$(6+\varepsilon)$-Approximation for Minimum Weight Dominating Set in Unit Disk Graphs*

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Abstract. It was a long-standing open problem whether the minimum weight dominating set in unit disk graphs has a polynomial-time constant-approximation. In 2006, Ambühl *et al* solved this problem by presenting a 72-approximation for the minimum weight dominating set and also a 89-approximation for the minimum weight connected dominating set in unit disk graphs. In this paper, we improve their results by giving a $(6+\varepsilon)$-approximation for the minimum weight dominating set and a $(10+\varepsilon)$-approximation for the minimum weight connected dominating set in unit disk graphs where ε is any small positive number.

Keywords: Unit Disk Graph, Approximation algorithm, Dominating Set.

1 Introduction

Consider a graph $G = (V, E)$. A subset A of V is called a *dominating set* if every node in $V - A$ is adjacent to a node in A, and furthermore, A is called a *connected dominating set* if the subgraph $G[A]$ induced by A is connected. A graph is called a *unit disk graph* if every node is associated with unit disk (a disk of diameter one) in the Euclidean plane and there is an edge between two nodes if and only if two corresponding disks have nonempty intersection. Therefore, when we place each node at the center of its associated disk, an edge (u, v) exists if and only if $d(u, v) \leq 1$. The unit disk graph is a mathematical model for wireless sensor networks when all sensors have the same communication range. Both the dominating set and the connected dominating set have important applications in the study of wireless sensor networks [3].

Given a unit-disk graph $G = (V, E)$ with node weight $c : V \to R^+$, find a dominating set with minimum total weight. This is an NP-hard problem [4] and it was open for a long time whether there exists a polynomial-time constant-approximation for this problem. In 2006, Ambühl *et al* [1] solved this problem by presenting a 72-approximation for the minimum weight dominating set and also a

* Support in part by National Science Foundation of USA under grants CCF-9208913 and CCF-0728851; and in part by NSFC (60603003) and XJEDU.
** This work was done when this author visited at University of Texas at Dallas.

X. Hu and J. Wang (Eds.): COCOON 2008, LNCS 5092, pp. 551–557, 2008.

89-approximation for the minimum weight connected dominating set in unit disk graphs. In this paper, we improve their results by giving a $(6+\varepsilon)$-approximation for the minimum weight dominating set and a $(10+\varepsilon)$-approximation for the minimum weight connected dominating set in unit disk graphs where ε is arbitrarily small positive number.

2 Preliminaries

Firstly, we set a fixed constant $0 < \mu < \sqrt{2}/2$. Suppose all the nodes of the given unit disk graph are contained in the interior area of a square with edge length $m\mu$. Then, we divide this square into a $m \times m$ grid such that each cell is a square with edge length μ. We may assume that no node lies on any cut-line of the grid since, if such a case occurs, we can always make a little move of the grid to have cut-lines away from nodes. For a cell e and a node subset D, let $D^+(e)$ denote the subset of all nodes each of which is able to dominate (i.e., adjacent to) a node in e and $D(e) = e \cap D$. Ambühl *et al* [1] studied the following subproblem:

> Subproblem-on-Cell: Find a minimum weight subset of $V^+(e)$ to dominate $V(e)$.

They found that this subproblem has a polynomial-time 2-approximation, which results in a 72-approximation for minimum weight dominating set in whole unit disk graph. A key lemma in establishing the 2-approximation is as follows.

Lemma 1. *Consider a set P of points lying inside a horizontal strip and a set $\mathcal{D}$ of disks with radius one and with centers either above or below the strip (Fig. 1). Give each disk with a nonnegative weight. Suppose the union of all disks covers P. Then the minimum weight subset of disks covering P can be computed in time $O(m^4 n)$ where $n = |P|$ and $m = |\mathcal{D}|$.*

This lemma is very useful in the whole paper.

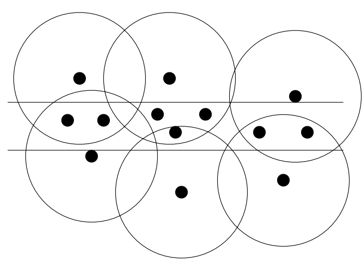

Fig. 1. A covering problem about a strip

3 Main Result

Our new results are based on several new arguments and new ideas.

Firstly, we set $\mu = \sqrt{2}/2$ and establish some new results about a cell e.

Let A, B, C, D be four vertices of e and divide outside of e into eight areas as shown in Fig. 2. For any node $p \in V(e)$, let $\angle p$ be a right angle at p such that two edges intersecting horizontal line AB each at an angle of $\pi/4$. Let $\Delta_{low}(p)$ denote the part of e lying inside of $\angle p$. Similarly, we can define $\Delta_{up}(p)$, $\Delta_{left}(p)$ and $\Delta_{right}(p)$ as shown in Fig. 3.

Lemma 2. *If p is dominated by a node u in area LM then every point in $\Delta_{low}(p)$ can be dominated by u. The similar statement holds for CL and $\Delta_{left}(p)$, CR and $\Delta_{right}(p)$, and UM and $\Delta_{up}(p)$.*

Proof. Since $\Delta_{low}(p)$ is a cover polygon, it sufficient to show that the distance from u to every vertex of $\Delta_{low}(p)$ is at most one.

Suppose v is a vertex of $\Delta_{low}(p)$ on BC (Fig. 4). Draw a line L' perpendicular to pv and equally divide pv. If u is below L', then we have $d(u,v) \leq d(u,p) \leq 1$. If u is above L', then $d(u,v) \geq d(u,p)$, then $\angle uvp < \pi/2$ and hence $\angle uvC < 3\pi/4$. It follows that $d(u,v) < \mu/\cos\pi/4 = 1$.

A similar argument can be applied in the case that the vertex v of $\Delta_{low}(p)$ is on DA or on AB. □

Consider two nodes $p, p' \in V(e)$. Suppose p is on the left of p'. Extend the left edge of $\angle p$ and the right edge of $\angle p'$ to intersect at point p''. Define $\Delta_{low}(p,p')$ to be the part of e lying inside of $\angle p''$ (Fig. 5). Similarly, we can define $\Delta_{up}(p.p')$.

Lemma 3. *Let K be a subset of $V^+(e) - V(e)$, which dominates $V(e)$. Suppose $p, p' \in V(e)$ are dominated by some nodes in $K \cap LM$ (or $K \cap UM$), but neither p nor p' is dominated by any node in $K \cap (CL \cup CR)$. Then every node in*

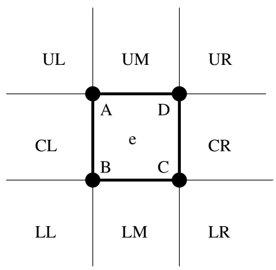

Fig. 2. Outside of e is divided into eight areas

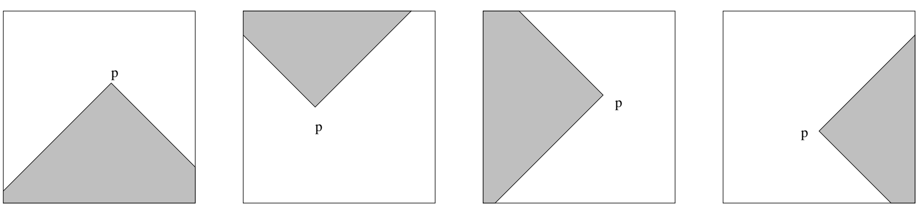

Fig. 3. $\Delta_{low}(p)$, $\Delta_{up}(p)$, $\Delta_{left}(p)$ and $\Delta_{right}(p)$

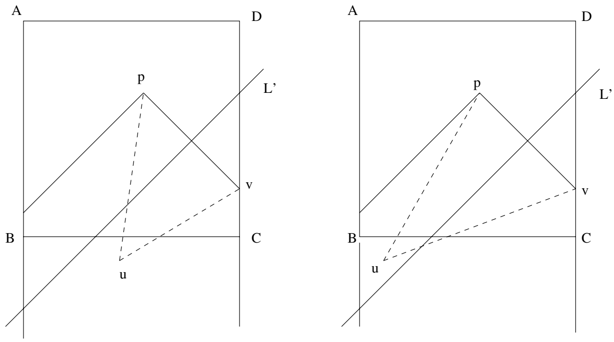

Fig. 4. The proof of Lemma 2

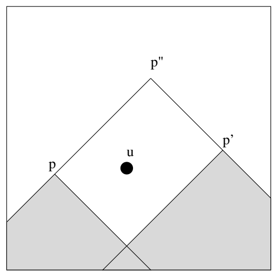

Fig. 5. $\Delta_{low}(p, p')$

$\Delta_{low}(p, p')$ can be dominated by nodes in $K \cap (U \cup L)$ where $U = UL \cup UM \cup UR$ and $L = LL \cup LM \cup LR$.

Proof. By Lemma 2, it suffices to consider a node u lying in $\Delta_{low}(p, p') \backslash (\Delta_{low}(p) \cup \Delta_{low}(p')$. For contradiction, suppose u is dominated by a node v in $K \cap (CL \cup CR)$. If $v \in CL$, then $\Delta_{left}(v)$ contains p and by Lemma 2, p is dominated by v, a contradiction. A similar contradiction can result from $v \in CR$. $\square$

Define $\Delta_{low}(\emptyset) = \emptyset$, and $\Delta_{low}(\{p\}) = \Delta_{low}(p)$. Also, define $\Delta_{low}(\{p, p'\}) = \Delta_{low}(p, p')$ if p is on the left of p and $\Delta_{low}(\{p, p'\}) = \Delta_{low}(p)$ if p and p' on a vertical line and p is higher than p'. Similarly, define $\Delta_{up}(W)$ for a subset W of at most two vertices in $V(e)$.

Now, consider an optimal solution *Opt* for the minimum weight dominating set and its total weight is denoted by *opt*. Let C be the set of all cells in the grid and $C' = \{e \in C \mid e \cap Opt \neq \emptyset\}$.

For $e \in C - C'$, choose p (p') to be the leftmost (rightmost) node in $V(e)$, which is dominated by a node in $Opt^+(e) \cap LM$, but not by any node in $Opt^+(e) \cap (CL \cup CR)$; if there are more than one such p (p'), then choose the highest one. Choose q (q') to be the leftmost (rightmost) node in $V(e)$, which is dominated by a node in $Opt^+(e) \cap UM$, but not by any node in $Opt^+(e) \cap (CL \cup CR)$; if there are more than one such q (q'), then choose the lowest one.

Define

$$W = \begin{cases} \{p, p'\} & \text{if } p \text{ and } p' \text{ exist} \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$W' = \begin{cases} \{q, q'\} & \text{if } q \text{ and } q' \text{ exist} \\ \emptyset & \text{otherwise,} \end{cases}$$

By Lemma 3, every node in $V_1(e) = \Delta_{low}(W) \cup \Delta_{up}(W)$ can be dominated by $Opt^+(e) \cup (U \cup L)$ and every node in $V_2(e) = V(e) \setminus V_1(e)$ can be dominated by $Opt^+(e) \cap (L^* \cup R)$ where $L^* = UL \cup CL \cup LL$ and $R = UR \cup CR \cup LR$.

Let $H_1, ..., H_m$ be m horizontal strips and $Y_1, ..., Y_m$ m vertical strips. For every $e \in C'$, choose a node $v_e \in e \cap Opt$. Let $U = \{v_e \mid e \in C'\}$ and Z_U be the set of nodes dominated by U. For each cell e, denote $V_1^+(e) = V^+(e) \cap (U \cup L^*)$ and $V_2^+(e) = V^+(e) \cap (L \cup R)$ and each strip H_i (Y_i), denote $Opt^+(H_i) = (\cup_{e \in (C-C') \cap H_i} Opt_1^+(e)) \setminus U$ $(Opt^+(Y_i) = (\cup_{e \in (C-C') \cap Y_i} Opt_2^+(e)) \setminus U)$.

For each strip H_i, we compute a minimum weight subset $OPT(H_i)$ from set $(\cup_{e \in (C-C') \cap H_i} V_1^+(e)) \setminus U$ to dominate $(\cup_{e \in (C-C') \cap H_i} V_1(e)) \setminus Z_U$ and for each strip Y_i, compute a minimum weight subset of $(\cup_{e \in (C-C') \cap Y_i} V_2^+(e)) \setminus U$ to dominate $(\cup_{e \in (C-C') \cap Y_i} V_2(e)) \setminus Z_U$. Putting U, $OPT(H_i)$ and $OPT(Y_i)$ together, we would obtain a dominating set with total weight at most

$$c(U) + \sum_{i=1}^{m} c(Opt_1^+(H_i)) + \sum_{i=1}^{m} c(Opt_2^+(Y_i)).$$

Note that a node can be in $Opt_1^+(H_i)$ or $Opt_2^+(Y_i)$ for at most six (horizontal or vertical) strips. Therefore,

$$c(U) + \sum_{i=1}^{m} c(Opt_1^+(H_i)) + \sum_{i=1}^{m} c(Opt_2^+(Y_i)) \leq 6opt.$$

This means that we obtain a dominating set with total weight at most $6opt$.

However, computing this dominating set has a trouble because C' and U are defined by Opt and for each cell, p, p', q, q' are also determined by Opt. Don't worry! This trouble can be removed by consider all possible C', all possible U and all possible p, p', q, q'. This idea gives the following approximation algorithms.

6-Approximation: Put input unit-disk graph $G = (V, E)$ in the interior of a $m\mu \times m\mu$ square S. Divide the square S into an $m \times m$ grid such that each cell is a $\mu \times \mu$ square. Let C be the set of m^2 cells. Let $C' \subseteq C$. For each cell $e \in C'$, choose a node $v_e \in V(e)$ and let $U = \{v_e \mid e \in C'$. For every subset C' and every U, compute a node subset $A(C', U)$ in the following way:

Step 1. For every cell $e \in C - C'$ and for every $W \subseteq V_{low}(e)$ with $|W| \leq 2$ and every $W' \subseteq V_{up}(e)$ with $|W'| \leq 2$, let $V_1(e) = \Delta_{low}(W) \cup \Delta_{low}(W')$ and $V_2(e) = V(e) - V_1(e)$.

Step 2.1. For each horizontal strip H, compute a minimum weight subset $OPT(H)$ of $(\cup_{e \in H \setminus C'} V_1^+(e)) \setminus U$ to dominate $(\cup_{e \in H \setminus C'} V_1(e)) \setminus Z_U$.

Step 2.2. For each vertical strip Y, compute a minimum weight subset $OPT(Y)$ of $\cup_{e\in Y\setminus C'} V_2^+(e)$ to dominate $(\cup_{e\in H\setminus C'} V_2(e)) \setminus Z_U$.
Step 2.3. Compute $O = (\cup_H OPT(H)) \cup (\cup_Y OPT(Y))$ to minimize the total weight $c(O)$ over all possible combinations of W, W' for all $e \in C - C'$.
Step 3. Set $A(C', U) = O(C') \cup U$.
Step 4. Finally, compute a $A = A(C', U)$ to minimize the total weight $c(A(C', U))$ for C' over all subsets of C and U over all choices.

We now estimate the time for computing A. There are $O(2^{m^2})$ possible subsets of C, $n^{O(m^2)}$ possible choices of U and $O(n^{4m^2})$ possible combinations of W and W' for all cells in $C - C'$. For each combination, computing all $OPT(H)$ and all $OPT(Y)$ needs time $O(n^5)$. Therefore, total computation time is $n^{O(m^2)}$. A good news is that when m is a constant, this is a polynomial-time. A bad news is that in general, m is not a constant. However, we can use partition again to make a constant m!

Theorem 1. *For any* $\varepsilon > 0$*, there exists* $(6+\varepsilon)$*-approximation with computation time* $n^{O(1/\varepsilon^2)}$ *for the minimum weight dominating set in unit disk graph.*

Proof. Choose $m = 12\max(1, \lceil 1/\varepsilon \rceil)$. Put input unit-disk graph G into a grid with each cell being an $m\mu \times m\mu$ square. For each cell e, solve the problem for subset of nodes within distance one to cell e to dominate the nodes in cell e with the 6-approximation algorithm. Union them together and denote this union by $A(P)$ for the partition P induced by this grid. Shift this grid in diagonal direction with distance one in each time. This results in m partitions $P_1, ..., P_m$. Choose $A = A(P_i)$ to be the one with the minimum weight among $A(P_1), ..., A(P_m)$. We claim $c(A(P_i)) \leq (6 + \varepsilon)opt$.

In fact, each disk with radius one can cross cutlines of at most four partition. When a disk with radius one and center at a vertex u crosses a cutline of a partition P_j, u may involve more than one, but at most four $m\mu \times m\mu$ cells' subproblems. It follows that

$$c(A) \leq (1 + 12/m)opt \leq (1 + \varepsilon)opt.$$

Now, the total computation time is $m \cdot n^{O(m^2)} = n^{O(1/\varepsilon^2)}$. □

Next, we study the minimum weight connected dominating set problem in unit disk graphs: Given a unit-disk graph $G = (V, E)$ with weight $c : V \to R^+$, find a connected dominating set with minimum total weight.

Theorem 2. *For any* $\varepsilon > 0$*, there exists a* $(10 + \varepsilon)$*-approximation with computation time* $n^{O(1/\varepsilon^2)}$ *for the minimum weight connected dominating set in unit disk graphs.*

Proof. We first compute a $(7 + \varepsilon)$-approximation D for the minimum weight dominating set and then connect D with nodes of total weight at most $4opt$

where *opt* is the minimum weight of connected dominating set. This can be done due to the following:

(1) Let OPT be the minimum weight connected dominating set for G. Then, we can find a minimum-length spanning tree T for $D \cup OPT$ such that every node in $OPT - D$ has degree five by the method in the proof of Lemma 1 in [2].

(2) Using method in [5], we can construct from T a spanning tree T' for D such that each edge (u, v) of T' is a path between u and v in T and each node in $OPT \setminus D$ appears in at most four edges of T'. If we assign the weight of edge (u, v) of T' equal to the total weight of nodes on the path between u and v. The total edge-weight of T' is at most $4opt$.

(3) We can compute a tree T^* with weight as small as that of T' in the following:

Step 1. Construct a complete graph H on D. For each edge (u, v), assign cost $w(u, v)$ with the minimum total weight of internal nodes on a path between u and v in graph G.

Step 2. Compute a minimum spanning tree T'' of H and map T'' back to G in order to obtain a tree T^* for connecting all connected component of D.

Then we can finished the whole proof. □

4 Conclusion

In this paper we gave an improved constant-factor approximation algorithm to compute minimum weight connected dominating set in unit disk graphs. We divided our algorithm into two parts. The first part selected a minimum weight dominating set for a given unit disk graph, and then the first part connects this dominating set by inserting several disks into the dominating set. The first part is a $(6+\varepsilon)$-approximation, while the whole algorithm is a $(10+\varepsilon)$-approximation.

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