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4-Regular Vertex-Transitive Tilings of $\mathbb{E}^3$

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Abstract. There exist precisely 149 topological types of semipolytopal tile-transitive tilings of $\mathbb{E}^3$ by "extetrahedra" (obtained from tetrahedra by introducing certain new vertices of degree 2). Dualization gives rise to 149 types of 4-regular vertex-transitive tilings. The 4-coordinated networks carried by these tilings are closely related to crystal structures such as zeolites or diamond. These results are obtained using "combinatorial tiling theory."

1. Introduction

In *Tilings and Patterns* [GS2], Grünbaum and Shephard present in detail the full range of problems and methods associated with (mainly two-dimensional) tilings and patterns and discuss in depth their relevance for art and science. They address the problem of tiling three-dimensional space in a number of papers including [GS1], [Gr2], [GMLS], and [DGS].

It seems obvious that a classification of periodic tilings of three-dimensional Euclidean space $\mathbb{E}^3$ will have applications in crystal chemistry, ideally by supplying an enumeration of all mathematically feasible crystal-structures of a given type, up to a certain degree of complexity.

However, the problem of classifying periodic tilings of $\mathbb{E}^3$ is considerably more difficult than the problem of classifying two-dimensional periodic tilings, and indeed touches on one of mathematics great open problems: the Poincaré Conjecture.

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More than 10 years ago Dress introduced the method of Delaney symbols [Dr1], [Dr2] and developed the foundation of what we propose to call *combinatorial tiling theory*. This has given rise to a number of papers that investigate different questions and aspects of this theory.

Based on combinatorial tiling theory, we have developed a computer-aided approach [DHM], [DH3] to the problem of classifying periodic tilings of $\mathbb{E}^3$, involving combinatorial topology [DH1], computational geometry [EM], [DH2], computational algebra [Sc$^+$], and other tools [Mc], [LMP$^+$], [NM]. Its viability was recently demonstrated in [DH3] by classifying all tilings of Euclidean space by combinatorial cubes, tetrahedra, or octahedra, establishing, in particular, the existence of precisely 11, 9, and 3 (respectively) topological types of tile-transitive tilings by such tiles.

One aim of the current paper is to show that our approach is not restricted to combinatorially regular tiles, but also applies in the case of combinatorially less regular ones. In a future paper we shall demonstrate that it can also be used to classify tilings with two or more types of tiles.

Crystal structures are often interpreted as atom-bond networks, or, topologically, simply as graphs embedded in $\mathbb{E}^3$. Given such a network, it is a highly nontrivial task to decide whether a periodic tiling exists that *carries* it in the sense that the edge-skeleton of the tiling is (topologically) the given network.

A zeolite is an aluminosilicate in which the Al and Si atoms occupy 4-coordinated (i.e., 4-valent) vertices of a three-dimensional network, and the oxygen atoms occupy 2-coordinated positions between the 4-coordinated vertices [Sm]. Neglecting the 2-valent oxygen atoms, zeolites are 4-valent networks, as is the diamond network, too. They have many important applications in chemistry [Sm].

Currently, the online version of the Atlas of Zeolite Structure Types [OMB] (see also [MO]) lists 121 approved zeolite structures. Precisely 18 of these are *uninodal*, i.e., have symmetry groups that act transitively on the set of 4-valent atoms. In turns out that precisely six of these are carried by duals of tile-transitive tilings by combinatorial tetrahedra [DH3] This inspires us to consider the following question: Do there exist periodic tilings that carry the remaining 12 uninodal zeolite structures?

In an attempt to answer this, we introduce the concept of an *extetrahedron* of level h, which is obtained by "extending" a tetrahedron by inserting new vertices of degree 2 into some of the original edges, up to h in each. We classify all tile-transitive tilings of Euclidean space by extetrahedra of level 1 and will see that by dualization we obtain carriers for all 18 uninodal zeolites.

Some further definitions are introduced in Section 2. We then give a short summary of our approach in Section 3. Finally, in Section 4, we describe our classification results in tabulated form and depict a number of interesting examples.

Combinatorial tiling theory and the methods and results described in this paper represent a major step forward toward the goal of systematically enumerating mathematically feasible crystal structures [DDH$^+$], [O'K].

2. Definitions

Although there is a common general understanding of what a tiling should be, definitions differ in their details. Within the framework of "combinatorial tiling theory," tilings are naturally and very generally defined in terms of their chamber systems as those subdivisions of space that possess a “Delaney-symbol” [Dr1], [Dr2]. For the purposes of this paper, we define a tiling of some d-dimensional manifold X without boundary as the collection of cells of a regular CW-complex with total space X. The cells are also called *faces* of the tiling. This definition is narrower in that every tiling that satisfies it possesses a Delaney symbol, but not vice versa.

We define the terms *vertex*, *edge*, *facet*, and *tile* in the usual way. Obviously, the set of tiles covers X. Two faces are said to be *incident* if one is included in (the boundary of) the other. Two nonincident faces are *adjacent* if their intersection is nonempty. If f is any fixed face, then the set of all faces contained in the boundary of f form a tiling of this boundary.

We define the *graph carried* by a tiling to be the graph naturally induced by the vertices and edges of the 1-skeleton of the tiling. As usual, a graph is called *polytopal* if it is isomorphic to the graph of a convex 3-polytope, i.e., if it is planar and 3-connected [Gr1]. We call a tiling of a topological 2-sphere a *combinatorial polytope* if its graph is polytopal, and we call it *semipolytopal* if its graph can be derived from a polytopal graph by subdivision of edges. We call a tiling of $\mathbb{E}^3$ *semipolytopal* if all its tiles and the tiles of its dual are semipolytopal.

The *degree* of a vertex is the number of edges incident to it and we call a tiling *n-regular* if all its vertices have degree n. The smallest degree that can appear in a semipolytopal tiling is 4.

We call two tilings *topologically equivalent* if there exists a homeomorphism between their total spaces that takes faces onto faces. The *symmetry group* of a tiling of some metric space such as $\mathbb{E}^3$ consists of all isometries of that space that map faces onto faces. A tiling is called *vertex-transitive* if for each pair of vertices there exists a symmetry that maps one onto the other. In general, *transitivity classes* of tiles, faces, edges, or vertices are to be understood with respect to the symmetry group of the tiling.

3. Methods

Ultimately we are interested in vertex-transitive 4-regular tilings. These arise by dualization from tile-transitive tilings by 4-faced tiles and we now focus on the latter.

There exist precisely 11 different topological types of extetrahedra of level 1, see Fig. 1. Each gives rise to one or more different *equivariant types*, which are distinguished by taking the possible symmetry groups into account, as in [DH3]. The 11 depicted topological types t, t_1, t_{2a}, t_{2b}, ... give rise to 11, 5, 2, 8, 2, 4, 4, 2, 8, 5, and 11 equivariant types, respectively.

For each equivariant type T, we apply the combinatorial enumeration approach described in [DH3] to classify all periodic tilings of $\mathbb{E}^3$ by tiles of type T. In total we obtain 1720 different topological types, including the nine types of tilings by tetrahedra. To be precise, in terms of combinatorial tiling theory, this produces a list of “maximal Delaney symbols” that describe the tilings uniquely up to topological equivalence.

For each such combinatorial description we are then faced with the task of constructing a geometric realization of the encoded tiling. In [De] and [DH3] we indicate how a straight-edge realization can often be obtained by first determining and

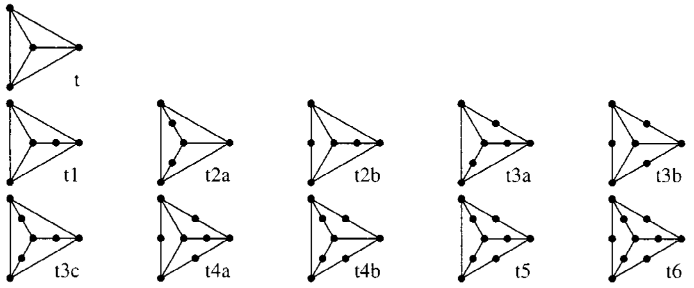


Fig. 1. The 11 extended tetrahedra of level 1.

parameterizing the linear hull of the space of admissible vertex positions and then using standard optimization techniques to find "preferable" parameter values.

We prefer parameters values that give rise to realizations with high volume (measured as the ratio of the volume of a fundamental domain and the cubed average edge length) and small variation of edge lengths. For a given positioning of vertices, each higher-dimensional face is constructed inductively as the linear cone on its boundary with apex at the center of gravity of its vertices. Some simple steps were taken to give the two-dimensional faces a smoother appearance in Figs. 2–4.

This simple form of optimization is not guaranteed always to produce correctly embedded tilings and indeed for 11 of the 149 cases listed in Table 2 (numbers 53, 58, 67, 68, 77, 78, 79, 100, 101, 102, and 145) it fails to do so. However, we emphasize that the existence of Euclidean Delaney symbols for these exceptional cases implies that they all possess geometric realizations, although not necessarily with straight edges.

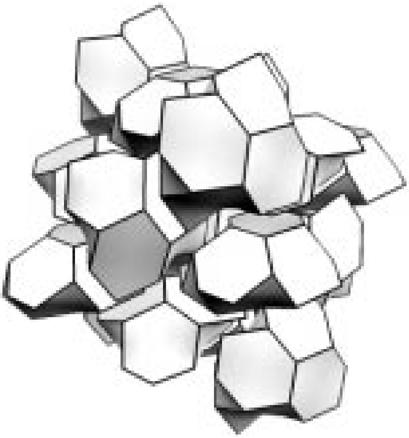
We remark that all tilings are realized with *full symmetry*, i.e., in such a way that all combinatorial symmetries are isometries. It follows from a nontrivial result in geometric topology [MS] that this is always possible for periodic tilings of $\mathbb{E}^3$ with "maximal Delaney symbols" [De].

4. Results

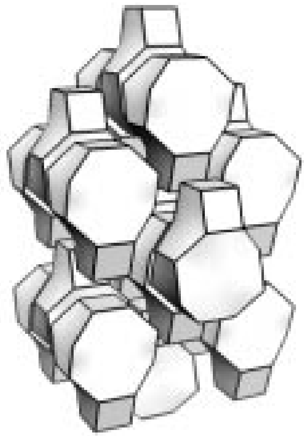
Using the approach indicated in the preceding section, we obtain the following result:

Theorem 4.1. *There exist precisely* 149 *topological types of semipolytopal tile-transitive tilings of three-dimensional Euclidean space by extetrahedra of level* 1, *of which exactly nine are by combinatorial regular tetrahedra. Their duals are summarized in Table* 2. *If tilings are not required to be semipolytopal, then there exist* 1571 *further types.*

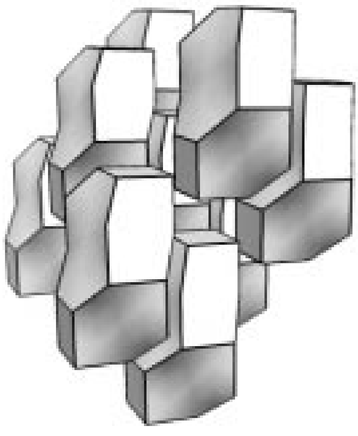
By dualization, Theorem 4 gives rise to 149 types of semipolytopal, vertex-transitive 4-regular tilings of $\mathbb{E}^3$. Of the 18 uninodal networks listed in the current online version of the Atlas of Zeolite Structure Types [OMB], 16 are carried by at least one of these



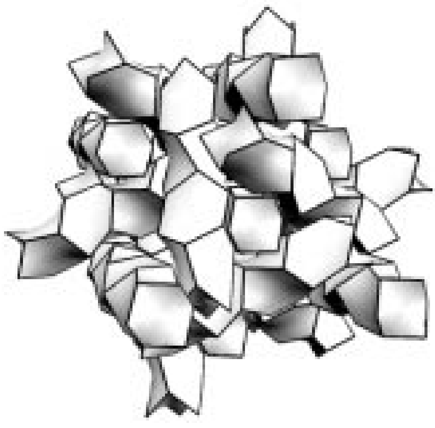
(a) Tiling #022



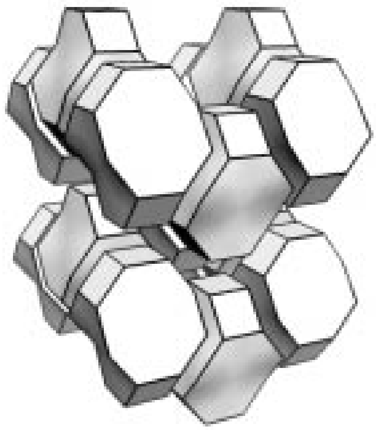
(b) Tiling #028



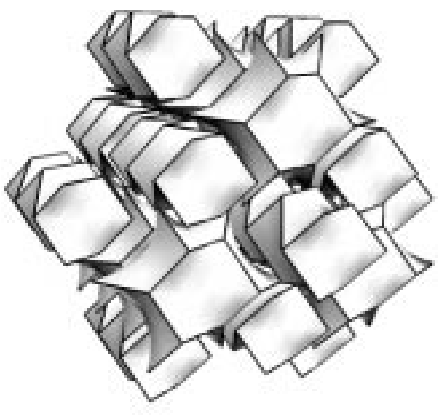
(c) Tiling #029



(d) Tiling #037

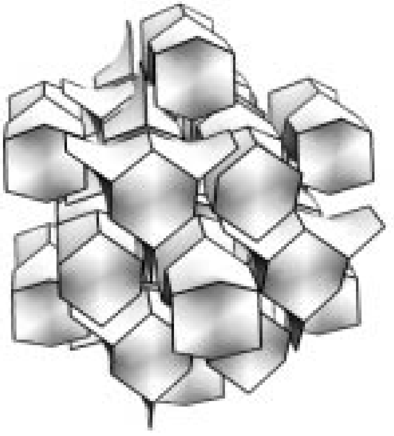


(e) Tiling #038

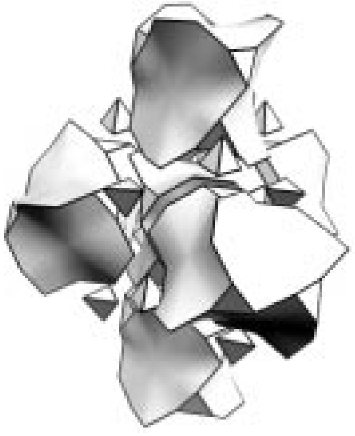


(f) Tiling #044

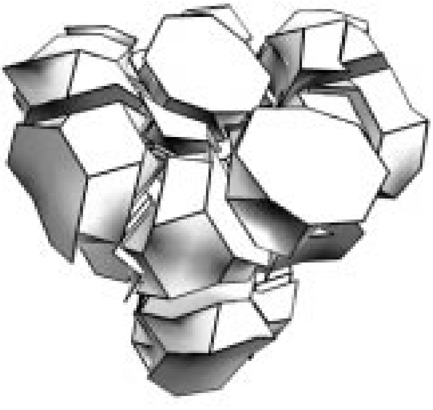
Fig. 2



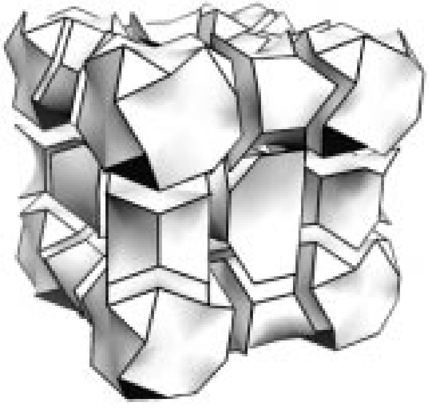
(a) Tiling #047



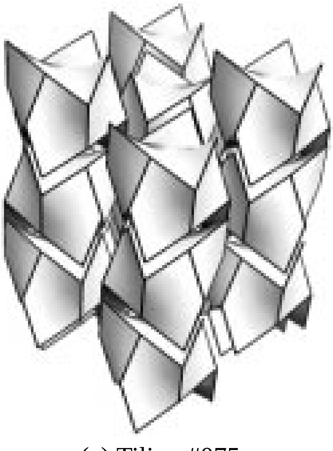
(b) Tiling #048



(c) Tiling #050



(d) Tiling #057



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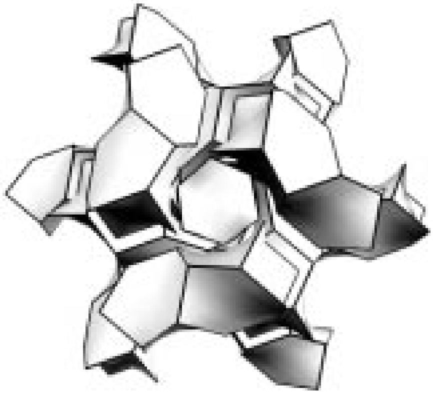
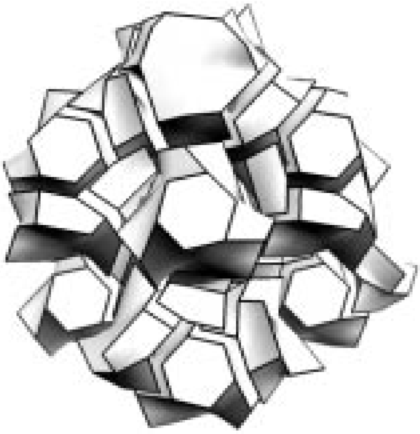
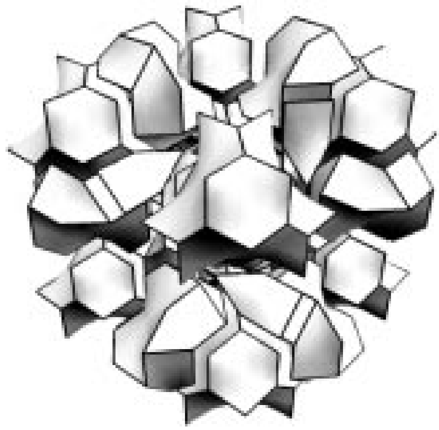


(f) Tiling #076

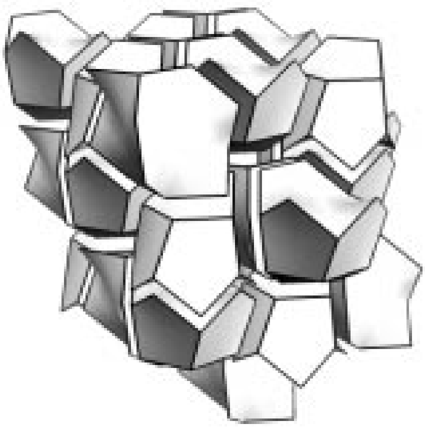
Fig. 3



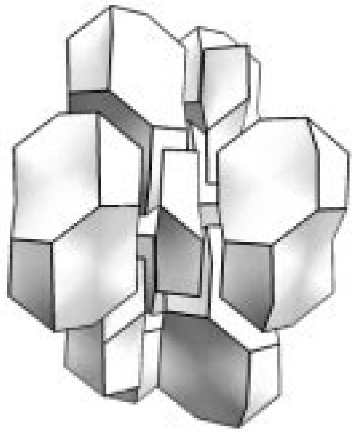
(a) Tiling #083



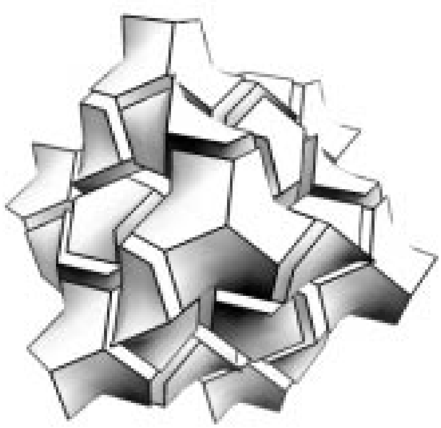
(b) Tiling #088



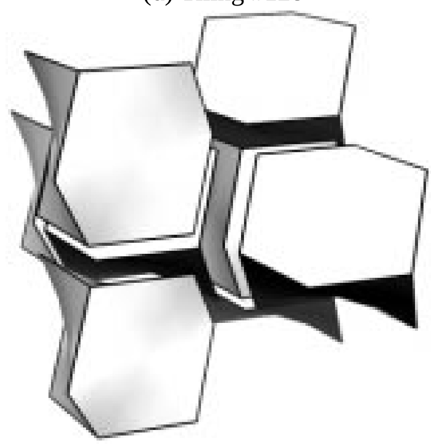
(c) Tiling #113



(d) Tiling #128



(e) Tiling #139



(f) Tiling #149

Fig. 4

Table 1. Conway's orbifold notation (i) and the standard crystallographic notation (ii) for crystallographic point-groups.

(i)	(ii)	(i)	(ii)	(i)	(ii)	(i)	(ii)
1	1	$2\times$	$\bar{4}$	432	432	$*322$	$\bar{6}2m$
$1*$	m	322	32	44	4	$*33$	$3m$
$1\times$	$\bar{1}$	33	3	$4*$	$4/m$	$*332$	$\bar{4}3m$
22	2	332	23	622	622	$*422$	$4/mmm$
222	222	$3*$	$3/m$	66	6	$*432$	$m\bar{3}m$
$2*$	$2/m$	$3*2$	$m\bar{3}$	$6*$	$6/m$	$*44$	$4mm$
$2*2$	$\bar{4}2m$	$3\times$	$\bar{3}$	$*22$	$mm2$	$*622$	$6/mmm$
$2*3$	$\bar{3}m$	422	422	$*222$	mmm	$*66$	$6mm$

tilings, as indicated in Table 2. Tilings that carry the remaining two uninodal zeolites ANA (Analcime) and DFT can be found among (the duals of) the 1571 additional tilings.

Our results clearly do not give a complete classification of all semipolytopal, vertex-transitive 4-regular tilings of $\mathbb{E}^3$, as we only considered extetrahedra of level 1. By results in combinatorial tiling theory [DHM], for any fixed level h, there exist only a finite number of types of tile-transitive tilings by extetrahedra of level h. We state the following open problem: Is there an upper bound for the possible number of additional vertices? In other words, do there exist only finitely many types of semipolytopal, tile-transitive tilings by extetrahedra of arbitrary level?

We depict a number of examples in Figs. 2–4. For each tiling a finite patch of tiles is shown. Tiles are shrunk slightly toward their centers to make their faces visible. Note that tiling number 149 (and all tilings that contain tiles of type 3k) represents the diamond network. A complete description of our results is available on the World Wide Web at `http://www.mathematik.uni-bielefeld.de/~delgado/tilings3d`.

We summarize the classification in Table 2. Each row describes one of the 149 topological types of tilings. The data in each column is:

(1) The number of the tiling.
(2) The *vertex type*, i.e., the topological type of the tiles of the dual tiling, as defined in Fig. 1. Tilings with the same vertex type are listed consecutively.
(3) The "orbifold name" [Co], [CH] of the vertex stabilizer, the group of all symmetries of the tiling that leave a given vertex fixed, see Table 1.
(4) A four-digit code listing the number of transitivity classes of vertices, edges, facets and tiles.
(5) The topological types of the tiles, as defined in Fig. 5.
(6) The international number and Hermann–Mauguin name for the symmetry group [Ha]. This refers to a representative of the topological class with maximal symmetry.
(7) The "fibrifold name" (in the case of reducible groups) or "Conway name" (in the case of the 35 irreducible groups) for the symmetry group [CDHT].
(8) For those tilings that carry a known zeolite, the "structure code" for the zeolite [OMB]. The first occurrence of each zeolite is underlined.

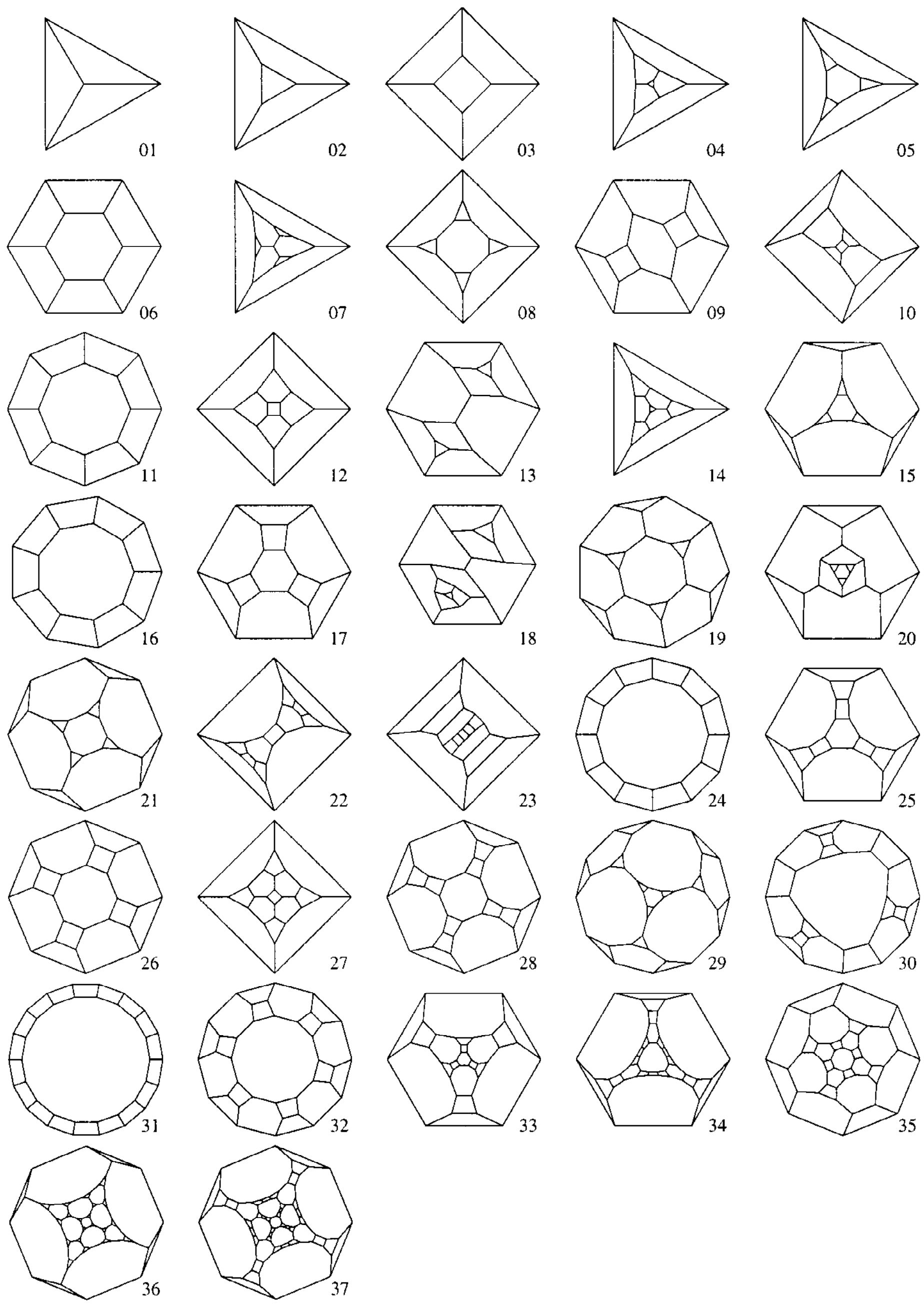


Fig. 5. The 149 types of tilings make use of 111 different topological types of tiles. These are all derived by subdivision from the 37 polytopes shown here. In Table 2 each tile is identified by a number 1–37 indicating which polytope it is derived from and a letter to distinguish it from other tiles derived from the same one.

Table 2

(1)	(2)	(3)	(4)	(5)	(6)		(7)	(8)
	Vert.	Vert.	Trans-	Tile	Space group		Fibrifold	Zeolite
No.	type	stab.	itivity	types	No.	Name	name	net
1	t	2*2	1121	27a	229.	Im$\bar{3}$m	$8^{\circ}{:}2$	SOD
2	t	1*	1343	21a, 35a, 5a	225.	Fm$\bar{3}$m	$2^{-}{:}2$	
3	t	1*	1343	27a, 35a, 3a	221.	Pm$\bar{3}$m	$4^{-}{:}2$	LTA
4	t	1*	1332	1a, 36a	229.	Im$\bar{3}$m	$8^{\circ}{:}2$	
5	t	22	1242	11a, 35a	229.	Im$\bar{3}$m	$8^{\circ}{:}2$	RHO
6	t	1	1453	11a, 37a, 3a	229.	Im$\bar{3}$m	$8^{\circ}{:}2$	
7	t	1	1453	27a, 34a, 6a	227.	Fd$\bar{3}$m	$2^{+}{:}2$	FAU
8	t	1	1453	28a, 35a, 6a	229.	Im$\bar{3}$m	$8^{\circ}{:}2$	KFI
9	t	1	1442	33a, 6a	166.	R$\bar{3}$m	$(*{\cdot}6\,3_1 2)$	CHA
10	t1	1	1463	25a, 6a, 6j	194.	P6_3/mmc	$[*{:}6{\cdot}3{\cdot}2]$	GME
11	t1	1	1463	11a, 28a, 3q	139.	I4/mmm	$[*{\cdot}4{\cdot}4{:}2]$	MER
12	t1	1	1452	17b, 6a	166.	R$\bar{3}$m	$(*{\cdot}6\,3_1 2)$	CHA
13	t1	1	1452	17a, 6i	164.	P$\bar{3}$m1	$(*6{\cdot}3{\cdot}2)$	CAN
14	t1	1	1452	26a, 3o	129.	P4/nmm	$(*4{\cdot}4{\cdot}2)$	ATN
15	t1	1	1452	27a, 6d	166.	R$\bar{3}$m	$(*{\cdot}6\,3_1 2)$	SOD
16	t1	1	1442	26a, 3o	119.	I$\bar{4}$m2	$(*4{\cdot}4\,2_1)$	ATN
17	t1	1	1453	11a, 27b, 6a	224.	Pn$\bar{3}$m	$4^{+}{:}2$	
18	t1	1	1453	35a, 3a, 3q	223.	Pm$\bar{3}$n	8°	
19	t1	1	1453	24a, 27c, 3a	224.	Pn$\bar{3}$m	$4^{+}{:}2$	
20	t1	1	1453	24a, 27b, 6a	224.	Pn$\bar{3}$m	$4^{+}{:}2$	
21	t1	1	1442	10a, 10i	141.	I4_1/amd	$(*4_1 4{\cdot}2)$	
22	t1	1	1331	12a	142.	I4_1/acd	$(*4_1 4{:}2)$	
23	t1	1	1452	22a, 3r	140.	I4/mcm	$[*{\cdot}4{:}4{:}2]$	
24	t1	1	1452	2f, 30a	193.	P6_3/mcm	$[*{\cdot}6{:}3{:}2]$	
25	t1	1	1452	26a, 3r	125.	P4/nbm	$(*4_0 4{\cdot}2)$	
26	t1	1	1452	2f, 32a	162.	P$\bar{3}$1m	$(*{\cdot}6\,3_0 2)$	
27	t1	1	1442	16a, 27d	211.	I432	$8^{+\circ}$	
28	t1	1	1441	10b	63.	Cmcm	$[2_0 2_1 *{\cdot}]$	
29	t1	1	1441	9b	12.	C2/m	$[2_0 2_0 2_1 2_1]$	ABW
30	t1	1	1431	9a	64.	Cmca	$[2_0 2_1 *{:}]$	
31	t1	1	1431	12a	68.	Ccca	$(*2_0 2{:}2{:}2)$	
32	t1	1	1431	12b	70.	Fddd	$(2\bar{*}2_0 2_1)$	
33	t1	1	1431	9c	70.	Fddd	$(2\bar{*}2_0 2_1)$	
34	t1	1	1442	23a, 3a	141.	I4_1/amd	$(*4_1 4{\cdot}2)$	
35	t1	1	1441	9b	53.	Pmna	$[2_0 2_0 *{:}]$	ABW
36	t1	1	1431	12a	98.	I$4_1$22	$(*4_3 4_1 2_0)$	
37	t1	1	1221	18a	205.	Pa$\bar{3}$	$2^{-}/4$	
38	t1	22	1231	10c	141.	I4_1/amd	$(*4_1 4{\cdot}2)$	GIS
39	t2a	1	1332	14a, 2a	212.	P$4_3$32	$2^{+}/4$	
40	t2a	1	1332	13a, 2f	212.	P$4_3$32	$2^{+}/4$	
41	t2a	1	1452	10a, 3p	141.	I4_1/amd	$(*4_1 4{\cdot}2)$	
42	t2a	1	1442	10f, 3d	137.	P4_2/nmc	$(*4{\cdot}4{:}2)$	
43	t2a	1	1432	12c, 3d	126.	P4/nnc	$(*4_0 4{:}2)$	
44	t2a	1	1432	4b, 6f	163.	P$\bar{3}$1c	$(*{:}6\,3_0 2)$	
45	t2a	1	1443	10h, 3a, 3q	119.	I$\bar{4}$m2	$(*4{\cdot}4\,2_1)$	ACO
46	t2a	1	1432	12c, 3b	97.	I422	$(*4_2 4_0 2_1)$	

Table 2 (*Continued*)

(1)	(2)	(3)	(4)	(5)	(6)		(7)	(8)
	Vert.	Vert.	Trans-	Tile	Space group		Fibrifold	Zeolite
No.	type	stab.	itivity	types	No.	Name	name	net
47	t2a	1	1432	4b, 6c	182.	$P6_322$	$(*6_33_02_1)$	
48	t2a	1	1332	15a, 1a	230.	$Ia\bar{3}d$	$8^{\circ}/4$	
49	t2a	1	1452	10i, 3c	141.	$I4_1/amd$	$(*4_14{\cdot}2)$	
50	t2a	1	1332	17c, 1d	230.	$Ia\bar{3}d$	$8^{\circ}/4$	
51	t2a	1	1442	31a, 6l	166.	$R\bar{3}m$	$(*{\cdot}6\,3_12)$	
52	t2a	1	1431	24c	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
53	t2a	1	1431	10g	68.	Ccca	$(*2_02{:}2{:}2)$	
54	t2a	1	1431	10d	70.	Fddd	$(2\bar{*}2_02_1)$	
55	t2a	1	1431	10d	68.	Ccca	$(*2_02{:}2{:}2)$	
56	t2a	1	1442	6b, 6i	166.	$R\bar{3}m$	$(*{\cdot}6\,3_12)$	ATO
57	t2b	1	1344	21a, 21a, 3o, 3r	226.	$Fm\bar{3}c$	4^{--}	
58	t2b	1	1331	8a	102.	$P4_2nm$	$(4_2*{\cdot}2)$	
59	t2b	1	1442	6b, 6i	166.	$R\bar{3}m$	$(*{\cdot}6\,3_12)$	ATO
60	t2b	1	1331	4d	199.	$I2_13$	$2^{\circ}/4$	
61	t2b	1	1322	4a, 4c	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
62	t2b	1	1222	4c, 4e	205.	$Pa\bar{3}$	$2^{-}/4$	
63	t2b	1	1322	4b, 6f	148.	$R\bar{3}$	$(6\,3_12)$	
64	t2b	1	1322	4b, 6c	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
65	t2b	1	1331	10e	86.	$P4_2/n$	$(4\,4_22)$	
66	t2b	1	1331	7a	86.	$P4_2/n$	$(4\,4_22)$	
67	t2b	1	1431	21b	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
68	t2b	1	1431	19a	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
69	t2b	22	1241	6d	166.	$R\bar{3}m$	$(*{\cdot}6\,3_12)$	SOD
70	t2b	22	1242	21a, 3q	223.	$Pm\bar{3}n$	8°	
71	t2b	22	1232	5a, 5e	210.	$F4_132$	2^{+}	
72	t2b	22	1232	21a, 3r	211.	I432	$8^{+\circ}$	
73	t2b	22	1221	4a	155.	R32	$(*3_03_13_2)$	
74	t2b	22	1221	4c	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
75	t2b	22	1221	6e	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
76	t2b	22	1121	4b	206.	$Ia\bar{3}$	$4^{-}/4$	
77	t3a	1	1231	20b	148.	$R\bar{3}$	$(6\,3_12)$	
78	t3a	1	1431	20c	212.	$P4_332$	$2^{+}/4$	
79	t3a	1	1331	20a	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
80	t3a	1	1453	3d, 3o, 3r	125.	P4/nbm	$(*4_04{\cdot}2)$	
81	t3a	1	1453	2f, 3d, 6i	162.	$P\bar{3}1m$	$(*{\cdot}6\,3_02)$	
82	t3a	1	1431	3k	52.	Pnna	$(2_02\bar{*}_1)$	
83	t3a	1	1333	1d, 5e, 6e	228.	$Fd\bar{3}c$	4^{++}	
84	t3a	1	1343	2f, 2f, 6c	176.	$P6_3/m$	$[6_33_02_1]$	
85	t3a	1	1343	2f, 2f, 6f	163.	$P\bar{3}1c$	$(*{:}6\,3_02)$	
86	t3a	1	1332	2f, 6f	148.	$R\bar{3}$	$(6\,3_12)$	
87	t3a	1	1332	2f, 6c	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
88	t3a	1	1333	1b, 5e, 6e	201.	$Pn\bar{3}$	$4^{\circ+}$	
89	t3a	1	1232	2f, 4b	205.	$Pa\bar{3}$	$2^{-}/4$	
90	t3a	1	1232	2e, 4c	205.	$Pa\bar{3}$	$2^{-}/4$	

(*continued*)

Table 2 (*Continued*)

(1) No.	(2) Vert. type	(3) Vert. stab.	(4) Trans-itivity	(5) Tile types	(6) Space group No.	(6) Space group Name	(7) Fibrifold name	(8) Zeolite net
91	t3a	1	1342	3d, 3n	58.	Pnnm	$[2_0 2_0 \times_1]$	
92	t3a	1	1331	3e	62.	Pnma	$(2_1 2 \bar{*} \cdot)$	
93	t3a	1	1331	3e	40.	Ama2	$(2_0 2_1 * \cdot)$	
94	t3a	1	1321	3k	152/154.	$P3_121$	$(3_1 * 3_1)$	
95	t3a	1	1331	3k	15.	C2/c	$(2_0 2_1 2\,2)$	
96	t3a	1	1321	3e	178/179.	$P6_122$	$(* 6_1 3_1 2_1)$	
97	t3a	1	1331	3l	56.	Pccn	$(2 \bar{*}{:}2{:}2)$	
98	t3a	22	1232	3d, 3o	119.	$I\bar{4}m2$	$(* 4 \cdot 4\, 2_1)$	
99	t3a	22	1221	3l	70.	Fddd	$(2 \bar{*} 2_0 2_1)$	
100	t3b	1	1442	24b, 3s	141.	$I4_1/amd$	$(* 4_1 4 \cdot 2)$	
101	t3b	1	1332	1b, 29b	201.	$Pn\bar{3}$	$4^{\circ +}$	
102	t3b	1	1322	1b, 29a	197.	I23	$4^{\circ\circ}$	
103	t3b	1	1431	11b	142.	$I4_1/acd$	$(* 4_1 4{:}2)$	
104	t3b	1	1463	3c, 3q, 3r	140.	I4/mcm	$[* \cdot 4{:}4{:}2]$	
105	t3b	1	1463	2f, 3c, 6j	193.	$P6_3/mcm$	$[* \cdot 6{:}3{:}2]$	
106	t3b	1	1453	3d, 3o, 3r	125.	P4/nbm	$(* 4_0 4 \cdot 2)$	
107	t3b	1	1453	2f, 3d, 6i	162.	$P\bar{3}1m$	$(* \cdot 6\, 3_0 2)$	
108	t3b	1	1462	3c, 3p	141.	$I4_1/amd$	$(* 4_1 4 \cdot 2)$	
109	t3b	1	1441	3f	64.	Cmca	$[2_0 2_1 *{:}]$	
110	t3b	1	1431	3l	54.	Pcca	$(2_0 2 \bar{*}_0)$	
111	t3b	1	1441	3j	98.	$I4_122$	$(* 4_3 4_1 2_0)$	
112	t3b	1	1331	3l	91/95.	$P4_122$	$(* 4_1 4_1 2_1)$	
113	t3b	1	1321	3i	152/154.	$P3_121$	$(3_1 * 3_1)$	
114	t3b	1	1453	2f, 3b, 6k	192.	P6/mcc	$[*{:}6{:}3{:}2]$	AFI
115	t3b	1	1453	2f, 3d, 6k	177.	P622	$(* 6_0 3_0 2_0)$	
116	t3b	1	1341	3j	142.	$I4_1/acd$	$(* 4_1 4{:}2)$	
117	t3b	1	1452	3c, 3p	63.	Cmcm	$[2_0 2_1 * \cdot]$	
118	t3b	1	1431	3e	57.	Pbcm	$(2_0 2 \bar{*} \cdot)$	
119	t3b	1	1442	3d, 3n	12.	C2/m	$[2_0 2_0 2_1 2_1]$	
120	t3b	1	1441	3g	64.	Cmca	$[2_0 2_1 *{:}]$	
121	t3b	1	1322	1d, 5c	88.	$I4_1/a$	$(4\, 4_1 2)$	
122	t3b	1	1431	3l	13.	P2/c	$(2_0 2_0 2\, 2)$	
123	t3b	1	1431	3l	54.	Pcca	$(2_0 2 \bar{*}_0)$	
124	t3b	1	1441	3j	68.	Ccca	$(* 2_0 2{:}2{:}2)$	
125	t3b	1*	1343	2a, 2d, 6j	194.	$P6_3/mmc$	$[*{:}6 \cdot 3 \cdot 2]$	
126	t3b	1*	1332	2b, 6i	164.	$P\bar{3}m1$	$(* 6 \cdot 3 \cdot 2)$	
127	t3b	1*	1332	3d, 3o	129.	P4/nmm	$(* 4 \cdot 4 \cdot 2)$	
128	t3b	1*	1322	1b, 5b	141.	$I4_1/amd$	$(* 4_1 4 \cdot 2)$	MON
129	t3b	1*	1321	3e	63.	Cmcm	$[2_0 2_1 * \cdot]$	
130	t3b	1*	1332	3b, 3r	140.	I4/mcm	$[* \cdot 4{:}4{:}2]$	
131	t3b	1*	1332	2f, 6c	193.	$P6_3/mcm$	$[* \cdot 6{:}3{:}2]$	
132	t3b	1*	1332	3d, 3r	125.	P4/nbm	$(* 4_0 4 \cdot 2)$	
133	t3b	1*	1332	2f, 6f	162.	$P\bar{3}1m$	$(* \cdot 6\, 3_0 2)$	
134	t3b	1*	1331	3h	74.	Imma	$(* 2_1 2 \cdot 2 \cdot 2)$	ABW
135	t3b	1*	1321	3k	53.	Pmna	$[2_0 2_0 *{:}]$	
136	t3b	*33	1222	3a, 3q	229.	$Im\bar{3}m$	$8^{\circ}{:}2$	ACO
137	t3b	*33	1222	1a, 5d	227.	$Fd\bar{3}m$	$2^{+}{:}2$	
138	t3b	*33	1211	3k	166.	$R\bar{3}m$	$(* \cdot 6\, 3_1 2)$	
139	t3b	33	1211	3m	206.	$Ia\bar{3}$	$4^{-}/4$	

Table 2 (*Continued*)

(1) No.	(2) Vert. type	(3) Vert. stab.	(4) Trans-itivity	(5) Tile types	(6) Space group No.	Name	(7) Fibrifold name	(8) Zeolite net
140	t3c	1	1432	1b, 6h	180/181.	$P6_222$	$(*6_23_22_0)$	
141	t3c	1	1342	2f, 6g	230.	$Ia\bar{3}d$	$8^\circ/4$	
142	t3c	33	1211	3k	212/213.	$P4_332$	$2^+/4$	
143	t4a	1	1432	1e, 6f	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
144	t4a	1	1432	1c, 6k	167.	$R\bar{3}c$	$(*{:}6\,3_12)$	
145	t4b	22	1221	4f	212.	$P4_332$	$2^+/4$	
146	t4b	2x	1121	2f	230.	$Ia\bar{3}d$	$8^\circ/4$	
147	t5	1	1432	1g, 2c	142.	$I4_1/acd$	$(*4_14{:}2)$	
148	t5	1*	1332	1b, 2g	141.	$I4_1/amd$	$(*4_14\cdot2)$	MON
149	t6	*332	1111	1f	227.	$Fd\bar{3}m$	$2^+{:}2$	

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