Quantitative Reasoning and the Development of Algebraic Reasoning

John P. Smith III

College of Education

Michigan State University

Patrick W. Thompson

Department of Teaching and Learning

Arizona State University

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RUNNING HEAD: Algebra from Quantitative Reasoning

Abstract

Recent discussions of algebra reform have focused on having more students take algebra courses and on improving their mathematical content. We believe that neither approach can be successful without substantial changes in the K-8 mathematics curriculum's current focus on numbers and arithmetic operations. This curriculum does not prepare students for the use of explicit, rule-governed notational systems to express, manipulate, and formalize ways of thinking about quantitative and numerical relationships. So students' difficulties with algebra result not only from algebra curricula that lack meaning and coherence, but also from elementary curricula that fail to develop students' abilities to reason about complex additive and multiplicative relationships. Without mathematical concepts and relationships to express and manipulate, many students find algebra a meaningless symbolic exercise. We argue that a focus on quantitative reasoning can develop students' abilities to conceptualize, reason about, and operate on quantities and relationships in sensible problem situations. We describe a broad view of quantitative reasoning as it relates to algebraic and arithmetical reasoning and show how it actually provides content for algebra.

> No doubt, it is difficult for a teacher to teach something which does not satisfy him entirely, but the satisfaction of the teacher is not the unique goal of teaching; one has at first to take care of what is the mind of the student and what one wants it to become. (Poincaré, 1904, p. 255)

Introduction

In keeping with the theme of this book—the early development of algebraic knowledge and reasoning—we describe how students might develop knowledge and ways of thinking in elementary and middle school that support their learning of algebra. When we use the term, "algebra," we are not referring to the content of the "Algebra I" course currently taught in most American middle and high schools. We mean the expression, manipulation, and formalization of mathematical concepts and structures mediated by explicit, rule-governed notational systems. As such, the content of algebra, to us, depends on ideas of coherence, representation, generalization, and abstraction. To address the development of algebraic reasoning, especially a meaningful and useful algebra, we must first address a more fundamental problem in mathematics teaching and learning.

For too many students and teachers, mathematics bears little useful relationship to their world. It is first a world of numbers and numerical procedures (arithmetic), and later a world of symbols and symbolic procedures (algebra). What is often missing is any linkage between numbers and symbols and the situations, problems, and ideas that they help us think about. Preparing students for algebra should not mean importing parts of an Algebra I course into the earlier grades. Rather, it should involve changing elementary and middle school curricula and teaching so that students come to use symbolic notation to represent, communicate, and generalize their reasoning.

The opening quote from Poincaré highlights our central point. As we design an "early algebra" program for elementary and middle school students, we must avoid the temptation to make it resemble the algebra familiar to us as adults. There are too many problems with traditional Algebra I in the US—perhaps the most serious of which is students' inability to find meaning and purpose in it—to use it as a model for our efforts (Silver, 1997). For this reason, a fresh approach is needed. Indeed, we must craft our expectations so that students build a kind of algebraic competence that is rich, generative, and multi-purpose.

For most of us, “algebra” means the content of traditional “Algebra I” and the courses that follow it. For authors of Algebra I textbooks, algebra is a tightly integrated system of symbolic procedures, each of which is closely connected with a particular problem type. The procedures are often introduced as the mathematical means to solve specific types of problems, but the focus quickly becomes learning how to manipulate symbolic expressions. These procedures are then practiced extensively and later applied to specific problem situations (that is, “word problems”). Teaching this content involves helping students to interpret various commands—“solve,” “reduce,” “factor,” “simplify”— as calls to apply memorized procedures that have little meaning beyond the immediate context. For many students, this reduces algebra to a set of rituals involving strings of symbols and rules for rewriting them instead of being a useful or powerful way to reason about situations and questions that matter to them. Consequently, many students limit their engagement with algebra and stop trying to understand its nature and purpose. In many cases, this marks more or less the end of their mathematical growth.

Many mathematics educators have recognized the deep problems of content and impact of Algebra I and have made introductory algebra a major site in curricular reform efforts (Chazan, 2000; Dossey, 1998; Edwards, 1990; Fey, 1989; Heid, 1995; Phillips & Lappan, 1998). In one class of proposals, algebra is presented as a set of tools for analyzing realistic problems that outstrip students’ arithmetic capabilities. In contrast to Algebra I, problem situations involving related quantities serve as the true source and ground for the development of algebraic methods, rather than mere pretext (Chazan, 2000; Lobato, Gamoran, & Magidson, 1993; Phillips

& Lappan, 1998). These introductions to algebra aim to develop students' abilities to use verbal rules, tables of values, graphs, and algebraic expressions to analyze the mathematical functions embedded in the problem situations, and centrally involve computer-based tools and graphing calculators to achieve these goals (Confrey, 1991; Demana & Waits, 1990; Heid, 1995; Schwartz & Yerushalmy, 1992).

Other proposals have emphasized the abstract and formal aspects of mathematical practice, suggesting that introductory algebra should develop students' abilities to identify and analyze abstract mathematical objects and systems. For example, Cuoco (1993, 1995) has characterized algebra as the study of numerical and symbolic calculations and, through the development of a theory of calculation, the study of operations, relations among them (e.g., distributivity), and mathematical systems structured by those operations. Cuoco's proposal reflects mathematicians' interest in the study of increasing abstract and general algebraic systems.

Two working groups, directed to chart algebra reform K–12, have proposed a more pluralistic approach (National Council of Teachers of Mathematics Algebra Task Force; 1993; National Council of Teachers of Mathematics Algebra Working Group, 1997). They identified four basic conceptual themes in current algebra reform proposals—functions and relations, modeling, structure, and representation and language—which in turn can be explored in various mathematical contexts, such as growth and change, number, pattern and regularity (National Council of Teachers of Mathematics Algebra Working Group, 1997). Rather than mandate one best introductory algebra, these educators anticipated different courses that emphasize different themes and draw from different contexts (see also, Bednarz, Kieran, & Lee, 1996). Kaput's (1995) characterization of algebra and algebra reform was similarly pluralistic, identifying five major strands of algebraic thinking which alternately focus on mathematical process (generalization, formalization, manipulation), content (structures, functions), and language.

Given this proliferation of alternatives, one approach to “early algebra” would pick one view of algebra and develop scaled-down introductory versions in earlier grades. For example, the current “Pre-Algebra” course common to middle schools is a scaled down version of Algebra I. We suggest this would be a mistake. We believe it is possible to prepare children for different views of algebra—algebra as modeling, as pattern finding, as the study of structure—by having them build ways of knowing and reasoning which make those mathematical practices different aspects of a more central and fundamental way of thinking. Alternative views of “what algebra is” can be seen as different emergent aspects of making sense of one’s world quantitatively. To invoke a biological metaphor, the development of quantitative reasoning can serve as the conceptual root stalk for many different approaches to “algebra.” Because the stalk can support multiple branches, wedding "early algebra" to one or another approach is unnecessary and limiting.

We advocate an early emphasis on developing children’s ability to conceive of, reason about, and manipulate complex ideas and relationships, as an equal complement to numerical reasoning and computation. Children who develop a rich capacity for reasoning about general relationships among quantities will possess the conceptual foundation for learning and making sense of different programs and views of algebra. This chapter describes a conceptual orientation toward “what is going on” in complex quantitative situations, showing how teachers can help students make mathematical sense of those situations. The key claim in our argument is: *If students are eventually to use algebraic notation and techniques to express their ideas and reasoning productively, then their ideas and reasoning must become sufficiently sophisticated to warrant such tools.*

There is a reciprocal relationship between the long-term development of students' algebraic abilities and the long-term development of their reasoning from which these abilities emerge. If algebra, meaning the use of representational practices that employ systematic use of symbols to express quantitative and structural relationships, is to become students' means of expressing and supporting their thinking, they must have experiences from whence the thinking that those practices support emerges. Likewise, if they are to develop thinking that calls for representational practices that employ systematic use of symbols to express quantitative and structural relationships, then the roots of those practices must be present in their early activities.

Algebra, Situations, and Quantities

For the mathematically sophisticated, the best approach to complex mathematical problems is to use the tools of algebra to help manage that complexity.[1] We move quickly away from the problem situations themselves, with all their complex relationships, toward the formality of algebraic and numerical expressions and manipulations. In appealing to algebraic methods to solve complex problems, we elect not to use less formal approaches that are tied more closely to the situation. We also tend to devalue this "informal" reasoning in comparison to algebraic methods. In this devaluation, we miss an essential connection between the two kinds of thinking: That more concrete, intuitive, and situation-specific patterns of reasoning, appropriately supported and nurtured over a period of years, can foster students' development of the algebraic reasoning we value so highly. If our goal is for students to understand and use algebra, then the success of an "early algebra" program will depend on supporting the development of formal reasoning from an informal foundation.

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[1] Mathematics educators use the term, "problem," in various different ways. In this chapter, our "problems" are mathematical tasks suggested by verbal descriptions of situations constituted by interrelated quantities. Such "problem situations" can generate many "problems." In students' terms, our "problems" are always "word

To illustrate the common separation of formal, algebraic reasoning and informal reasoning, compare a traditional algebraic solution to the following problem to one that more directly involves the quantities and relationships in the problem situation.

Problem 1. *I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetski, 1976, p. 160)*

A typical algebraic solution to this problem involves assigning variables, writing algebraic expressions, and eventually stating and solving an equation. If t represents the number of minutes I have walked, then whenever I have walked t minutes my brother will have walked $t + 6$ minutes. If d stands for the number of miles from home to school, and if I and my brother travel at constant speeds, then my walking speed is $d/30$ miles per minute and my brother's is $d/40$ miles per minute. Using the general relationship that "rate multiplied by time equals distance" ($d= r*t$), these expressions can be stated in the equation, $(t +6)d/40 = t\, d/30$, which can be easily solved from its equivalent form, $(t+6)/40 = t/30$. Related motion problems like Problem 1 are common in algebra textbooks because algebraic methods are presumed necessary to solve them. As long as no difficulties arise, this algebraic approach makes few explicit references to how speeds, times and distances are related in the situation. The basic idea is to move out of the situation and its constituent quantities and into the world of symbolic expressions and equations.

But this problem can also be solved by reasoning about the relationships among distances, walking rates, and times of travel without the support of variable assignments or algebraic expressions. Here is one example of this approach, which we will call "quantitative reasoning."

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problems." In addition, we follow Schoenfeld's (1985) view that "problems," in contrast to "exercises," cannot be solved by routine, well-practiced methods. They require thinking.

- *I imagine myself walking behind my brother, seeing him ahead of me. What matters in catching up with him is the distance between us and how long it takes for that distance to become zero.*
- *The distance between us shrinks at a speed that is the difference of our walking speeds.*
- *I take 3/4 as long as brother to walk to school, so I walk 4/3 as fast as brother.*
- *Since I walk 4/3 as fast as brother, the distance between us shrinks at the rate of 1/3 of brother's speed.*
- *The time required for the distance between us to vanish will therefore be 3 times as long as it took brother to walk it in the first place (6 minutes).*
- *Therefore, I will overtake brother in 18 minutes.*

Like the algebraic solution, this reasoning is quite sophisticated, requiring a rich understanding of how times, speeds, and distances are related, how those relationships can be used to draw inferences, and how numerical values can be inferred from those that are given. It also has the same level of potential generality as the algebraic solution. If different initial numerical values were given, the calculations in the solution might become more cumbersome, but the logic of the reasoning would not change. The two solutions differ most visibly in their use of algebraic symbols. However, they differ more deeply in the former's focus on translating relationships into symbols and the latter's focus on expressing and working directly with those relationships.

In proposing quantitative reasoning as a root for algebraic thinking we acknowledge that the former does not develop easily or quickly. In fact, the student who produced this solution achieved his proficiency from a wide variety of experiences over several years. Our thesis is that students' quantitative reasoning is worth years of attention and development, both because it increases the likelihood of success with algebra *and* because it makes arithmetic and algebraic knowledge more meaningful and productive.

Algebraic reasoning is characterized by its generality and by the role that symbolic expressions play in stating general relationships, comparing and manipulating them, and facilitating many numerical evaluations. Quantitative reasoning, when developed throughout children's elementary and middle school years, develops mathematical ideas of similar generality that students will eventually find sensible to express in algebraic notation. Put simply, quantitative reasoning provides conceptual content for powerful forms of representation and manipulation in algebra.

Before we proceed, it is important to emphasize that we are *not* using the terms "quantity" and "quantitative reasoning" as synonyms for "number" and "numerical reasoning." Indeed, our central purpose below will be to show how the elementary years can be used to support the development of students' quantitative reasoning by focusing their attention away from thinking strictly about numbers and numerical operations. In our view, conceiving of and reasoning about quantities in situations does not require knowing their numerical value (e.g., how many there are, how long or wide they are, etc.). Quantities are attributes of objects or phenomena that are measurable; it is our *capacity* to measure them—whether we have carried out those measurements or not—that makes them quantities (Thompson, 1989; 1993; 1994). In this sense, we follow Piaget's meaning of quantity and quantification (Piaget, 1952; 1970). But as we do, we also acknowledge that other analysts draw much closer associations between quantity and number (e.g., Fey, 1990; Fuson, et al., 1997).

Relationships Between Quantitative Reasoning and Algebraic Reasoning

The two prior solutions to Problem 1 suggest a stark contrast. One translated the relationships in the situation into traditional algebraic expressions and looked like "algebra;" the other directly manipulated the relationships among the quantities in the situation—elapsed times, walking speeds, and walking distances. While this contrast exemplifies the character of quantitative reasoning as a distinct form of mathematical thinking, we stress the connections between quantitative reasoning and algebra, as well as their differences. Before we consider how such sophisticated quantitative reasoning can be nurtured over the years, we return to Problem 1 and examine three solutions in greater detail. These solutions (the two given above and one more) show how quantitative reasoning can underlie and motivate reasoning with symbols.

The traditional algebraic solution involved generating and solving the equation, $(6+t)\frac{d}{40} = t\frac{d}{30}$. But what sort of thinking could motivate the initial variable assignments and the symbolic expressions for times and speeds found in that equation? If the equation writer understood the problem situation (rather than memorized a script for this problem type), his/her reasoning might have had some of the following character.[2]

a. *Since we both begin from home, I will catch my brother when we both have walked the same distance from home.*[3] *We both walk any distance by traveling at some speed for some amount of time.*
b. *I do not know how far it is from home to school, but I can think of it as some number of miles, which I will designate by d. My brother walks d miles in 40 minutes, so his speed is* $\frac{d}{40}$ *miles per minute. I walk d miles in 30 minutes, so my walking speed is* $\frac{d}{30}$ *miles per minute.*
c. *At any moment in my walk, I have walked* $t\frac{d}{30}$ *miles in t minutes. After I start, brother will have walked for 6 minutes longer than I, so when I have walked t minutes, brother*

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[2] We do not claim that all cases of reasoning sensibly with algebraic symbols follow these exact steps. This is only one example of sensible algebraic reasoning on this problem.

[3] The reasoner assumes that the walkers follow the same path.

will have walked (t+6) mnutes. Therefore, he will have walked $(t+6)\frac{d}{40}$ *miles when I have walked t minutes .*

d. *I will catch brother when he and I have walked exactly the same distance from home. At that moment, our two distances will be the same, so the formula for his distance,* $(t+6)\frac{d}{40}$ *, and the formula for my distance,* $t\frac{d}{30}$ *, will have the same value. So, I am looking for values of t that make the sentence* $(6+t)\frac{d}{40} = t\frac{d}{30}$ *true for any value of d.*

We consider this reasoning a good example of using algebra with "understanding." The main content of that understanding is a solid conceptual grasp of how the quantities of walking speeds, times traveled, and distanced traveled from home are interrelated. This elaborated algebraic solution differs from the first "bare-bones" one in that it (1) restates the problem in terms of the reasoner's own experience with relative motion, (2) sketches the logic for transforming walking times into speeds, (3) sees the expressions for distanceas complete and continuous descriptions of motion, and (4) generates an equation to determine where those distance expressions produce the same value. Indeed, every step in the solution expresses some conceptual relationship between two or more quantities in the situation, and it is these relationships that motivate and justify the various algebraic expressions. One role for quantitative reasoning in complex problem solving is therefore to provide the content for algebraic expressions so that the power of that notation can be exploited.

Another role of quantitative reasoning is to support reasoning that is flexible and general in character but does not necessarily rely on symbolic expressions. We return to the non-algebraic, quantitative solution of Problem 1 and unpack it to show how such sophisticated reasoning might grow and how it shares the generality that characterizes algebraic reasoning.

- *I imagine myself and my brother walking. What matters in catching up with him is the distance between us and how long it takes for that distance to vanish.*

The reasoner projects herself into the problem situation, adopting the perspective of actually looking at her brother walking ahead of her. From this perspective, her distance and brother's distance from home are irrelevant, and the only thing that matters is the distance between them. It is this step of imagining oneself into the problem situation that so often eludes students.

- *The distance between us shrinks at a rate that is the difference of our walking speeds.*

This is quite a sophisticated inference. When two quantities change at constant rates, in the same direction, and we consider how rapidly their measures move apart, we are asking at what rate the difference between them changes. If we consider each one changing for a unit of time, then the added difference will be the difference of their rates. Thus, the rate at which the excess of one over the other changes is the *difference* of the two quantities' rates. In terms of distance and speed, the rate at which the distance between the walkers changes is the difference of their walking speeds. We hasten to add that by *difference* we do not mean the result of subtracting. Rather, we mean the distance that is created by comparing how much one distance exceeds or falls short of the other.

- *I take 3/4 as long as brother to walk to school, so I walk 4/3 as fast as brother.*

This is another sophisticated inference. It exploits a general understanding of speed as a rate of change of distance with respect to changes in time. Because we walk the same distance, our walking speeds are different only because our travel times are different. Longer travel times mean slower walking speeds. If I walk the same distance as brother in 3/4 the time, then I would walk 1/3 again as far as brother in the same amount of time (Figure 1). So I walk 4/3 as fast as brother because I walk four one-thirds (4/3) as far as brother in the same amount of time.

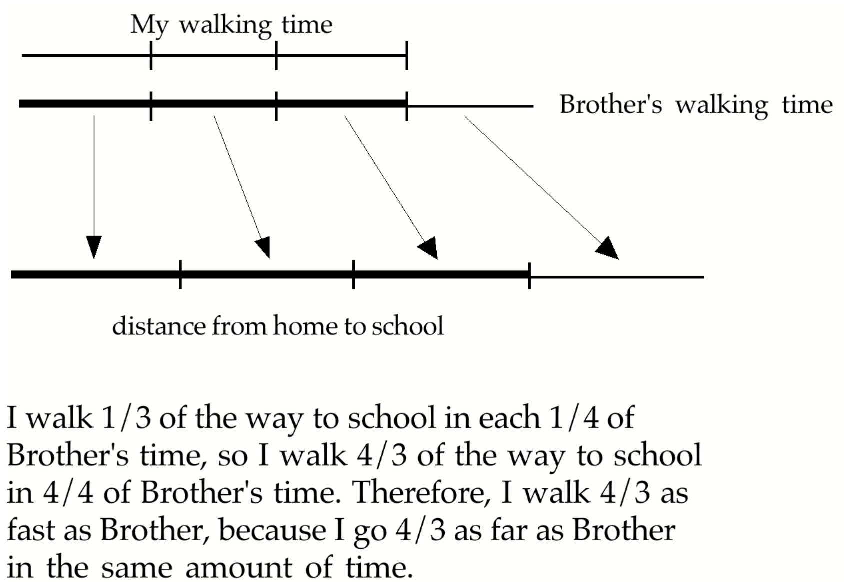

Figure 1. Relating my speed to brother's speed based on the relationship of my time to his.

- *Since I walk 4/3 as fast as brother, the distance between us shrinks at the rate of 1/3 of brother's speed.*

Walking 4/3 as fast as brother means that my speed is 1/3 greater than my brother's speed. So, in a given amount of time, I not only walk as far as brother, I walk an extra one-third of the distance he has walked. Therefore, the distance between us shrinks at the rate of one-third of brother's speed.

- *The time required for the distance between us to vanish will therefore be 3 times as long as it took brother to walk it in the first place (which was 6 minutes).*

If brother took some amount of time to walk some distance, and another person walked at one-third of brother's speed for the same amount of time, then that person will walk one-third of the distance brother walked. So, in 6 minutes, the amount of time brother used to get ahead of me, the distance between us will shrink by 1/3 of the distance he walked in 6 minutes. I need to shrink this distance 3 times, so it takes me 3 times 6 minutes, or 18 minutes, to catch brother.

- *Therefore, I will overtake brother in 18 minutes.*

This solution illustrates some important features of quantitative reasoning and its origins. First, quantitative reasoning draws heavily on everyday experience. The basic approach—studying how the distance between the walkers decreases—depends on the reasoner projecting herself into the situation and invoking the visual imagery of "catching up." Once framed in that way, the solution proceeds by drawing on relationships among speeds, times, and distances. The change in distance-between-walkers is cast as a rate of change and expressed in terms of the brothers' walking speeds, and numerous quantitative manipulations of the speed-time-distance relationship support numerical inferences about the value of various quantities. Finally, though it is grounded in everyday experience, it is difficult to imagine how students could develop this level of facility without focused instruction which draws on and stretches their abilities to state general relationships and make inferences from them.

We do not offer this solution as paradigmatic of "quantitative reasoning." Indeed, quantitative reasoning does not typically follow any standard pattern or routine like the variable assignment and equation solving in traditional algebraic problem solving. Quantitatively-oriented solutions tend to vary more widely than algebraic solutions to the same problem, primarily because they are grounded in *how students conceive of situations*, and there is tremendous range in these conceptions. To illustrate this variety we present another quantitative solution to Problem 1 produced by a less mathematically mature student. This student's reasoning was less general but grounded in a solid, concrete understanding of constant speed as a rate of change.

- *Imagine the distance from home to school cut up into 30 pieces. Each piece is how far I walk in one minute.*
- *Imagine the distance from home to school also cut up into 40 pieces. Each of these pieces is how far brother walks in one minute.*

- *Brother's one-minute-distance piece will be 3/4 the length of my one-minute-distance piece.*

This statement directly compares the length of the one-minute-distance to brother's one-minute-distance.

- *In six minutes, Brother will travel six [of his one-minute distances], so he will travel 18/4 of my one-minute-distance piece. That is how far ahead of me brother is.*

Six iterations of 3/4 of my one-minute-distance piece is 18/4 of my one-minute-piece.

- *When I start walking, I will move closer to brother by 1/4 of my one-minute-distance piece each minute.*

If for every one-minute-distance piece I move brother moves 3/4 of that one-minute-distance piece, then I am gaining by the difference each minute.

- *I will make up 18/4 (eighteen one-fourths) of my one-minute-distance piece in 18 minutes when I gain on brother at the rate of 1/4 one-minute-distance piece each minute.*

Several features of this solution are worth highlighting. Though it may appear entirely "arithmetical" and "concrete," it involves quantities whose actual values are unknown (distance from home to school, "my" one-minute-distance, brother's one-minute-distance), yet from which the reasoner derives essential information ("in one minute, brother travels 3/4 the distance I do"). Her study of one-minute-distances and how they build up over time also coordinates distances and times without the appeal to speed-as-rate as a mediator, whereas a rate conception of speed was central to the previous quantitative solution. This contrast underscores the inappropriateness of expecting particular statements of conceptual relationships in quantitative solutions. Quantitative relationships—especially complicated, multiplicative ones—can be expressed in many ways. This reasoning describes in verbal terms what was expressed in symbols in the "bare-bones" algebraic solution. Brother's one-minute-distance piece is equivalent to the formula $\frac{d}{40}$ if d were used to represent the distance from home to school. Finally, ideas of functional co-variation— how one quantity varies in relation to the variation of another—are central to this student's reasoning. Segmenting the total distance into one-minute-distance intervals provides a framework for comparing and coordinating distances and eventually to quantify how much she is gaining.

But reasoning of this sort, even if less sophisticated than the previous solution, does not spring forth either quickly or spontaneously. It must be carefully nurtured over many years. On the one hand, it requires positive support from curricula and pedagogy that extend children's existing abilities. The mental operations used in quantitative reasoning must be built in many contexts and over relatively long periods of time. On the other, it means avoiding the classroom orientation that quickly shifts the focus away from making sense of situations and toward calculation (Thompson, Philipp, Thompson, & Boyd, 1994). It opposes the prevailing view that mathematics is about getting "the answer," in numerical or symbolic form. Senseless patterns of thinking emerge for students once meaning and purpose in mathematics disappear, and students' expectation of making sense is therefore difficult to restore. In the next two sections, we attempt to illustrate how richer capacities for quantitative reasoning can be nurtured in the elementary and middle school years, beginning with additive situations.

Quantitative Reasoning and Arithmetic Reasoning

We now look more closely at what we mean by quantities and relationships between them. Just as we contrasted the emphasis on symbolic procedures with reasoning about quantities and relationships in Problem 1, we emphasize here the difference between reasoning about numbers and calculations and reasoning about quantities in a problem situation typically seen as "arithmetic."

Problem 2. *At some time in the future John will be 38 years old. At that time he will be three times as old as his daughter Sally. Sally is now 7 years old. How old is John now? [adapted from Thompson, et al., 1994]*

A Numerical/Computational Solution

From one perspective, Problem 2 is a three-step arithmetic word problem. To solve it, students must first divide 38 by 3 to determine that Sally's age in the future is 12 2/3 years; then subtract 7 from 12 2/3 to determine that the difference between her age "then" and "now" is 5 2/3 years; and finally subtract 5 2/3 from 38 to determine John is 32 1/3 years old now. Most middle grades students know that there are three numbers they must use (38, 3, and 7) and four operations (addition, subtraction, multiplication, and division) to choose from. They know they must find the right sequence of operations on the right pairs of numbers (including intermediate results like 12 2/3) to produce the correct final answer. From this perspective, classroom discussions usually center on those issues—which numbers, which operations, and in what order? Though some attention might be given to justifying the operations (e.g., noting that "three times as old" is a clue to multiply), the primary focus is numerical and computational.

A Quantitative/Conceptual Solution

A quite different approach is also possible—one that centers on what many people call "understanding the problem" (see Riley, Greeno, & Heller [1983] for a very detailed example of this general view). From this perspective, the problem concerns quantities, their properties, and relationships among them, and its solution involves reasoning about those relationships and eventually linking them to numerical operations. This perspective focuses on helping students conceptualize situations irrespective of the numerical information they are presented and the calculations they can produce.

To illustrate this approach we first identify four quantities in Problem 2: "John's age at some future time," "Sally's age at that same future time," "Sally's age now," and "John's age now." We can think about these ages and many relationships between them (e.g., one person is older, their ages change at the same rate) without knowing their numerical values. The fact that we happen to know the values of two of them—"John's age at some future time; 38 years" and "Sally age now; 7 years"—is incidental to our ability to think about their ages changing over time.

These four quantities by themselves do not represent "what is going on" in Problem 2. To comprehend fully we need to recognize three important relationships that integrate the quantities into a coherent structure. First, there is the temporal relationship that time "moves on" from "now" to "then" which relates John's ages and Sally's ages (Figure 2).4

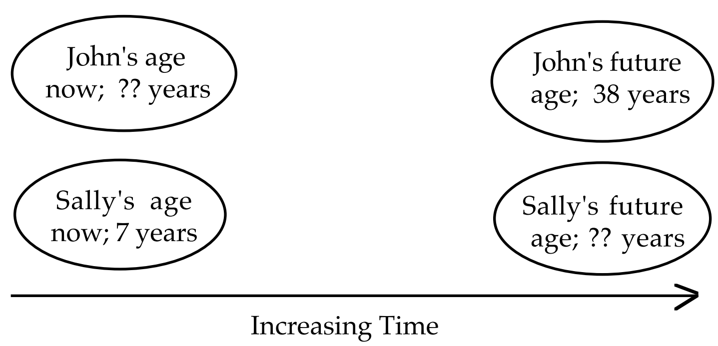

Figure 2. Everyone ages as time passes.

Second, we need to recognize that John's ages (now and in the future) and Sally's ages (now and in the future) stand in a specific quantitative relationship to each other, commonly called a difference. The difference between John's two ages, for example, is the amount of time

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[4] We use ovals to represent quantities and place inside those ovals all relevant information about those quantities—their "name," their units of measure, and any numerical value or expression that is given or can be inferred—here and in all subsequent Figures. This is a convenient notation, but there is nothing mathematically or cognitively unique or essential about it.

by which his age in the future exceeds his age now. Though we cannot immediately determine the value of this difference, we can imagine it. We also know that however much Sally grows older, John (and every other person) will grow older by the same amount, so the difference between Sally's present and future ages will be the same as the difference in John's present and future ages. We represent these relationships as new quantities (the differences in Figure 3) linked to the age quantities.

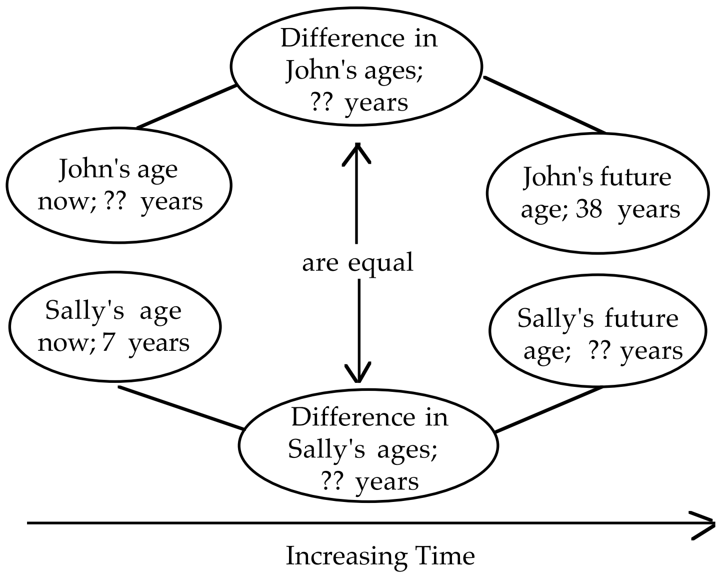

Figure 3. Time passes in equal amounts for everyone.

A third relationship provides a crucial link between John's and Sally's ages. At some point in the future, John's age will be three times as great as Sally's. This relationship is typically called a ratio. We use that term to indicate a quantity that expresses a multiplicative comparison of two other quantities. A ratio's measure describes how many times as great the measure of one quantity is as the measure of the other. As with differences, we refer to ratios as quantities. Both are born of comparison and therefore have dual existences: They express relationships between two quantities (either a multiplicative [ratio] or additive [difference] comparison), and they are quantities in their own right (as a measurable attribute of such comparisons).

When we include the ratio between John's and Sally's future ages to the structure presented in Figure 3, we have identified, analyzed, and related the quantitative information relevant to answering the question of how old John is now. In traditional terms, we have built an "understanding" of the problem which will support and justify our arithmetical reasoning (Figure 4).

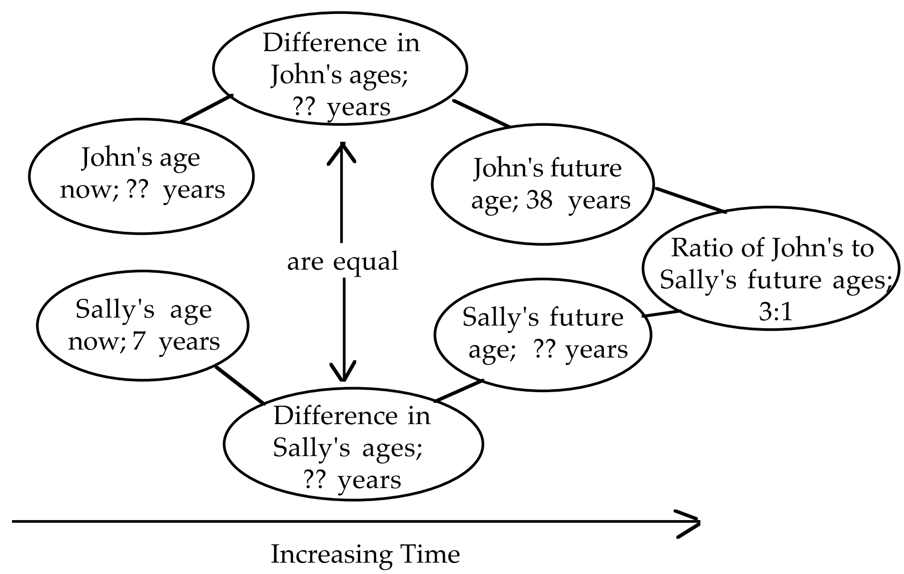

Figure 4. Relationships among quantities.

With such an understanding, it is easy to decide (and justify) what calculations are needed. When John's future age is three times as large as Sally's future age, Sally's age at that moment will be 1/3 as large as John's. Thus, it makes sense to divide "John's age in the future" by 3 to determine the value of "Sally's age in the future" (Figure 5). Knowing the values of Sally's current and future ages, we can determine by how much older she grows, which is 38/3 - 7, or 17/3, years. Since we know that John grows older by the same amount as Sally (17/3 years), and that in 17/3 years he will be 38, we can make our final computation to find that John's age now is 32 1/3 years (Figure 5).

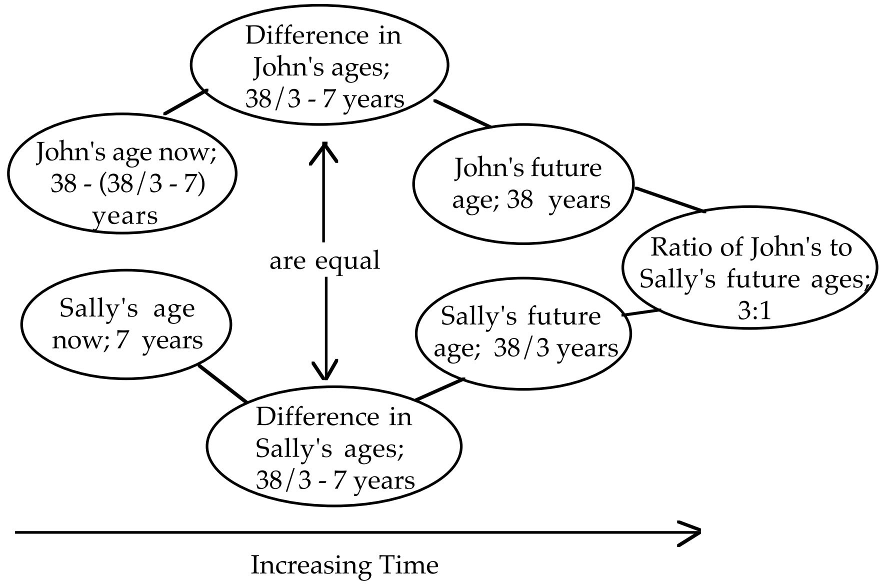

Figure 5. Computing within relationships among quantities.

So, What's the Point?

Our purpose in these extended examples is to emphasize the richness that mathematical reasoning can have when we focus on quantities and relationships among them instead of on numbers and arithmetic operations. Sowder (1988) showed that when students do not attend to quantities and relationships, their problem solving quickly becomes a matter of ungrounded debate about choosing numbers and operations. An emphasis on the quantitative aspects of situations reorients students' mathematical focus in three important ways, affecting both the development of their arithmetic reasoning and their future prospects in algebra.

First, the quantitative/conceptual approach makes thinking about the quantities and their relationships a central and explicit focus of "solving the problem." The resulting conceptual structure, that we have represented in a series of diagrams, can be used to explain and justify both quantitative inferences and numerical computations. Such diagrams create public frameworks that students and teachers can use to think about situations, making it less likely that students will see numerical computations as materializing from nowhere. They also create contexts for examining mathematical issues that are unlikely to arise in the numerical/computation approach. For example, a teacher could raise the question of whether the ratio between John's and Sally's ages remains constant as they get older. Questioning how ratios of people's ages change over time could lead to an examination of (a) how ratios change as the related quantities increase in equal increments and (b) how those increments themselves must change for ratios to remain constant.

Second, this focus on thinking about and representing general relationships between quantities (i.e., the *relationships* inherent in the quantities themselves, not the specific numerical values they take on) supports the kind of conceptual development that will eventually make algebra a sensible tool for thinking and problem solving. We say this because quantities are inherently indeterminate. We can imagine comparing two heights without knowing their specific measures. Heights are quantities that we understand as being measurable, but knowing their measures does not add to conceptualizing the comparison. Rather, knowing their measures simply adds information about the comparison. Algebraic notation and methods are powerful tools for stating, analyzing, and manipulating general relationships, but without ideas of substantial generality to express, students will find little sense in and little use for algebra.

Third, the quantitative/conceptual approach also suggests an early route to algebraic symbols in its focus on representing the general numerical relationships, rather than specific computations. If students write their calculations in open form (e.g., "38/3 - 7" instead of "$5\frac{2}{3}$") and focus on the nature of that calculation rather than its result, they can adjust more easily to using expressions in place of computed values. The purpose of writing open expressions is to record a chain of reasoning clearly—a rationale that is useful in arithmetic as well as algebraic reasoning. Open expressions make it much easier for students to think about the effects of changing a given numerical value in the situation and therefore support the shift in focus from particular to general relationships (Mason, this volume). Also, if students regularly use expressions to represent values, it is a much smaller step to using formulas to represent values.[5] Of course, it is incumbent upon teachers to draw children's attention consistently to the nature and purpose of their activity—that they are in fact representing a number without actually having to compute it and that they are reasoning about many similar problems all at one time.

In emphasizing the importance of open numerical expressions as an entry point to algebraic formalism, we recognize that they are a standard early topic in the traditional Algebra I and Pre-Algebra curricula. However, with students' prior experiences dominated by numerical computation, most of them do not understand what purpose open expressions serve. After all, if you can complete a computation, why state it in incomplete terms? When the focus is on grasping, stating, and exploring general relationships between quantities, open expressions serve a clearer purpose: To connect general relationships (like differences) to specific situations, quantities, and numerical values.

The comparison of the two solutions to Problem 2 also illustrates other features of quantitative reasoning that we attempt to elaborate in the balance of the chapter. Specifically:

- Some quantities arise directly from measuring things; others, like differences and ratios, arise from quantitative operations—operations on other quantities. Quantitative operations (e.g., multiplicative comparison) are not the same as numerical operations (e.g., multiplication) despite the frequent similarity in terminology.
- Quantities that result from quantitative operations exist in two different senses, as quantities in their own right and as relationships between the two quantities. It can be conceptually demanding to reason and communicate about such quantities because

[5] With just a slight change in information, children could engage in the same pattern of inferences as depicted in Figure 5, with a symbol to represent a numerical value. Activities designed with this end in mind—to generalize a pattern of inferences instead of a pattern of numbers—lead much more naturally to generating symbolic expressions as models of quantitative relationships.

we must distinguish and coordinate these two senses, and, when necessary, shift between them.

- Quantitative reasoning produces essential non-numerical inferences about quantities and how they relate in the problem situation. It is often the "glue" that holds arithmetic reasoning and algebraic reasoning together.
- Cognitive resources other than spoken language are useful (and often necessary) in managing quantitative reasoning in complex problem settings. One class of resources are diagrams that represent relationships in sensible, public ways.
- Mathematicians and educators lack a standardized, accepted terminology for quantities. Some terms used in this chapter, like difference, represent relatively standard usage; others, e.g., ratio and rate, are less standard (see Thompson, 1994). But the names used to designate types of quantities are less important than the way people think about them.

Quantitative Reasoning in Complex Additive Situations

Situations that involve complex additive relationships (i.e., more than three related quantities) can be an important site for the early development of students' quantitative reasoning. Developing young students' abilities to reason with additive situations prepares them for algebra in multiple ways. It provides occasions for them to think about situations systematically, initially ignoring matters of calculation and instead focusing on "what is going on." When additive situations include large numbers of interrelated quantities, this complexity presses students' abilities to understand, represent, and express those relationships and therefore develops their non-numerical mathematics skills. If we want students to learn and use algebra as a sensible tool for expressing their thinking and solving problems, then work with complex problems must come first.

Students who have mastered addition and subtraction as operations on numbers may have much more to learn about additive relationships among quantities. For example, consider the following two problems:

Problem 3. *Thomas has 38 baseball cards and 13 more than his friend Alex, How many baseball cards does Alex have?*

Problem 4. *Jim, Sue, and Tom played marbles. Sue won 6 marbles from Jim and 5 from Tom. Jim won 3 marbles from Tom and 4 from Sue. Tom won 12 marbles from Jim and 2 from Sue. (Compare Tom's number of marbles before and after the game.) [adapted from Thompson, 1993]*

By grade 3, Problem 3 is not difficult for most students, despite the fact that "more" could be mistaken as a cue to add 38 and 13. But even older students struggle with Problem 4 because they must manage many quantities (collections of marbles and changes in those collections), construct relationships among those quantities, and reason about those relationships, instead of simply choosing numbers from the problem statement and computing. (We placed parentheses around the last sentence in Problem 4 to emphasize that situations can be the instructional focus without becoming "problems"—that is, without asking for a quantity's value.)

Problem 4 illustrates some of the quantitative issues that can receive attention when the main goal is to help students understand what is going on in situations. A teacher might ask, "Is it possible for Jim to win 3 marbles from Tom if Tom won 12 marbles from Jim?" and then make their understanding of that possibility the focus of her lesson. The ensuing discussion, when oriented in this way, could move in the direction of considering how the changes in the various collections of marbles interrelate (see Thompson [1993] for an example).

Developing students' ability to reason with complex additive relationships means rethinking the notion of "problem" in elementary mathematics. We cheat our students if our problems are always requests for calculations. In this section and the next, we illustrate a different sense of "problems"—that of situations where students conceptualize and reason about relationships between quantities. If students' experience of problems changes, so can the kinds of questions that teachers can ask, the kinds of assessment that make sense, and the character of students' capabilities. We must guard against underestimating students' quantitative reasoning abilities. They come to school fully capable of learning to reason with simple additive relationships and that reasoning can develop in impressive ways with thoughtful instruction (Carpenter & Moser, 1984; Carpenter, Moser, & Romberg, 1982; Fennema, Carpenter, & Peterson, 1989; Fuson et al., 1997). A major task of the elementary curriculum should be to build on that competence.

Sources of Differences

In this section we describe how differences—a key component of additive problem situations—can arise in a variety of ways. We also examine the kinds of questions that can be posed in complex additive situations; and discuss some general aspects of students' quantitative reasoning in these situations and how teachers can support and extend it. This section does not even begin to sketch out a K-8 curriculum in additive reasoning, but it does provide examples of how traditional word problems can be adapted to support the development of quantitative reasoning.

Many complex additive situations centrally involve one or more differences, quantities that measure how much one quantity exceeds or falls short of another. Differences arise in situations in at least three ways. In some cases, they emerge when actions physically change quantities in the situation. For example, in Problem 4, Sue's act of winning 6 marbles from Jim created the difference, "the 'new' marbles in Sue's collection."[6] Differences also result from comparisons of two quantities that remain wholly intact in the situation, e.g., between two people's height (how much taller?), between size of classes (how many more children?), between two people's driving speed (how much faster?). Third, differences can emerge when quantities change over time but without any physical act of transfer. For example, we can conceive of the daily fluctuations in the price of some commodity relative to its "base" price as a sequence of differences, one for each day.

The basic case. In the conceptually simplest case, a difference compares two quantities that are not themselves the result of other quantitative operations (i.e., they are not differences, ratios, or rates). Conceptualizing a difference means thinking about three quantities in relationship to each other.

The situations in Problems 2–4 included examples of basic differences, such as the difference between John's age now and Sally's age now (Problem 2) and the difference between Sue's marbles before and after playing Jim (Problem 4). In the first example, the values of the difference and one quantity were known, so the value of the third quantity could easily be calculated. In the second, only the value of the difference was known so there were many possibilities for the number of marbles in Sue's collection before and after playing Jim. It is not necessary—*and this is a crucial point*—to know the values of *either* quantity to conceptualize their difference and reason about it. Conceptualizing a difference only requires thinking about the excess (or deficit) of one quantity over another. As Carraher and Schliemann (2000) have shown, even primary age children can conceptualize differences and reason about them in relation to their constituent quantities.

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[6] Alternatively, the same act can be seen as creating the difference, "the marbles that are no longer in Jim's collection." The character of the difference depends on how the reasoner conceives the situation.

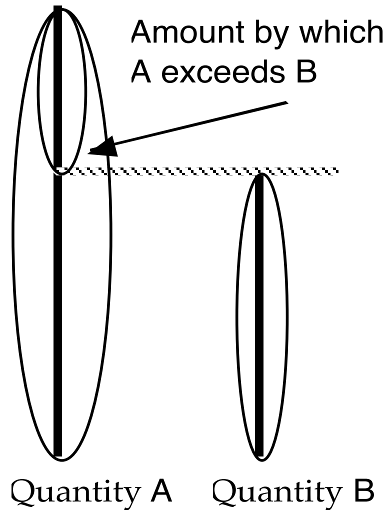

Figure 6. Difference of two quantities.

Operations on differences. In some situations two or more basic differences are present, and reasoning about those situations can involve additive comparisons or combinations of those differences.[7] Conceptualizing a difference of differences or a combination of differences means coordinating the relationships among seven quantities (though not necessarily simultaneously).

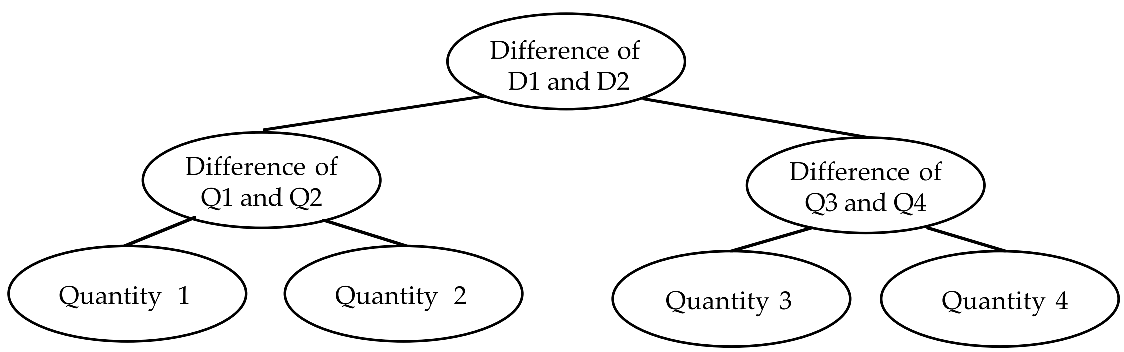

Figure 7. Difference of differences.

In, "Allen is 27 years older than Alva and Alva is 13 years younger than Denise," we can think about the relationship between Allen's and Denise's ages as a comparison: how much older than Alva is Allen than Denise? Relative to Alva's age, Allen is "more older" than Denise, by exactly 14 years. In, "the price of regular unleaded gas at Sam's station increased 5.9¢ one month, decreased the next month, and then was 3.7¢ lower at the end of those two months," the change during the second month can be thought of as a combination of differences (Figure 8).

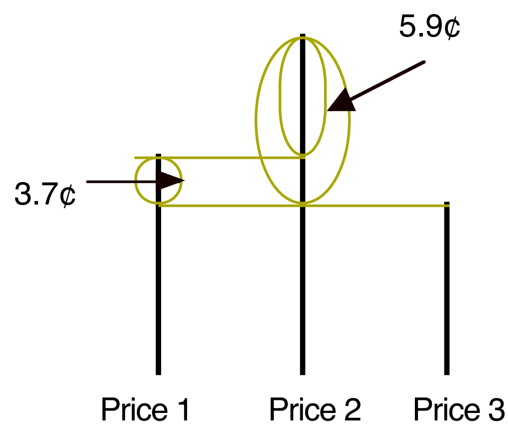

Figure 8. The difference between the 2nd and 3rd months' prices is seen as a combination of two differences.

As in the basic case, conceptualizing a difference or a combination of differences, however, does not require or necessarily follow from knowing the specific values of the differences that are compared. In the statement, "Toni compared her height to her brother's height and Melissa compared her height to her brother's height. They found that Toni is taller than her brother by more than Melissa is taller than hers," we know neither the specific value of any of the three differences nor the specific values of the four people's heights. Yet, we can imagine a comparison of Toni's and Melissa's differences and think about that difference as a quantity. Even if the value of the difference of height differences were known, many different heights and height differences could produce it. For this reason, situations involving differences of differences are often complicated to sort out, make sense of, and speak about. There are more quantities to keep track of; they participate in multiple relationships; and assigning or changing their numerical value may have complicated effects.

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[7] We could just as easily consider differences of ratios or any other quantities that are themselves the result of a quantitative operation. Since the context is additive situations, we only discuss a difference of differences.

Patterns of differences. Other situations involve additive comparisons of two quantities which change many times (even continuously) in the situation. Repeated or continuous additive change creates the possibility of conceiving a pattern of differences. Conceptualizing a pattern of differences means grasping the *collection* of changes as an object of consideration. This approximates thinking about differences as a function of two variable quantities.[8] In many cases, changes take place over time, so differences can be arranged and thought about as a temporal sequence. For example, the performance of a business is often conceptualized as a series of differences between revenues and expenses over some units of time, say months (Figure 9).

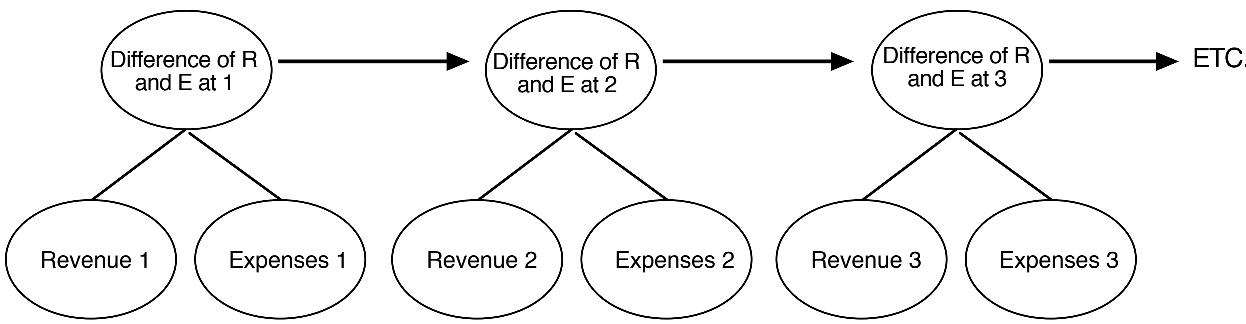

Figure 9. Chain of differences.

Curriculum: Problems and Problem Situations

In developing a curriculum to support students' additive reasoning capabilities, it is important to consider the kind and complexity of the situations presented. Since the overall goal is to prepare students to use mathematics to think about their world, it is sensible to choose and/or develop situations and quantities for which they have rich, everyday experience. This general principle does not imply a strict realism (e.g., "kids can only think about quantities they have directly experienced"). Rather, it means that the situations where students can draw upon broadest reservoirs of personal experience, thinking, and talk (e.g., motion, growth, physical characteristics) are dependable places to start. They are contexts where students' reasoning may

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[8] If one quantity did not vary, the pattern of differences would be similar to a function of one variable. If both

be initially the most developed. But other features also contribute to the complexity and difficulty of situations. In general, additive situations become more complex as the total number of quantities increases, the level of interrelation among quantities increases, and the number of quantities with known values is decreases.[9]

Students' abilities to explain and reason are clearly dependent on and strongly influenced by the kind of questions teachers pose. In most elementary classrooms, students are asked to find the value of one quantity from the given values of other quantities, often in conceptually simple situations. A curriculum of quantitative reasoning must include different sorts of questions about more complex situations. Sometimes these questions will be calls to find a value of a quantity (or, perhaps more fruitfully, a range of values – see below); at other times they will be questions about the nature or behavior of a quantity. Problems 5 and 6 provide two examples.

Problem 5. *Sam and Joseph each had a shorter sister, and they argued about who was more taller than his sister. Sam won the argument by 14 centimeters. He was 186 cm tall; his sister was 87 cm; and Joseph was 193 cm tall. How tall was Joseph's sister? [adapted from Thompson, 1993]*

In this case, even though the right series of three numerical subtractions will produce the "answer," the complexity of Problem 5 makes it important to think about and make sense of the situation by representing and sorting out the quantities and their relationships. It makes reference to four people's heights, two brother/sister height differences, and a difference between those differences. Without making sense of the situation as a set of related relationships, the connection between the difference of differences (whose value is 14 centimeter) and Sam's

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quantities vary, the differences would a function of two variables.

family difference (whose value is 99 centimeters) can be quite mysterious. So one strategy for developing students' quantitative reasoning is to pose problems that are too complex for them to apply an overlearned strategy for solving simpler problems.

A second curricular design strategy to highlight quantitative structure is to include problems that permit many numerical "answers." For example, Problem 5 can be adapted as follows.

Problem 6. *Sam and Joseph each had a shorter sister, and they argued about who stood taller over their sister. Sam won the argument by 14 centimeters. He was __ cm tall; his sister was __ cm; Joseph was __ cm tall; and his sister was __ cm tall. What numbers can you put in the blanks so that everything works out? [adapted from Thompson, 1993]*

Because only one quantity has a given value, this form of the situation shifts attention further toward the structure of the additive relationships. Six of the seven quantities can take on different values, and students must coordinate their choices of height values to generate appropriate values of the three differences. This strategy of asking students to reason *from* a difference rather than *to* a difference can be applied to generate a wide range of interesting additive problems.

Student Reasoning and Pedagogical Considerations

All students have the capacity, both existing and potential, to develop significantly better quantitative reasoning skills than is currently likely in most elementary mathematics classrooms (Carpenter, Moser, & Romberg, 1982; Nemirovsky, Tierney, & Ogonowski, 1993; Vergnaud,

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[9] These considerations should be treated as general guidelines for preparing and selecting situations for the classroom, not hard and fast rules. Children and classrooms differ in their reactions to situations and teachers must be prepared to experiment with and, if necessary, adjust their problem situations in response to their students.

1982; 1983). But that development depends on students' committing to the goal of describing situations, quantities, and relationships as clearly as possible *as they see them*. Students, particularly those who only see mathematics as finding numerical answers by arithmetic, are unlikely to accept that goal if (a) clear reasoning is not valued and rewarded, (b) appropriate support for clearer thinking and communication is not provided, and (c) multiple perspectives on situations are not welcomed. Each of the requirements makes specific demands on teachers and the norms they maintain in their classrooms (Wood, Cobb, Yackel, & Dillon, 1993).

In comprehending and communicating quantitative situations in their own terms, students have two primary means of expression: verbal descriptions and various sorts of external diagrams.[10] Producing a conceptually clear verbal description does not require using the "right" terminology; it is a matter of taking the task of clear description for others seriously, listening to others' reactions, and clarifying and refining when necessary. When students' verbal descriptions lose clarity and do not communicate well—either to teachers or their peers, they can be asked to show their thinking in a diagram. As with verbal descriptions, the nature of drawings will vary with the situation and the student, but here is a representative example. To manage the quantitative complexity of Problem 6, one fifth-grade student used pencils placed side-by-side (Figure 10) to represent heights and basic differences (see Thompson, [1993] for a more complete account).

[10] We also acknowledge the role of gesture in mathematical reasoning. We focus our discussion on verbal descriptions and diagrams because they are usually more permanent forms of expression (e.g., when words or phrases are written on the board) and therefore more accessible foci of group discussion and understanding.

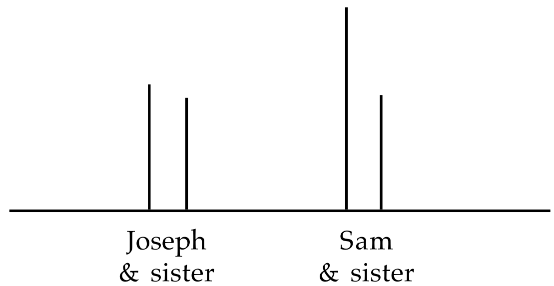

Figure 10. Concrete comparison of differences.

Such "difference diagrams" can provide useful grounding for discussions of problems like Problem 6, where the value of a difference is known but the values of the other quantities are not. More broadly, and perhaps more algebraically, they can help students to think about quantitative relationships at a level of generality beyond the immediate problem. With some experience and practice they can also easily be adjusted and annotated to fit a wider range of problems. For example, diagrams may make it easier for students to explore whether the "winner" in Problem 6 could be the shorter boy, when he stands taller over his sister by more than the other boy (Figure 11).

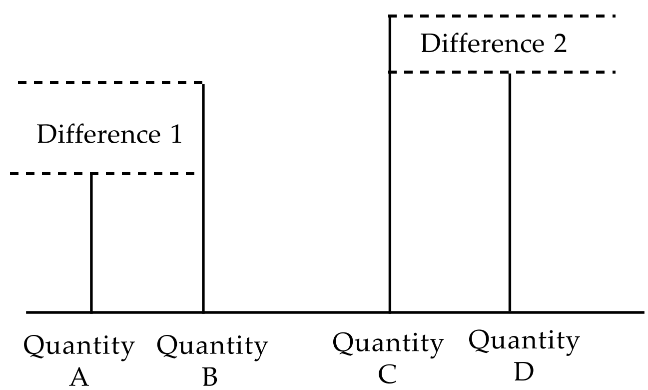

Figure 11. A general scheme of additive comparisons.

Teacher Questions and Classroom Discussion

Teaching quantitative reasoning involves two main components: (1) Choosing a sequence of situations and (2) providing appropriate support for students' reasoning. Teachers can prepare in advance for the help they provide their students. For each situation, they should first decide for themselves what quantities are involved, how they are related, and how *they* would describe the situation quantitatively. Then they should imagine how their students might describe the situation differently and what conceptual difficulties might be lodged in their descriptions (e.g., the common event of confusing the value of a difference with one of its constituent quantities). Though students' interpretations cannot be completely predicted in advance (in fact, presuming to know exactly which errors are coming can lead to terrible problems), teachers do not need to wait for the discussion to start thinking about how they might respond.

Because the central goal is to focus on quantities and how they relate in situations, and because this represents a major mathematical change of focus for many students, it is important to open discussions with questions that lead to discussions of *quantities*, not *numbers*. A useful opening question can be a general one, like "what is going on here?" The goal is to get students to describe situations as they see them. In supporting quantitative discussions, the most central skill is careful listening. Using their own knowledge of the situations, teachers can listen for which quantities are mentioned, which are central for particular students, and how students see relationships between those quantities. For situations that express additive relationships, teachers should especially listen for how students are discriminating differences from other quantities in the situation.

In teaching fifth graders about complex additive situations, one of us opened the discussion of Problem 5 above as follows (Thompson, 1993).

Teacher (T): What are they doing?

Several students: Arguing.

T: What are they arguing about?

S1 & S2: Who's taller.

S3: Who's taller than Sam's sister and Joseph's sister.

T: Are they arguing about who's taller?

Several students: No.

S4: Who was taller than their sisters.

T: Who was taller than their sister?

Several students: Yeah.

T: Are they both taller than their sisters?

Several students: Yes.

S1: Who was *more* taller.

T: Who was *more* taller? What does that mean?

(Students move quickly to modeling the situation using four pencils; they also try adjusting their model, at the teacher's request, so that the shorter brother is actually the winner.)

This is one example of students reasoning quantitatively (note the absence of references to numbers) with appropriate teacher support. The opening question led to increasingly accurate descriptions of the comparisons in the situation. The teacher was able to point to where students needed to revise their views without being explicit about how they should change or doing it for them. Students were still "in charge" of their thinking.[11]

Perhaps the most challenging task in supporting quantitative reasoning is to listen for and respect alternative descriptions of the quantities in the situation while pressing all participants for clarity. This is challenging work when two or more quite different views of the situation have been expressed. For example, in Problem 5, one student might emphasize the differences while

[11] It is worthwhile to note that, in contexts like this, we recommend *against* moving too quickly to simplify the situation by asking children to work with numbers. The point of this example is to illustrate how a teacher can support students' thinking about the situation without drawing their attention to calculating an answer.

another the relationships between the constituent quantities (the heights). When quite different perspectives arise, teachers can help their students by highlighting the contrasts between them and then managing when each view gets "worked on." Leading the discussion to elaborate only one view at a time may help the class come to see different descriptions as simply different perspectives of the same situation.

Two Final Examples: Developing a Quantitative Focus

We have stressed that the current K-8 mathematics curriculum, with its emphasis on numbers and arithmetic, falls well short of adequately developing students' quantitative reasoning as a foundation for algebra. To achieve a better curricular balance between quantitative/situational reasoning and numerical reasoning, many additive situations currently expressed as numerical "word problems" can be adapted to focus on quantities and relationships. We close this section with two examples that illustrate how simple this process can be.

A Single difference. Many word problems in the primary curriculum are situated in quantitative change, the gains and losses of everyday objects. For example,

Problem 7. *Tony had 11 marbles but he lost 4 marbles to Marguerite in a game. How many marbles did Tony have after the game?*

Instead of following up Problem 7 with many others with the same structure, Problem 8 can be given.

Problem 8. *Sharon lost 6 marbles to Philip in a game. What can we say about the number of Sharon's marbles before and after the game?*

Problem 8 asks students to reason from a difference, rather than to combine the values of a difference and one constituent quantity to find the missing value of the other. Though students may initially object that no "answer" exists, once five or six pairs of values have been generated for Sharon's marbles, they can explore the properties of those pairs. For example, they can determine that the smallest number of marbles Sharon could start the game with was 6. Of course, if this problem is presented in isolation, students will likely not accept it as a "problem" or understand its role in the development of their thinking. However, when it is one small piece in a more deliberate, multi-year quantitative curriculum that challenges and extends their reasoning capacities, students' reaction may be quite different.

Coordinating two (or more) differences. The upper elementary curriculum contains many complex, multi-step additive word problems, like Problem 9.

Problem 9. *An elementary school has two 4th grade classes, Room 5 and 7, and two 5th grade classes, Room 6 and 8. There are 52 4th graders in the school, which is 7 more students than are in the 5th grade. There are 22 students in Room 6. How many students are in Room 8?*

While this situation is certainly more complex than Problems 7 and 8, the reasoning required of students is minimized by the structure of the quantities and their known values. The value of the difference between the combined sizes of 4th-grade classes and 5th-grade classes determines the total number of 5th graders (52 - 7 = 45), and the given number of students in Room 6 determines the number of students in Room 8 (45 - 22 = 23). Only one difference appears in this situation; it compares the sizes of the two combined classes and has a value of 7 children.

Problem 10 describes a similar situation with a slightly different structure to provoke greater attention to quantitative relationships.

Problem 10. The same elementary school has two 1st grade classes, Rooms 1 and 2, and two 2nd grade classes, Rooms 3 and 4. Rooms 1 and 2 together have 50 students, and Rooms 3 and 4 together have 46 students. Room 1 has 6 more students than Room 4 and Room 2 has 2 fewer students than Room 3. How many students can be in each room? Is there only one possible size of each class?

The reasoning generated by Problem 10 will likely be quite different for a number of reasons. Since the size of the individual classes is only restricted by the size of grades (total 1st and total 2nd graders), many different values will "work." There is also the interesting and important relationship between the differences between classes (e.g., Room 1 has 6 more children than Room 4) and the difference between grades (the 1st grade has 4 more children than the 2nd grade). If students do not raise the issue, teachers can ask if the relationship between three differences ("6 more" combined with "2 less" is equivalent to "4 more" overall) is coincidence or not. Diagrams like Figure 12 can help support students' analysis of this situation. Also, as we have suggested before, they provide teachers with a convenient structure for generating new problems by varying the given numerical values and helping students to trace the impact of these changes and look for generality in the situation.

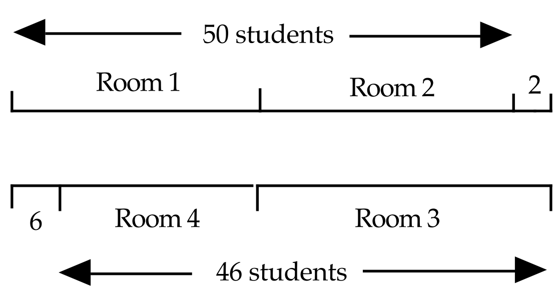

Figure 12. Complex arrangement of differences and combinations.

Conclusion

We have attempted to sketch out one proposal for “early algebra”: An approach to elementary and middle school mathematics that both readjusts the current K-8 focus on arithmetic (numbers and operations) and supports the development of algebraic reasoning. We question the content of Algebra I as the presumed standard for early algebra development and with it, the presumed developmental linkage between arithmetic and algebra. We suggest instead that elementary and middle school curricula be reconceptualized in terms of students’ quantitative, arithmetic, and algebraic reasoning. On the other hand, we recognize that we have not outlined a curriculum in quantitative reasoning. While that is a pressing and important task, it is not one that can be addressed in a single chapter. Instead, we have tried to show how simple adaptations of current curricula, when taught with a different emphasis, can make students’ mathematical experiences much richer quantitatively.

In developing this position we have made the following claims.

Quantitative reasoning is a central dimension of students’ mathematical development.

It is related to, but in an important sense independent of, both arithmetic and algebra. It is also foundational for both arithmetic and algebra, providing content and meaning for numerical and symbolic expression and computation. The common “arithmetic to algebra” framework is too limiting and narrow; discussions of early algebra should be framed in terms of connections among quantitative reasoning, arithmetic reasoning, and emergent algebraic reasoning (Figure 13). In each area, we should consider reasonable goals for students' learning, available curricula, useful teaching tools, and research on students' capabilities.

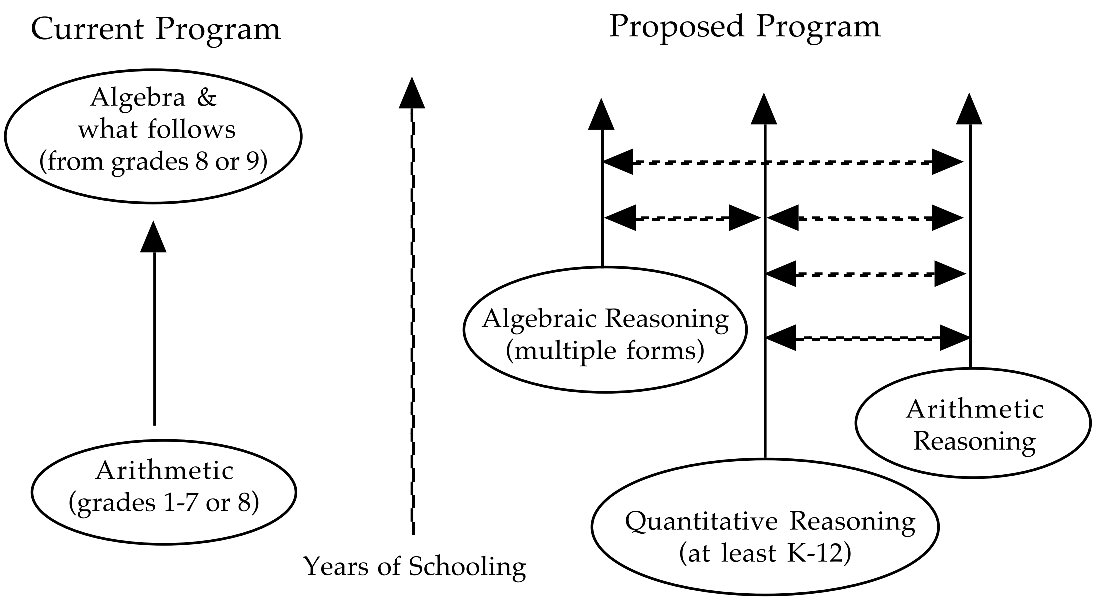

Figure 13. Two views of the introduction of algebra *The current emphasis on numerical and symbolic expression and manipulation is fundamentally flawed.*

It is flawed largely because it fails to substantively connect mathematics to students' experiential world. The implications of this disconnection are varied, profound, and negative for too many students by the middle school years. For students to learn mathematics that is powerful and productive, more attention must be given to the development of quantitative reasoning. Developing students' abilities to conceptualize and reason about situations in quantitative terms is no less important that developing their abilities to compute.

A rich program of quantitative reasoning spurs the development of students' conceptual and representational capacities as it connects mathematics to the world of objects and situations, measurement, and change.

It pushes students to examine, articulate, and represent general relationships among and between quantities. Without the understanding of such general, conceptual relationships, students find little need or sense in learning the tools of algebra. If on the other hand, students develop mathematical ideas of sufficient complexity—among them complex quantities and relationships between quantities, their expression, manipulation, and further abstraction in algebraic notation can become a more meaningful and sensible activity.

Building sophisticated quantitative reasoning skills for the majority of students is not a one or two year program; it requires development throughout the elementary and middle school years.

Students often come to school with substantial quantitative competence in additive relationships and build that competence, in and outside of classrooms. However, their development of skilled quantitative reasoning will depend on instructional programs that recognize and extend students' existing abilities. These programs will require work on more complex additive situations and relationships and, even more centrally, on developing students' abilities to conceptualize and reason about multiplicative quantities and relationships (Harel & Confrey, 1994; Vergnaud, 1983, 1988).

Some important components of such a program include, (a) suitably chosen situations and problems, (b) greater instructional focus on “making sense” of quantities and relationships in those situations than on finding “answers,” (c) emphasis on reasoning and expression that is neither numerical or symbolic (in the sense of traditional algebraic symbols), and (d) support for representing and communicating reasoning clearly and publicly in diagrams and open-form expressions.

In closing, we address two potential objections to our position, one concerning content balance in the K-8 curriculum and other concerning the claim that quantitative reasoning prepares students for diverse approaches to algebra.

Throughout this chapter, we have argued for better curricular balance between teaching and learning about number/operation and quantity/quantitative reasoning. But while quantity and number are central categories of elementary mathematics, they do not comprise a comprehensive

K-8 curriculum. We believe that students should also experience extended work with metric and non-metric geometry, data and statistics, as well as introductory probability. Each can provide important ideas to represent and reason about in greater generality with algebraic symbols and methods (see Boester & Lehrer [this volume] for examples from geometry). We acknowledge but cannot resolve the fundamental dilemma that there is more worthwhile mathematics to learn than there is space in the school curriculum to teach and learn it. Competition between ideas and subfields of mathematics is a necessary, perhaps not wholly negative, result. Our push to shift from a "number only" orientation (K-8) to a "number and quantity" orientation reflects our present task of reconceptualizing early algebra as something more than the common view of algebra as generalized arithmetic.

Likewise, we do not believe that a strict conceptual distinction between number and quantity is psychologically defensible or educationally useful. Children can find wonder and engage deeply in the nature of numbers qua numbers and in quantities and situations. Any good mathematics curriculum should recognize and nurture both interests. Moreover, it can be difficult to decide if a person's reasoning is "more numerical" or "more quantitative," particularly when the quantities are numerically specific. In most cases, there is a natural dialogue between the two mathematical dimensions. But softening the distinction between number and quantity does not undermine our fundamental argument that much more attention should be given to thinking about quantities, relationships, and situations.

We argued initially that a K-8 program that gives more attention to quantitative reasoning can support numerous conceptually different and sensible introductions to algebra, as replacements for the current US Algebra I course. But our proposal could be seen as supporting, most directly, the more "applied" view of algebra, algebra as modeling. We have, after all, repeatedly emphasized the importance of making sense of problem situations and that an important kind of mathematical reasoning examines general relationships between quantities. Indeed we think that introducing algebra as a set of tools for expressing reasoning about complex situations and for generalizing their solutions has substantial promise, especially when the assumption is that all students will take and learn algebra (Silver, 1997). But we also believe that this quantitative, “applied” introduction can easily support the subsequent shift toward the more formal, structural side of algebra. Algebraic knowledge that has grown from a quantitative root stalk can serve as the basis for moves toward increasing abstraction and focus on abstract structure, making them abstractions from students experience and in students’ reasoning about that experience.

Finally, we note that while we were familiar with the Davydov and El'konin approach to basing early mathematics instruction on ideas of quantity (Davydov, 1975, 1982; El’konin & Davydov, 1975), we have only recently become aware of curricular materials being developed and researched that binds ideas of quantity with inscriptional practices that mean to provide a bridge between quantitative and symbolic reasoning (Dougherty, this volume). In this approach, children are asked to think about quantities' measures and relationships among them, and to attend to what one can deduce from those relationships. For example, the Davydov and El'konin curriculum intends that children read "A/B = 5" as "Quantity A, measured in units of Quantity B, has a measure of 5." It is then a small step to deducing that "A is 5 times as large as B", which introduces the mathematical notion of fraction as reciprocal relationships of relative size (Thompson & Saldanha, 2003) while respecting the starting point that numbers are measures and symbolic statements capture quantitative relationships. Clearly much work remains to be done.

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Author Note

John P. Smith III, Department of Counseling, Educational Psychology, and Special Education, Michigan State University; Patrick W. Thompson, Department of Teaching and Learning, Vanderbilt University.

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Correspondence concerning this chapter should be directed to John P. Smith III, 513H Erickson Hall, Michigan State University, East Lansing, MI, 48824. E-mail: jsmith@msu.edu.

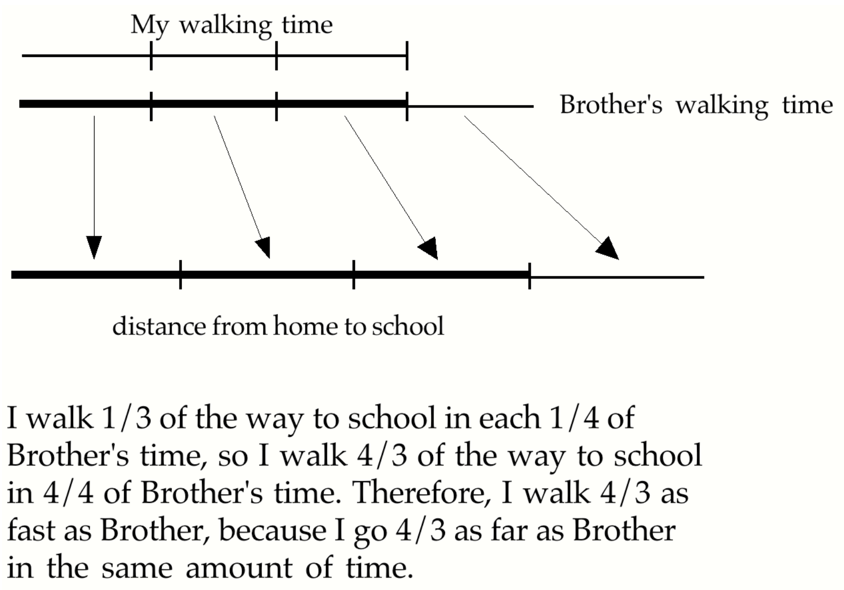

Figure 1. Relating my speed to brother’s speed based on the relationship of my time to his.

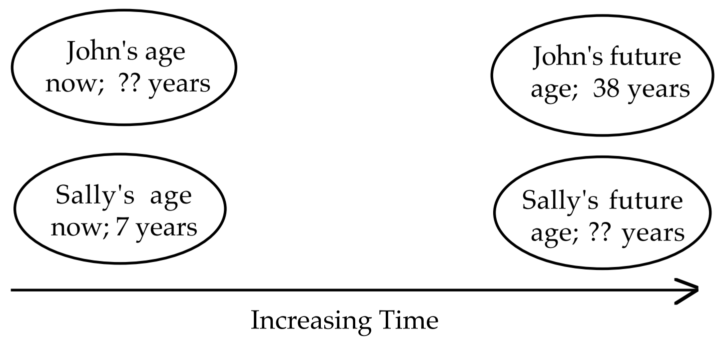

Figure 2. Everyone ages as time passes.

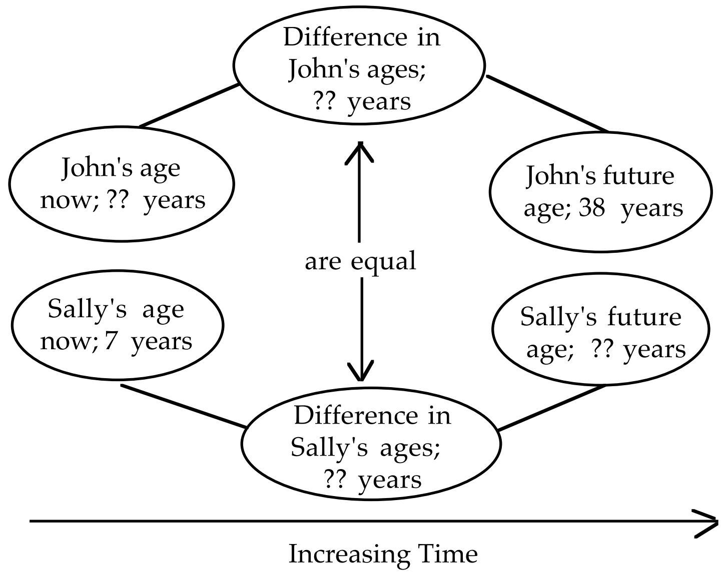

Figure 3. Time passes in equal amounts for everyone.

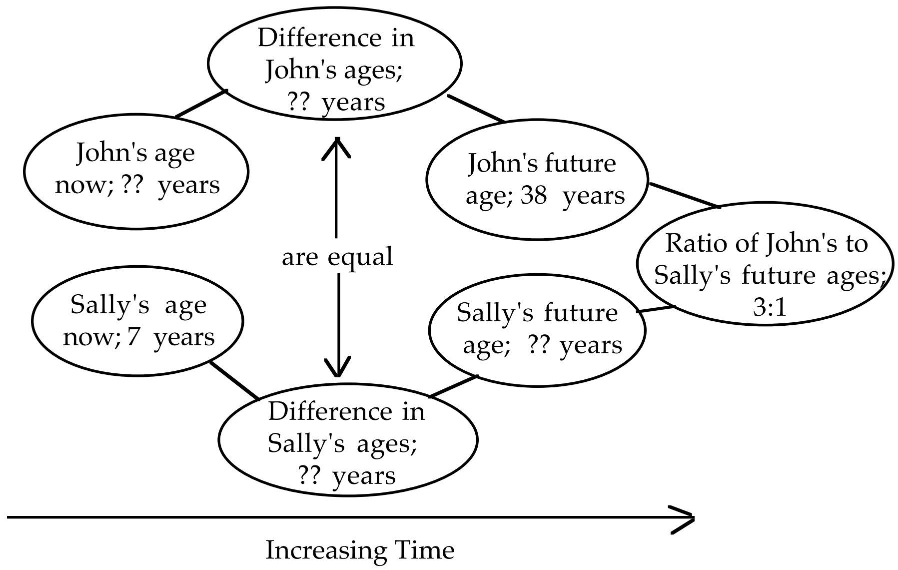

Figure 4. Relationships among quantities.

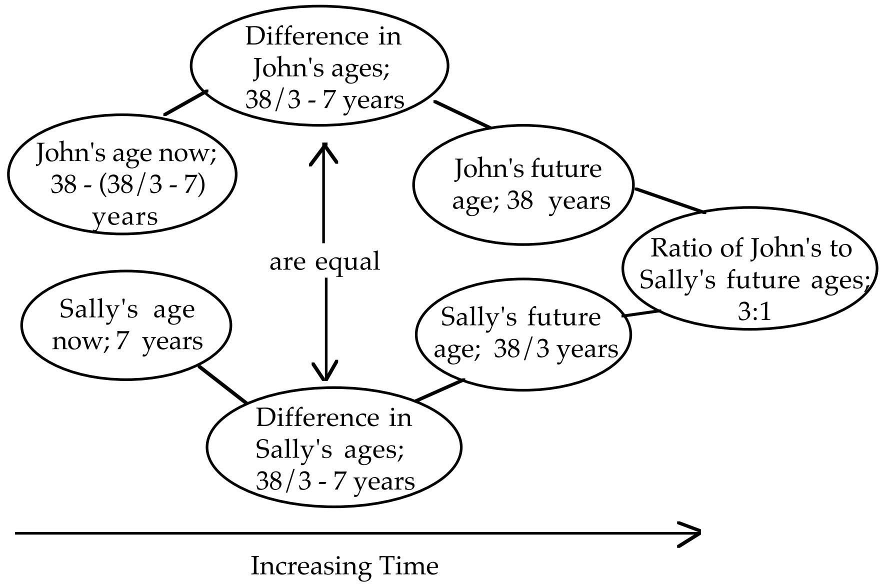

Figure 5. Computing within relationships among quantities.

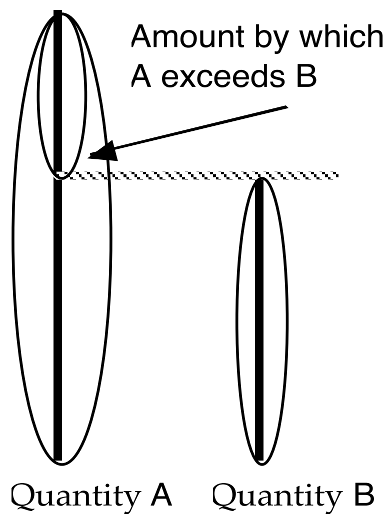

Figure 6. Difference of two quantities.

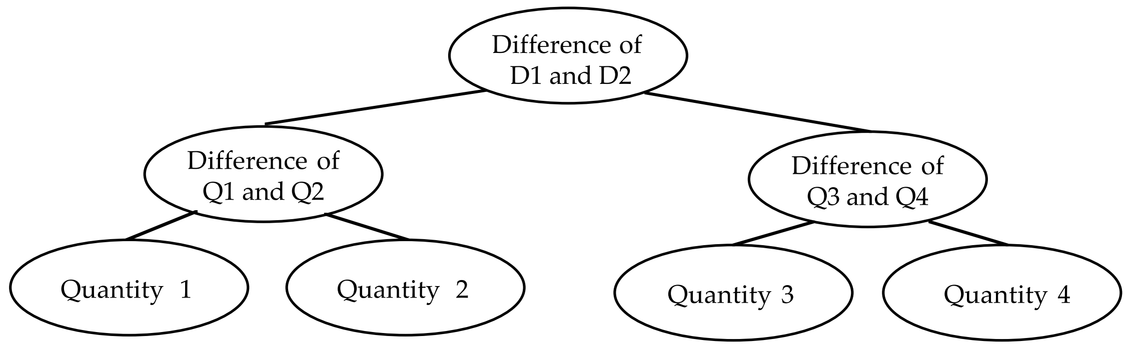

Figure 7. Difference of differences.

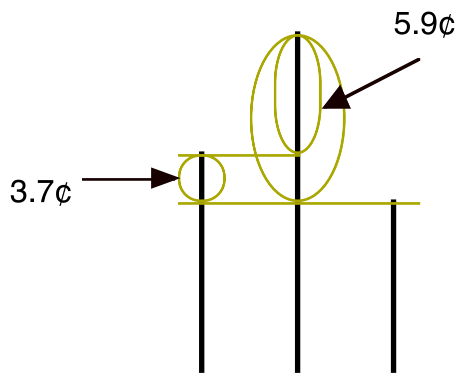

Figure 8. The difference between the 2nd and 3rd months' prices is seen as a combination of two differences.

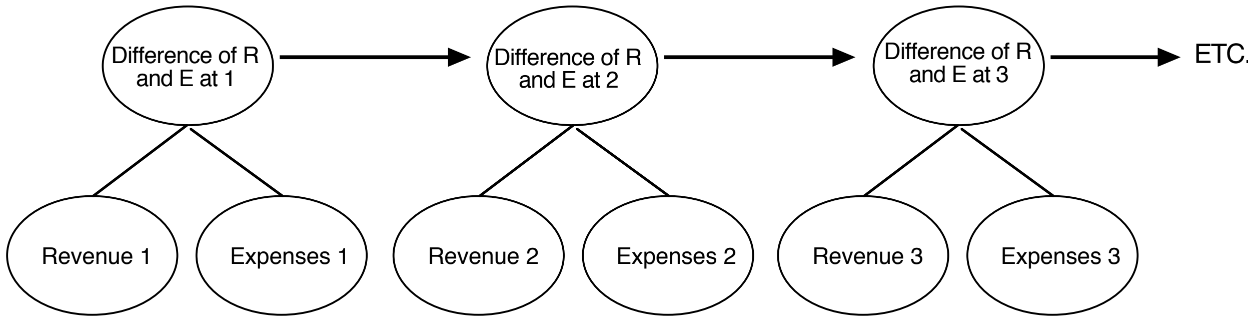

Figure 9. Chain of differences.

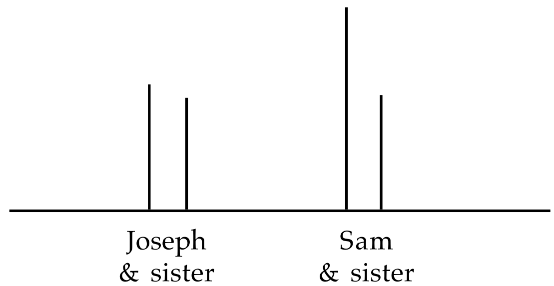

Figure 10. Concrete comparison of differences.

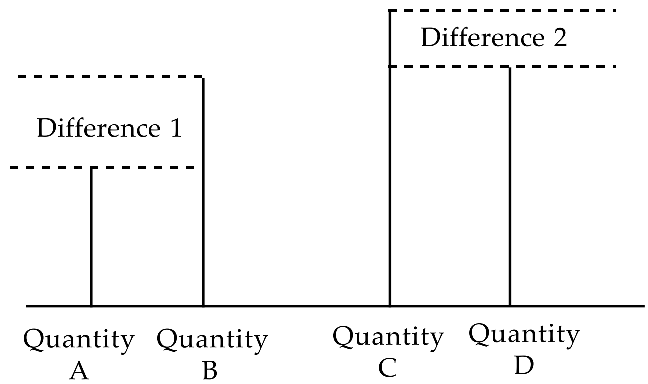

Figure 11. A general scheme of additive comparisons.

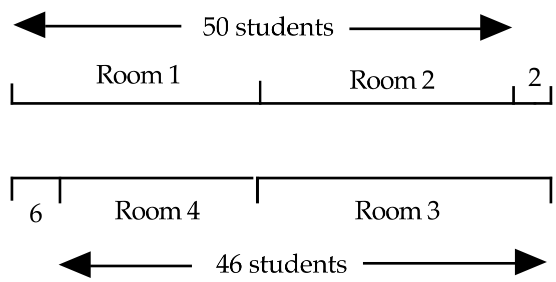

Figure 12. Complex arrangement of differences and combinations.

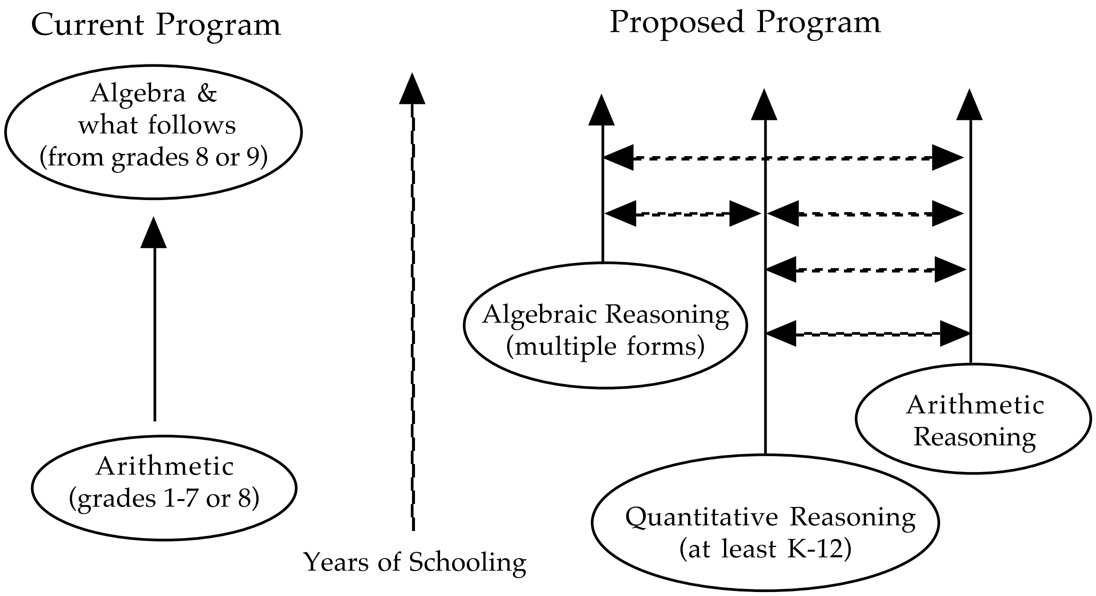

Figure 13. Two views of the introduction of algebra