows from the results for competitive systems. It is not hard to see that if (3.4) satisfies the conditions above then after a change of variables in (3.4), the resulting system is monotone with respect to the standard ice-cream cone.

For applications of competitive and cooperative systems, see for example Benaïm [15], Benaïm and Hirsch [16,17], Hirsch [69,74] Hofbauer and Sandholm [81], Hsu and Waltman [84], Smith [194,196], Smith and Waltman [202].

4. Delay differential equations

4.1. *The semiflow*

The aim of the present section is to apply the theory developed in Sections 1 and 2 to differential equations containing delayed arguments. Such equations are often referred to as delay differential equations or functional differential equations. Since delay differential equations contain ordinary differential equations as a special case, when all delays are zero, the treatment is quite similar to the previous section. The main difference is that a delay differential equation generally can't be solved backward in time and therefore there is not a well-developed theory of competitive systems with delays.

Delay differential equations generate infinite-dimensional dynamical systems and there are several choices of state space. We restrict attention here to equations with bounded delays and follow the most well-developed theory (see Hale and Verduyn Lunel [61]). If r denotes the maximum delay appearing in the equation, then the space $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$ is a natural choice of state space. Given a cone K in $\mathbb{R}^n$, $\mathcal{C}_K$ contains the cone of functions which map $[-r, 0]$ into K. The section begins by identifying sufficient conditions on the right hand side of the delay differential equation for the semiflow to be monotone with respect to the ordering induced by this cone. This quasimonotone condition reduces to the quasimonotone condition for ordinary differential equations when no delays are present. Our main goal is to identify sufficient conditions for a delay differential equation to generate an eventually strongly monotone semiflow so that results from Sections 1 and 2 may be applied.

In order to motivate fundamental well-posedness issues for delay equations, it is useful to start with a consideration of a classical example that has motivated much research in the field (see, e.g., Krisztin et al. [105] and Hale and Verduyn Lunel [61]), namely the equation

$$x'(t) = -x(t) + h\big(x(t-r)\big), \quad t \geqslant 0, \tag{4.1}$$

where h is continuous and $r > 0$ is the delay. It is clear that $x(t)$ must be prescribed on the interval $[-r, 0]$ in order that it be determined for $t \geqslant 0$. A natural space of initial conditions is the space of continuous functions on $[-r, 0]$, which we denote by $\mathcal{C}$, where $n = 1$ in this case. $\mathcal{C}$ is a Banach space with the usual uniform norm $|\phi| = \sup\{|\phi(\theta)|\colon -r \leqslant \theta \leqslant 0\}$. If $\phi \in \mathcal{C}$ is given, then it is easy to see that the equation has a unique solution $x(t)$ for $t \geqslant 0$ satisfying

$$x(\theta) = \phi(\theta), \quad -r \leqslant \theta \leqslant 0.$$

If the state space is $\mathcal{C}$, then we need to construct from the solution $x(t)$, an element of the space $\mathcal{C}$ to call the state of the system at time t. It should have the property that it uniquely determines $x(s)$ for $s \geqslant t$. The natural choice is $x_t \in \mathcal{C}$, defined by

$$x_t(\theta) = x(t+\theta), \quad -r \leqslant \theta \leqslant 0.$$

Then, $x_0 = \phi$ and $x_t(0) = x(t)$.

The general autonomous functional differential equation is given by

$$x'(t) = f(x_t), \tag{4.2}$$

where $f : D \to \mathbb{R}^n$, D is an open subset of $\mathcal{C}$ and f is continuous. In the example above, f is given by $f(\phi) = -\phi(0) + h(\phi(-r))$ for $\phi \in \mathcal{C}$. Observe that (4.2) includes the system of ordinary differential equations

$$x' = g(x),$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$, as a special case. Simply let $f(\phi) = g(\phi(0))$ so that $f(x_t) = g(x_t(0)) = g(x(t))$.

It will always be assumed that (4.2), together with the initial condition $x_0 = \phi \in D$ has a unique, maximally defined solution, denoted by $x(t, \phi)$, on an interval $[0, \sigma)$. The state of the system is denoted by $x_t(\phi)$ to emphasize the dependence on the initial data. Uniqueness of solutions holds if, for example, f is Lipschitz on compact subsets of D (see Hale and Verduyn Lunel [61]). This holds, for example, if $f \in C^1(D)$ has locally bounded derivative. If uniqueness of solutions of initial value problems hold, then the map $(t, \phi) \to x_t(\phi)$ is continuous. Therefore, a (local) semiflow on D can be defined by

$$\Phi_t(\phi) = x_t(\phi). \tag{4.3}$$

In contrast to the case of ordinary differential equations, $x(t, \phi)$ cannot usually be defined for $t \leqslant 0$ as a solution of (4.2) and consequently, Φ_t need not be one-to-one.

It will be convenient to have notation for the natural embedding of $\mathbb{R}^n$ into $\mathcal{C}$. If $x \in \mathbb{R}^n$, let $\hat{x} \in \mathcal{C}$ be the constant function equal to x for all values of its argument. The set of equilibria for (4.2) is given by

$$E = \{\hat{x} \in D \colon x \in \mathbb{R}^n \text{ and } f(\hat{x}) = 0\}.$$

4.2. *The quasimonotone condition*

Given that $\mathcal{C}$ is a natural state space for (4.2), we now consider what sort of cones in $\mathcal{C}$ will yield useful order relations. The most natural such cones are those induced by cones in $\mathbb{R}^n$. Let K be a cone in $\mathbb{R}^n$ with nonempty interior and K^* denote the dual cone. All inequalities hereafter are assumed to be those induced on $\mathbb{R}^n$ by K. The cone K induces a cone $\mathcal{C}_K$ in the Banach space $\mathcal{C}$ defined by

$$\mathcal{C}_K = \{\phi \in \mathcal{C} \colon \phi(\theta) \geqslant 0, \ -r \leqslant \theta \leqslant 0\}.$$

It has nonempty interior in $\mathcal{C}$ given by $\operatorname{Int} \mathcal{C}_K = \{\phi \in \mathcal{C}_K \colon \phi(\theta) \gg 0, \theta \in [-r, 0]\}$. The usual notation $\leqslant, <, \ll$ will be used for the various order relations on $\mathcal{C}$ generated by $\mathcal{C}_K$. In particular, $\phi \leqslant \psi$ holds in $\mathcal{C}$ if and only if $\phi(s) \leqslant \psi(s)$ holds in $\mathbb{R}^n$ for every $s \in [-r, 0]$. The same notation will also be used for the various order relations on $\mathbb{R}^n$ but hopefully the context will alert the reader to the appropriate meaning. Cones in $\mathcal{C}$ that are not induced by a cone in $\mathbb{R}^n$ have also proved useful. See Smith and Thieme [198,200,194].

An immediate aim is to identify sufficient conditions on f for the semiflow Φ to be a monotone semiflow. The following condition should seem natural since it generalizes the condition (QM) for ordinary differential equations in the previous section. We refer to it here as the *quasimonotone condition*, (QMD) for short. "D" in the notation, standing for delay, is used so as not to confuse the reader with (QM) of the previous section. We follow this pattern in several definitions in this section.

(QMD) $\phi, \psi \in D$, $\phi \leqslant \psi$ and $\eta(\phi(0)) = \eta(\psi(0))$ for some $\eta \in K^*$, implies $\eta(f(\phi)) \leqslant \eta(f(\psi))$.

For the special case $K = \mathbb{R}^n_+$, (QMD) becomes:

$$\phi, \psi \in D, \ \phi \leqslant \psi \text{ and } \phi_i(0) = \psi_i(0) \quad \text{implies} \quad f_i(\phi) \leqslant f_i(\psi).$$

As in Section 3, it is convenient to consider the nonautonomous equation

$$x'(t) = f(t, x_t), \tag{4.4}$$

where $f \colon \Omega \to \mathbb{R}^n$ is continuous on Ω, an open subset of $\mathbb{R} \times \mathcal{C}$. Given $(t_0, \phi) \in \Omega$, we write $x(t, t_0, \phi, f)$ and $x_t(t_0, \phi, f)$ for the maximally defined solution and state of the system at time t satisfying $x_{t_0} = \phi$. We assume this solution is unique, which will be the case if f is Lipschitz in its second argument on each compact subset of Ω. We drop the last argument f from $x(t, t_0, \phi, f)$ when no confusion over which f is being considered will result.

$f : \Omega \to \mathbb{R}^n$ is said to satisfy (QMD) if $f(t, \cdot)$ satisfies (QMD) on $\Omega_t \equiv \{\phi \in C\colon (t, \phi) \in \Omega\}$ for each t.

The next theorem not only establishes the desired monotonicity of the semiflow Φ but also allows comparisons of solutions between related functional differential equations. It generalizes Theorem 3.2 of Chapter 3 to functional differential equations and is a generalization of Proposition 1.1 of [190] and Theorem 5.1.1 of [194] where $K = \mathbb{R}^n_+$ is considered. The quasimonotone condition for delay differential equations seems first to have appeared in the work of Kunisch and Schappacher [109], Martin [128], and Ohta [152].

THEOREM 4.1. *Let $f, g : \Omega \to \mathbb{R}^n$ be continuous, Lipschitz on each compact subset of Ω, and assume that either f or g satisfies* (QMD). *Assume also that $f(t, \phi) \leqslant g(t, \phi)$ for all $(t, \phi) \in \Omega$. Then*

$$\phi, \psi \in \Omega_{t_0},\ \phi \leqslant \psi,\ t \geqslant t_0, \quad \Longrightarrow \quad x(t, t_0, \phi, f) \leqslant x(t, t_0, \psi, g)$$

for all t for which both are defined.

PROOF. Assume that f satisfies (QMD), a similar argument holds if g satisfies (QMD). Let $e \in \mathbb{R}^n$ satisfy $e \gg 0$, $g_\epsilon(t, \phi) := g(t, \phi) + \epsilon e$ and $\psi_\epsilon := \psi + \epsilon \hat{e}$, for $\epsilon \geqslant 0$. If $x(t, t_0, \psi, g)$ is defined on $[t_0 - r, t_1]$ for some $t_1 > t_0$, then $x(t, t_0, \psi_\epsilon, g_\epsilon)$ is also defined on this same interval for all sufficiently small positive ϵ and

$$x(t, t_0, \psi_\epsilon, g_\epsilon) \to x(t, t_0, \psi, g), \quad \epsilon \to 0,$$

for $t \in [t_0, t_1]$ by Hale and Verduyn Lunel [61, Theorem 2.2.2]. We will show that $x(t, t_0, \phi, f) \ll x(t, t_0, \psi_\epsilon, g_\epsilon)$ on $[t_0 - r, t_1]$ for small positive ϵ. The result will then follow by letting $\epsilon \to 0$. If the assertion above were false for some ϵ, then applying the remark below Proposition 3.1, there exists $s \in (t_0, t_1]$ such that $x(t, t_0, \phi, f) \ll x(t, t_0, \psi_\epsilon, g_\epsilon)$ for $t_0 \leqslant t < s$ and $\eta(x(s, t_0, \phi, f)) = \eta(x(s, t_0, \psi_\epsilon, g_\epsilon))$ for some nontrivial $\eta \in K^*$. As $\eta(x(t, t_0, \phi, f)) < \eta(x(t, t_0, \psi_\epsilon, g_\epsilon))$ for $t_0 \leqslant t < s$, by Proposition 3.1, we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=s}\eta(x(s, t_0, \phi, f)) \geqslant \frac{\mathrm{d}}{\mathrm{d}t}|_{t=s}\eta(x(s, t_0, \psi_\epsilon, g_\epsilon))$. But

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} \eta\big(x(s, t_0, \psi_\epsilon, g_\epsilon)\big) &= \eta\big(g\big(s, x_s(t_0, \psi_\epsilon, g_\epsilon)\big)\big) + \epsilon\eta(e) \\
&> \eta\big(f\big(s, x_s(t_0, \psi_\epsilon, g_\epsilon)\big)\big) \\
&\geqslant \eta\big(f\big(s, x_s(t_0, \phi, f)\big)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} \eta\big(x(s, t_0, \phi, f)\big),
\end{aligned}$$

where the last inequality follows from (QMD). This contradiction implies that no such s can exist, proving the assertion. $\square$

In the case of the autonomous system (4.2), taking $f = g$ in Theorem 1.1 implies that $x_t(\phi) \leqslant x_t(\psi)$ for $t \geqslant 0$ such that both solutions are defined. In other words, the semiflow Φ defined by (4.3) is monotone. In contrast to Theorem 3.2 of the previous section, if $\phi < \psi$ we cannot conclude that $x(t,\phi) < x(t,\psi)$ or $x_t(\phi) < x_t(\psi)$ since Φ_t is not generally one-to-one. A simple example is provided by the scalar equation (4.2) with $r = 1$ and $f(\phi) := \max \phi$, which satisfies (QMD). Let $\phi < \psi$ be strictly increasing on $[-1, -1/2]$, $\phi(-1) = \psi(-1) = 0$, $\phi(-1/2) = \psi(-1/2) = 1$, and $\phi(\theta) = \psi(\theta) = -2\theta$ for $-1/2 < \theta \leqslant 0$. It is easy to see that $x(t,\phi) = x(t,\psi)$ for $t \geqslant 0$.

It is useful to have sufficient conditions for the positive invariance of K. By this we mean that $t_0 \in J$ and $\phi \geqslant 0$ implies $x(t, t_0, \phi) \geqslant 0$ for all $t \geqslant t_0$ for which it is defined. The following result provides the expected necessary and sufficient condition. The proof is similar to that of Theorem 4.1; the result is the delay analog of Proposition 3.3.

THEOREM 4.2. *Assume that $J \times K \subset \Omega$ where J is an open interval. Then K is positively invariant for* (4.4) *if and only if for all $t \in J$*

(PD) $\phi \geqslant 0, \lambda \in K^*$ *and* $\lambda(\phi(0)) = 0$ *implies* $\lambda(f(t,\phi)) \geqslant 0$

holds.

Let $L : J \to L(\mathcal{C}, \mathbb{R}^n)$ be continuous, where $L(\mathcal{C}, \mathbb{R}^n)$ denotes the space of bounded linear operators from $\mathcal{C}$ to $\mathbb{R}^n$, and consider the initial value problem for the linear nonautonomous functional differential equation

$$x' = L(t)x_t, \quad x_{t_0} = \phi. \tag{4.5}$$

Observing that (PD) and (QMD) are equivalent for linear systems, we have the following corollary.

COROLLARY 4.3. *Let $x(t, t_0, \phi)$ be the solution of* (4.5). *Then $x(t, t_0, \phi) \geqslant 0$ for all $t \geqslant t_0$ and all $\phi \geqslant 0$ if and only if for each $t \in J$,* (PD) *holds for $L(t)$.*

As in the case of ordinary differential equations, a stronger condition than (PD) for linear systems is that for every $t \in J$, there exists $\alpha \in \mathbb{R}$ such that $L(t)\phi + \alpha\phi(0) \geqslant 0$ whenever $\phi \geqslant 0$.

It is useful to invoke the Riesz Representation Theorem [171] in order to identify $L(t)$ with a matrix of signed Borel measures $\eta(t) = (\eta(t)_{ij})$:

$$L(t)\phi = \int_{-r}^{0} \mathrm{d}\eta(t)\phi. \tag{4.6}$$

The Radon–Nikodym decomposition of η_{ij} with respect to the Dirac measure δ with unit mass at 0 gives $\eta_{ij}(t) = a_{ij}(t)\delta + \tilde{\eta}_{ij}(t)$ where a_{ij} is a scalar and $\tilde{\eta}_{ij}(t)$ is mutually singular with respect to δ. In particular, the latter assigns zero mass to $\{0\}$. Therefore,

$$L(t)\phi = A(t)\phi(0) + \tilde{L}(t)\phi, \quad \tilde{L}(t)\phi := \int_{-r}^{0} \mathrm{d}\tilde{\eta}(t)\phi. \tag{4.7}$$

Continuity of the map $t \to A(t)$ follows from continuity of $t \to L(t)$. The decomposition (4.7) leads to sharp conditions for (PD) to hold for $L(t)$.

PROPOSITION 4.4. (PD) *holds for* $L(t)$ *if and only if*
(a) $A(t)$ *satisfies* (P) *of Proposition* 3.3, *and*
(b) $\tilde{L}(t)\phi \geqslant 0$ *whenever* $\phi \geqslant 0$.

PROOF. If (a) and (b) hold, $\phi \geqslant 0$, $\lambda \in K^*$ and $\lambda(\phi(0)) = 0$ then $\lambda(L(t)\phi) = \lambda(A(t)\phi(0)) + \lambda(\tilde{L}(t)\phi) \geqslant 0$ because each summand on the right is nonnegative.

Conversely, if (PD) holds for $L(t)$, $v \in \partial K$, $\lambda \in K^*$, and $\lambda(v) = 0$, define $\phi_n(\theta) = e^{n\theta} v$ on $[-r, 0]$. Then $\phi_n \geqslant 0$ and ϕ_n converges point-wise to zero, almost everywhere with respect to $\tilde{\eta}(t)$. By (PD),

$$\lambda(L(t)\phi_n) = \lambda(A(t)v + \tilde{L}(t)\phi_n) \geqslant 0.$$

Letting $n \to \infty$, we get $\lambda(A(t)v) \geqslant 0$ implying that (P) holds for $A(t)$. Let $\phi \geqslant 0$ be given and define $\phi_n(\theta) = [1 - e^{n\theta}]\phi(\theta)$ on $[-r, 0]$, $n \geqslant 1$. ϕ_n converges point-wise to $\phi\chi$, where χ is the indicator function of the set $[-r, 0)$, and $\phi\chi = \phi$ almost everywhere with respect to $\tilde{\eta}(t)$. If $\lambda \in K^*$, then $\lambda(\phi_n(0)) = 0$ so applying (PD) we get $0 \leqslant \lambda(L(t)\phi_n) = \tilde{L}(t)\phi_n$. Letting $n \to \infty$ we get (b). $\square$

For the remainder of this section, we suppose that $\Omega = J \times D$ where J is a nonempty open interval and $D \subset \mathcal{C}$ is open. Suppose that $\frac{\partial f}{\partial \phi}(t, \psi)$ exists and is continuous on $J \times D$ to $L(\mathcal{C}, \mathbb{R}^n)$. In that case, $x(t, t_0, \phi)$ is continuously differentiable in its last argument and $y(t, t_0, \chi) = \frac{\partial x}{\partial \phi}(t, t_0, \phi)\chi$ satisfies the variational equation

$$y'(t) = \frac{\partial f}{\partial \phi}\big(t, x_t(t, \phi)\big)y_t, \quad y_{t_0} = \chi. \tag{4.8}$$

See Theorem 2.4.1 of Hale and Verduyn Lunel [61]. We say that f (or (4.4)) is *K-cooperative* if for all $(t, \chi) \in J \times D$ the function $\psi \to \frac{\partial f}{\partial \phi}(t, \chi)\psi$ satisfies (PD). By Corollary 4.3 applied to the variational equation we have the following analog of Theorem 3.5 for functional differential equations. The proof is essentially the same.

THEOREM 4.5. *Let* $\frac{\partial f}{\partial \phi}(t, \psi)$ *exist and be continuous on* $J \times D$. *If* (QMD) *holds for* (4.4), *then* f *is K-cooperative. Conversely, if* D *is p-convex and* f *is K-cooperative, then* (QMD) *holds for* f.

Consider the nonlinear system

$$x'(t) = g\big(x(t), x(t - r_1), x(t - r_2), \ldots, x(t - r_m)\big), \tag{4.9}$$

where $g(x, y^1, y^2, \ldots, y^m)$ is continuously differentiable on $\mathbb{R}^{(m+1)n}$ and $r_{j+1} > r_j > 0$. Then

$$\frac{\partial f}{\partial \phi}(\psi) = \frac{\partial g}{\partial x}(x, Y)\delta + \sum_k \frac{\partial g}{\partial y^k}(x, Y)\delta_{-r_k}, \tag{4.10}$$

where δ_{-r_k} is the Dirac measure with unit mass at $\{-r_k\}$ and $x = \psi(0)$, $y^k = \psi(-r_k)$ and $(x, Y) := (x, y^1, y^2, \ldots, y^m)$. By Theorem 4.5, Corollary 4.3, and Proposition 4.4, (QMD) holds if and only if for each (x, Y), $\frac{\partial g}{\partial x}(x, Y)$ satisfies condition (P) and $\frac{\partial g}{\partial y^k}(x, Y)$ is K-positive. If $K = \mathbb{R}^n_+$, the condition becomes $\frac{\partial g_i}{\partial x_j}(x, Y) \geqslant 0$, for $i \neq j$ and $\frac{\partial g_i}{\partial y^k_j}(x, Y) \geqslant 0$ for all i, j, k; if, in addition, $n = 1$ then $\frac{\partial g}{\partial y^k}(x, Y) \geqslant 0$ for all k suffices.

4.3. *Eventual strong monotonicity*

We begin by considering the linear system (4.5). The following hypothesis for the continuous map $L: J \to L(C, \mathbb{R}^n)$ reduces to (ST) of the previous section when $r = 0$:

(STD) for all $t \in J$ and $\phi \geqslant 0$ with $\phi(0) \in \partial K$ satisfying one of the conditions
(a) $\phi(-r) > 0$ and $\phi(0) = 0$, or
(b) $\phi(s) > 0$ for $-r \leqslant s \leqslant 0$,
there exists $\nu \in K^*$ such that $\nu(\phi(0)) = 0$ and $\nu(L(t)\phi) > 0$.

The following result is the analog of Theorem 3.6 of the previous section for delay differential equations.

THEOREM 4.6. *Let linear system* (4.5) *satisfy* (PD) *and* (STD) *and let* $t_0 \in J$. *Then*

$$\phi > 0,\ t \geqslant t_0 + 2r \implies x(t, t_0, \phi) \gg 0.$$

In particular, $x_t(t_0, \phi) \gg 0$ *for* $t \geqslant t_0 + 3r$.

PROOF. By Corollary 4.3, we have that $x(t) := x(t, t_0, \phi) \geqslant 0$ for all $t \geqslant t_0$ that belong to J. There exists $t_1 \in (t_0, t_0 + r)$ such that $x(t_1 - r) = \phi(t_1 - r) = x_{t_1}(-r) > 0$ since $\phi > 0$. If $x(t_1) = 0$, then (STD)(a) implies the existence of $\nu \in K^*$ such that $\nu(L(t_1)x_{t_1}) > 0$. As $\nu(x(t)) \geqslant 0$ for $t \geqslant t_0$ and $\nu(x(t_1)) = 0$ we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_1}\nu(x(t)) \leqslant 0$. But $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_1}\nu(x(t)) = \nu(L(t_1)x_{t_1}) > 0$, a contradiction. Therefore, $x(t_1) > 0$.

Now, by (4.7)

$$x' = A(t)x + \tilde{L}(t)x_t$$

from which we conclude

$$x(t) = X(t, t_1)x(t_1) + \int_{t_1}^{t} X(t, r)\tilde{L}(r)x_r\,\mathrm{d}r,$$

where $X(t, t_0)$ is the fundamental matrix for $y' = A(t)y$ satisfying $X(t_0, t_0) = I$. From (a) of Proposition 4.4 and Corollary 3.4, it follows that $X(t, t_0)$ is K positive for $t \geqslant t_0$. This, the fact that $x_r \geqslant 0$, and (b) of Proposition 4.4 imply that the integral belongs to K so we conclude that

$$x(t) \geqslant X(t, t_1)x(t_1) > 0, \quad t \geqslant t_1.$$

We claim that $x(t) \gg 0$ for $t \geqslant t_1 + r$. If not, there is a $t_2 \geqslant t_1 + r$ such that $x(t_2) = x_{t_2}(0) \in \partial K$ but $x_{t_2}(s) > 0$ for $-r \leqslant s \leqslant 0$. Then (STD) implies the existence of $\nu \in K^*$ such that $\nu(x(t_2)) = 0$ and $\nu(L(t_2)x_{t_2}) > 0$. Since $\nu(x(t)) \geqslant 0$ for $t \geqslant t_0$ we must have $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_2}\nu(x(t)) \leqslant 0$. But $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_2}\nu(x(t)) = \nu(L(t_2)x_{t_2}) > 0$, a contradiction. We conclude that $x(t) \gg 0$ for $t \geqslant t_1 + r$. □

In a sense, (STD)(a) says that r has been correctly chosen; (STD)(b) is more fundamental. The next result gives sufficient conditions for it to hold.

PROPOSITION 4.7. *If $L(t)$ satisfies* (PD) *and either*
(a) *$A(t)$ satisfies* (ST), *or*
(b) $\phi > 0 \Longrightarrow \tilde{L}(t)\phi \gg 0$
then (STD)(b) *holds.*

PROOF. This is immediate from the definitions, the decomposition (4.7), Proposition 4.4, and the expression $\nu(L(t)\phi) = \nu(A(t)\phi(0)) + \nu(\tilde{L}(t)\phi)$. □

Theorem 4.6 leads immediately to a result on eventual strong monotonicity for the nonlinear system (4.4) where we assume that $\Omega = J \times D$ as above.

THEOREM 4.8. *Let D be p-convex, $\frac{\partial f}{\partial \phi}(t, \psi)$ exist and be continuous on $J \times D$ to $L(\mathcal{C}, \mathbb{R}^n)$, and f be K-cooperative. Suppose that* (STD) *holds for $\frac{\partial f}{\partial \phi}(t, \psi)$, for each $(t, \psi) \in J \times D$. Then*

$$\phi_0, \phi_1 \in D,\ \phi_0 < \phi_1 \quad \Longrightarrow \quad x(t, t_0, \phi_0) \ll x(t, t_0, \phi_1)$$

for all $t \geqslant t_0 + 2r$ for which both solutions are defined.

PROOF. By Theorem 4.5, we have $x(t, t_0, \phi_0) \leqslant x(t, t_0, \phi_1)$ for $t \geqslant t_0$ for which both solutions are defined. We apply the formula

$$x(t, t_0, \phi_1) - x(t, t_0, \phi_0) = \int_0^1 \frac{\partial x}{\partial \phi}\big(t, t_0, s\phi_1 + (1-s)\phi_0\big)(\phi_1 - \phi_0)\,\mathrm{d}s.$$

Here, for $\psi \in D$ and $\beta \in \mathcal{C}$, $y(t, t_0, \beta) := \frac{\partial x}{\partial \phi}(t, t_0, \psi)\beta$ satisfies the variational equation (4.5) where $\phi = \beta$ and $L(t) = \frac{\partial f}{\partial \phi}(t, x_t(t_0, \psi))$. See Theorem 2.4.1 of Hale and Verduyn Lunel [61]. The desired conclusion will follow if we show that $y(t, t_0, \beta) \gg 0$ for

$t \geqslant t_0 + 2r$ for $\psi = s\phi_1 + (1-s)\phi_0$ and $\beta = \phi_1 - \phi_0 > 0$. By Theorem 4.6, it suffices to show that $L(t)$ satisfies (PD) and (STD). But this follows from our hypotheses. □

In the next result, Theorem 4.8 is applied to system (4.9). We make use of notation introduced below Theorem 4.5.

COROLLARY 4.9. *Let $g : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ be continuously differentiable and satisfy*

(a) $\frac{\partial g}{\partial x}(x, Y)$ *satisfies* (P) *for each* $(x, Y) \in Z$;

(b) *for each* k, $\frac{\partial g}{\partial y^k}(x, Y)$ *is* K *positive*;

(c) *either* $\frac{\partial g}{\partial x}(x, Y)$ *satisfies* (ST) *or some* $\frac{\partial g}{\partial y^k}(x, Y)$ *is strongly positive on* K.

Then the hypotheses of Theorem 4.8 *hold for* (4.9).

PROOF. Recalling (4.10), it is evident that (a) and (b) imply that (4.9) is K-cooperative. Hypothesis (c) and Proposition 4.7 imply that (STD) holds. □

In the special case that (4.9) is a scalar equation, $m = 1$ and $K = \mathbb{R}_+$, then $\frac{\partial g}{\partial y}(x, y) > 0$ suffices to ensure an eventually strongly monotone semiflow.

4.4. *K is an orthant*

Our results can be improved in the case that K is a product cone such as $\mathbb{R}^n_+ = \prod_{i=1}^n \mathbb{R}_+$, i.e., an orthant. The following example illustrates the difficulty with our present set up.

$$\begin{aligned} x_1'(t) &= -x_1(t) + x_2(t - 1/2), \\ x_2'(t) &= x_1(t-1) - x_2(t). \end{aligned}$$

Observe that (PD) holds for the standard cone. For initial data, take $\phi = (\phi_1, \phi_2) \in \mathcal{C}$ $(r = 1)$ where $\phi_1 = 0$ and $\phi_2(\theta) > 0$ for $\theta \in (-1, -2/3)$ and $\phi_2(\theta) = 0$ elsewhere in $[-1, 0]$. The initial value problem can be readily integrated by the method of steps of length 1/2 and one sees that $x(t) = 0$ for all $t \geqslant -2/3$. In the language of semiflows, $\phi > 0$ yet $\Phi_t(\phi) = \Phi_t(0) = 0$ for all $t \geqslant 0$. The problem is that $C([-1,0], \mathbb{R}^2)$ is not the optimal state space; a better one is the product space $X = C([-1,0],\mathbb{R}) \times C([-1/2, 0], \mathbb{R})$. Obviously, an arbitrary cone in $\mathbb{R}^2$ will not induce a cone in the product space X.

For the remainder of this section we focus on the standard cone but the reader should observe that an analogous construction works for any orthant $K = \{x\colon\ (-1)^{m_i} x_i \geqslant 0\}$. Motivated by the example in the previous paragraph, let $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_+$ be a vector of delays, $R = \max r_i$ and define

$$\mathcal{C}_r = \prod_{i=1}^n C([-r_i, 0], \mathbb{R}).$$

Note that we allow some delays to be zero. We write $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ for a generic point of $\mathcal{C}_r$. $\mathcal{C}_r$ is a Banach space with the norm $|\phi| = \sum |\phi_i|$. Let

$$\mathcal{C}_r^+ = \prod_{i=1}^{n} C([-r_i, 0], \mathbb{R}_+)$$

denote the cone of functions in $\mathcal{C}_r$ with nonnegative components. It has nonempty interior given by those functions with strictly positive components. As usual, we use the notation $\leqslant, <, \ll$ for the corresponding order relations on $\mathcal{C}_r$ induced by $\mathcal{C}_r^+$. If $x_i(t)$ is defined on $[-r_i, \sigma)$, $1 \leqslant i \leqslant n$, $\sigma > 0$ then we may redefine $x_t \in \mathcal{C}_r$ as $x_t = (x_t^1, x_t^2, \ldots, x_t^n)$ where $x_t^i(\theta) = x_i(t + \theta)$ for $\theta \in [-r_i, 0]$. Notice that now, the subscript signifying a particular component will be raised to a superscript when using the subscript "t" to denote a function.

If $D \subset \mathcal{C}_r$ is open, J is an open interval and $f : J \times D \to \mathbb{R}^n$ is given, then the standard existence and uniqueness theory for the initial value problem associated with (4.4) is unchanged. Furthermore, Theorems 4.1 and 4.2, and Corollary 4.3 remain valid in our current setting where, of course, we need only make use of the coordinate maps $\eta(x) = x_i$, $1 \leqslant i \leqslant n$ in (QMD) and (PD). Our goal now is to modify (STD) so that we may obtain a result like Theorem 4.6 that applies to systems such as the example above. We begin by considering the linear system (4.5) where $L : J \to L(\mathcal{C}_r, \mathbb{R}^n)$ is continuous and let $L_i(t)\phi := \langle e_i, L(t)\phi \rangle$, $1 \leqslant i \leqslant n$.

In our setting, $L(t)$ satisfies (PD) if and only if:

$$\phi \geqslant 0 \text{ and } \phi_i(0) = 0 \quad \text{implies} \quad L_i(t)\phi \geqslant 0.$$

THEOREM 4.10. *Let linear system* (4.5) *satisfy* (PD) *and*

(i) $t \in J, r_j > 0, \phi \geqslant 0, \phi_j(-r_j) > 0 \Longrightarrow L_i(t)\phi > 0$ *for some* i;

(ii) *for every proper subset* I *of* $N := \{1, 2, \ldots, n\}$, *there exists* $j \in N \setminus I$ *such that* $L_j(t)\phi > 0$ *whenever* $\phi \geqslant 0$, $\phi_i(s) > 0$, $-r_i \leqslant s \leqslant 0$, $i \in I$.

Then $x(t, \phi, t_0) \gg 0$ *if* $\phi > 0$ *for all* $t \geqslant t_0 + nR$.

PROOF. By (PD) and Corollary 4.3 we have $x(t) \geqslant 0$ for $t \geqslant t_0$. An application of the Riesz Representation Theorem and Radon–Nikodym Theorem implies that for $i = 1, 2, \ldots, n$, we have

$$L_i(t)\phi = a_i(t)\phi_i(0) + \sum_{j=1}^{n} \int_{-r_j}^{0} \phi_j(\theta)\, \mathrm{d}_\theta \eta_{ij}(t, \theta) = a_i(t)\phi_i(0) + \bar{L}_i(t)\phi,$$

where $\eta_{ij}(t)$ is a positive Borel measure on $[-r_j, 0]$, $a_i(t) \in \mathbb{R}$ and $\bar{L}_i(t)\phi \geqslant 0$ whenever $\phi \geqslant 0$. Moreover, $t \to \eta_{ij}(t)$ and $t \to a_i(t)$ are continuous. See Smith [190,194] for details. The representation of L_i in terms of signed measures, $\bar{\eta}_{ij}$, is standard; (PD) implies that $\eta_{ij} := \bar{\eta}_{ij}$ must be positive for $i \neq j$ and that $\bar{\eta}_{ii}$ has the Lebesgue decomposition $\bar{\eta}_{ii} = a_i\delta + \eta_{ii}$ with respect to δ, the Dirac measure of unit mass at zero, and η_{ii} is a positive measure which is mutually singular with respect to δ.

If $x_i(t_1) > 0$ for some i and $t_1 > t_0$ then from $x_i'(t) = a_i(t)x_i(t) + \bar{L}_i(t)x_t \geqslant a_i(t)x_i(t)$, we conclude from standard differential inequality arguments that $x_i(t) > 0$ for $t \geqslant t_1$.

As $\phi > 0$, there exists j such that $\phi_j > 0$. If $r_j = 0$ then $x_j(t_0) > 0$; if $r_j > 0$ then $x_j(t_1 - r_j) > 0$ for some $t_1 \in (t_0, t_0 + r_j)$. In this case, it follows from (i) that $x_i'(t_1) = L_i(t_1)\phi > 0$ for some i and hence $x_i(t_1) > 0$. Hence, $x_i(t) > 0$ for $t \geqslant t_1$ by the previous paragraph. Applying (ii) with $I = \{i\}$ and $t = t_2 = t_1 + r_i$ we may find $k \neq i$ such that $x_k'(t_2) = L_k(t_2)x_{t_2} > 0$ because $x_{t_2}^i(s) > 0$, $-r_i \leqslant s \leqslant 0$. Therefore, we must have $x_k(t_2) > 0$ and hence $x_k(t) > 0$ for $t \geqslant t_2$. Obviously, we may continue in this manner until we have all components positive for $t \geqslant t_0 + nR$ as asserted. □

Theorem 4.10 leads directly to a strong monotonicity result for the nonlinear nonautonomous delay differential equation (4.4) in the usual way. We extend the definition of K-cooperativity of f to our present setup with state space $\mathcal{C}_r$ exactly as before.

THEOREM 4.11. *Let $D \subset \mathcal{C}_r$ be p-convex, $\frac{\partial f}{\partial \phi}(t, \psi)$ exist and be continuous on $J \times D$ to $L(\mathcal{C}_r, \mathbb{R}^n)$, and f be K-cooperative. Suppose that for all $(t, \psi) \in J \times D$, $L(t) := \frac{\partial f}{\partial \phi}(t, \psi)$ satisfies the conditions of Theorem* 4.10. *Then*

$$\phi_0, \phi_1 \in D,\ \phi_0 < \phi_1,\ t \geqslant t_0 + nR \quad \Longrightarrow \quad x(t, t_0, \phi_0) \ll x(t, t_0, \phi_1).$$

The biochemical control circuit with delays, modeled by the system

$$\begin{aligned} x_1'(t) &= g\bigl(x_n(t - r_n)\bigr) - \alpha_1 x_1(t), \\ x_j'(t) &= x_{j-1}(t - r_{j-1}) - \alpha_j x_j(t), \quad 2 \leqslant j \leqslant n \end{aligned} \tag{4.11}$$

with decay rates $\alpha_j > 0$ and delays $r_i \geqslant 0$ with $R > 0$ provides a good application of Theorem 4.11 which cannot be obtained by Theorem 4.8 if the delays are distinct. We assume the $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable and $g' > 0$. Equation (4.11) is an autonomous system for which $\mathcal{C}_r^+$ is positively invariant by Theorem 4.2. See Smith [191, 194] for more on this application.

4.5. *Generic convergence for delay differential equations*

The aim of this section is to apply Theorem 4.8 and Theorem 4.11 to the autonomous delay differential equation (4.2) to conclude that the generic solution converges to equilibrium. To Φ, defined by (4.3), we associate C, S and E, denoting respectively the sets of convergent, stable and equilibrium points. The main result of this section is the following.

THEOREM 4.12. *Let $f \in C^1(D)$,* (4.2) *be cooperative on the p-convex open subset D of $\mathcal{C}$ or $\mathcal{C}_r$ and satisfy:*

(a) *The hypotheses of Theorem* 4.8 *or of Theorem* 4.11 *hold;*

(b) *Every positive semiorbit of Φ has compact closure in D and $D = AC \cup BC$.*

Then

(i) *$C \cap S$ contains a dense open subset of D, consisting of points whose trajectories converge to equilibria;*
(ii) *If E is compact there is a stable equilibrium, and an asymptotically stable equilibrium when E is finite.*

PROOF. For definiteness, suppose that (4.2) is cooperative on the p-convex open subset D of $\mathcal{C}$ and that the hypotheses of Theorem 4.8 hold. The other case is proved similarly. Assumption (a) ensures that Φ is eventually strongly monotone. Moreover, the derivative of $\Phi_t(\phi)$ with respect to ϕ exists and $\Phi'_\tau(\phi)\chi = y_\tau(t_0, \chi)$, where $y(t, t_0, \chi)$ is the solution of the variational equation (4.8). As our hypotheses ensure that $L(t) = \frac{\partial f}{\partial \phi}(x_t(\phi))$ satisfies (STD), we conclude from Theorem 4.6 that $\Phi'_\tau(\phi)$ is strongly positive for $\tau \geqslant 3r$. Compactness of $\Phi'_\tau(\phi): \mathcal{C} \to \mathcal{C}$ for $\tau \geqslant r$ follows from the fact that a bound for $y_\tau(t_0, \chi)$, uniform for χ belonging to a bounded set $B \subset \mathcal{C}$, can be readily obtained so, using (4.8), we may also find a uniform bound for $y'(t, t_0, \chi)$, $\tau - r \leqslant t \leqslant \tau$. See, e.g., Hale [58, Theorem 4.1.1] for more detail.

The hypotheses of Theorem 2.26, with $X = D$, are fulfilled: D is normally ordered and $D = BC \cup AC$; while (M) and (D^*) hold as noted above. Therefore Theorem 2.26 implies the conclusion. $\square$

In the special case that (4.2) is scalar ($n = 1$) we note that the set E of equilibria is totally ordered in $\mathcal{C}_r$ or $\mathcal{C}$ so the set of quasiconvergent points coincides with the set of convergent points: $Q = C$. The classical scalar delay differential equation (4.1) has been thoroughly investigated in the case of monotone delayed feedback ($f(0) = 0$ and $f' > 0$) by Krisztin et al. [105]. They characterize the closure of the unstable manifold of the trivial solution in case it is three-dimensional and determine in remarkable detail the dynamics on this invariant set.

Smith and Thieme [198,200,194] introduce an exponential ordering, not induced by a cone in $\mathbb{R}^n$, that extends the scope of application of the theory described here. One of the salient results from this work is that a scalar delay equation for which the product of the delay r and the Lipschitz constant of f is smaller than e^{-1} generates an eventually strongly monotone semiflow with respect to the exponential ordering and therefore the generic orbit converges to equilibrium: the dynamics mimics that of the associated ordinary differential equation obtained by ignoring the delay. See also work of Pituk [159].

We have considered only bounded delays. Systems of delay differential equations with unbounded and even infinite delay are also of interest. See Wu [234] for extensions to such systems. Wu and Freedman [235] and Krisztin and Wu [106–108] extend the theory to delay differential equations of neutral type.

5. Monotone maps

5.1. *Background and motivating examples*

One of the chief motivations for the study of monotone maps is their importance in the study of periodic solutions to periodic quasimonotone systems of ordinary differential equations. See for example the monograph of Krasnosel'skii [99], the much cited paper of de Mottoni and Schiaffino [42], Hale and Somolinos [60], Smith [188,189], Liang and Jiang [121], and Wang and Jiang [229–231]. To fix ideas, let $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be a locally Lipschitz function and consider the ordinary differential equation

$$x' = f(t, x). \tag{5.1}$$

As usual, denote by $x(t, t_0, x_0)$ the noncontinuable solution of the initial value problem $x(t_0) = x_0$, which for simplicity we assumed is defined for all t. If f is periodic in t of period one: $f(t+1, x) = f(t, x)$ for all (t, x), then it is natural to consider the *period map* $T:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$T(x_0) = x(1, 0, x_0). \tag{5.2}$$

Its fixed points (periodic points) are in one-to-one correspondence with the periodic (subharmonic) solutions of (5.1). If K is a cone in $\mathbb{R}^n$ for which f satisfies the quasimonotone condition (QM), then it follows from Theorem 3.2 that T is a monotone map: $x \leqslant y$ implies $Tx \leqslant Ty$. Moreover, T has the important property, not shared with general monotone maps, that it is an orientation-preserving homeomorphism.

In a similar way, periodic solutions for second order parabolic partial differential equations with time-periodic data can be analyzed by considering period maps in appropriate function spaces. Here monotonicity comes from classical maximum principles. Hess [63] remains an up-to-date survey. See also Alikakos et al. [3] and Zhao [245]. Remarkable results are known for equations on a compact interval with standard boundary conditions. Chen and Matano [23] show that every forward (backward) bounded solution is asymptotic to a periodic solution; Brunovsky et al. [22] extend the result to more general equations. Chen et al. [24] give conditions for the period map to generate Morse–Smale dynamics and thus be structurally stable. Although monotonicity of the period map is an important consideration in these results, it is not the key tool. The fact that the number of zeros on the spatial interval of a solution of the linearized equation is non-increasing in time is far more important. See Hale [59] for a nice survey.

A different theme in order-preserving dynamics originates in the venerable subject of nonlinear elliptic and parabolic boundary value problems. The 1931 edition of Courant and Hilbert's famous book [34] refers to a paper of Bieberbach in *Göttingen Nachrichten*, 1912 dealing with the elliptic boundary value problem $\Delta u = \mathrm{e}^u$ in Ω, $u|\partial\Omega = f$, in a planar region Ω. A solution is found by iterating a monotone map in a function space. Courant and Hilbert extended this method to a broad class of such problems. Out of this technique grew the method of "upper and lower solutions" (or "supersolutions and subsolutions") for solving, both theoretically and numerically, second order elliptic PDEs (see Amann [4], Keller and Cohen [95], Keller [93,94], Sattinger [176]). Krasnosel'skii and Zabreiko [101] trace the use of positivity in functional analysis—closely related to monotone dynamics—to a 1924 paper by Uryson [222] on concave operators. The systematic use of positivity in PDEs was pioneered Krasnosel'skii and Ladyzhenskaya [100] and Krasnosel'skii [98].

Amann [5] showed how a sequence $\{u_n\}$ of approximate solutions to an elliptic problem can be viewed as the trajectory $\{T^n u_0\}$ of u_0 under a certain monotone map T in a suitable function space incorporating the boundary conditions, with fixed points of T being solutions of the elliptic equation. The dynamics of T can therefore be used to investigate the equation. Thus when T is globally asymptotically stable, there is a unique solution; while if T has two asymptotically stable fixed points, in many cases degree theory yields a third fixed point. As Amann [6] emphasized, a few key properties of T—continuity, monotonicity and some form of compactness—allow the theory to be efficiently formulated in terms of monotone maps in ordered Banach spaces.

Many questions in differential equations are framed in terms of eigenvectors of linear and nonlinear operators on Banach spaces. The usefulness of operators that are positive in some sense stems from the theorem of Perron [158] and Frobenius [51], now almost a century old, asserting that for a linear operator on $\mathbb{R}^n$ represented by a matrix with positive entries, the spectral radius is a simple eigenvalue having a positive eigenvector, and all other eigenvalues have smaller absolute value and only nonpositive eigenvectors. In 1912 Jentzsch [85] proved the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel.

In 1935 the topologists Alexandroff and Hopf [2] reproved the Perron–Frobenius theorem by applying Brouwer's fixed-point theorem to the action of a positive $n \times n$ matrix on the space of lines through the origin in $\mathbb{R}^n_+$. This was perhaps the first explicit use of the dynamics of operators on a cone to solve an eigenvalue problem. In 1940 Rutman [173] continued in this vein by reproving Jentzsch's theorem by means of Schauder's fixed-point theorem, also obtaining an infinite-dimensional analog of Perron–Frobenius, known today as the Krein–Rutman theorem [104,214]. In 1957 G. Birkhoff [20] initiated the dynamical use of Hilbert's projective metric for such questions.

The dynamics of cone-preserving operators continues to play an important role in functional analysis; for a survey, see Nussbaum [149,150]. One outgrowth of this work has been a focus on purely dynamical questions about such operators; some of these results are presented below. Polyhedral cones in Euclidean spaces have lead to interesting quantitative results, including *a priori* bounds on the number of periodic orbits. For recent work see Lemmens et al. [117], Nussbaum [151], Krause and Nussbaum [102], and references therein.

Monotone maps frequently arise as mathematical models. For example, the discrete Lotka–Volterra competition model (see May and Oster [136]):

$$\begin{aligned}(u_{n+1}, v_{n+1}) &= T(u_n, v_n)\\ &:= \bigl(u_n \exp\bigl[r(1 - u_n - bv_n)\bigr], v_n \exp\bigl[s(1 - cu_n - v_n)\bigr]\bigr)\end{aligned}$$

generates a monotone dynamical system relative to the fourth-quadrant cone only when the intrinsic rate of increase of each population is not too large ($r, s \leqslant 1$) and then only on the order interval $[0, r^{-1}] \times [0, s^{-1}]$ (Smith [192]). Fortunately in this case, every point in the first quadrant enters and remains in this order interval after one iteration. As is typical in ecological models, the Lotka–Volterra map is neither injective nor orientation-preserving or orientation-reversing. For monotone maps as models for the spread of a gene or an epidemic through a population, see Thieme [218], Selgrade and Ziehe [181], Weinberger [232], Liu [123] and the references therein.

5.2. *Definitions and basic results*

A continuous map $T : X \to X$ on the ordered metric space X is *monotone* if $x \leqslant y \Rightarrow Tx \leqslant Ty$, *strictly monotone* if $x < y \Rightarrow Tx < Ty$, *strongly monotone* if $x < y \Rightarrow Tx \ll Ty$, and *eventually strongly monotone* if whenever $x < y$, there exists $n_0 \geqslant 1$ such that $T^n x \ll T^n y$. We call T *strongly order-preserving* (*SOP*) if T is monotone, and whenever $x < y$ there exist respective neighborhoods U, V of x, y and $n_0 \geqslant 1$ such that $n \geqslant n_0 \Rightarrow T^n U \leqslant T^n V$.[1] As with semiflows, eventual strong monotonicity implies the strong order preserving property.

The orbit of x is $O(x) := \{T^n x\}_{n \geqslant 0}$, and the omega limit set of x is $\omega(x) := \bigcap_{k \geqslant 0} \overline{O(T^k x)}$. If $O(x)$ has compact closure, $\omega(x)$ is nonempty, compact, invariant (that is, $T\omega(x) = \omega(x)$) and *invariantly connected*. The latter means that $\omega(x)$ is not the disjoint union of two closed invariant sets [116].

If $T(x) = x$ then x is a *fixed point* or *equilibrium*. E denotes the set of fixed points. More generally, if $T^k x = x$ for some $k \geqslant 1$ we call x *periodic*, or k-periodic. The minimal such k is called the period of x (and $O(x)$).

Let Y denotes an ordered Banach space with order cone Y_+. A linear operator $A \in L(Y)$ is called *positive* if $A(Y_+) \subset Y_+$ (equivalently, A is a monotone map) and *strongly positive* if $A(Y_+ \setminus \{0\}) \subset \operatorname{Int} Y_+$ (equivalently, A is a strongly monotone map).

The following result is useful for proving smooth maps monotone or strongly monotone:

LEMMA 5.1. *Let $X \subset Y$ be a p-convex set and $f : X \to Y$ a locally C^1 map with quasiderivative $h : U \to L(Y)$ defined on an open set $U \subset Y$. If the linear maps $h(x) \in L(Y)$ are positive* (*respectively, strongly positive*) *for all $x \in U$, then f is monotone* (*respectively, strongly monotone*).

PROOF. By p-convexity it suffices to prove that every $p \in X$ has a neighborhood N such that $f | N \cap X$ is monotone (respectively, strongly monotone). We take N to be an open ball in U centered at p. Suppose $p + z \in X \cap N$, $z > 0$. By p-convexity, $X \cap N$ contains the line segment from p to $p + z$. The definition (above Lemma 2.15) of locally C^1 implies that the map $g : [0, 1] \to Y$, $t \mapsto f(p + tz)$ is C^1 with $g'(t) = h(tz)z$. Therefore

$$f(p + z) - f(z) = g(0) - g(1) = \int_0^1 g'(t)\,\mathrm{d}t = \int_0^1 h(tz)z\,\mathrm{d}t.$$

Because $h(tz) \in L(Y)$ is positive and $z > 0$, we have $h(tz)z \in Y_+$, therefore $f(p + z) - f(p) \geqslant 0$. If the operators $h(tz)$ are strongly positive, $f(p + z) - f(p) \gg 0$. □

PROPOSITION 5.2 (Nonordering of Periodic Orbits). *A periodic orbit of a monotone map is unordered.*

[1] Our use of "strongly order-preserving" conflicts with Dancer and Hess [38], who use these words to mean what we have defined as "strongly monotone". Our usage is consistent with that of several authors. Takáč [208,209] uses "strongly increasing" for our SOP.

PROOF. If not, there exists x in the orbit such that $T^k(x) > x$ for some $k > 0$. Induction on n shows that $T^{nk}(x) > x$ for all $n > 0$. But if x has period $m > 0$, induction on k proves that $T^{mk}(x) = x$. □

LEMMA 5.3 (Monotone Convergence Criterion). *Assume T is monotone and $O(z)$ has compact closure. If $m \geqslant 1$ is such that $T^m z < z$ or $T^m z > z$ then $\omega(z)$ is an m-periodic orbit.*

PROOF. Consider first the case $m = 1$. Compactness of $\overline{O(z)}$ implies the decreasing sequence $\{T^k z\}$ converges to a point $p = \omega(x)$. Now suppose $m > 1$. Applying the case just proved to the map T^m, we conclude that $\{T^{km} z\}$ converges to a point $p = T^m(p)$. It follows that $\omega(z) = \{p, Tp, T^2 p, \ldots, T^{m-1} p\}$. □

Lemma 5.3 yields information on one-sided stability of compact limit sets when T is SOP; see Hirsch [70].

In order to state the following lemma succinctly, we call a set $J \subset \mathbb{N}$ an *interval* if it is nonempty and contains all integers between any two of its members. For $a, b \in \mathbb{N}$ we set $[a, b] = \{j \in \mathbb{N}\colon\ a \leqslant j \leqslant b\}$ (there will be no confusion with real intervals). Two intervals *overlap* if they have more than one point in common.

Let $J \subset \mathbb{N}$ be an interval and $f : J \to X$ be a map. A subinterval $[a, b] \subset J$, $a < b$ is *rising* if $f(a) < f(b)$, and *falling* if $f(b) < f(a)$.

THEOREM 5.4. *A trajectory of a monotone map cannot have both a rising interval and a falling interval.*

PROOF. Follows from Theorem 1.6. □

LEMMA 5.5. *If T is monotone, $\omega(z)$ cannot contain distinct points having respective neighborhoods U, V such that $T^r(U) \leqslant T^r(V)$ for some $r \geqslant 0$.*

PROOF. Follows from Theorem 5.4 (see proof of Lemma 1.7). □

The next result is fundamental to the theory of monotone maps:

THEOREM 5.6 (Nonordering Principle). *Let $\omega(z)$ be an omega limit set for a monotone map T.*

(i) *No points of $\omega(z)$ are related by $\ll$.*

(ii) *If $\omega(z)$ is a periodic orbit or T is SOP, no points of $\omega(z)$ are related by $<$.*

PROOF. Follows from Proposition 5.2 and Lemma 5.5 (see the proof of Theorem 1.8). □

Call x *convergent* if $\omega(x)$ is a fixed point, and *quasiconvergent* if $\omega(x) \subset E$. Just as for semiflows, Proposition 5.6 leads immediately to a convergence criterion:

COROLLARY 5.7. *Assume Φ is SOP.*

(i) *If an omega limit set has a supremum or infimum, it reduces to a single fixed point.*

(ii) *If the fixed point set is totally ordered, every quasiconvergent point with compact orbit closure is convergent.*

PROOF. Part (i) follows from Theorem 5.6(ii), since the supremum or infimum, if it exists, belongs to the limit set. Part (ii) is a consequence (i). . □

5.2.1. *Failure of the limit set dichotomy* We now point out a significant difference between strongly monotone maps and semiflows:

The Limit Set Dichotomy fails for strongly monotone maps.

Recall that for an SOP semiflow with compact orbit closures, the dichotomy (Theorem 1.16) states:

If $a < b$, either $\omega(a) < \omega(b)$ or $\omega(a) = \omega(b) \subset E$.

Takáč [211, Theorem 3.10], gives conditions on strongly monotone maps under which $a < b$ implies that either $\omega(a) \cap \omega(b) = \emptyset$ or $\omega(a) = \omega(b)$. He also gives a counterexample showing that $\omega(a) \cap \omega(b) = \emptyset$ does not imply $\omega(a) < \omega(b)$, nor does $\omega(a) = \omega(b)$ imply that these limit sets consist of fixed points (they are period-two orbits in his example). However, the mapping in his example is defined on a disconnected space.

For any map T in a Banach space, having an asymptotically stable periodic point p of period > 1, the Limit Set Dichotomy as formulated above must fail: take a point $q > p$ so near to p that $O(p) = \omega(p) = \omega(q)$. Clearly $\omega(p)$, being a nontrivial periodic orbit, contains no fixed points. Thus the second assertion of the Limit Set Dichotomy fails in this case.

Dancer and Hess [38] gave a simple example in $\mathbb{R}^k$ for prime k of a strongly monotone map with an asymptotically stable periodic point of period k which we describe below. Therefore the second alternative of the Limit Set Dichotomy can be no stronger than that $\omega(a) = \omega(b)$ is a periodic orbit.

The Limit Set Dichotomy fails even for strictly monotone maps in $\mathbb{R}^2$. Let $f(x) = 2\arctan(x)$, let $a > 0$ be its unique positive fixed point, and note that $0 < f'(a) < 1$. Define $T_0 : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_0(x, y) := (f(y), f(x))$. Then $E = \{(-a, -a), (0, 0), (a, a)\}$ since f has no points of period 2. The fixed points of T_0^2 are the nine points obtained by taking all pairings of $-a, 0, a$. An easy calculation shows that $\{(-a, a), (a, -a)\}$ is an asymptotically stable period-two orbit of T_0 because the Jacobian matrix of T_0^2 is $f'(a)^2$ times the identity matrix. T^0 is strictly monotone but not strongly monotone. Now consider the perturbations $T_\epsilon(x, y) := T_0(x, y) + (\epsilon x, \epsilon y)$. It is easy to see that T_ϵ is strongly monotone for $\epsilon > 0$; and by the implicit function theorem, for small $\epsilon > 0$, T_ϵ has an asymptotically stable period-two orbit $O(p_\epsilon)$ with p_ϵ near $(-a, a)$. As noted in [38], this example can be generalized to $\mathbb{R}^k$ for prime k.

Takáč [212] shows that linearly stable periodic points can arise for the period map associated with monotone systems of ordinary and partial differential equations. Other counterexamples for low-dimensional monotone maps can be found in Smith [192,195].

As we have shown, asymptotically stable periodic orbits that are not singletons can exist for monotone, even strongly monotone maps. Later we will show that the generic orbit of a smooth, dissipative, strongly monotone map converges to a periodic orbit. Here, we show that every attractor contains a stable periodic orbit.

Recall that a point p is *wandering* if there exists a neighborhood U of p and a positive integer n_0 such that $T^n(U) \cap U = \emptyset$ for $n > n_0$. The *nonwandering set* Ω, consisting of all points q that are not wandering, contains all limit sets. In the following, we assume that X is an open subset of the strongly ordered Banach space Y and $T : X \to X$ is monotone with compact orbit closures. The following result is adapted from Hirsch [71, Theorems 4.1, 6.3].

THEOREM 5.8. *If T is strongly monotone and K is a compact attractor, then K contains a stable periodic orbit.*

The proof relies on the following result that does not use strong monotonicity nor that K attracts uniformly:

THEOREM 5.9. *Let $p \in K$ be a maximal (resp., minimal) nonwandering point. Then p is periodic, and every neighborhood of p contains an open set $W \gg p$ (resp., $W \ll p$) such that $\omega(x) = O(p)$ for all $x \in W$.*

PROOF. Suppose K attracts the open neighborhood U of K and fix $y \gg p$, $y \in U$. Since p is nonwandering there exists a convergent sequence $x_i \to p$ and a sequence $n_i \to \infty$ such that $T^{n_i} x_i \to p$. For all large i, $x_i \leqslant y$. Passing to a subsequence, we assume that $T^{n_i} y \to q$. By monotonicity and $x_i \leqslant y$ for large i, we have $q \geqslant p$. But $q \in K \cap \Omega$ and the maximality of p requires $q = p$. Since $p \ll y$ and $T^{n_i} y \to p$ it follows that $T^m y \ll y$ for some m. Lemma 5.3 implies that $\omega(y)$ is an m-periodic orbit containing p. As this holds for every $y \gg p$, the result follows. □

LEMMA 5.10. *Let $p, q \in K$ be fixed points such that $p \ll q$, p is order stable from below, and q is order stable from above. Then $K \cap [p, q]$ contains a stable equilibrium.*

PROOF. Let R be a maximal totally ordered set of fixed points in $K \cap [p, q]$. An argument similar to the one in the proof of Theorem 1.30 shows that the fixed point

$$e := \inf\{z \in E \cap R\colon\ z \text{ is order stable from above}\}$$

is order stable. That e is stable follows from the analog of Proposition 1.28. □

PROOF OF THEOREM 5.8. Theorem 5.9 shows that some iterate T^n, $n \geqslant 0$ has fixed points p, q as in Lemma 5.10, which result therefore implies Theorem 5.8. □

Jiang and Yu [90, Theorem 2] implies that if T is analytic, order compact with strongly positive derivative, then K must contain an asymptotically stable periodic orbit.

5.3. *The order interval trichotomy*

In this section we assume that X is a subset of an ordered Banach space Y with positive cone Y_+, with the induced order and topology. Much of the early work on monotone maps on ordered Banach spaces focused on the existence of fixed points for self maps of order intervals $[a, b]$ such that $a, b \in E$; see especially Amann [6]. The following result of Dancer and Hess [38], quoted without proof, is crucial for analyzing such maps.

Let u, v be fixed points of T. A doubly-infinite sequence $\{x_n\}_{n\in\mathbb{Z}}$ ($\mathbb{Z}$ is the set of all integers) in Y is called an *entire orbit from u to v* if

$$x_{n+1} = T(x_n), \qquad \lim_{n\to-\infty} x_n = u, \ \lim_{n\to\infty} x_n = v.$$

If $x_n \leqslant x_{n+1}$ (respectively, $x_n < x_{n+1}$), the entire orbit is *increasing* (respectively, *strictly increasing*). If $x_n \geqslant x_{n+1}$ (respectively, $x_n > x_{n+1}$), the entire orbit is *decreasing* (respectively, *strictly decreasing*). If the entire orbit $\{x_n\}$ is increasing but not strictly increasing, then $x_n = v$ for all sufficiently large n; and similarly for decreasing.

Consider the following hypothesis:

(G) $X = [a, b]$ where $a, b \in Y$, $a < b$ and $Ta = a$, $Tb = b$. The map $T : X \to X$ is monotone and $T(X)$ has compact closure in X.

THEOREM 5.11 (The Order Interval Trichotomy). *Under hypothesis* (G), *at least one of the following holds*:

(a) *there is a fixed point c such that $a < c < b$*;
(b) *there exists an entire orbit from a to b that is increasing, and strictly increasing if T is strictly monotone*;
(c) *there exists an entire orbit from b to a that is decreasing, and strictly decreasing if T is strictly monotone*.

An extension of Theorem 5.11 to allow additional fixed points on the boundary of $[a, b]$ is carried out in Hsu et al. [83]. Wu et al. [236] weaken the compactness condition. See Hsu et al. [83], Smith [192], and Smith and Thieme [201] for applications to generalized two-species competition dynamics. For related results see Hess [63], Matano [133], Poláčik [162], Smith [184,194].

A fixed point q of T is *stable* if every neighborhood of q contains a positively invariant neighborhood of q. An immediate corollary of the Order Interval Trichotomy is:

COROLLARY 5.12. *Assume hypothesis* (G), *and let a and b be stable fixed points. Then there is a third fixed point in* $[a, b]$.

Corollary 5.14 establishes a third fixed point under different assumptions.

In general, more than one of the alternatives (a), (b), (c) may hold (see [83]). The following complement to the Order Interval Trichotomy gives conditions for exactly one to hold; (iii) is taken from Proposition 2.2 of [83].

Consider the following three conditions:

(a′) there is a fixed point c such that $a < c < b$;
(b′) there exists an entire orbit from a to b;
(c′) there exists an entire orbit from b to a.

PROPOSITION 5.13. *Assume hypothesis* (G).

(i) *If T is strongly order-preserving, exactly one of* (a′), (b′), (c′) *can hold. More precisely: Assume $a < y < b$ and y has compact orbit closure. Then $\omega(y) = \{b\}$ if there is an entire orbit from a to b, while $\omega(y) = \{a\}$ if there is an entire orbit from b to a.*
(ii) *If $a \ll b$, at most one of* (b′), (c′) *can hold.*
(iii) *Suppose $a \ll b$, and $E \cap [a, b] \setminus \{a, b\} \neq \emptyset$ implies $E \cap [[a, b]] \setminus \{a, b\} \neq \emptyset$. Then at most one of* (a′), (b′), (c′) *can hold.*

PROOF. For (i), consider an entire orbit $\{x_n\}$ from a to b. There is a neighborhood U of a such that $T^k U \leqslant T^k y$ for sufficiently large k. Choose $x_j \in U$. Then $T^k x_j \leqslant T^k y \leqslant b$ for all large k. As $\lim_{k\to\infty} T^k x_j = b$ and the order relation is closed, b is the limit of every convergent subsequence of $\{T^k y\}$. The case of an entire orbit from b to a is similar.

In (ii), choose neighborhoods U, V of a, b respectively such that $U \ll V$. Fix j so that $x_j \in U$. If $y \in V$ then an argument similar to the proof of (i) shows that $\omega(y) = \{b\}$. Hence there cannot be an entire orbit from b to a, since it would contain a point of V.

Assume the hypothesis of (iii), and note that (ii) makes (b′) and (c′) incompatible. If (a′), there is a fixed point $c \in [[a, b]]$, and arguments similar to the proof of (ii) show that neither (b′) nor (c′) holds. □

COROLLARY 5.14. *In addition to hypothesis* (G), *assume T is strongly order preserving with precompact image. If some trajectory does not converge, there is a third fixed point.*

PROOF. Follows from the Order Interval Trichotomy 5.11 and Proposition 5.13(i). □

A number of authors have considered the question of whether *a priori* knowledge that every fixed point is stable implies the convergence of every trajectory. See Alikakos et al. [3], Dancer and Hess [38], Matano [133] and Takáč [209] for such results. The following theorem is adapted from [38].

A set $A \subset X$ is a *uniform global attractor* for the map $T : X \to X$ if $T(A) = A$ and $\mathrm{dist}(T^n x, A) \to 0$ uniformly in $x \in X$.

THEOREM 5.15. *Let $a, b \in Y$ with $a < b$. Assume $T : [a, b] \to [a, b]$ is strongly order preserving with precompact image, and every fixed point is stable. Then E is a totally ordered arc J that is a uniform global attractor, and every trajectory converges.*

PROOF. We first show that there exists a totally ordered arc of fixed points; this will not use the SOP property. $O(a)$ is an increasing sequence converging to the smallest fixed point in $[a, b]$. Similarly, $O(b)$ is a decreasing sequence converging to the largest fixed point in $[a, b]$. By renaming a and b as these fixed points, we may as well assume that $a, b \in E$. The stability hypothesis and Corollary 5.12 implies there is a fixed point c satisfying $a < c < b$.

The same reasoning applies to $[a, c]$ and $[c, b]$, and can be repeated indefinitely to show that every maximal totally ordered set of fixed points is compact and connected, hence an arc (Wilder [233, Theorem I.11.23]). Thus by Zorn's Lemma there is a totally ordered arc $J \subset E$ joining a to b.

Next we prove: *Every unordered compact invariant set K is a point of J.* This will not use precompactness of $T([a,b])$. Set $q = \inf\{e \in J\colon\ K \leqslant e\}$. It suffices to prove $q \in K$, for then K, being unordered, reduces to $\{q\}$. If $q \notin K$ then $q > k$. By SOP and invariance of K there is a are neighborhoods V of p and $n > 0$ such that $K = T^n(K) \leqslant T(V)$, hence $K \leqslant T^n(V \cap J) = V \cap J$. This gives the contradiction $K \leqslant \inf(V \cap J) < q$.

Every $\omega(x)$ is compact by the precompactness assumption, and unordered by the Nonordering Principle 5.6(ii). Total ordering of J therefore implies $\omega(x)$ is a point of J. This proves every trajectory converges.

To show that J is a global attractor, let N be the open ϵ-neighborhood of J for an arbitrary $\epsilon > 0$. The stability hypothesis implies N contains a positively invariant open neighborhood W of J. It suffices to prove $T^n(X) \subset W$ when n is sufficiently large. Convergence of all trajectories implies that for every $x \in X$ there exists an open neighborhood $U(x)$ of x and $n(x) > 0$ such that $T^n(x) \in W$ for all $n \geqslant n(x)$. Precompactness of $T(X)$ implies $T(X) \subset \bigcup U(x_i)$ for some finite set $\{x_i\}$. Hence $T^n(X) \subset W$ provided $n > \max\{n(x_i)\}$. □

If the map T in Theorem 5.15 is C^1 and strongly monotone, then E is a smooth totally ordered arc by a result of Takáč [211].

5.3.1. *Existence of fixed points* Dancer [37] obtained remarkable results concerning the dynamics of monotone maps with some compactness properties: Limit sets can always be bracketed between two fixed points, and with additional hypotheses these fixed points can be chosen to be stable. The next two theorems are adapted from [37].

A map $T : Y \to Y$ is *order compact* if it takes each order interval, and hence each order bounded set, into a precompact set.

THEOREM 5.16. *Let X be an order convex subset of Y. Assume that $T : X \to X$ is monotone and order compact, with every orbit having compact closure in X and every omega limit set order bounded. Then for all $z \in Y$ there are fixed points f, g such that $f \leqslant \omega(z) \leqslant g$.*

PROOF. There exists $u \in X$ such that $u \geqslant \omega(z)$ because omega limit sets order bounded. Since $T(\omega(z)) = \omega(z)$, it follows that $\omega(z) \leqslant T^i u$ for all i, hence $\omega(z) \leqslant \omega(u)$. Similarly, there exists $s \in X$ such that $\omega(u) \leqslant \omega(s)$. The set $F := \{x \in Y\colon\ \omega(z) \leqslant x \leqslant \omega(s)\}$ is the intersection of closed order intervals, hence closed and convex, nonempty because it contains $\omega(u)$, and obviously order bounded. Moreover $F \subset X$ because X is order convex. Therefore $T(F)$ is defined and is precompact. Monotonicity of T and invariance of $\omega(z)$ and $\omega(s)$ imply $T(F) \subset F$. It follows from the Schauder fixed point theorem that there is a fixed point $g \in F$, and $g \geqslant \omega(z)$ as required. The existence of f is proved similarly. □

The cone Y_+ is *reproducing* if $Y = Y_+ - Y_+$. This holds for many function spaces whose norms do not involve derivatives. If Y_+ has nonempty interior, it is reproducing: any $x \in Y$

can be expressed as $x = \lambda e - \lambda(e - \lambda^{-1}x) \in Y_+ - Y_+$, where $e \gg 0$ is arbitrary and $\lambda > 0$ is a sufficiently large real number.

THEOREM 5.17. *Let $X \subset Y$ be order convex. Assume $T : X \to X$ is monotone, completely continuous, and order compact. Suppose orbits are bounded and omega limits sets are order bounded.*

(i) *For all $z \in X$ there are fixed points f, g such that $f \leqslant \omega(z) \leqslant g$.*

(ii) *Assume Y_+ is reproducing, $X = Y$ or Y_+, and E is bounded. Then there are fixed points $e_M = \sup E$ and $e_m = \inf E$, and all omega limit sets lie in $[e_m, e_M]$. Moreover, if $x \leqslant e_m$ then $\omega(x) = \{e_m\}$, while if $x \geqslant e_M$ then $\omega(x) = \{e_M\}$.*

(iii) *Assume Y_+ is reproducing, $X = Y$ or Y_+, E is bounded, and T is strongly order preserving. Suppose $z_0 \in Y$ is not convergent. Then there are three fixed points $f < p < g$ such that $f < \omega(z_0) < g$. If T is strongly monotone, f and g can be chosen to be stable.*

PROOF. We prove all assertions except for the stability in (iii). Complete continuity implies that every positively invariant bounded set is precompact. Therefore orbit closures are compact and omega limit sets are compact and nonempty, so (i) follows from Theorem 5.16.

To prove (ii), note that E is compact because it is bounded invariant and closed. Choose a maximal element $e_M \in E$ (Lemma 1.1). We must show that $e_M \geqslant e$ for every $e \in E$. Since the order cone is reproducing, $e_M - e = v - w$ with $v, w \geqslant 0$. Set $u := e + v + w$. Then $u \in X$, $u \geqslant e$, and $u \geqslant e_M$. Monotonicity implies $e_M = T^i e_M \leqslant T^i u$ for all $i \geqslant 0$, hence $e_M \leqslant \omega(u)$. By Theorem 5.16 there exists $g \in E$ such that $\omega(u) \leqslant g$. Hence $e_M \leqslant g$, whence $e_M = g$ by maximality. We now have $e_M \leqslant \omega(u) \leqslant g = e_M$, so $\omega(u) = \{e_M\}$. Monotonicity implies (as above) $e \leqslant \omega(u)$, therefore $e \leqslant e_M$ as required. This proves $e_M = \sup E$, and the dual argument proves $e_m = \inf E$. If $x \leqslant e_m$ then $\omega(x) \leqslant e_m$ by monotonicity; but $\omega(x) \geqslant e_m$ by (i), so $\omega(x) = \{e_m\}$. Similarly for the case $x \geqslant e_M$.

To prove the first assertion of (iii), note that $e_m < \omega(z) < e_M$ by (i) and the Nonordering Principle 5.6(ii). Monotonicity and order compactness of T imply $[e_m, e_M]$ is positively invariant with precompact image. As T is SOP, there is a third fixed point in $[e_m, e_M]$ by Corollary 5.14. □

5.4. *Sublinearity and the cone limit set trichotomy*

Motivated by the problem of establishing the existence of periodic solutions of quasimonotone, periodic differential equations defined on the positive cone in R^n, Krasnosel'skii pioneered the dynamics of sublinear monotone self-mappings of the cone [99]. We will prove Theorem 5.20 below, adapted from the original finite-dimensional version of Krause and Ranft [103].

Let Y denote an ordered Banach space with positive cone Y_+. Denote the interior (possibly empty) of Y_+ by P. A map $T : Y_+ \to Y_+$ is *sublinear* (or "subhomogeneous") if

$$0 < \lambda < 1 \implies \lambda T(x) \leqslant T(\lambda x),$$

and *strongly sublinear* if

$$0 < \lambda < 1,\ x \gg 0 \quad \Longrightarrow \quad \lambda T(x) \ll T(\lambda x).$$

Strong sublinearity is the strong concavity assumption of Krasnosel'skii [99]. It can be verified by using the following result from that monograph:

LEMMA 5.18. *$T : P \to Y$ is strongly sublinear provided T is differentiable and $Tx \gg T'(x)x$ for all $x \gg 0$.*

PROOF. Let $F(s) = s^{-1}T(sx)$ for $s > 0$ and some fixed $x \gg 0$. Then $F'(s) = -s^{-2}T(sx) + s^{-1}T'(sx)x \ll 0$ by our hypothesis. So for $0 < \lambda < 1$, we have

$$\phi\big(Tx - \lambda^{-1}T(\lambda x)\big) = \phi\big(F(1)\big) - \phi\big(F(\lambda)\big) < 0$$

for every nontrivial $\phi \in Y_+^*$, the dual cone in Y^*, because $\frac{\mathrm{d}}{\mathrm{d}s}\phi(F(s)) < 0$. The desired conclusion follows from Proposition 3.1. □

COROLLARY 5.19. *Assume Y is strongly ordered. A continuous map $T : Y_+ \to Y$ is sublinear provided T is differentiable in P and $Tx \geqslant T'(x)x$ for all $x \gg 0$.*

PROOF. By continuity it suffices to prove $T|P$ is sublinear. Fix $e \gg 0$. For each $\delta > 0$ the map $P \to Y$, $x \mapsto Tx + \delta e$ is strongly sublinear by Lemma 5.18. Sending δ to zero implies T is sublinear. □

Krause and Ranft [103] have results establishing sublinearity of some iterate of T, which is an assumption used in Theorem 5.20 below.

The following theorem demonstrates global convergence properties for order compact maps that are monotone and sublinear in a suitably strong sense.

THEOREM 5.20 (Cone Limit Set Trichotomy). *Assume $T : Y_+ \to Y_+$ is continuous and monotone and has the following properties for some $r \geqslant 1$:*
(a) *T^r is strongly sublinear;*
(b) *$T^r x \gg 0$ for all $x > 0$;*
(c) *T^r is order compact.*
Then precisely one of the following holds:
(i) *each nonzero orbit is order unbounded;*
(ii) *each orbit converges to 0, the unique fixed point of T;*
(iii) *each nonzero orbit converges to $q \gg 0$, the unique nonzero fixed point of T.*

A key tool in the proofs of such results is Hilbert's projective metric and the related part metric due to Thompson [219]. We define the part metric $\mathsf{p}(x, y)$ here in a very limited way, as a metric on P (which is the "part"). For $x, y \gg 0$, define

$$\mathsf{p}(x, y) := \inf\{\rho > 0\colon \mathrm{e}^{-\rho}x \ll y \ll \mathrm{e}^{\rho}y\}.$$

The family of open order intervals in P forms a base for the topology of the part metric. It is easy to see that the identity map of P is continuous from the original topology on P to that defined by the part metric.

When $Y \doteq \mathbb{R}^n$ with vector ordering, with $P = \mathrm{Int}(\mathbb{R}^n_+)$, the part metric is isometric to the max metric on $\mathbb{R}^n$, defined by $d_{\max}(x, y) = \max_i |x_i - y_i|$, *via* the homeomorphism $\mathrm{Int}(\mathbb{R}^n_+) \approx \mathbb{R}^n$, $x \mapsto (\log x_1, \ldots, \log x_n)$. Restricted to compact sets in $\mathrm{Int}(\mathbb{R}^n_+)$, the part metric and the max metric are equivalent in the sense that there exist $\alpha, \beta > 0$ such that $\alpha \mathsf{p}(x, y) \leqslant |x - y|_{\max} \leqslant \beta \mathsf{p}(x, y)$.

The usefulness of the part metric in dynamics stems from the following result. Recall map T between metric spaces is a *contraction* if it has a Lipschitz constant < 1, and it is *nonexpansive* if it has Lipschitz constant 1. We say T is *strictly nonexpansive* if $\mathsf{p}(Tx, Ty) < \mathsf{p}(x, y)$ whenever $x \neq y$.

PROPOSITION 5.21. *Let $T : P \to P$ be a continuous, monotone, sublinear map.*
- (i) *T is nonexpansive for the part metric.*
- (ii) *If T is strongly sublinear, T is strictly nonexpansive for the part metric.*
- (iii) *If T is strongly monotone, $A \subset P$, and no two points of A are linearly dependent, then $T|_A$ is strictly nonexpansive for the part metric.*
- (iv) *Under the assumptions of* (ii) *or* (iii), *if $L \subset A$ is compact* (*in the norm topology*) *and $T(L) \subset L$, then the set $L_\infty = \bigcap_{n>0} T^n(L)$ is a singleton.*

PROOF. Fix distinct points $x, y \in A$ and set $e^{\mathsf{p}(x,y)} = \lambda > 1$, so that $\lambda^{-1}x \leqslant y \leqslant \lambda x$ and λ is the smallest number with this property. By sublinearity and monotonicity,

$$\lambda^{-1}Tx \leqslant T(\lambda^{-1}x) \leqslant Ty \leqslant T(\lambda x) \leqslant \lambda Tx \tag{5.3}$$

which implies $\mathsf{p}(Tx, Ty) \leqslant \mathsf{p}(x, y)$.

If T is strongly sublinear, the first and last inequalities in (5.3) can be replaced by $\ll$, which implies $\mathsf{p}(Tx, Ty) < \mathsf{p}(x, y)$.

When x and y are linearly independent, $\lambda^{-1}x < y < \lambda x$. If also T is strongly monotone, (5.3) is strengthened to

$$\lambda^{-1}Tx \leqslant T(\lambda^{-1}x) \ll Ty \ll T(\lambda x) \leqslant \lambda Tx$$

which also implies $\mathsf{p}(Tx, Ty) < \mathsf{p}(x, y)$.

To prove (iv), observe first that if L is compact in the norm metric, it is also compact in the part metric. In both (ii) and (iii) T reduces the diameter in the part metric of every compact subset of L. Since T maps L_∞ onto itself but reduces its part metric diameter, (iv) follows. □

PROOF OF THE CONE LIMIT SET TRICHOTOMY 5.20. We first work under the assumption that $r = 1$. In this case Proposition 5.21 shows that every compact invariant set in P reduces to a fixed point, and there is at most one fixed point in P. It suffices to consider the orbits of points $x \in P$, by (b).

Suppose there is a fixed point $q \gg 0$. There exist numbers $0 < \lambda < 1 < \mu$ such that $x \in [\lambda q, \mu q] \subset P$. For all n we have

$$0 \ll \lambda q = \lambda T^n q \leqslant T^n(\lambda q) \leqslant T^n x \leqslant T^n(\mu q) \leqslant \mu T^n q = \mu q.$$

Hence $O(x) \subset [\lambda q, \mu q]$, so $O(Tx)$ lies in $T([\lambda p, \mu q])$, which is precompact by (c). Therefore $\omega(x)$ is a compact unordered invariant set in P. Proposition 5.21(iii) implies that $\omega(x) = \{q\}$. This verifies (iii).

Case I: If some orbit $O(y)$ is order unbounded, we prove (i). We may assume $y \gg 0$. There exists $0 < \gamma < 1$ such that $\gamma y \ll x$. Then $\gamma T^n y \leqslant T^n(\gamma y) \leqslant T^n x$, implying $O(x)$ is unbounded.

Case II: If $0 \in \omega(y)$ for some y, we prove (ii). We may assume $y \gg 0$. Fix $\mu > 1$ with $x \ll \mu y$. Then $0 \leqslant T^n x \leqslant T^n(\mu y) \leqslant \mu T^n y \to 0$. Therefore $\overline{O(x)}$ is compact and $T^n x \to 0$.

Case III: If the orbit closure $\overline{O(x)} \subset [a, b] \subset P$, then (iii) holds. For $\overline{O(x)}$ is compact by (c), so $\omega(x)$ is a nonempty compact invariant set. Because $\omega(x) \subset \overline{O(x)} \subset P$, Case I implies (iii).

Cases I, II and III cover all possibilities, so the proof for $r = 1$ is complete. Now assume $r > 1$. One of the statements (i), (ii), (iii) is valid for T^r in place of T. If (i) holds for T^r, it obviously holds for T. Assume (ii) holds for T^r. If $x > 0$ then $\omega(x) = \{0, T(0), \ldots, T^{r-1}(0)\}$. As this set is compact and T^r invariant, it reduces to $\{0\}$, verifying (ii) for T. A similar argument shows that if (iii) holds for T^r, it also holds for T. □

The conclusion of the Cone Limit Trichotomy can fail for strongly monotone sublinear maps—simple linear examples in the plane have a line of fixed points. But the following holds:

THEOREM 5.22. *Assume*:

(a) *$T : Y_+ \to Y_+$ is continuous, sublinear, strongly monotone, and order compact*;

(b) *for each $x > 0$ there exists $r \in \mathbb{N}$ such that $T^r x \gg 0$.*

Then:

(i) *either $O(x)$ is not order bounded for all $x > 0$, or $O(x)$ converges to a fixed point for all $x \geqslant 0$;*

(ii) *the set of fixed points > 0 has the form $\{\lambda e\colon a \leqslant \lambda \leqslant b\}$ where $e \gg 0$ and $0 \leqslant a \leqslant b \leqslant \infty$.*

PROOF. Let $y > 0$ be arbitrary. If $O(y)$ is not order bounded, or $0 \in \omega(y)$, the proof of (i) follows Cases I and II in the proof of the Cone Limit Set Trichotomy 5.20. If $\overline{O(x)} \subset [a, b] \subset P$, then $\omega(y)$ is a compact invariant set in P, as in Case III of 5.20. As $\omega(y)$ is unordered, every pair of its elements are linearly independent. Therefore Proposition 5.21(iv) implies $\omega(y)$ reduces to a fixed point, proving (i). The same reference shows that all fixed points lie on a ray $R \subset Y_+$ through the origin, which must pass through some

$e \gg 0$ by (b). Suppose p, q are distinct fixed points and $0 \ll p \ll x << \ll q$. There exist unique numbers $0 < \mu < 1 < \nu$ such that $x = \mu p = \mu q$. Then

$$Tx \geqslant \mu Tp = \mu p = x, \qquad Tx \leqslant \nu Tq = \nu q = x$$

proving $Tx = x$. This implies (ii). □

Papers related to sublinear dynamics and the part metric include Dafermos and Slemrod [35], Krause and Ranft [103], Krause and Nussbaum [102], Nussbaum [149,150], Smith [183], and Takáč [208,215]. For interesting applications of sublinear dynamics to higher order elliptic equations, see Fleckinger-Pellé and Takáč [45,46].

5.5. *Smooth strongly monotone maps*

Smoothness together with compactness allows one to settle questions of stability of fixed points and periodic points by examining the spectrum of the linearization of the mapping. Let $T : X \to X$ where X is an open subset of the ordered Banach space Y with cone Y_+ having nonempty interior in Y. Assume that T is a completely continuous, C^1 mapping with a strongly positive derivative at each point. Then T is strongly monotone by Lemma 5.1 and $T'(x)$ is a Krein–Rutman operator so the Krein–Rutman Theorem 2.17 holds for $T'(p)$, $p \in E$. Let ρ be the spectral radius of $T'(p)$, which the reader will recall is a simple eigenvalue which dominates all others in modulus and for which the generalized eigenspace is spanned by an eigenvector $v \gg 0$. Let V_1 be the span of v in Y. There is a complementing closed subspace V_2 such that $Y = V_1 \oplus V_2$ satisfying $T'(p)V_2 \subset V_2$ and $V_2 \cap Y_+ = \{0\}$. Let P denote the projection of Y onto V_2 along v. Finally, let τ denote the spectral radius of $T'(p)|V_2 : V_2 \to V_2$, which obviously satisfies $\tau < \rho$. Mierczyński [139] exploits this structure of the linearized mapping to obtain very detailed behavior of the orbits of points near p. In order to describe his results, define $K := \{x \in X\colon\ T^n x \to p\}$ to be the basin of attraction of p. Let $M_- := \{x \in X\colon\ T^{n+1} \ll T^n x, n \geqslant n_0,\ \text{some}\ n_0\}$ be the set of eventually decreasing orbits, $M_+ := \{x \in X\colon\ T^n x \ll T^{n+1}x, n \geqslant n_0,\ \text{some}\ n_0\}$ be the set of eventually increasing orbits, and $M := M_- \cup M_+$ be the set of eventually monotone (in the strong sense) orbits.

The following result is standard but nonetheless important.

THEOREM 5.23 (Principle of Linearized Stability). *If $\rho < 1$, there is a neighborhood U of p such that $T(U) \subset U$ and constants $c > 0$, $\kappa \in (\rho, 1)$ such that for each $x \in U$ and all n*

$$\|T^n x - p\| \leqslant c\kappa^n \|x - p\|.$$

In the more delicate case that $\rho \leqslant 1$, Mierczyński [139] obtains a smooth hypersurface C, which is an analog for T of the codimension-one linear subspace V_2 invariant under the linearized mapping $T'(p)$:

THEOREM 5.24. *If $\rho \leqslant 1$ there exists a codimension-one embedded invariant manifold $C \subset X$ of class C^1 having the following properties*:

(i) *$C = \{x + Pw + R(w)v\colon w \in O\}$ where $R : O \to \mathbb{R}$ is a C^1 map defined on the relatively open subset O of V_2 containing 0, satisfying $R(0) = R'(0) = 0$. In particular, C is tangent to V_2 at p.*

(ii) *C is unordered.*

(iii) *$C = \{p \in X\colon \|T^n x - p\|/\kappa^n \to 0\} = \{x \in X\colon \|T^n x - p\|/\kappa^n$ is bounded$\}$, for any κ, $\tau < \kappa < \rho$. In particular, $C \subset K$.*

(iv) *$K \setminus C = \{x \in K\colon \|T^n x - p\|/\kappa^n \to \infty\} = \{x \in K\colon \|T^n x - p\|/\kappa^n$ is unbounded$\}$, for any κ, $\tau < \kappa < \rho$.*

(v) *$K \setminus C = K \cap M$.*

Conclusion (v) implies most orbits converging to p do so monotonically, but more can be said. Indeed, $K \cap M_+ = \{x \in K\colon (T^n x - p)/\|T^n x - p\| \to -v\}$ and a similar result for $K \cap M_-$ with v replacing $-v$ holds. The manifold C is a local version of the unordered invariant hypersurfaces obtained by Takáč in [209].

Corresponding to the space V_1 spanned by $v \gg 0$ for $T'(p)$, a locally forward invariant, one dimensional complement to the codimension one manifold C is given in the following result.

THEOREM 5.25. *There is $\epsilon > 0$ and a one-dimensional locally forward invariant C^1 manifold $W \subset B(p; \epsilon)$, tangent to V at p. If $\rho > 1$, then W is locally unique, and for each $x \in W$ there is a sequence $\{x_{-n}\} \subset W$ with $Tx_{-n} = x_{-n+1}$, $x_0 = x$, and $\kappa^n \|x_{-n} - p\| \to 0$ for any κ, $1 < \kappa < \rho$.*

Here $B(p; \epsilon)$ is the open ϵ-ball centered at p. Local forward invariance of W means that $x \in W$ and $Tx \in B(x; \epsilon)$ implies $Tx \in W$. Related results are obtained by Smith [184]. In summary, the above results assert that the dynamical behavior of the nonlinear map T behaves near p like that of its linearization $T'(p)$. Obviously, the above results can be applied at a periodic point p of period k by considering the map T^k which has all the required properties.

Mierczyński [139] uses the results above to classify the convergent orbits of T. Similar results are obtained by Takáč in [210].

It is instructive to consider the sort of stable bifurcations that can occur from a linearly stable fixed point, or a linearly stable periodic point, for a one parameter family of mappings satisfying the hypotheses of the previous results, as the parameter passes through a critical value at which $\rho = 1$. The fact that there is a simple positive dominant eigenvalue of $(T^k)'(p)$ ensures that period-doubling bifurcations from a stable fixed point or from a stable periodic point, as a consequence of a real eigenvalue passing through -1, cannot occur. In a similar way, a Neimark–Sacker [113] bifurcation to an invariant closed curve cannot occur from a stable fixed or periodic point. These sorts of bifurcations can occur from unstable fixed or periodic points but then they will "be born unstable."

The generic orbit of a smooth strongly order preserving semiflow converges to fixed point but such a result fails to hold for discrete semigroups, i.e., for strongly order preserving mappings. Indeed, such mappings can have attracting periodic orbits of period exceeding one as we have seen. However, Tereščák [217], improving earlier joint work with Poláčik [164,165], and [65], has obtained the strongest result possible for strongly monotone, smooth, dissipative mappings.

THEOREM 5.26 (Tereščák, 1996). *Let $T: Y \to Y$ be a completely continuous, C^1, point dissipative map whose derivative is strongly positive at every point of the ordered Banach space Y having cone Y_+ with nonempty interior. Then there is a positive integer m and an open dense set $U \subset Y$ such that the omega limit set of every point of U is a periodic orbit with period at most m.*

The map P is *point dissipative* (see Hale [58]) provided there is a bounded set B with the property that for every $x \in X$, there is a positive integer $n_0 = n_0(x)$ such that $P^n x \in B$ for all $n \geqslant n_0$. We note that the hypothesis that $T'(x)$ is strongly positive implies that T is strongly monotone by Lemma 5.1.

5.6. *Monotone planar maps*

A remarkable convergence result for planar monotone maps was first obtained by de Mottoni and Schiaffino [42]. They focused on the period-map for the two-species, Lotka–Volterra competition system of ordinary differential equations with periodic coefficients. The full generality of their arguments was recognized and improved upon by Hale and Somolinos [60] and Smith [188,189,192]. We follow the treatment Smith in [192].

In addition to the usual order relations on $\mathbb{R}^2$, $\leqslant, <, \ll$, generated by $\mathbb{R}^2_+$, we have the "southeast ordering" ($\leqslant_K$), generated by the fourth quadrant $K = \{(u, v)\colon\ u \geqslant 0, v \leqslant 0\}$. The map T is *cooperative* if it is monotone relative to $\leqslant$ and *competitive* if it is monotone relative to K.

Throughout this subsection, we assume that $T: A \to A$ is a continuous competitive map on the subset A of the plane. Further hypotheses concerning A will be made below. As noted above, all of the results have obvious analogs in the case of cooperative planar maps (just interchange cones). Competitive planar maps preserve the order relation $\leqslant_K$ by definition, but they also put constraints on the usual ordering, as we show below.

LEMMA 5.27. *Let $T: A \to A$ be a competitive map on $A \subset \mathbb{R}^2$. If $x, y \in A$ satisfy $Tx \ll Ty$, then either $x \ll y$ or $y \ll x$.*

PROOF. If neither $x \ll y$ nor $y \ll x$ hold, then $x \leqslant_K y$ or $y \leqslant_K x$ holds. But $x \leqslant_K y$ implies $Tx \leqslant_K Ty$ which is incompatible with $Tx \ll Ty$. A similar contradiction is obtained from $y \leqslant_K x$. □

Lemma 5.27 suggests placing one of the following additional assumptions on T.

(O_+) If $x, y \in A$ and $Tx \ll Ty$, then $x \leqslant y$.
(O_-) If $x, y \in A$ and $Tx \ll Ty$, then $y \leqslant x$.

As we shall soon see, if T is orientation preserving, then (O_+) holds and if it is orientation reversing, then (O_-) holds. A sequence $\{x_n = (u_n, v_n)\} \subset \mathbb{R}^2$ is *eventually componentwise monotone* if there exists a positive integer N such that either $u_n \leqslant u_{n+1}$ for all $n \geqslant N$ or $u_{n+1} \leqslant u_n$ for all $n \geqslant N$ and similarly for v_n.

In the case of orientation-preserving maps, the following result was first proved by de Mottoni and Schiaffino [42] for the period map of a periodic competitive Lotka–Volterra system of differential equations.

THEOREM 5.28. *If T is a competitive map for which (O_+) holds then for all $x \in A$, $\{T^n x\}_{n\geqslant 0}$ is eventually component-wise monotone. If the orbit of x has compact closure in A, then it converges to a fixed point of T. If, instead, (O_-) holds then for all $x \in A$, $\{T^{2n} x\}_{n\geqslant 0}$ is eventually component-wise monotone. If the orbit of x has compact closure in A, then its omega limit set is either a period-two orbit or a fixed point.*

PROOF. We first note that if T is competitive and (O_-) holds then T^2 is competitive and (O_+) holds (use Lemma 5.27) so the second conclusion of the theorem follows from the first.

Suppose that (O_+) holds. If $T^n x \leqslant_K T^{n+1} x$ or $T^{n+1} x \leqslant_K T^n x$ holds for some $n \geqslant 1$, then it holds for all larger n so the conclusion is obvious. Therefore, we assume that this is not the case. It follows that for each $n \geqslant 1$ either (a) $T^n x \ll T^{n+1} x$ or (b) $T^{n+1} x \ll T^n x$. We claim that either (a) holds for all n or (b) holds for all n. Assume $x \ll Tx$ (the argument is similar in the other case). If the claim is false, then there is an $n \geqslant 1$ such that $x \ll Tx \ll \cdots \ll T^{n-1} x \ll T^n x$ but $T^{n+1} x \ll T^n x$. But (O_+) implies $T^n x \leqslant T^{n-1} x$ contradicting the displayed inequality. □

Orbits may not converge to a fixed point if (O_-) holds. Consider the map $T : I \to I$ where $I = [-1, 1]^2$ and $T(u, v) = (-v, -u)$ reflects points through the line $v = -u$. It is easy to see using Lemma 5.1 that T is competitive and that (O_-) holds (see below). Fixed points of T lie on the above-mentioned line but all other points in I are period-two points.

The hypotheses (O_+) and (O_-) on T are global in nature and therefore can be difficult to check in specific examples. We now give sufficient conditions for them to hold that may be easier to verify in applications. *A contains order intervals* if $x, y \in A$ and $x \ll y$ implies that $[x, y] \subset A$. Clearly, $A = [a, b]$ contains order intervals. If $A \subset \mathbb{R}^2$ and $T : A \to \mathbb{R}^2$, we say that T is C^1 if for each $a \in A$ there is an open set U in $\mathbb{R}^2$ and a continuously differentiable function $F : U \to \mathbb{R}^2$ that coincides with T on $U \cap A$. We will have occasion to make certain hypotheses concerning $T'(x)$ even though it is not necessarily uniquely defined. What we mean by this is that there exists an F as above such that $T'(x) = DF(x)$ has the desired properties. This abuse of language will lead to no logical difficulties in the arguments below. In the applications, A will typically be $\mathbb{R}^2_+$ or some order interval $[a, b]$ where $a \ll b$ in which case T' is uniquely defined.

Consider the following hypothesis:

(H_+) (a) A contains order intervals and is p-convex with respect to $\leqslant_K$.
(b) $\det T'(x) > 0$ for $x \in A$.

(c) $T'(x)(K) \subset K$ for $x \in A$.
(d) T is injective.

Hypothesis (H$_-$) is identical except the inequality is reversed in (b).

LEMMA 5.29. *If $T : A \to A$ satisfies* (H$_+$), *then T is competitive and* (O$_+$) *holds. If* (H$_-$) *holds, then T is competitive and* (O$_-$) *holds.*

PROOF. T is competitive by hypothesis (c) since A is p-convex with respect to $\leqslant_K$. Assuming that (H_+) holds, $x, y, Tx, Ty \in A$ and $Tx \ll Ty$, we will show that $x \ll y$. According to Lemma 5.27, the only alternative to $x \ll y$ is $y \ll x$ so we assume the latter for contradiction. Let a and b be the northwest and southeast corners of the rectangle $[y, x] \subset A$ so that $a \ll_K b$ and $[y, x] = [a, b]_K$. Since T is competitive on A, $T([y, x]) \subset [Ta, Tb]_K$ and $Tx \ll Ty$ implies that $Ta \ll_K Tb$. Consider the oriented Jordan curve forming the boundary of $[y, x]$ starting at a and going horizontally to x, then going vertically down to b, horizontally back to y and vertically up to a. As T is injective on A, the image of this curve is an oriented Jordan curve. Monotonicity of T implies that the image curve is contained in $[Ta, Tb]_K$, begins at Ta and moves monotonically with respect to $\leqslant_K$ (southwest) through Tx and then monotonically to Tb before moving monotonically (decreasing or northwest) from Tb through Ty and on to Ta. (H$_+$)(b) implies that T is locally orientation preserving, so upon traversing the first half of the image curve from Ta to Ty to Tb, the curve must make a "right turn" at Tb before continuing on to Tx and to Ta. As the image curve cannot intersect itself, we see that $Tx \ll Ty$ cannot hold, a contradiction. □

In specific examples it is often difficult to check that T is injective. It automatically holds if A is compact and connected and there exists $z \in T(A)$ such that the set $T^{-1}(z)$ is a single point. This is because the cardinality of $T^{-1}(w)$ is finite and constant for $w \in T(A)$ by Chow and Hale [26, Lemma 2.3.4].

The following is an immediate corollary of Theorem 5.28 and Lemma 5.29.

COROLLARY 5.30. *If $T : A \to A$ satisfies* (H$_+$), *then $\{T^n x\}$ is eventually component-wise monotone for every $x \in A$. In this case, if an orbit has compact closure in A, then it converges to a fixed point of T. If T satisfies* (H$_-$), *then $\{T^{2n} x\}$ is eventually component-wise monotone for every $x \in A$. In this case, if an orbit has compact closure in A, then its omega limit set is either a fixed point or a period-two orbit.*

As an application of Corollary 5.30, we recall the celebrated results of de Mottoni and Schiaffino [42] for the periodic Lotka–Volterra system

$$\begin{aligned} x' &= x\big[r(t) - a(t)x - b(t)y\big], \\ y' &= y\big[s(t) - c(t)x - b(t)y\big], \end{aligned} \tag{5.4}$$

where r, s, a, b, c, d are periodic of period one and $a, b, c, d \geqslant 0$. The period map $T : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, defined by (5.2) for (5.4), is strictly monotone relative to the fourth quadrant cone K by virtue of Theorem 3.5. Indeed, (5.4) is a competitive system relative to the cone $\mathbb{R}^2_+$ (the off-diagonal entries of the Jacobian $J = J(t, x, y)$ of the right-hand side are nonpositive), and every such system is monotone relative to K. Observe that $J + \alpha I$, for large enough $\alpha > 0$, has nonnegative diagonal entries so $(J + \alpha I)(u, v)^T \in K$ if $(u, v)^T \in K$ (i.e., $u \geqslant 0, v \leqslant 0$). T is strongly monotone relative to K in $\operatorname{Int} \mathbb{R}^2_+$ if $b, c > 0$ by Corollary 3.11. Because T is injective and orientation preserving by Liouville's theorem, (H$_+$) holds. Orbits are seen to be bounded by simple differential inequality arguments, e.g., applied to $x' \leqslant x[r(t) - a(t)x]$. Consequently, by Corollary 5.30, all orbits $O(T)$ converge to a fixed point; equivalently, every solution of (5.4) is asymptotic to a period-one solution.

System (5.4) is most interesting when each species can survive in the absence of its competitor, i.e. the time average of r and s are positive. In that case, aside from the trivial fixed point $E_0 := (0, 0)$, there are unique fixed points of type $E_1 := (e, 0)$ and $E_2 := (0, f)$. Of course, $e, f > 0$ give initial data corresponding to the unique nontrivial one-periodic solutions of the scalar equations: $x' = x[r(t) - a(t)x]$ and $y' = y[s(t) - d(t)y]$. The dynamics of the period map for these equations is described by alternative (iii) of Theorem 5.20.

It is shown by de Mottoni and Schiaffino that there is a monotone, relative to K, T-invariant curve joining E_1 to E_2 which is the global attractor for the dynamics of T in $\mathbb{R}^2_+ \setminus \{E_0\}$. This work has inspired a very large amount of work on competitive dynamics. See Hale and Somolinos [60], Smith [188,189], Hess and Lazer [64], Hsu et al. [83], Smith and Thieme [201], Wang and Jiang [230,231,229], Liang and Jiang [121], Zanolin [238].

6. Semilinear parabolic equations

The purpose of this section is to analyze the monotone dynamics in a broad class of second order, semilinear parabolic equations.

For basic theory and further information on many topics we refer the reader to books of Amann [11], Henry [62] Cholewa and Dlotko [25], Hess [63], Lunardi [124] and Martin [125], the papers of Amann [7–10], and the survey article of Poláčik [163].

Solution processes for semilinear parabolic problems have been obtained by many authors; see for example [3,8,38,62,63,124,125,134,147,163,194,174,208,246]. We briefly outline the general procedure, due to Henry, with important improvements by Mora and Lunardi.

To balance the sometimes conflicting goals of order, topology and dynamics, the domain of a solution process must be chosen carefully. We rely on results of Mora [147], refined by Lunardi [124], for solution processes in Banach subspaces $C^k_B(\overline{\Omega}) \subset C^k(\overline{\Omega})$, $k = 0, 1$ determined by the boundary operator B.

6.1. *Solution processes for abstract ODEs*

If Y and X are spaces such that Y is a subset of X and the inclusion map $Y \to X$ is continuous, we write $Y \hookrightarrow X$. When Y and X are ordered Banach space structures, this notation tacitly states that Y is a linear subspace of X and $Y_+ = Y \cap X_+$.

The domain and range of any map h are denoted by $\mathsf{D}(h)$ and $\mathsf{R}(h)$.

6.1.1. *Processes* Let Z be a topological space and set $\widehat{Z} := \{(t, t_0, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z\colon t \geqslant t_0\}$. A *process* in Z is a family $\Theta = \{\Theta_{t,t_0}\}_{0 \leqslant t_0 < t}$ of continuous maps

$$\Theta_{t,t_0} : D_{t,t_0} \to Z, \quad D_{t,t_0} \text{ open in } Z,$$

where the set $\{(t, t_0, z) \in \widehat{Z}\colon z \in D(t, t_0)\}$ is open in $\widehat{Z}$ containing $\{(t, t, z)\colon t \geqslant 0, z \in Z\}$, with the properties:

- the map $(t, t_0, z) \mapsto \Theta_{t,t_0}(z)$ is continuous from $\widehat{Z}$ to Z.
- the *cocycle identities* hold:

$$t \geqslant t_1 \geqslant t_0 \implies \Theta_{t,t_1} \circ \Theta_{t_1,t_0} = \Theta_{t,t_0}, \quad \Theta_{t,t} = \text{identity map of } Z.$$

Equivalently: there is a local semiflow Λ on $\mathbb{R}_+ \times Z$ such that $\Lambda_t(t_0, u_0) = (t + t_0, \Theta_{t,t_0}(u_0))$. It follows that for each (t_0, z) there is a maximal $\tau := \tau(t_0, z) \in (t_0, \infty]$ such that $z \in D_{t,t_0}$ for all $t \in [t_0, \tau)$. The *trajectory* of (t_0, z) is the parametrized curve $[t_0, \tau) \to Z$, $t \mapsto \Theta_{t,t_0}(z)$, whose image is the *orbit* of (t_0, z). A subset $S \subset Z$ is *positively invariant* if it contains the orbit of every point in $\mathbb{R}_+ \times S$.

A trajectory is *global* if it is defined on $[t_0, \infty)$. The process is called global when all trajectories are global.

Let S be a space such that $S \hookrightarrow Z$. It may be that S is positively invariant under the process Θ, and the maps $\Theta_{t,t_0} : S \cap D(t, t_0) \to S$ are continuous respecting the topology on S and furthermore, the map $(t, t_0, s) \to \Theta_{t,t_0}s$ is continuous from $\widehat{S}$ to S. In this case these maps form the *induced process* Θ^S in S.

A process Θ in an ordered space is called (locally) monotone, SOP, Lipschitz, compact, and so forth, provided every map Θ_{t,t_0}, $t > t_0$ has the corresponding property.

6.1.2. *Solution processes* Let X be a Banach space. $\mathcal{A}$ denotes a linear operator (usually unbounded) in X with domain $\mathsf{D}(\mathcal{A}) \subset X$, that is *sectorial* in the following strong sense:

- $\mathcal{A}$ is a densely defined, closed operator generating an analytic semigroup $\{e^{t\mathcal{A}}\}_{t \geqslant 0}$ in $L(X)$, and the resolvent operators $(\lambda I - \mathcal{A})^{-1} \in L(X)$ are compact for sufficiently large $\lambda \geqslant 0$.

The latter property ensures that $e^{t\mathcal{A}}$ is compact for $t > 0$ [156, Theorem 2.3.3].

We make $\mathsf{D}(\mathcal{A})$ into a Banach space with the graph norm $\|x\|_{\mathsf{D}(\mathcal{A})} = \|x\| + \|\mathcal{A}x\|$, or any equivalent norm. Then $\mathcal{A} : \mathsf{D}(\mathcal{A}) \to X$ is bounded, and $\mathsf{D}(\mathcal{A}) \hookrightarrow X$.

For $0 \leqslant \alpha \leqslant 1$ we define the fractional power domain of $\mathcal{A}^\alpha$ to be $X^\alpha = X^\alpha(\mathcal{A}) := \mathsf{D}(\mathcal{A}^\alpha)$. Thus we have [62]

$$\mathsf{D}(\mathcal{A}) \hookrightarrow X^\alpha \hookrightarrow X, \qquad \overline{\mathsf{D}(\mathcal{A})} = X.$$

Let $F : [0, \infty) \times X^\alpha \to X$ be a continuous map that is *locally Lipschitz in the second variable*, i.e.:

- $F|[0, \tau] \times B(r)$ has Lipschitz constant $L(\tau, r)$ in the second variable whenever $[0, \tau] \subset [0, \infty)$ and $B(r)$ is the closed ball of radius r in X^α.

Locally Hölder in the first variable is defined analogously. We say F is C^1 *in the second variable* if $\partial_w F(t, w)$ is continuous.

The data $(X, \mathcal{A}, F)$ determine the abstract initial value problem

$$\begin{cases} u'(t) = \mathcal{A}u(t) + F(t, u(t)) & (t > t_0), \\ u(t_0) = u_0 \in X. \end{cases} \tag{6.1}$$

A continuous curve $u : [t_0, \tau) \to X$, $t_0 < \tau \leqslant \infty$ is a (classical) *solution through* (t_0, u_0) if $u(t) \in \mathrm{D}(\mathcal{A})$ for $t_0 < t < \tau$ and (6.1) holds. It is well known (e.g., Lunardi [124, 4.1.2]) that every solution is also a *mild solution*, i.e., it satisfies the integral equation

$$u(t) = \mathrm{e}^{(t-t_0)\mathcal{A}}u_0 + \int_{t_0}^{t} \mathrm{e}^{(t-s)\mathcal{A}}F\big(s, u(s)\big)\,\mathrm{d}s \quad (t_0 \leqslant t < \tau). \tag{6.2}$$

Moreover, every mild solution is a solution provided F is locally Hölder in t (Lunardi [124, Proposition 7.1.3]).

A classical or mild solution is *maximal* if it does not extend to a classical or mild solution on a larger interval in $[t_0, \infty)$; it is then referred to as a *trajectory* at (t_0, u_0), and its image is an *orbit*. When such a trajectory is unique it is denoted by $t \mapsto u(t, t_0, u_0)$. In this case the *escape time* of (t_0, u_0) is $\tau(t_0, u_0) := \tau$. If $\tau = \infty$ the trajectory is called *global*.

The following basic result means that Eq. (6.1) is well-posed in a strong sense, and that solutions enjoy considerable uniformity and compactness.

THEOREM 6.1. *Let $(t_0, u_0) \in \mathbb{R}_+ \times X^\alpha$. There is a unique mild trajectory at (t_0, u_0), and it is a classical trajectory provided $F(t, u)$ is locally Hölder in t. If $t_0 < t_1 < \tau(t_0, u_0)$, there is a neighborhood U of x_0 in X^α and $M > 0$ such that*

$$\|u(t, t_0, u_1) - u(t, t_0, u_2)\|_{X^\alpha} \leqslant M\|u_1 - u_2\|_{X^\alpha}, \quad u_1, u_2 \in U.$$

There exist $C > 0$, $t_0 < t_1 < \tau(t_0, u_0)$, a bounded neighborhood N of u_0 in X and a continuous map

$$\Psi : [t_0, t_1] \times N \to X, \qquad (t, v) \mapsto u(t, t_0, v),$$

where $u(t, t_0, v)$ is a mild solution, such that the following hold. If $s, t \in (t_0, t_1]$, $0 \leqslant \alpha < 1$ and $v, w \in N$:

(i) $\|\Psi(s, v) - \Psi(s, w)\| \leqslant C\|v - w\|$;
(ii) $\|\Psi(s, v) - \Psi(s, w)\|_{X^\alpha} \leqslant (s - t_0)^{-\alpha}C\|v - w\|$;
(iii) *$\Psi([s, t_1] \times N)$ is precompact in X^α*;
(iv) *$u(\cdot, t_0, v) : (t_0, t] \to X^\alpha$ and $u(\cdot, t_0, v) : [t_0, t] \to X$ are continuous*;
(v) *trajectories bounded in X^α are global.*

PROOF. Lunardi [124, Theorems 7.1.2, 7.1.3 and 7.1.10] proves the first assertion. Items (i), (ii) and (iv) follow from [124, Theorem 7.1.5], and (v) follows from Theorem 7.1.8 (see also Henry [62, 3.3.4]). Fix β with $\alpha < \beta < 1$. As N is bounded in X, $\Psi(s \times N)$ is bounded in X^β by (ii) (with α in (ii) replaced by β). Therefore $\Psi(s \times N)$ is precompact in X^α, and (iii) follows because Ψ defines a local semiflow on $\mathbb{R}_+ \times X^\alpha$. □

Equation (6.1) induces a *solution process* Θ in X, defined by $\Theta_{t,t_0}(u_0) := u(t, t_0, u_0)$. Its restriction to X^α defines an induced solution process on that space. When Eq. (6.1) is autonomous, i.e., $F(t, u) = F(u)$, this solution process boils down to a local semiflow Φ in X^α, defined by $\Phi_t(t_0, u_0) = u(t + t_0, t_0, u_0)$.

When $F(t, u)$ has period $\lambda > 0$ in t, the solution process is λ-*periodic*: $\Theta_{t,t_0} \equiv \Theta_{t+\lambda, t_0+\lambda}$. In this case Θ reduces to a local semiflow on $\mathbf{S}^1 \times X^\alpha$, the dynamics of which are largely determined by the Poincaré map $T := \Theta_{\lambda,0}$ which maps an open subset of X^α continuously into X^α.

Let S be a set and Z a Banach space. We use expressions such as "S is bounded in Z" or "$S \subset Z$ is bounded" to mean $S \subset Z$ and $\sup_{u\in S} \|u\|_Z < \infty$. Note that S may also be unbounded in other Banach spaces.

A map defined on a metric space is *compact* if every bounded set in its domain has precompact image. It is *locally compact* if every point of the domain has a neighborhood with precompact image.

A Banach space Y is *adapted* to the data $(X, \mathcal{A}, F)$ if the following two conditions hold:

$$X^\alpha \hookrightarrow Y \hookrightarrow X \tag{6.3}$$

and the map $(t, u_0) \mapsto \Theta_{t,t_0} u_0$ from $[t_0, \tau) \times D(t, t_0) \cap Y$ to Y is continuous. The solution process Θ determines the *induced solution process* Θ^Y in Y. The domain of Θ^Y_{t,t_0} is the open subset $D^Y(t, t_0) := D_{t,t_0} \cap Y$ of Y.

Rather than work with fractional power spaces, one can assume that $F : [0, \infty) \times K \to X$ where K is a suitable subset of X. The subset $K \subset X$ is *locally closed* in the Banach space X if for each $x \in K$ there exists $r > 0$ such that $\{y \in K\colon \|x - y\| \leqslant r\}$ is closed in X. Closed and open subsets K of X are locally closed. Note that the following result gives existence and uniqueness of mild solutions while at the same time giving positive invariance. It is a special case of Theorems VIII.2.1 and VIII.3.1 in Martin [125]. Assumptions on the semigroup $e^{t\mathcal{A}}$ remain as above.

THEOREM 6.2. *Let K be a nonempty locally closed subset of a Banach space X and let $F : [0, \infty) \times K \to X$ be continuous and satisfy: For each $R > 0$ there are $L_R > 0$ and $\gamma \in (0, 1]$ such that for $x, y \in K$, $\|x\|, \|y\| \leqslant R$, $0 \leqslant s, t \leqslant R$*

$$\big\|F(t, x) - F(s, y)\big\| \leqslant L_R\big(|t - s|^\gamma + \|x - y\|\big). \tag{6.4}$$

Suppose also that:

(a) $e^{t\mathcal{A}}(K) \subset K$ *for all* $t \geqslant 0$, *and*

(b) $\liminf_{h\searrow 0} \frac{1}{h} \operatorname{dist}(x + hF(t, x), K) = 0$ *for* $(t, x) \in [0, \infty) \times K$.

Then for each $(t_0, u_0) \in [0, \infty) \times K$, there is a unique classical trajectory $u(t, t_0, u_0)$ of (6.2) *defined on a maximal interval $[t_0, \tau)$, and $u(t) \in K$ for $t_0 \leqslant t < \tau$.*

This result is useful for parabolic systems when $X = C^k(\overline{\Omega})$, $k = 0, 1$ but not when $X = L^p(\Omega)$. The substitution operators are well-behaved in the former cases but require very stringent growth conditions for the latter; see Martin [125]. By virtue of the uniqueness assertions of Theorem 6.1 and Theorem 6.2, the solution processes given by the two results agree on K if (6.4) holds.

Hypothesis (a) is obviously required for the positive invariance of K in case $F = 0$. Hypothesis (b), called the *subtangential condition*, is easily seen to be a necessary condition for the positive invariance of K if $\mathcal{A} = 0$. See Martin [125, Theorem VI.2.1]. Both hypotheses are trivially satisfied if $K = X$.

The following result is a special case of [125, Proposition VIII.4.1]:

PROPOSITION 6.3. *Let $F : [0, \infty) \times X \to X$ be continuous and satisfy* (6.4) *with $K = X$ and let $u(t) = u(t, t_0, x_0)$ be the unique classical trajectory defined on a maximal interval $[t_0, \tau)$ guaranteed by Theorem* 6.2. *If $\tau < \infty$ then $\lim_{t\to\tau} \|u(t)\| = \infty$.*

6.1.3. *Monotone processes* Given our interest in establishing monotonicity properties of solution processes induced by parabolic systems in various functions spaces, there are two approaches one may take. One is to establish the properties on spaces of smooth functions such as fractional power spaces X^α for $\alpha < 1$ near unity and then try to extend the monotonicity to larger spaces, e.g., $C^0(\overline{\Omega})$, by approximation. An alternative is to establish the monotonicity properties on the larger spaces first and then get corresponding properties on the smaller spaces by restriction. We give both approaches here, beginning with the former.

A process Θ is *very strongly order preserving* (= VSOP) if it is monotone and has the following property: Given $t_0 \geqslant 0$, $u > v$, and $\epsilon > 0$, there exist $s \in (t_0, t_0 + \epsilon]$ and neighborhoods U, V of u, v respectively such that

$$t \geqslant s \implies \Theta_{t,t_0}(U \cap D_{t,t_0}) > \Theta_{t,t_0}(V \cap D_{t,t_0}).$$

This implies Θ is SOP and strictly monotone.

THEOREM 6.4. *Assume X is an ordered Banach space and $Y \hookrightarrow X$ an ordered Banach space such that Y is dense in X and the order cone $Y_+ := Y \cap X_+$ is dense in X_+. Let Θ be a process in X that induces a monotone process Θ^Y in Y. Then*:

(a) *Θ is monotone.*

(b) *Assume $\mathsf{R}(\Theta_{t,t_0}) \subset Y$ for all $t > t_0 \geqslant 0$. Then Θ is strictly monotone if Θ^Y is strictly monotone, and Θ is VSOP provided Θ^Y is strongly monotone and $\Theta_{t,t_0} : D(t, t_0) \to Y$ is continuous for $t > t_0$.*

PROOF. (a) Fix u and $v > u$ in X. The closed line segment $\overline{uv}$ spanned by u and v is compact, hence there exists $\rho > t_0$ with $\overline{uv} \subset D_{t_0,\rho}$. By the density assumptions there exist convergent sequences $u_n \to u$, $v_n \to v$ in $D_{t_0,\rho}$ such that $u_n, v_n \in Y$ and $u_n < v_n$. As Θ^Y is induced from Θ, it follows that $u_n, v_n \in D^Y_{t_0,\rho}$. For all $t \in [t_0, \rho)$,

$$\Theta_{t,t_0}(u_n) = \Theta^Y_{t,t_0}(u_n) \leqslant \Theta^Y_{t,t_0}(v_n) = \Theta_{t,t_0}(v_n).$$

Taking limits as $n \to \infty$ proves $\Theta_{t,t_0}(u) \leqslant \Theta_{t,t_0}(v)$. Thus Θ is monotone.

(b) Assume now that Θ^Y is strictly monotone. We show that Θ is strictly monotone. Let $u(t), v(t)$ be local trajectories with $u(t_0) < v(t_0)$. If $r \in (t_0, t_1]$ is sufficiently near t_0, then $u(r), v(r)$ are distinct points of Y, and $u(r) < v(r)$ by (a). Hence $u(t_1) < v(t_1)$ by strict monotonicity of Θ^Y.

To prove Θ is VSOP, let $u(t), v(t)$ be as above with $u(t_0), v(t_0) \in D_{t,t_0}$. If $t_0 < s < r < t$, strict monotonicity implies $u(s) > v(s)$. These points are in Y, Θ^Y is strongly monotone, and Θ agrees with Θ^Y in Y. Therefore there are disjoint neighborhoods $U_1, V_1 \subset Y$ of $u(s), v(s)$ respectively, such that

$$\Theta_{r,s}(U_1 \cap D_{r,s}) \gg \Theta_{r,s}(V_1 \cap D_{r,s})$$

and strict monotonicity implies that

$$t > r \quad \Longrightarrow \quad \Theta_{t,r}(U_1 \cap D_{t,r}) > \Theta_{t,r}(V_1 \cap D_{t,r}). \tag{6.5}$$

As $\Theta_{r,t_0} : D(r, t_0) \to Y$ is continuous, we may define neighborhoods $U, V \subset X$ of $u(t_0), v(t_0)$ respectively by

$$U = \Theta_{r,t_0}^{-1}(U_1), \qquad V = \Theta_{r,t_0}^{-1}(V_1).$$

By (6.5) and the cocycle identities,

$$t > r \quad \Longrightarrow \quad \Theta_{t,t_0}(U \cap D_{t,t_0}) > \Theta_{t,t_0}(V \cap D_{t,t_0}). \qquad \square$$

Let X be an ordered Banach space with positive cone X_+ and K a locally closed subset. The mapping $F : K \to X$ is said to be *quasimonotone* (relative to X_+) if:

(QM) For all $(t, x), (t, y) \in [0, \infty) \times K$ satisfying $x \leqslant y$ we have:

$$\lim_{h \searrow 0} \frac{1}{h} \operatorname{dist}\big(y - x + h\big[F(t, y) - F(t, x)\big], X_+\big) = 0.$$

The next result is due to [125, Proposition VIII.6.1 and Lemma 6.3] (see also [129] in case of abstract delay differential equations).

THEOREM 6.5. *Assume the hypotheses of Theorem* 6.2 *hold, F is quasimonotone, and $e^{t\mathcal{A}}$ is a positive operator for $t \geqslant 0$. In addition, suppose one of the following*:

(i) *K is open.*
(ii) *$K + X_+ \subset K$.*
(iii) *X is a Banach lattice and $K = [u, v]$ for some $u, v \in X \cup \{-\infty, \infty\}$, $u \leqslant v$.*

Then

$$x, y \in K, x \leqslant y \quad \Longrightarrow \quad u(t, x) \leqslant u(t, y) \quad \big(0 \leqslant t \leqslant \min\{\tau_x, \tau_y\}\big).$$

By $[-\infty, v]$, $v \in X$, is meant the set $\{x \in X\colon x \leqslant v\}$; similarly for other intervals involving $\pm\infty$. Of course, $-\infty \leqslant v \leqslant \infty$ for every $v \in X$. Observe that $K = [u, \infty]$ is covered by both (ii) and (iii).

REMARK 6.6. If F has the property that for each $x, y \in K$ with $x \leqslant y$, there exists $\lambda > 0$ such that $F(t,x) + \lambda x \leqslant F(t,y) + \lambda y$ then F is quasimonotone because

$$\begin{aligned} y - x + h\big[F(t,y) - F(t,x)\big] = {} & (1 - \lambda h)(y - x) \\ & + h\big[F(t,y) + \lambda y - F(t,x) - \lambda x\big] \in X_+ \end{aligned}$$

when $h < \lambda^{-1}$.

REMARK 6.7. It is well-known that $e^{t\mathcal{A}}$ is a positive operator if and only if $(\lambda I - \mathcal{A})^{-1}$ is a positive operator for all large positive λ. See, e.g., [11, Theorem II 6.4.1] or [125, Proposition 7.5.3]. Indeed, if K is a closed convex subset of X, then $e^{t\mathcal{A}} K \subset K$ if and only if $(\lambda I - \mathcal{A})^{-1} K \subset K$ for all large positive λ.

A Banach space X is a *Banach lattice* if for each $x, y \in X$, $x \vee y := \sup\{x, y\}$ exists and the norm is monotone in the sense:

$$|x| \leqslant |y| \implies \|x\| \leqslant \|y\|,$$

where $|x|$ denotes the absolute value of x: $|x| := (-x) \vee x$ (see Vulikh [225]). Banach lattices are easy to work with due to simple formulas such as

$$\operatorname{dist}(x, X_+) = \|x - x_+\| = \|x_-\|,$$

where $x_+ := x \vee 0$ and $x_- = -(-x)_+$. The requirement that X be a Banach lattice is a rather strong hypothesis which essentially restricts applicability to $X = L^p(\Omega)$, $C^0(\overline{\Omega})$ or $C_0^0(\overline{\Omega})$. However, the latter two will be important for reaction–diffusion systems.

6.2. *Semilinear parabolic equations*

Let $\Omega \subset \mathbb{R}^n$ be the interior of a compact n-dimensional manifold with C^2 boundary $\partial\Omega$. We consider the semilinear system of m coupled equations ($1 \leqslant i \leqslant m$):

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = (A_i u_i)(t,x) + f_i(t,x,u,\nabla u) && (x \in \Omega,\ t > t_0), \\ &(B_i u_i)(t,x) = 0 && (x \in \partial\Omega,\ t > t_0), \\ &u_i(t_0, x) = v_{0,i}(x) && (x \in \overline{\Omega}). \end{aligned} \tag{6.6}$$

Here the unknown function is $u = (u_1, \ldots, u_m): \overline{\Omega} \to \mathbb{R}^m$, and $\nabla u := (\nabla u_1, \ldots, \nabla u_m) \in (\mathbb{R}^n)^m$ lists the spatial gradients ∇u_i of the u_i, i.e., $\nabla u_i := (\frac{\partial u_i}{\partial x_1}, \ldots, \frac{\partial u_i}{\partial x_n})$. Each $A_i(x)$ is a second order, elliptic differential operator of the form

$$A_i(x) = \sum_{l,j=1}^{n} C_{lj}^i(x) \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j^i(x) \frac{\partial}{\partial x_j} \tag{6.7}$$

with uniformly continuous and bounded coefficients. Each $n \times n$ matrix $C^i(x) := [C^i_{lj}(x)]$ is assumed positive definite:

$$0 < \inf\big(\langle C^i(x)y, y\rangle\big) \quad \big(x \in \Omega, y \in \mathbb{R}^n, |y| = 1\big),$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

The function

$$f := (f_1, \dots, f_m) : \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}^m \times \big(\mathbb{R}^n\big)^m \to \mathbb{R}^m$$

is continuous, and $f(t, x, u, \xi)$ is locally Lipschitz in $(u, \xi) \in \mathbb{R}^m \times (\mathbb{R}^n)^m$.

Each boundary operator B_i acts on sufficiently smooth functions $v : [t_0, \tau) \times \overline{\Omega} \to \mathbb{R}$ in one of the following ways, where $x \in \partial\Omega$:

$$\begin{aligned}
&\textit{Dirichlet}: && (B_i v)(t,x) = v(t,x);\\
&\textit{Robin}: && (B_i v)(t,x) = \gamma_i v(t,x) + \frac{\partial v}{\partial \xi_i}(t,x);\\
&\textit{Neumann}: && (B_i v)(t,x) = \frac{\partial v}{\partial \xi_i}(t,x),
\end{aligned}$$

where $\gamma_i : \overline{\Omega} \to [0, \infty)$ is continuously differentiable, and $\xi_i : \overline{\Omega} \to \mathbb{R}^n$ is a continuously differentiable vector field transverse to $\partial\Omega$ and pointing outward from Ω. Note that Neumann is a special case of Robin.

We rewrite (6.6) as an initial-boundary value problem for an unknown vector-valued function $u := (u_1, \dots, u_m) : [t_0, \tau) \times \overline{\Omega} \to \mathbb{R}^m$,

$$\begin{aligned}
&\frac{\partial u}{\partial t} = (Au)(t,x) + f(t,x,u,\nabla u) && (x \in \Omega,\ t > t_0),\\
&(Bu)(t,x) = 0 && (x \in \partial\Omega,\ t > t_0),\\
&u(t_0, x) = u_0(x) && (x \in \overline{\Omega}),
\end{aligned} \tag{6.8}$$

where the operators $A := A_1 \times \cdots \times A_m$ and $B := B_1 \times \cdots \times B_m$ act componentwise on $u = (u_1, \dots, u_m)$. By a *solution process* for Eq. (6.8) we mean a process in some function space on $\overline{\Omega}$, whose trajectories are solutions to (6.8).

Of special interest are *autonomous* systems, for which $f = f(x, u, \nabla u)$; and the *reaction–diffusion* systems, characterized by $f = f(t, x, u)$.

Assume $n < p < \infty$. To Eq. (6.8) we associate an abstract differential Eq. (6.1) in $L^p(\Omega, \mathbb{R}^m)$. The pair of operators (A_i, B_i) has a sectorial realization $\mathcal{A}_i$ in $L^p(\Omega)$ with domain $\mathcal{D}(\mathcal{A}_i) \hookrightarrow L^p(\Omega)$ (Lunardi [124, 3.1.3]). The operator $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ is sectorial on $X := L^p(\Omega, \mathbb{R}^m) = [L^p(\Omega)]^m$.

For $\alpha \in [0, 1)$ set $X^\alpha := X^\alpha(\mathcal{A})$. We choose α so that f defines a continuous *substitution operator*

$$F : \mathbb{R}_+ \times X^\alpha \to X, \qquad F(t, u)(x) := f\big(t, x, u(x), \nabla u(x)\big).$$

It suffices to take α

$$1 > \alpha > \frac{1}{2}\left(1 + \frac{n}{p}\right), \tag{6.9}$$

for then $X^\alpha \hookrightarrow C^1(\overline{\Omega}, \mathbb{R}^m)$ by the Sobolev embedding theorems.

The data $(\mathcal{A}, F)$ thus determine an abstract differential equation $u' = \mathcal{A}u + F(t,u)$ in X, whose trajectories $u(t)$ correspond to solutions $u(t,x) := u(t)(x)$ of (6.6). The assumptions on f make $F(t,u)$ locally Lipschitz in $u \in X^\alpha$.

By Theorem 6.1 and the Sobolev embedding theorem we have:

PROPOSITION 6.8. *Equation* (6.8) *defines a solution process* Θ *on* $X := L^p(\Omega, \mathbb{R}^m)$ *which induces a solution process in* X^β *for every* $\beta \in [0,1)$ *with* $\beta \geqslant \alpha$.

We quote a useful condition for globality of a solution:

PROPOSITION 6.9. *Assume there are constants* $C > 0$ *and* $0 < \epsilon \leqslant 1$ *such that*

$$\|f(t,x,v,\xi)\| \leqslant C\left(1 + \|v\| + \|\xi\|^{2-\epsilon}\right) \quad \text{for all } (t,x,v,\xi) \in \mathbb{R}_+ \times \overline{\Omega} \times S \times \mathbb{R}^n. \tag{6.10}$$

If $u : [t_0, \tau) \to L^p(\Omega, \mathbb{R}^m)$ *is a trajectory such that*

$$\limsup_{t \to \tau-} \|u(t)\|_{L^p(\Omega, \mathbb{R}^m)} < \infty \tag{6.11}$$

then $\tau = \infty$.

PROOF. Follows from Amann [9, Theorem 5.3(i)], taking the constants of that result to be $m = k = p_0 = \gamma_0 = 1$, $\kappa = s_0 = 0$, $\gamma_1 = 2 - \epsilon$. □

Solutions $u : [t_0, \tau) \times \overline{\Omega} \to \mathbb{R}^m$ to (6.8) enjoy considerable smoothness. For example, if the data $\partial\Omega_i, f_i, A_i, B_i$ are smooth of class $C^{2+2\epsilon}$, $0 < 2\epsilon < 1$, then $u \in C^{1+\epsilon, 2+2\epsilon}([t_1, t_2] \times \overline{\Omega}, \mathbb{R}^m)$ for all $t_0 < t_1 < t_2 < \tau$ (Lunardi [124, 7.3.3(iii)]).

While useful for many purposes, solution processes in the spaces X^α suffer from the drawback that X^α and its norm are defined implicitly, leaving unclear the domains of solutions and the meaning of convergence, stability, density and similar topological terms. In addition, the topology of X^α might be unsuitable for a given application. To overcome these difficulties we could appeal to results of Colombo and Vespri [29], Lunardi [124] and Mora [147], establishing induced processes in Banach spaces of continuous, smooth or L^p functions; or we can apply Theorem 6.2. We now define these spaces.

For $r \in \mathbb{N}$ let $C^r(\overline{\Omega})$ denotes the usual Banach space of C^r functions on $\overline{\Omega}$. Set

$$C_0^r(\overline{\Omega}) := \{v \in C^r(\overline{\Omega}) \colon v|\partial\Omega = 0\}.$$

With γ, ξ as in a Robin boundary operator and $r \geqslant 1$, define

$$C^r_{\gamma,\xi}(\overline{\Omega}) := \left\{ v \in C^r(\overline{\Omega}) \colon \gamma(x)v(x) + \frac{\partial v}{\partial \xi}(x) = 0 \ (x \in \partial\Omega) \right\}.$$

It is not hard to show that:

- $C^0(\overline{\Omega})$, $C^1(\overline{\Omega})$ and $C^1_{\gamma,\xi}(\overline{\Omega})$ are strongly ordered, with $u \gg 0$ if and only if $u(x) > 0$ for all $x \in \overline{\Omega}$;
- $C^1_0(\overline{\Omega})$ is strongly ordered, with $u \gg 0$ if and only if $u(x) > 0$ for all $x \in \Omega$ and $\partial u/\partial \nu > 0$ where $\nu : \partial\Omega \to \mathbb{R}^n$ is the unit vector field inwardly normal to $\partial\Omega$;
- $C^0_0(\overline{\Omega})$ is not strongly ordered. Both $C^0_0(\overline{\Omega})$ and $C^0(\overline{\Omega})$ are Banach lattices.

In terms of the boundary operators B_i, for $k = 0, 1$ we define Banach spaces

$$C^k_{B_i}(\overline{\Omega}) := \begin{cases} C^k_0(\overline{\Omega}) & \text{if } B_i \text{ is Dirichlet,} \\ C^k_{\gamma,\xi}(\overline{\Omega}) & \text{if } B_i \text{ is Robin and } k = 1, \\ C^0(\overline{\Omega}) & \text{if } B_i \text{ is Robin and } k = 0. \end{cases}$$

Note that $C^1_{B_i}(\overline{\Omega})$ is strongly ordered, while $C^0_{B_i}(\overline{\Omega})$ is strongly ordered if and only if B_i is Robin; $C^0_{B_i}(\overline{\Omega})$ is a Banach lattice. The ordered Banach space

$$C^k_B(\overline{\Omega}, \mathbb{R}^m) := \Pi_i C^k_{B_i}(\overline{\Omega}),$$

with the product order cone, is strongly ordered if $k = 1$, or $k = 0$ and no B_i is Dirichlet. The order cone $L^p(\Omega, \mathbb{R}^m)_+$ is the subset of $L^p(\Omega, \mathbb{R}^m)$ comprising equivalence classes represented by functions $\Omega \to \mathbb{R}^m_+$. Note that $L^p(\Omega, \mathbb{R}^m)$ is normally ordered but not strongly ordered.

It is known that the pair of operators (A_i, B_i) has a sectorial realization $\mathcal{A}_i$ on $C^k(B_i)$ and therefore the product operator $\mathcal{A}$ is sectorial on $C^k_B(\overline{\Omega}, \mathbb{R}^m)$. See Corollary 3.1.24, Theorems 3.1.25, 3.1.26 in [124].

LEMMA 6.10. *For $X = L^p(\Omega, \mathbb{R}^m)$ or $C^k_B(\overline{\Omega}, \mathbb{R}^m)$, the analytic semigroup $\mathrm{e}^{t\mathcal{A}}$ is a positive operator for $t \geqslant 0$ with respect to the cone of componentwise nonnegative functions in X.*

PROOF. As noted in Remark 6.7, it suffices to show that $(\lambda I - \mathcal{A})^{-1}$ is positive for large $\lambda > 0$, or equivalently, that for each i and $f_i \geqslant 0$, the solution $g_i \in D(\mathcal{A}_i)$ of $f_i = \lambda g_i - \mathcal{A}_i g_i$ satisfies $g_i \geqslant 0$. The existence of g_i is not the issue but rather it's positivity. Thus it boils down to $\lambda g_i - A_i g_i \geqslant 0 \Longrightarrow g_i \geqslant 0$. But these follow from standard maximum principle arguments. See Lemma 3.1.4 in [155]. □

With $X = L^p(\Omega, \mathbb{R}^m)$ and $\mathcal{A}$ and α as above, we have a chain of continuous inclusions of ordered Banach spaces

$$\mathsf{D}(\mathcal{A}) \hookrightarrow X^\alpha \hookrightarrow C^1_B(\overline{\Omega}, \mathbb{R}^m) \hookrightarrow C^0_B(\overline{\Omega}, \mathbb{R}^m) \hookrightarrow L^p(\Omega, \mathbb{R}^m),$$

with a solution process in $L^p(\Omega,\mathbb{R}^m)$ and an induced solution process in X^α.

PROPOSITION 6.11. *Let Θ be the solution process in $L^p(\Omega,\mathbb{R}^m)$ for Eq.* (6.1) *with $n < p < \infty$.*

(a) *For all $t > t_0$, Θ_{t,t_0} maps D_{t,t_0} continuously into $C^1_B(\overline{\Omega},\mathbb{R}^m)$.*
(b) *Θ induces a solution process Θ^1 in $C^1_B(\overline{\Omega},\mathbb{R}^m)$.*
(c) *Θ induces a solution process Θ^0 in $C^0_B(\overline{\Omega},\mathbb{R}^m)$ provided $f = f(t,x,u)$.*

PROOF. By uniqueness of solutions it suffices to establish induced solution processes in $C^1_B(\overline{\Omega},\mathbb{R}^m) \hookrightarrow L^p(\Omega,\mathbb{R}^m)$, and in $C^0_B(\overline{\Omega},\mathbb{R}^m) \hookrightarrow L^p(\Omega,\mathbb{R}^m)$ when $f = f(t,x,u)$. This is done in Lunardi [124, Proposition 7.3.3] for $m = 1$, and the general case is similar. Part (c) follows from Theorem 6.2. $\square$

Henceforth Θ^k, $k \in \{0,1\}$, denotes the process Θ^0 or Θ^1 as in Proposition 6.11.

6.2.1. *Dynamics in spaces X_Γ* For any set $\Gamma \subset \mathbb{R}^m$ and $k = 0, 1$ define

$$\begin{aligned} X^k_\Gamma &:= \{u \in C^k_B(\overline{\Omega},\mathbb{R}^m)\colon u(\overline{\Omega}) \subset \Gamma\}, \\ X_\Gamma &:= \{u \in L^p(\Omega,\mathbb{R}^m)\colon u(\Omega) \subset \Gamma\}. \end{aligned} \tag{6.12}$$

A *rectangle* in $\mathbb{R}^m$ is a set of the form $J = J_1 \times \cdots \times J_m$ where each $J_i \subset \mathbb{R}$ is a nondegenerate closed interval. $\mathbb{R}^m$, $\mathbb{R}^m_+$ and closed order intervals $[a,b]$, $a \leqslant b$ are rectangles.

PROPOSITION 6.12. *Let $J := \Pi^m_{i=1} J_i$ be a rectangle in $\mathbb{R}^m$ such that either $0 \in J_i$ or B_i is Neumann, and the following hold for all $x \in \overline{\Omega}$, $u \in \partial J$:*

$$f_i(t,x,u,0) \geqslant 0 \quad \text{if } u_i = \inf J_i, \qquad f_i(t,x,u,0) \leqslant 0 \quad \text{if } u_i = \sup J_i. \tag{6.13}$$

Then:

(i) *In the reaction–diffusion case, X_J is positively invariant for Θ and X^k_J is positively invariant for Θ^k ($k = 0, 1$).*
(ii) *Suppose $k = m = 1$ and $J \subset \mathbb{R}$ is an interval. Then X_J is positively invariant for Θ and X^1_J is positively invariant for Θ^1.*

PROOF. For the reaction–diffusion case we sketch a proof that X^0_J is Θ^0-positively invariant using Theorem 6.2. The proof that X_J is Θ-positively invariant follows from this since $\Theta_{t,t_0}(u)$ is the L^p limit $\lim_k \Theta^0_{t,t_0}(u_k)$ where $u_k \in X^0_J$ approximates $u \in X_J$ in L^p and the facts: $\Theta^0 = \Theta$ on X^0_J, a dense subset of the closed subset X_J. In order to verify the subtangential condition for X^0_J, it suffices to verify the subtangential condition for J:

$$\liminf_{h \searrow 0} \frac{1}{h} \operatorname{dist}\bigl(u + hf(t,x,u), J\bigr) = 0 \tag{6.14}$$

for each $(t, x, u) \in [0, \infty) \times \overline{\Omega} \times J$ by Martin [125, Proposition IX.1.1]. But (6.14) is a necessary condition for J to be positively invariant for the ODE

$$v' = f(t, x, v),$$

where x is a parameter. See, e.g., [125, Theorem VI.2.1]. It is well-known and easy to prove that condition (6.13) implies the positive invariance of J for the ODE (see, e.g., Proposition 3.3, Smith and Waltman [203, Proposition B.7], or Walter [227, Chapter II, Section 12, Theorem II]). It follows that (6.14) holds. Therefore the subtangential condition for X_J^0 holds. Finally, we must verify that $e^{t\mathcal{A}} X_J^0 \subset X_J^0$ or, equivalently, that $e^{t\mathcal{A}_i} C^0_{B_i}(\overline{\Omega}, J_i) \subset C^0_{B_i}(\overline{\Omega}, J_i)$. This follows from Remark 6.7 and standard maximum principle arguments. It also follows from standard comparison principles for parabolic equations. See, e.g., Pao [155, Lemma 2.1] or Smith [194, Corollary 2.4].

The case $k = m = 1$ is a special case of [227, Chapter IV, Section 25, Theorem II, Section 31, Corollaries IV and V]. □

Consider the case that (6.6) is autonomous:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= A_i u_i + f_i(x, u, \nabla u) \quad && (x \in \overline{\Omega},\ t > t_0), \\ B_i u_i &= 0 && (x \in \partial\Omega,\ t > t_0), \end{aligned} \tag{6.15}$$

$i = 1, \ldots, m$. The solution processes Θ, Θ^1, Θ^0 reduce to local semiflows.

We introduce a mild growth condition, trivially satisfied in the reaction–diffusion case:

$$\begin{aligned} &\text{For each } s > 0 \text{ there exists } C(s) > 0 \text{ such that} \\ &|v| \leqslant s \implies \left|f(x, v, \xi)\right| \leqslant C(s)\left(1 + |\xi|^{2-\epsilon}\right). \end{aligned} \tag{6.16}$$

The following result gives sufficient conditions for solution processes in X_Γ to be global, and to admit compact global attractors:

PROPOSITION 6.13. *Assume system* (6.15) *satisfies* (6.16). *Let $\Gamma \subset \mathbb{R}^m$ be a nonempty compact set such that X_Γ is positively invariant for* (6.15). *Then*:

(a) *There are solution semiflows Φ, Φ^1 in X_Γ, X_Γ^1 respectively. Φ^1 is compact.*
(b) *Assume* (6.15) *is reaction–diffusion. Then there is also a solution semiflow Φ^0 in X_Γ^0. The semiflows Φ, Φ^0, Φ^1 are compact and order compact. There is a compact set $K \subset X_\Gamma^1$ which is the global attractor for all three semiflows.*

PROOF. (a) Let Γ lie in the open ball of radius $R > 0$ about the origin in $\mathbb{R}^m$ and let $h : \mathbb{R}^m \to \mathbb{R}^m$ be any smooth bounded function that agrees with the identity on the open ball of radius R. Define g by $g(x, v, \xi) = f(x, h(v), \xi)$. Every trajectory in X_Γ of (6.15) is also a trajectory of the analogous system in which f is replaced by g (compare Poláčik [163, pp. 842–843]). Nonlinearity g satisfies (6.16) with $C(s)$ constant so (6.10) holds. As

$$\limsup_{t \to \tau -} \|u(t)\|_{C^0(\overline{\Omega}, \mathbb{R}^m)} \leqslant R,$$

which implies (6.11), all trajectories are global by Proposition 6.9. Thus, the restrictions of Ψ, Ψ^1 in X and $C^1_B(\overline{\Omega}, \mathbb{R}^m)$ respectively to X_Γ and X^1_Γ define semiflows Φ and Φ^1. As Ψ^1 is compact by Hale [58], Theorem 4.2.2, Φ^1 is compact because X^1_Γ is closed in $C^1_B(\overline{\Omega}, \mathbb{R}^m)$.

(b) In the reaction–diffusion case a similar argument establishes a compact solution semiflow Φ^0 in X^0_Γ; and Φ^0 is order compact because order intervals in X^0_Γ are bounded. To prove Φ^0 order compact, let N be an order interval in X_Γ. For every $t = 2s > 0$, Φ_s maps N continuously into an order interval N' of X^0_Γ. Precompactness in X_Γ of $\Phi_t N$ follows from the precompactness in X^0_Γ of $\Phi^0_s N'$, already established, and the continuous inclusion $\Phi_t N = \Phi^0_s \circ \Phi_s N \subset \Phi^0_s N'$.

To prove order compactness of Φ^1, let $N_1 \subset X^1_\Gamma$ be an order interval. N_1 is contained in an order interval N_0 of X^0_Γ. Let C^k denote closure in X^k_Γ. For all $t > 0$ we have $\mathrm{C}^1(\Phi^1_t N_1) = \mathrm{C}^1(\Phi^0_t N_1) \subset \mathrm{C}^1\mathrm{C}^0(\Phi^0_t N_0)$, and the latter set is compact because Φ^0 is order compact. This proves $\Phi^1_t N_1$ is precompact in X^1_Γ.

X^0_Γ is closed and bounded in X^0, hence $\Phi^0_t X^0_\Gamma$ is precompact in X^0_Γ for all $t > 0$ by (a). Therefore $K := \bigcap_{t>0} \overline{\Phi^0_t X^0_\Gamma}$ is a compact global attractor for Φ^0. Similarly, K (with the same topology) is a compact global attractor for Φ.

We rely on the identity $\Phi^1_t = \Phi^0_t | X^1_\Gamma$ and continuity of $\Phi^0_t : X^0_\Gamma \to X^1_\Gamma$ for all $t > 0$. As K is invariant under Φ^0, it follows that K is a compact subset of X^1_Γ. To prove K a global attractor for Φ^1, it suffices to prove: For arbitrary sequences $\{x(i)\}$ in X^1_Γ, and $t(i) \to \infty$ in $\mathbb{R}_+$ with $t(i) > \epsilon > 0$, there is a sequence $i_k \to \infty$ in $\mathbb{N}$ such that $\{\Phi^1_{t(i_k)} x(i_k)\}$ converges in X^1_Γ to a point of K. Choose $\{i_k\}$ so that $\Phi^0_{t(i_k)-\epsilon}(i_k)$ converges X^0_Γ as $k \to \infty$ to $p \in K$; this is possible because K is a compact global attractor for Φ^0. Then $\Phi^1_{t(i_k)} x(i_k) = \Phi^0_\epsilon \circ \Phi^0_{t(i_k)-\epsilon} x(i_k)$, which converges in X^1_Γ as $k \to \infty$ to $\Phi^0_\epsilon p \in K$. □

EXAMPLE. Let the u_i denote the concentrations or densities of entities such as chemicals or species. Such quantities are inherently positive, so taking the state space to be $L^p(\Omega, \mathbb{R}^m_+)$ or $C^k_B(\overline{\Omega}, \mathbb{R}^m_+)$ is appropriate. We make the plausible assumption that sufficiently high density levels must decrease. Modeling this situation by (a) and (b) below, we get the following result.

PROPOSITION 6.14. *In Eq.* (6.15) *assume* $f = f(x,u)$ *and let the following hold for* $i = 1, \ldots, m$:

(a) $f_i(x,u) \geqslant 0$ *if* $u_i = 0$;

(b) *there exists* $\kappa > 0$ *such that* $f_i(x,u) < 0$ *if* $u_i \geqslant \kappa$.

Then for $k = 0, 1$ *solution processes in the order cones* $L^p(\Omega, \mathbb{R}^m_+)$, $C^k_B(\overline{\Omega}, \mathbb{R}^m_+)$ *are defined by semiflows* Φ, Φ^k *respectively; and there is a compact set* $K \subset X^k_{[0,\kappa]^m}$ *that is the global attractor for* Φ, Φ^0 *and* Φ^1.

PROOF. Proposition 6.12 and (a) proves $L^p(\Omega, \mathbb{R}^m_+)$ and $C^k_B(\overline{\Omega}, \mathbb{R}^m_+)$ are positively invariant under the solution process.

Consider the compact rectangles $J(c) := [0, c\kappa]^m \subset \mathbb{R}^m, c \geqslant 1$. Assumption (b) and Proposition 6.12 entail positive invariance of $X_{J(c)}$. Proposition 6.13 shows that there are solution semiflows in $X_{J(c)}$ and $X^k_{J(c)}$ having a compact global attractor $K_c \subset X^1_{J(c)}$ in common. As the $J(c)$ are nested and exhaust $\mathbb{R}^m_+$, these semiflows come from solution semiflows Φ, Φ^k as required. Moreover, all the attractors K_c coincide with the compact set $K := K_1 \subset X^1_{J(1)}$. It is easy to see that K is the required global attractor. $\square$

Results on global solutions and positively invariant sets can be found in many places. See for example Amann [9,10], Cholewa and Dlotko [25], Cosner [33], Lunardi [124], Poláčik [163], Smith [194], Smoller [205].

6.2.2. *Monotone solution processes for parabolic equations* We restrict attention here to monotonicity properties with respect to the standard point-wise and component-wise ordering of functions $\Omega \to \mathbb{R}^m$: $f \leqslant g$ if and only if $f_i(x) \leqslant g_i(x)$ for all x and all i. The natural ordering on $L^p(\Omega, \mathbb{R}^m)$ is defined on equivalence classes by the condition on representatives that $f_i(x) \leqslant g_i(x)$ almost everywhere.

Orderings induced by orthants in $\mathbb{R}^n$ other than the positive orthant can be handled easily by change of variables. See Mincheva [144] and [145] for results in the case of polyhedral cones in $\mathbb{R}^n$.

Consider the case $m = 1$ in Eq. (6.8).

THEOREM 6.15. *In Eq.* (6.8), *assume $m = 1$ and f is C^1. Then*:

(i) *Θ is VSOP on $L^p(\Omega, \mathbb{R}^m)$.*

(ii) *Θ^1 is strongly monotone in $C^1_B(\overline{\Omega})$.*

(iii) *If $f = f(t, x, u)$ the induced process Θ^0 on $C^0_B(\overline{\Omega})$ is VSOP, and strongly monotone if all boundary operators are Robin.*

PROOF. Let $u, v : [t_0, t_1] \times \overline{\Omega} \to \mathbb{R}$ be solutions with $v(t_0, x) - u(t_0, x) \geqslant 0$ for all x and > 0 for some x. Then $w := v - u$ is the solution to the problem

$$\begin{aligned} &\frac{\partial w}{\partial t} = Aw + \sum_{j=1}^{n} b_j \frac{\partial w}{\partial x_j} + cw \quad && (x \in \overline{\Omega},\ t > t_0), \\ &Bw(t, x) = 0 && (x \in \partial\Omega,\ t > t_0), \\ &w(t_0, x) \geqslant 0, \quad w(t_0, x) \not\equiv 0 && (x \in \overline{\Omega}) \end{aligned} \tag{6.17}$$

where $b_j = b_j(t, x)$ and $c_j = c_j(t, x)$ are obtained as follows. Evaluate u, v and their spatial gradients at (t, x), and for $s \in [0, 1]$ set

$$\begin{aligned} &Z(s) = (1 - s)(t, x, u, \nabla u) + s(t, x, v, \nabla v), \\ &b(t, x) = \big(b_1(t, x), \ldots, b_n(t, x)\big) = \int_0^1 D_4 f\big(Z(s)\big)\, \mathrm{d}s, \\ &c(t, x) = \int_0^1 D_3 f\big(Z(s)\big)\, \mathrm{d}s \end{aligned}$$

where $D_4 f$ and $D_3 f$ denote respectively the derivatives of $f(t,x,y,\xi)$ with respect to $\xi \in \mathbb{R}^n$ and $y \in \mathbb{R}$. By Taylor's theorem

$$f(t,x,v,\nabla v) - f(t,x,u,\nabla u) = b(t,x)(\nabla u - \nabla v) + c(t,x)(u-v),$$

whence (6.17) follows.

The parabolic maximum principle and boundary point lemma ([194, Theorems 7.2.1, 7.2.2]) imply that the function $w(t_1,\cdot)$, considered as an element of $C^1_B(\overline{\Omega})$, is $\gg 0$. This proves (ii), and the first assertion of (iii) follows from Theorem 6.4(b). The proof of strong monotonicity for Robin boundary conditions is similar to the arguments given above. Part (i) follows from strong monotonicity of Θ^1, Theorem 6.4 and continuity of $\Theta_{t,t_0} : L^p(\Omega,\mathbb{R}^m) \to X^\alpha \hookrightarrow C^1_B(\overline{\Omega},\mathbb{R}^m)$. □

For $m \geqslant 2$ we impose further conditions on system (6.6) in order to have a monotone solution process: it must be of reaction–diffusion type, and the vector fields $f(t,x,\cdot)$ on $\mathbb{R}^m$ must be *cooperative*. In other words, $f(t,x,u)$ is C^1 in u and $\partial f_i/\partial u_j \geqslant 0$ for all $i \neq j$. (The latter condition holds vacuously if $m=1$). When this holds then the system is called cooperative. If in addition, there exists $\bar{x} \in \Omega$ such that the $m \times m$ Jacobian matrix $[\partial f_i/\partial u_j(t,\bar{x},u)]$ is irreducible for all (t,u), we call the system *cooperative and irreducible*

THEOREM 6.16. *If system* (6.15) *is cooperative, then* Θ, Θ^k, $k=0,1$ *are monotone. If the system is also irreducible, then*:

(i) Θ *is VSOP on* $L^p(\Omega,\mathbb{R}^m)$.
(ii) Θ^1 *is strongly monotone in* $C^1_B(\overline{\Omega},\mathbb{R}^m)$.
(iii) Θ^0 *is VSOP in* $C^0_B(\overline{\Omega},\mathbb{R}^m)$ *and is strongly monotone when all boundary operators are Robin.*

PROOF. Monotonicity in $C^0_B(\overline{\Omega},\mathbb{R}^m)$ follows directly from Theorem 6.5 and Remark 6.6. Indeed, let $u \leqslant v$ in $C^0_B(\overline{\Omega},\mathbb{R}^m)$ and t be fixed. Then

$$\begin{aligned}&\big[F(t,v) - F(t,u) + \lambda(v-u)\big](x)\\ &\quad = \int_0^1 \left(\frac{\partial f}{\partial u}\big(t,x,su(x)+(1-s)v(x)\big) + \lambda I\right) \mathrm{d}s (v-u)(x) \geqslant 0\end{aligned}$$

for some $\lambda > 0$ and all $x \in \overline{\Omega}$ by cooperativity of f and compactness of $\overline{\Omega}$. This implies that (QM) holds. The positivity of $\mathrm{e}^{t\mathcal{A}}$ follows from Lemma 6.10. Monotonicity of Θ in $L^p(\Omega,\mathbb{R}^m)$ follows from monotonicity of Θ^0 and Theorem 6.4.

The proof of VSOP and strong monotonicity for Robin boundary conditions in $C^0_B(\overline{\Omega},\mathbb{R}^m)$ is like that of Theorem 6.15(i), exploiting the maximum principle for weakly coupled parabolic systems (Protter and Weinberger [166, Chapter 3, Theorems 13, 14, 15 and pp. 192, Remark (i)]). See Smith [194, Section 7.4] for a similar proof.

Monotonicity of Θ^1 follows from monotonicity of Θ^0. Strong monotonicity of Θ^1, in the case of Dirichlet boundary conditions, requires exploiting the maximum principle as in the previous case (the same references apply). VSOP of $\dot{\Theta}$ follows from strong monotonicity of Θ^1, Theorem 6.4 and continuity of the composition $\Theta_{t,t_0} : L^p(\Omega, \mathbb{R}^m) \to X^\alpha \to C_B^1(\overline{\Omega}, \mathbb{R}^m)$. □

6.3. *Parabolic systems with monotone dynamics*

We now treat autonomous systems (6.15) having monotone dynamics. Our goal is Theorem 6.17, a sample of the convergence and stability results derivable from the general theory.

In addition to the assumptions for (6.6), we require the following conditions to hold for the solution process Θ in $X := L^p(\Omega, \mathbb{R}^m)$, with p satisfying (6.9) and X_Γ^k defined in (6.12):

(SP) If $m \geqslant 2$ in system (6.15) then $f = f(x, u)$ and the system is cooperative and irreducible. $\Gamma \subset \mathbb{R}^m$ is a nonempty set, either an open set or the closure of an open set. The solution process induces semiflows Φ, Φ^1 in X_Γ, X_Γ^1 respectively, and Φ^0 in X_Γ^0 for the reaction–diffusion case. These semiflows are assumed to have compact orbit closures.

Simple conditions implying (SP) can be derived from Propositions 6.13.

The following statements follow from (SP), assertions about Φ^0 having the implied hypothesis $f = f(x, u)$:

- X_Γ^1 is dense in X_Γ^0 and in X_Γ.
- Φ and Φ^0 agree on X_Γ^0, and Φ, Φ^0 and Φ^1 agree on X_Γ^1.
- Φ_t (respectively, Φ_t^0) maps X_Γ (respectively, X_Γ^0) continuously into X_Γ^1 for $t > 0$ (Proposition 6.11).
- Φ, Φ^1 and Φ^0 have the same omega limit sets, compact attractors and equilibria.
- If Γ is open or order convex and $f(x, u, \xi)$ is C^1 in (u, ξ), the Improved Limit Set Dichotomy (ILSD) holds for Φ^1 by Theorem 2.16, and for Φ and Φ^0 by Proposition 2.21.
- If Γ is compact then Φ^1 is compact. In the reaction–diffusion case with Γ compact, Φ, Φ^1 and Φ^0 are compact and order compact, and a common compact global attractor (Propositions 6.13).
- Φ^1 is strongly monotone; Φ^0 is VSOP, and strongly monotone if all boundary operators are Robin; Φ is VSOP (Theorem 6.16).

The sets of quasiconvergent, convergent and stable points for any semiflow Ψ are denoted respectively by $Q(\Psi)$, $C(\Psi)$, $S(\Psi)$. References to intrinsic or extrinsic topology of these sets (e.g., closure, density) for Φ, Φ^1 or Φ^0 are to be interpreted in terms of the topology of the corresponding domain X_Γ, X_Γ^1 or X_Γ^0.

THEOREM 6.17. *If system* (6.15) *satisfies hypothesis* (SP), *then*:

(i) *The sets* $Q(\Phi)$, $Q(\Phi^0)$ *and* $Q(\Phi^1)$ *are residual.*

(ii) *Assume* Γ *is open or order convex and* $f(x, u, \xi)$ *is* C^1 *in* (u, ξ). *Then the sets* $C(\Phi) \cap S(\Phi)$, $C(\Phi^0) \cap S(\Phi^0)$ *and* $C(\Phi^1)$ *have dense interiors.*

(iii) *Assume $f = f(x, u)$ and Γ is compact. Then the semiflows Φ, Φ^0, Φ^1 are compact and order compact, and they have a compact global attractor in common.*

(iv) *Assume Γ is open or order convex and E_Γ is compact. Then some $p \in E_\Gamma$ is stable for Φ. Every such p is also stable for Φ^1, and for Φ^0 in the reaction–diffusion case. When E_Γ is finite, the same holds for asymptotically stable equilibria.*

PROOF. (i) follows from Theorem 1.21.

(ii) for Φ and Φ^0 follows from Theorem 2.25(b). For Φ^1, (ii) follows from Theorem 2.26(a).

(iii) is a special case of Proposition 6.13(b).

In (iv), to find a $p \in E_\Gamma$ having the asserted stability properties for Φ, it suffices to verify the hypotheses of Theorem 1.30: (a) follows from (i), while (b) and (c) holds by the assumptions on Γ and compactness of E. Similarly for Φ^0 in the reaction–diffusion case.

To prove the stability properties for p under Φ^1, it suffices by Theorem 1.31 to show that p has a neighborhood in X^1_Γ that is attracted to a compact set. By (i) and the assumptions on Γ, there are sequences $\{u_k\}$, $\{v_k\}$ in $Q(\Phi^1)$ converging to p in X_Γ, such that

$$u_k \leqslant u_{k+1} \leqslant p \leqslant v_{k+1} \leqslant v_k$$

and

$$p \neq \inf X_\Gamma \implies u_k < u_{k+1} < p, \qquad p \neq \sup X_\Gamma \implies p < v_{k+1} < v_k.$$

Replacing u_k, v_k by their images under Φ_{ϵ_k} for sufficiently small $\epsilon_k > 0$, we see from strong monotonicity of Φ^1 that we can assume:

$$p \neq \inf X_\Gamma \implies u_k \ll u_{k+1} \ll p, \qquad p \neq \sup X_\Gamma \implies p \ll v_{k+1} \ll v_k.$$

The sets $N_k := [[u_k, v_k]]_X \cap X_\Gamma$ are positively invariant and form a neighborhood basis at p in X_Γ.

Fix k_0 such that N_k is bounded in X^1_Γ for all $k \geqslant k_0$. By Theorem 6.1(iii), for every $s > 0$ there exists $j \geqslant k_0$ such that $\Phi_s(N_j)$ is precompact in X^α, hence in X^1_Γ. Fix such numbers s and j and let P denote the closure of $\Phi_s(N_j)$ in X^1_Γ. Being compact and positively invariant, P contains the compact global attractor $K := \bigcap_{t>0} \overline{\Phi_t P}$ for the semiflow in $\Phi^1|P$. Then N_j is a neighborhood p in X^1_Γ that is attracted under Φ^1 to K. □

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