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7-DIMENSIONAL NILPOTENT LIE ALGEBRAS

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ABSTRACT. All 7-dimensional nilpotent Lie algebras over $\mathbb{C}$ are determined by elementary methods. A multiplication table is given for each isomorphism class. Distinguishing features are given, proving that the algebras are pairwise nonisomorphic. Moduli are given for the infinite families which are indexed by the value of a complex parameter.

1. INTRODUCTION

Historically, a marked difference is noted between the classification theory of semisimple Lie algebras and the classification theories of solvable or nilpotent Lie algebras. The semisimple theory can best be described as beautiful, while the others lack anything resembling elegance. For semisimple algebras over the complex numbers one has the Killing form, Dynkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula for finite-dimensional representations, and much more [11, 12]. In the theory of solvable algebras one has the theorems of Lie and Engel along with Malcev's reduction of the classification problem to the same problem for nilpotent algebras (and the determination of their derivation algebras) [15]. There does not seem to be any nice way to classify nilpotent algebras (such as a graph or diagram for each algebra). Indeed, the results of this paper offer evidence that none will be found.

Progress towards a complete classification of nilpotent algebras has been quite slow. Umlauf, a student of Engel, classified nilpotent algebras through dimension six [28]. He, like most others making advances in this subject, made mistakes such as the inclusion of several algebras more than once because he was unaware of isomorphisms among them. Given the nature of the subject matter, this is to be expected: one has little sense of how a correct classification should look. Since then, several attempts have been made to develop some machinery whereby the classification problem can be reformulated. Morozov observed that there is a lower bound for the dimension of a maximal abelian ideal I of a nilpotent Lie algebra. This suggests an inductive method for studying algebras. One must know all smaller nilpotent algebras, and their finite-dimensional representations [17]. Morozov's method is to consider L as a noncentral extension of L/I, where the abelian ideal I is a nontrivial L/I-module. Apparently

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by using this approach, and knowledge of low-dimensional representations of 2- and 3-dimensional algebras, Safiullina compiled a list of seven-dimensional algebras in his 1964 thesis [19]. For Morozov's method to be implemented in higher dimensions, one must also be able to classify (large) indecomposable finite-dimensional representations of all known algebras, and be able to detect isomorphisms among the resulting new algebras. Gauger studied metabelian Lie algebras as part of his doctoral dissertation in 1971 [8]. ("Metabelian" means that L/Z is abelian.) He set up some machinery whereby he studied orbits in exterior algebras of vector spaces, but his method did not easily extend to more general nilpotent algebras. He was, however, able to classify metabelian algebras in dimension 7 (with a slight mistake), and to get partial information in dimensions 8 and 9. Umlauf also classified a small subset of the algebras in dimensions seven, eight, and nine [28]. He found that in these dimensions, infinite families of nonisomorphic nilpotent algebras occur. This phenomenon has been exhibited by Chao, Gauger, Santharoubane, and others without providing very much specific information about the overall classification [7, 8, 21].

More recently, Skjelbred and Sund have reduced the classification of nilpotent algebras in a given dimension to the study of orbits under the action of a group on the space of second degree cohomology classes (of a smaller Lie algebra) with coefficients in a trivial module [27]. Specifically, one considers L as a central extension of L/Z, using the usual homological tools. This approach is difficult to use in practice, because orbits under the automorphism group of L/Z are not easily identified even after the cohomology groups are known. In fact, these authors claimed as an application only to have discovered another version of some of Umlauf's results. (Their examples are in dimensions less than six.) However, their method offers an improvement in that it requires less information about the representation theory of smaller algebras than does the Morozov method.

Magnin has introduced a different inductive approach to the study of nilpotent algebras: enlarge a smaller algebra by adjoining a derivation. He classified algebras up to dimension 6 over the real field, and obtained partial information in dimension 7 this way [14].

During the past few years there has been a good deal of activity in the subject of nilpotent Lie algebras. Since Safiullina's first attempt to classify all 7-dimensional nilpotent Lie algebras there have been a number of works in that direction, including this one. Various tactics have been implemented [1, 14, 18]. Second, in characteristic p, the Baker-Campbell-Hausdorff formula puts a group structure on a nilpotent Lie algebra. A project similar to this one has been carried out in all positive characteristics using linear methods with the help of computers [29]. Third, Carles and several others have studied the deformation theory of nilpotent and solvable Lie algebras [6, 9, 10, 22, 23]. Fourth, work of Laudal, Pfister, Bjar, Yau, and the author has begun to uncover relationships between the deformation theory of singularities and the deformation theory of Lie algebras [5, 13, 26, 30].

The classification of nilpotent Lie algebras in higher dimensions remains a vast open area. Again, as was the case after Umlauf's pioneering work, the most efficient way of sorting through the mess of facts in higher dimensions is not obvious. These days computers offer some hope of being able to deal with the kind of messy details which haunt this subject. Along with the classification problem is that of determining all degenerations among algebras of a given dimension. There are other unanswered questions in higher dimensions. Following the discussion of *modality* in [2], let us consider the minimum number F_n such that any (small) open subset in the space of all multiplication tables is covered by a finite number of F_n-parameter orbits. Thus, $F_n = 1$ for $n \leq 6$, and it will be seen below that $F_7 = 1$. That is, there are 1-parameter families of nonisomorphic nilpotent Lie algebras (but no 2-parameter families) in dimension 7. The examples of Chao and Santharoubane show that $F_n \geq 1$ for $n \geq 7$. (This fact can be deduced from Umlauf's work; the modern examples have some other nice properties.) It is now known that $F_{2n} \geq \frac{1}{6}(n-1)^3$ for $2n \geq 8$ [24]. The growth of this function should be of some interest. Other questions involve generic behavior of nilpotent Lie algebras. For instance, I suspect that most algebras in high enough dimensions do not have semisimple derivations. No doubt, there are other interesting phenomena in higher dimensions of which we are as yet unaware. Further study, by the primitive methods used here or by any other available methods, should shed some light on these questions.

2. Methods

Among the algebras with central dimension 2 or greater there are 31 decomposable nilpotent Lie algebras in dimension 7. They are direct sums of two or more of the indecomposable algebras of lower dimension. All except $1, 3 \oplus 1, 2, 4$ have a central (abelian) summand.

Lemma 1. *In a decomposition of a finite-dimensional Lie algebra as a direct sum of indecomposable ideals, the isomorphism classes of the ideals are unique. If $L = A_1 \oplus \cdots \oplus A_r$ and $L = C_1 \oplus \cdots \oplus C_s$ are two such decompositions, then $r = s$. After reordering the indices the derived parts A_i' and C_i' are equal, $A_i \cong C_i$, and a set of generators for A_i equals a set of generators for C_i modulo adding to each generator a vector in the center of L.*[1]

Assuming knowledge of all smaller algebras, we need then consider only indecomposable algebras.[2] The sequence of upper central series dimensions is an invariant of a nilpotent Lie algebra. It is easy to see that none of the centers of a nilpotent Lie algebra of dimension 7 can be 6-dimensional. (The same is true of the first derived algebra L'.) There are other sequences which do not occur as central dimensions of indecomposable algebras. For example the sequence of central dimensions cannot end in $3, 4, 7$ or $1, 2, 7$ because (looking at brackets modulo Z) this would yield a space L/Z^2 of odd dimension with a nondegenerate alternating form.

There are usually many nonisomorphic algebras sharing the same central dimensions. For a typical sequence of central dimensions some further invariants must be identified. I found no uniform way of doing this. If the first center is small, an inductive approach makes sense: L/Z contains a lot of information about L. From the multiplication for L/Z one knows the entire multiplication

[1]The proof of the lemma turns out to be somewhat complicated. It is done by toying around with the inclusions and projections associated with the two decompositions, starting with one of the nonabelian A_i's.

[2]Cf. [23] and especially [3] for several labelling schemes in smaller dimensions.

table of L, except for the coefficients of central vectors in all bracket relations. The Jacobi identities among elements of $L \backslash Z$ restrict these coefficients somewhat, and one can then proceed to sort out various possibilities. If the first few central dimensions are large (i.e., if $\dim(Z) \geq 2$), very little information about L is contained in L/Z. Then, such invariants as the number of vectors having an adjoint image which is smaller than expected, or the vanishing of the brackets among certain special vectors, provide a way to compare possible isomorphism classes. A good basis, from the point of view of this classification, is roughly one with the fewest possible nonzero structure constants. This makes comparisons easier. It is convenient to have a basis $\{a, b, c, d, e, f, g\}$ whose last n_1 elements are in the center Z, whose last n_2 elements are in the second center Z^2, and so forth. The Lie bracket of two vectors is thus a linear combination of vectors which occur alphabetically later. Assigning variable names to some of the structure constants, and trying to change basis vectors to normalize them one at a time, offers a way of sorting into cases and subcases. Whichever starting direction one takes, one must eventually do a lot of menial algebra to simplify a given multiplication table or to distinguish nonisomorphic algebras. The notes following the lists below should indicate the flavor of this kind of work.

A proof that all nilpotent Lie algebras of dimension 7 are included in the following list is available from the author. Because of its length it is omitted from this paper.

3. List of 7-dimensional nilpotent Lie algebras

A multiplication table for each algebra is given below (nonzero brackets only). There are 161 tables, including the 130 indecomposables and 31 decomposable algebras—six of these represent infinite families parametrized by a single complex variable. The numbers 130 and 161 are not to be treated as absolute in any sense; some of the algebras which could have been included in each infinite family have been singled out and listed separately for various reasons. The name given to each algebra is the list of central series dimensions. For example, the algebras having a center Z of dimension 2, a second center Z^2 of dimension 4, and a third center Z^3 of dimension 7 are listed as $2, 4, 7_A$, $2, 4, 7_B$, and so forth. $2 \oplus 1, 3, 5$ denotes the direct sum of the 2-dimensional abelian Lie algebra with the unique algebra whose upper central dimensions are $1, 3, 5$. Distinguishing features are listed for those indecomposable algebras sharing central dimensions with at least one other algebra. There $\langle a, b, c \rangle$ will denote the linear span of the vectors a, b, and c. The bracket operation $y \mapsto [x, y]$ will be referred to as ad_x. Also $L' = [L, L]$ and $L'' = [L, L']$, etc.

For the six infinite families, a variable ξ is used to denote a structure constant which can take on arbitrary complex values. An invariant $K(\xi)$ is given for each family in which multiple values of ξ yield isomorphic algebras.

Decomposable algebras with large centers

7
7-dimensional abelian

$2 \oplus 2, 5$
$[a, b] = d$
$[a, c] = e$

$4 \oplus 1, 3$
$[a, b] = c$

$1 \oplus 3, 6$
$[a, b] = d [b, c] = f$
$[a, c] = e$

$3 \oplus 1, 2, 4$
$[a, b] = c$
$[a, c] = d$

$2 \oplus 2, 3, 5$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d$

Central series dimensions 3, 7

Decomposables

$2 \oplus 1, 5$
$[a, b] = e \quad [c, d] = e$

$1 \oplus 2, 6$
$[a, b] = e \quad [c, d] = e$
$[a, c] = f$

$1 \oplus 1, 3 \oplus 1, 3$
$[a, b] = e \quad [c, d] = f$

Indecomposables

$3, 7_A$
$[a, b] = e \quad [b, c] = f$
$[b, d] = g$

$3, 7_B$
$[a, b] = e \quad [b, c] = f \quad [c, d] = g$

$3, 7_C$
$[a, b] = e \quad [b, c] = f \quad [c, d] = e$
$[b, d] = g$

$3, 7_D$
$[a, b] = e \quad [b, d] = g \quad [c, d] = e$
$[a, c] = f$

The $3, 7$ algebras can be distinguished by the number of vectors x, linearly independent (mod Z), such that $\dim([x, L]) = 1$. There are three in A, two in B, one in C, and none in D.

Central series dimensions 3, 5, 7

Decomposables

$2 \oplus 1, 3, 5$
$[a, b] = c \quad [b, c] = e$
$[a, d] = e$

$1 \oplus 2, 4, 6_A$
$[a, b] = c$
$[a, c] = e$
$[a, d] = f$

$1 \oplus 2, 4, 6_B$
$[a, b] = c \quad [b, d] = f$
$[a, c] = e$

$1 \oplus 2, 4, 6_C$
$[a, b] = c \quad [b, d] = e$
$[a, c] = e$
$[a, d] = f$

$1 \oplus 2, 4, 6_D$
$[a, b] = c \quad [b, c] = f$
$[a, c] = e$
$[a, d] = f$

$1 \oplus 2, 4, 6_E$
$[a, b] = c \quad [b, c] = f$
$[a, c] = e$
$[a, d] = e$

Indecomposables

$3, 5, 7_A$
$[a, b] = c \quad [b, d] = f$
$[a, c] = e$
$[a, d] = g$

$3, 5, 7_B$
$[a, b] = c \quad [b, c] = f$
$[a, c] = e$
$[a, d] = g$

$3, 5, 7_C$
$[a, b] = c \quad [b, c] = f$
$[a, c] = e \quad [b, d] = e$
$[a, d] = g$

$\mathrm{Dim}(L'') = 1$ in A, and $= 2$ in B, C. In B, $\exists c \in L'$ such that $\dim([c, L]) = 2$.

Central series dimensions 3, 4, 5, 7

Decomposables

$2 \oplus 1, 2, 3, 5_A$
$[a, b] = c$
$[a, c] = d$
$[a, d] = e$

$2 \oplus 1, 2, 3, 5_B$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d$
$[a, d] = e$

$1 \oplus 2, 3, 4, 6$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d$
$[a, d] = e$

Central series dimensions 2, 7

Indecomposables

$2, 7_A$
$[a, e] = f \quad [b, e] = g \quad [c, d] = f$

$2, 7_B$
$[a, d] = f \quad [b, e] = g \quad [c, d] = f + g$
$[c, e] = f + g$

In A, there are four linearly independent vectors (modulo Z) such that $\dim([x, L]) = 1$. There are only three in B.

Central series dimensions 2, 5, 7

Decomposables

$1 \oplus 1, 4, 6$
$[a, b] = c \quad [d, e] = f$
$[a, c] = f$

$1, 2, 4 \oplus 1, 3$
$[a, b] = c \quad [d, e] = g$
$[a, c] = f$

Indecomposables

$2, 5, 7_A$
$[a, b] = c \quad [b, d] = f$
$[a, c] = f$
$[a, e] = g$

$2, 5, 7_B$
$[a, b] = c \quad [b, d] = g$
$[a, c] = f$
$[a, e] = g$

$2, 5, 7_C$
$[a, b] = c \quad [b, d] = f$
$[a, c] = f \quad [b, e] = g$

$2, 5, 7_D$
$[a, b] = c \quad [b, d] = f$
$[a, c] = f \quad [b, e] = g$
$[a, d] = g$

$2, 5, 7_E$
$[a, b] = c \quad [b, d] = g \quad [d, e] = f$
$[a, c] = f$

$2, 5, 7_F$
$[a, b] = c \quad [d, e] = f$
$[a, c] = f$
$[a, d] = g$

$2, 5, 7_G$
$[a, b] = c \quad [b, d] = g \quad [d, e] = f$
$[a, c] = f$
$[a, e] = g$

$2, 5, 7_H$
$[a, b] = c \quad [b, d] = f \quad [d, e] = g$
$[a, c] = f$

$2, 5, 7_I$
$[a, b] = c \quad [b, c] = g$
$[a, c] = f$
$[a, d] = f$
$[a, e] = g$

$2, 5, 7_J$
$[a, b] = c \quad [b, c] = g$
$[a, c] = f \quad [b, d] = f$
$[a, e] = g$

$2, 5, 7_K$
$[a, b] = c \quad [b, c] = g \quad [d, e] = f$
$[a, c] = f$

$2, 5, 7_L$
$[a, b] = c \quad [b, c] = g \quad [d, e] = f$
$[a, c] = f$
$[a, d] = g$

$\mathrm{Dim}(L'') = 1$ in $ABCDEFGH$. For $ABCD$, $[Z^2, Z^2] = 0$. For EFG, $[Z^2, Z^2] = L''$. In $ABCDEF$, there is a unique b (up to scalar multiple modulo Z^2) such that $[b, L] = 0$. In A, $[b, Z^2] = L''$. In B, $0 \neq [b, Z^2] \neq L''$. In CD, $[b, Z^2] = Z$. In CE, $\exists a \notin \langle b \rangle \oplus Z^2$ such that $\dim([a, Z^2]) = 1$. In F, $[b, Z^2] = 0$. For H, $0 \neq [Z^2, Z^2] \neq L''$.

$\dim(L'') = 2$ in algebras $IJKL$. For K and L, $[Z^2, Z^2] \neq 0$. In K, but not L, there are d, e, linearly independent $(\mathrm{mod}\ Z)$, such that $\dim([d, L]) = \dim([e, L]) = 1$.

For I and J, $[Z^2, Z^2] = 0$. Any ad_x, for $x \in Z^2 \backslash Z$, induces a mapping $L/Z^2 \to Z$. There is a vector $c \in Z^2 \cap L'$, unique $(\mathrm{mod}\ Z)$; ad_c induces a linear isomorphism $\widetilde{ad_c}: \langle a, b \rangle \to \langle f, g \rangle$. Thus for any other $x \in Z^2/Z$ the map $(\widetilde{ad_c}^{-1}) \circ (\widetilde{ad_x})$ is an endomorphism of $\langle a, b \rangle$, which is determined up to conjugation in $M_{2\times 2}(\mathbb{C})$. The vectors c, d, e determine a 3-dimensional subspace of $M_{2\times 2}(\mathbb{C})$, again up to conjugacy. There is a 1-dimensional orthogonal complement with respect to the bilinear form trace($M^t N$) (again, up to conjugacy). The Jordan form of any nonzero vector in this 1-dimensional complement is determined up to scalar multiple. Algebra I is of type $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^\perp$ and algebra J is of type $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^\perp$.

Central Series Dimensions 2, 4, 7

Decomposables

$1 \oplus 1, 3, 6_A$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e$

$1 \oplus 1, 3, 6_B$
$[a, b] = d \quad [c, e] = f$
$[a, c] = e$
$[a, d] = f$

Indecomposables

$2, 4, 7_A$
$[a, b] = d$
$[a, c] = e$
$[a, d] = f$
$[a, e] = g$

$2, 4, 7_B$
$[a, b] = d \quad [c, e] = g$
$[a, c] = e$
$[a, d] = f$

$2, 4, 7_C$
$[a, b] = d \quad [c, e] = g$
$[a, c] = e$
$[a, d] = g$
$[a, e] = f$

$2, 4, 7_D$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e$
$[a, e] = g$

$2, 4, 7_E$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e$
$[a, d] = g$
$[a, e] = g$

$2, 4, 7_F$
$[a, b] = d \quad [b, d] = f \quad [c, e] = g$
$[a, c] = e$

$2, 4, 7_G$
$[a, b] = d \quad [b, d] = f \quad [c, e] = g$
$[a, c] = e$
$[a, d] = g$

$2, 4, 7_H$
$[a, b] = d \quad [b, d] = f \quad [c, e] = g$
$[a, c] = e$
$[a, d] = g$
$[a, e] = f$

$2, 4, 7_I$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e \quad [c, e] = g$

$2, 4, 7_J$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e \quad [c, e] = g$
$[a, d] = f$

$2, 4, 7_K$
$[a, b] = d \quad [b, e] = f \quad [c, d] = f$
$[a, c] = e \quad [c, e] = g$
$[a, d] = g$

$2, 4, 7_L$
$[a, b] = d \quad [b, c] = f$
$[a, c] = e$
$[a, d] = f$
$[a, e] = g$

$2, 4, 7_M$
$[a, b] = d \quad [b, c] = f$
$[a, c] = e \quad [b, d] = g$
$[a, e] = f$

$2, 4, 7_N$
$[a, b] = d \quad [b, c] = f$
$[a, c] = e \quad [b, d] = g$
$[a, e] = g$

$2, 4, 7_O$
$[a, b] = d \quad [b, c] = f$
$[a, c] = e \quad [b, d] = g$
$[a, d] = f$
$[a, e] = g$

$2, 4, 7_P$
$[a, b] = d \quad [b, c] = f \quad [c, d] = g$
$[a, c] = e \quad [b, e] = g$

$2, 4, 7_Q$
$[a, b] = d \quad [b, c] = f \quad [c, d] = g$
$[a, c] = e \quad [b, e] = g$
$[a, e] = f$

$2, 4, 7_R$
$[a, b] = d \quad [b, c] = f \quad [c, d] = g$
$[a, c] = e \quad [b, e] = g$
$[a, d] = f$
$[a, e] = f$

The $2, 4, 7$ algebras are distinguished by examining properties of a basis $\{a, b, c\}$ for L/Z (by abuse of notation: $a, b, c \in L$). In $ABCDEFGHIJK$, b and c can be chosen so that $[b, c] = 0$. A has a 6-dimensional abelian ideal $\langle b, c, d, e, f, g\rangle$. In BC, b can be chosen so that $[b, Z^2] = 0$. In B, a can be chosen so that $\dim([a, Z^2]) = 1$. For $DEFGH$, $\dim([b, Z^2]) = \dim([c, Z^2] = 1$. For DE, $[b, [b, L]] = [c, [c, L]] = 0$ also. In D, $[a, [a, b]] = 0$; in F, $[a, Z^2] = 0$; in G, $\dim([a, Z^2]) = 1$; in I, $[a, Z^2] = 0$; and in J, $[a, Z^2] = [b, Z^2]$.

In $LMNOPQR$, b and c can be chosen so that $[b, c] \in Z$. In L, $[b, Z2] = [c, Z2] = 0$. For MNO, $[c, Z2] = 0$. In MN, $[a, [a, b]] = 0$. In N these choices for a, b, c allow $[a, [a, L]] \subset [b, Z^2]$. In P, $[a, Z^2] = 0$, and in In Q, $[a, [a, b]] = 0$.

Central series dimensions 2, 4, 5, 7

Decomposables

$1 \oplus 1, 3, 4, 6_A$
$[a, b] = c \quad [b, e] = f$
$[a, c] = d$
$[a, d] = f$

$1 \oplus 1, 3, 4, 6_B$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = f$

$1 \oplus 1, 3, 4, 6_C$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d \quad [b, d] = f$
$[a, e] = f$

Indecomposables

$2, 4, 5, 7_A$
$[a, b] = c$
$[a, c] = d$
$[a, d] = f$
$[a, e] = g$

$2, 4, 5, 7_B$
$[a, b] = c \quad [b, e] = f$
$[a, c] = d$
$[a, d] = g$

$2, 4, 5, 7_C$
$[a, b] = c \quad [b, e] = f$
$[a, c] = d$
$[a, d] = f$
$[a, e] = g$

$2, 4, 5, 7_D$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = f$
$[a, e] = g$

$2, 4, 5, 7_E$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = g$

$2, 4, 5, 7_F$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d$
$[a, d] = f$
$[a, e] = g$

$2, 4, 5, 7_G$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d$
$[a, d] = g$
$[a, e] = f$

$2, 4, 5, 7_H$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = g$
$[a, d] = g$

$2, 4, 5, 7_I$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = g$
$[a, d] = f$

$2, 4, 5, 7_J$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = g$
$[a, d] = f + g$

$2, 4, 5, 7_K$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, e] = g$
$[a, d] = g$
$[a, e] = f$

$2, 4, 5, 7_L$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d \quad [b, e] = g$
$[a, d] = f$

$2, 4, 5, 7_M$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d \quad [b, d] = f$
$[a, d] = g$
$[a, e] = f$

These algebras can be distinguished by choosing a basis $\{a, b\}$ for L/Z^3 so that $[b, Z^3] \subset Z$ and so that a has some other nice property.

In $ABCDEFG$, $[b, Z^3] \subset Z$ but $Z \not\subset [b, L]$. In A, $[b, Z^3] = 0$; in BC, $[b, [a, b]] = 0$; in B, $\dim([a, Z^3]) = 1$; in D, $L''' \subset [b, Z^2]$; in AFG, $[b, Z^2] = 0$; and in F, $[a, [a, b]] \in [b, Z^3]$.

In $HIJK$, $[b, Z^3] \subset Z$ and $Z \subset [b, L]$; in H, $\dim([L, Z^2]) = 1$; in I, $L''' \subset [b, L']$; and in J, $[a, Z^2] \neq Z$.

In LM, $[b, Z^3] \not\subset Z$ and $Z \not\subset [b, L]$, and in L, but not M, $[b, [a, [a, b]]] = 0$.

Central series dimensions 2, 3, 5, 7

Decomposables

$1 \oplus 1, 2, 4, 6$
$[a, b] = c \quad [b, d] = e \quad [c, d] = f$
$[a, c] = e$
$[a, e] = f$

Indecomposables

$2, 3, 5, 7_A$
$[a, b] = c \quad [b, d] = e + g \quad [c, d] = f$
$[a, c] = e$
$[a, e] = f$

$2, 3, 5, 7_B$
$[a, b] = c \quad [b, d] = e \quad [c, d] = f$
$[a, c] = e$
$[a, d] = g$
$[a, e] = f$

$2, 3, 5, 7_C$
$[a, b] = c \quad [b, c] = g \quad [c, d] = f$
$[a, c] = e \quad [b, d] = e$
$[a, e] = f$

$2, 3, 5, 7_D$
$[a, b] = c \quad [b, c] = g \quad [c, d] = f$
$[a, c] = e \quad [b, d] = e + g$
$[a, e] = f$

In AB, there are basis vectors a, b for L/Z^3 (by abuse of notation) such that $\dim([b, Z^3]) = 1$. In A, one can choose a so that $\dim([a, Z^3]) = 2$. It can be shown that $C \ncong D$. One simply finds the most general choice of basis vectors in D which preserve all bracket relations common to both C and D. It can then be shown that $[b, d] = e$ is inconsistent. The actual proof (about 15 lines) is omitted.

Central series dimensions 2, 3, 4, 5, 7

Decomposables

$1 \oplus 1, 2, 3, 4, 6_A$
$[a, b] = c$
$[a, c] = d$
$[a, d] = e$
$[a, e] = f$

$1 \oplus 1, 2, 3, 4, 6_B$
$[a, b] = c \quad [b, c] = f$
$[a, c] = d$
$[a, d] = e$
$[a, e] = f$

$1 \oplus 1, 2, 3, 4, 6_C$
$[a, b] = c \quad [b, e] = f \quad [c, d] = -f$
$[a, c] = d$
$[a, d] = e$

$1 \oplus 1, 2, 3, 4, 6_D$
$[a, b] = c \quad [b, c] = e$
$[a, c] = d \quad [b, d] = f$
$[a, d] = e$
$[a, e] = f$

$1 \oplus 1, 2, 3, 4, 6_E$
$[a, b] = c \quad [b, c] = e \quad [c, d] = -f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = e$

Indecomposables

$2, 3, 4, 5, 7_A$
$[a, b] = c \quad [b, c] = g$
$[a, c] = d$
$[a, d] = e$
$[a, e] = f$

$2, 3, 4, 5, 7_B$
$[a, b] = c \quad [b, c] = g \quad [c, d] = -f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = e$

$2, 3, 4, 5, 7_C$
$[a, b] = c \quad [b, e] = g \quad [c, d] = -g$
$[a, c] = d$
$[a, d] = e$
$[a, e] = f$

$2, 3, 4, 5, 7_D$
$[a, b] = c \quad [b, c] = f \quad [c, d] = -g$
$[a, c] = d \quad [b, e] = g$
$[a, d] = e$
$[a, e] = f$

$2, 3, 4, 5, 7_E$
$[a, b] = c \quad [b, c] = e + g$
$[a, c] = d \quad [b, d] = f$
$[a, d] = e$
$[a, e] = f$

$2, 3, 4, 5, 7_F$
$[a, b] = c \quad [b, c] = e + g \quad [c, d] = -f$
$[a, c] = d \quad [b, e] = f$
$[a, d] = e$

$2, 3, 4, 5, 7_G$
$[a, b] = c \quad [b, c] = e \quad [c, d] = -g$
$[a, c] = d \quad [b, d] = f$
$[a, d] = e \quad [b, e] = g$
$[a, e] = f$

For $ABCD$, $L/Z \cong 1, 2, 3, 5_A$. In A, $[Z^4, Z^4]$; in B, $\dim([L, Z^2]) = 1$; and in C, there is $b \notin Z^4$ such that $\dim([b, Z^4]) = 1$.

For EFG, $L/Z \cong 1, 2, 3, 5_B$. in EF, $[L, Z^2] \neq Z$; in E, $[Z^4, Z^4] = 0$; and in G, $[L, Z^2] = Z$.

Central series dimensions 1, 7

$1, 7$
$[a, b] = g \quad [c, d] = g \quad [e, f] = g$

Central series dimensions 1, 5, 7

$1, 5, 7$
$[a, b] = c \quad [b, d] = g \quad [e, f] = g$
$[a, c] = g$

Central series dimensions 1, 4, 7

$1, 4, 7_A$
$[a, b] = d \quad [b, e] = g \quad [c, d] = g$
$[a, c] = e$
$[a, f] = g$

$1, 4, 7_B$
$[a, b] = d \quad [b, f] = g \quad [c, e] = g$
$[a, c] = e$
$[a, d] = g$

$1, 4, 7_C$
$[a, b] = d \quad [b, c] = e \quad [c, d] = \frac{1}{2}g$
$[a, c] = -f \quad [b, f] = \frac{1}{2}g$
$[a, e] = -g$

$1, 4, 7_D$
$[a, b] = d \quad [b, c] = e \quad [c, d] = \frac{1}{2}g$
$[a, c] = -f \quad [b, f] = \frac{1}{2}g \quad [c, f] = g$
$[a, e] = -g$

$1, 4, 7_E : K(\xi) = \frac{(\xi^2-\xi+1)^3}{(\xi-2)^2(\xi+1)^2(\xi-\frac{1}{2})^2}$
$\xi \neq 2, -1, \frac{1}{2}$
$[a, b] = d \quad [b, c] = e$
$[a, c] = -f \quad [b, f] = \xi g$
$[a, e] = -g \quad [c, d] = (1 - \xi)g$

In AB, $\dim(L') = 3$; in A, there is a such that $[a, L''] = 0$ and $[a, L] = L'$; and in CDE, $\dim(L') = 4$. C is a special case of E for which the modulus $K(\xi) = \infty$.

That $K(\xi)$ is an invariant of the E algebras is shown as follows. The vectors a, b, c and their scalar multiples are special $\pmod{Z^2}$ in that $ad_a^2 = ad_b^2 = ad_c^2 = 0$ for any value of ξ. Assume that in $1, 4, 7_\xi$ a basis is chosen so that $[a, b] = d$, $[a, c] = -f$, $[b, c] = e$, and $ad_a^2 = ad_b^2 = ad_c^2 = 0$. This choice of a, b, c can differ from the original choice of a, b, c only by permutation or scalar multiplication. We must have, for the new choice of basis, $[a, e] = \rho g$, $[b, f] = \sigma g$, and $[c, d] = \tau g$ for some ordered triple of nonzero complex numbers (ρ, σ, τ). Permuting a and b while replacing g by $-g$ yields (σ, ρ, τ) instead of (ρ, σ, τ), etc. Multiplying a and g by a scalar λ leaves (ρ, σ, τ) unchanged. By scaling g alone we can make $\rho = -1$. The Jacobi identity of a, b, c implies then that $\tau = 1 - \sigma$. Now $(\rho, \sigma, \tau) = (-1, \sigma, 1 - \sigma)$. Permuting a, b, c (and renormalizing g) yields the following set of values for the second variable: $\sigma, \frac{1}{\sigma}, \frac{\sigma}{\sigma-1}, \frac{\sigma-1}{\sigma}, 1-\sigma, \frac{1}{1-\sigma}$. This shows that $1, 4, 7_E : K(\xi) \cong 1, 4, 7_E : K(\sigma)$ iff $\sigma \in \{\xi, \frac{1}{\xi}, \frac{\xi}{\xi-1}, \frac{\xi-1}{\xi}, \frac{1}{1-\xi}, 1-\xi\}$. The equality

$$K(\xi) = \frac{(\xi^2 - \xi + 1)^3}{(\xi - 2)^2(\xi + 1)^2(\xi - \frac{1}{2})^2}$$

is a generically 6-to-1 function which depends only on this set, i.e., $K(\xi) = K(1 - \xi) = K(\xi^{-1})$, etc.[3]

Central series dimensions 1, 4, 5, 7

$1, 4, 5, 7_A$
$[a, b] = c$ $\quad [e, f] = g$
$[a, c] = d$
$[a, d] = g$

$1, 4, 5, 7_B$
$[a, b] = c$ $\quad [b, c] = g$ $\quad [e, f] = g$
$[a, c] = d$
$[a, d] = g$

A has 5-dimensional abelian ideal $\langle b, c, d, f, g \rangle$.

Central series dimensions 1, 3, 7

$1, 3, 7_A$
$[a, b] = e$ $\quad [c, d] = f$
$[a, e] = g$ $\quad [c, f] = g$

$1, 3, 7_B$
$[a, b] = e$ $\quad [b, c] = g$ $\quad [c, d] = f$
$[a, e] = g$ $\quad [c, f] = g$

$1, 3, 7_C$
$[a, b] = e$ $\quad [c, d] = e$ $\quad [d, f] = -g$
$[a, c] = f$
$[a, e] = g$

$1, 3, 7_D$
$[a, b] = e$ $\quad [b, c] = g$ $\quad [c, d] = e$
$[a, c] = f$
$[a, e] = g$ $\quad [d, f] = -g$

For A, $L/Z \cong 1, 3 \oplus 1, 3$, and for BC, $L/Z \cong 2, 6$. In B, but not C, there is a vector $b \notin Z^2$ such that $\dim([b, L]) = 1$.

Central series dimensions 1, 3, 5, 7

$1, 3, 5, 7_A$
$[a, b] = c$ $\quad [b, d] = e$ $\quad [c, d] = g$
$[a, c] = e$ $\quad [b, f] = g$
$[a, e] = g$

$1, 3, 5, 7_B$
$[a, b] = c$ $\quad [b, d] = e$ $\quad [c, d] = g$
$[a, c] = e$
$[a, e] = g$ $\quad [d, f] = g$

$1, 3, 5, 7_C$
$[a, b] = c$ $\quad [b, c] = g$ $\quad [c, d] = g$
$[a, c] = e$ $\quad [b, d] = e$
$[a, e] = g$ $\quad [d, f] = g$

$1, 3, 5, 7_D$
$[a, b] = c$ $\quad [b, f] = g$ $\quad [c, d] = -g$
$[a, c] = e$
$[a, d] = f$
$[a, e] = g$

[3]This invariant $K(\xi)$ was suggested by work of Saito on the singularities $\widetilde{E_7}$: $f(x, y) = x^4 + y^4 + \sigma x^2 y^2$ [20, 26].

$1, 3, 5, 7_E$

$[a, b] = c$ $\quad [d, f] = g$
$[a, c] = e$
$[a, d] = f$
$[a, e] = g$

$1, 3, 5, 7_F$

$[a, b] = c$ $\quad [b, c] = g$ $\quad [d, f] = g$
$[a, c] = e$
$[a, d] = f$
$[a, e] = g$

$1, 3, 5, 7_G$

$[a, b] = c$ $\quad [b, d] = f$
$[a, c] = e$ $\quad [b, f] = g$
$[a, e] = g$

$1, 3, 5, 7_H$

$[a, b] = c$ $\quad [b, d] = f$ $\quad [c, d] = g$
$[a, c] = e$ $\quad [b, f] = g$
$[a, e] = g$
$[a, f] = g$

$1, 3, 5, 7_I$

$[a, b] = c$ $\quad [b, d] = f$ $\quad [d, f] = g$
$[a, c] = e$
$[a, e] = g$

$1, 3, 5, 7_J$

$[a, b] = c$ $\quad [b, c] = g$ $\quad [d, f] = g$
$[a, c] = e$ $\quad [b, d] = f$
$[a, e] = g$

$1, 3, 5, 7_K$

$[a, b] = c$ $\quad [b, d] = e$ $\quad [c, d] = \frac{1}{2}g$
$[a, c] = e$ $\quad [b, f] = \frac{1}{2}g$
$[a, d] = f$
$[a, e] = g$

$1, 3, 5, 7_L$

$[a, b] = c$ $\quad [b, c] = g$ $\quad [c, d] = \frac{1}{2}g$
$[a, c] = e$ $\quad [b, d] = e$
$[a, d] = f$ $\quad [b, f] = \frac{1}{2}g$
$[a, e] = g$

$1, 3, 5, 7_M : \xi$

$\xi \neq 0, \frac{1}{2}$ (cf. 2357_B)
$[a, b] = c$ $\quad [b, d] = e$ $\quad [c, d] = (1 - \xi)g$
$[a, c] = e$ $\quad [b, f] = \xi g$
$[a, d] = f$
$[a, e] = g$

$1, 3, 5, 7_N : \xi$

$[a, b] = c$ $\quad [b, c] = \xi g$ $\quad [c, d] = g$
$[a, c] = e$ $\quad [b, d] = e$
$[a, d] = f$ $\quad [d, f] = g$
$[a, e] = g$

$1, 3, 5, 7_O$

$[a, b] = c$ $\quad [b, c] = f$
$[a, c] = e$ $\quad [b, e] = g$
$[a, d] = f$
$[a, f] = g$

$1, 3, 5, 7_P$

$[a, b] = c$ $\quad [b, c] = f$ $\quad [c, d] = -g$
$[a, c] = e$ $\quad [b, f] = g$
$[a, d] = f$
$[a, e] = g$

$1, 3, 5, 7_Q$

$[a, b] = c + d$ $\quad [b, c] = f$
$[a, d] = e$ $\quad [b, f] = g$
$[a, e] = g$

$1, 3, 5, 7_R$

$[a, b] = c + d$ $\quad [b, c] = f$ $\quad [c, d] = -g$
$[a, d] = e$ $\quad [b, e] = g$
$[a, f] = g$

$1, 3, 5, 7_S : \xi$

$[a, b] = c + d$ $\quad [b, c] = f$ $\quad [c, d] = -g$
$[a, d] = e$ $\quad [b, e] = g$
$[a, e] = g$ $\quad [b, f] = \xi g$
$[a, f] = g$

For ABC, $L/Z \cong 1 \oplus 1, 3, 5$. In A, $[Z^3, Z^2] = 0$; and in BC, $[Z^3, Z^2] = Z$. B has a vector $b \notin Z^2$ such that $Z \not\subset [b, L]$.

For DEF, $L/Z \cong 2, 4, 6_A$. In D, $[Z^3, Z^2] = 0$, and in EF, $[Z^3, Z^2] = Z$. E has a vector $b \notin Z^3$ such that $Z \not\subset [b, Z^3]$.

For $GHIJ$, $L/Z \cong 2, 4, 6_B$. In G, $[Z^3, Z^3] = 0$; in HIJ, $[Z^3, Z^3] = Z$; and in H, but not IJ, $[Z^3, Z^2] = 0$. I has a vector $b \notin Z^3$ such that $[b, L'] = 0$.

For $KLMN$, $L/Z \cong 2, 4, 6_C$. In KLM, $[Z^3, Z^2] = 0$, and in KM, there is a vector $b \notin Z^3$ such that $[b, [b, L]] = 0$ and $[b, Z^3] \neq Z$. K is the case $\xi = \frac{1}{2}$ of M. Again, one can show that the algebras $M : \xi$ and the algebras $N : \xi$ are pairwise nonisomorphic. The proofs are straightforward (about eight lines apiece) and are omitted.

For OP, $L/Z \cong 2, 4, 6_D$; in O, $[Z^3, Z^3] = 0$. For QRS, $L/Z \cong 2, 4, 6_E$; in Q, $[Z^3, Z^3] = 0$. R has vectors a, b linearly independent $(\operatorname{mod} Z^3)$ such that $ad_a^3 = ad_b^3 = 0$ and $[a, Z^3] \neq Z \neq [b, Z^3]$. The $S : \xi$ algebras are pairwise nonisomorphic. The only choices for a and b which preserve the bracket relations common to all the S algebras are permutations and scalar multiples of the original a and b $(\operatorname{mod} Z^3)$. The rest follows easily.

Central series dimensions 1, 3, 4, 5, 7

$1, 3, 4, 5, 7_A$

$[a, b] = c \quad [b, f] = g$
$[a, c] = d$
$[a, d] = e$
$[a, e] = g$

$1, 3, 4, 5, 7_B$

$[a, b] = c \quad [b, c] = g$
$[a, c] = d \quad [b, f] = g$
$[a, d] = e$
$[a, e] = g$

$1, 3, 4, 5, 7_C$

$[a, b] = c \quad [b, e] = g \quad [c, d] = -g$
$[a, c] = d$
$[a, d] = e$
$[a, f] = g$

$1, 3, 4, 5, 7_D$

$[a, b] = c \quad [b, c] = e$
$[a, c] = d \quad [b, d] = g$
$[a, d] = e \quad [b, f] = g$
$[a, e] = g$

$1, 3, 4, 5, 7_E$

$[a, b] = c \quad [b, c] = e \quad [c, d] = -g$
$[a, c] = d \quad [b, e] = g$
$[a, d] = e$
$[a, f] = g$

$1, 3, 4, 5, 7_F$

$[a, b] = c \quad [b, c] = f$
$[a, c] = d \quad [b, f] = g$
$[a, d] = e$
$[a, e] = g$

$1, 3, 4, 5, 7_G$

$[a, b] = c \quad [b, c] = f \quad [c, d] = -g$
$[a, c] = d \quad [b, d] = g$
$[a, d] = e \quad [b, e] = g$
$[a, f] = g$

$1, 3, 4, 5, 7_H$

$[a, b] = c \quad [b, c] = f \quad [c, d] = -g$
$[a, c] = d \quad [b, d] = g$
$[a, d] = e \quad [b, e] = g$
$[b, f] = g$

$1, 3, 4, 5, 7_I$

$[a, b] = c \quad [b, c] = f \quad [c, d] = -g$
$[a, c] = d \quad [b, d] = g$
$[a, d] = e \quad [b, e] = g$
$[a, f] = g \quad [b, f] = g$

For ABC, $L/Z \cong 1 \oplus 1, 2, 3, 5_A$; in AB, $[Z^4, Z^3] = 0$. A has a vector $b \notin Z^4$ such that $[b, L'] = 0$. For DE, $L/Z \cong 1 \oplus 1, 2, 3, 5_B$; in D, $[Z^4, Z^3] = 0$. For $FGHI$, $L/Z \cong 2, 3, 4, 6$. In F, $[Z^4, Z^3] = 0$. G has a vector $b \notin Z^4$ such that $[b, Z^4] \subset Z^2$ and $ad_b^3 = 0$, and H has a vector $a \notin Z^4$ such that $[a, Z^2] = 0$.

Central series dimensions 1, 2, 4, 5, 7

$1, 2, 4, 5, 7_A$

$[a, b] = c \quad [b, e] = f \quad [c, e] = g$
$[a, c] = d$
$[a, d] = f$
$[a, f] = g$

$1, 2, 4, 5, 7_B$

$[a, b] = c \quad [b, e] = f + g \quad [c, e] = g$
$[a, c] = d$
$[a, d] = f$
$[a, f] = g$

$1,2,4,5,7_C$

$[a,b]=c \quad [b,e]=f \quad [c,d]=-g$
$[a,c]=d \quad [b,f]=g$
$[a,d]=f$

$1,2,4,5,7_D$

$[a,b]=c \quad [b,e]=f \quad [c,d]=-g$
$[a,c]=d \quad [b,f]=g$
$[a,d]=f$
$[a,e]=g$

$1,2,4,5,7_E$

$[a,b]=c \quad [b,c]=f \quad [c,e]=g$
$[a,c]=d \quad [b,d]=g$
$[a,d]=f \quad [b,e]=f$
$[a,f]=g$

$1,2,4,5,7_F$

$[a,b]=c \quad [b,c]=f \quad [c,d]=-g$
$[a,c]=d \quad [b,e]=f$
$[a,d]=f \quad [b,f]=g$

$1,2,4,5,7_G$

$[a,b]=c \quad [b,c]=f \quad [c,d]=-g$
$[a,c]=d \quad [b,e]=f$
$[a,d]=f \quad [b,f]=g$
$[a,e]=g$

$1,2,4,5,7_H$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f$
$[a,e]=f$
$[a,f]=g$

$1,2,4,5,7_I$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f$
$[a,e]=f \quad [b,e]=g$
$[a,f]=g$

$1,2,4,5,7_J$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f$
$[a,d]=g \quad [b,e]=g$
$[a,e]=f$
$[a,f]=g$

$1,2,4,5,7_K$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f$
$[a,d]=g$
$[a,e]=f$
$[a,f]=g$

$1,2,4,5,7_L$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f \quad [c,e]=-g$
$[a,e]=f \quad [b,f]=g$
$[a,f]=g$

$1,2,4,5,7_M$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f \quad [c,e]=-g$
$[a,d]=g \quad [b,f]=g$
$[a,e]=f$
$[a,f]=g$

$1,2,4,5,7_N : K(\xi)=\xi+\xi^{-1}$
$\xi \neq 0$

$[a,b]=c \quad [b,c]=e \quad [c,d]=g$
$[a,c]=d \quad [b,d]=f \quad [c,e]=-g$
$[a,d]=g \quad [b,e]=\xi g$
$[a,e]=f \quad [b,f]=g$
$[a,f]=g$

Note. Aut$(1,2,4,5,7_N : K(\xi))$ consists of unipotent automorphisms. (There are no semisimple automorphisms.)

For $ABCD$, $L/Z \cong 1,3,4,6_A$. In AB, there is a vector $b \notin Z^4$ such that $[b, Z^2]=0$ and $[b,Z^4] \subset Z^2$; in A, there are vectors $a \notin Z^4$ and $e \in Z^3 \backslash L'$ such that $ad_a^3(b)=[b,e]$; and in C, there is $e \in Z^3\backslash L'$ such that $Z \not\subset [e,L]$. For EFG, $L/Z \cong 1,3,4,6_B$. In E, there is a vector $b \notin Z^4$ such that $[b,Z^2]=0$ and $[b,Z^4]\subset Z^2$; and in F, there is a vector $e \in Z^3\backslash L'$ such that $Z \not\subset [e,L]$. For $HIJKLMN$, $L/Z \cong 1,3,4,6_C$ and the vectors a and b are unique up to scalar multiplication (mod Z^4) in that Image$(ad_a^3) \subset Z$ and Image$(ad_b^3)\subset Z$. For HL, $ad_a^3 = ad_b^3 = 0$. In H, $[b,Z^2]=0$; in I, $ad_a^3=0$ and $[b,Z^2]=0$; and in K, $ad_b^3=0$ and $[b,Z^2]=0$. For M, $ad_a^3=0$ and $[a,Z^2]=[b,Z^2]=Z$. For JN, $ad_a^3 \neq 0 \neq ad_b^3$. In J, $[b,Z^2]=0$. $N:\xi \cong N:\xi^{-1}$ as can be seen by multiplying the basis vectors a,b,c,d,e,f,g by $\xi, \xi, -\xi^2, -\xi^3, -\xi^3, -\xi^4, -\xi^5$ respectively. The proof that only these values of ξ yield isomorphic algebras is omitted (about eight lines).

Central series dimensions 1, 2, 3, 5, 7

$1,2,3,5,7_A$

$$\begin{array}{lll} [a,b]=c & [b,d]=e & [c,d]=f \\ [a,c]=e & & \\ [a,e]=f & & [d,e]=-g \\ [a,f]=g & & \end{array}$$

$1,2,3,5,7_B$

$$\begin{array}{lll} [a,b]=c & [b,d]=e+g & [c,d]=f \\ [a,c]=e & & \\ [a,e]=f & & [d,e]=-g \\ [a,f]=g & & \end{array}$$

$1,2,3,5,7_C$

$$\begin{array}{lll} [a,b]=c & [b,c]=g & [c,d]=f \\ [a,c]=e & [b,d]=e & \\ [a,e]=f & & [d,e]=-g \\ [a,f]=g & & \end{array}$$

In AB, there is a vector $b \notin Z^4$ such that $[b, L'] = 0$. In A, there are vectors $a \notin Z^4$ and $d \in Z^4 \backslash L'$ such that $[a, d] = 0$ and $[a, [a, b]] = [b, d]$ with b as above.

Central series dimensions 1, 2, 3, 4, 5, 7

$1,2,3,4,5,7_A$

$$\begin{array}{l} [a,b]=c \\ [a,c]=d \\ [a,d]=e \\ [a,e]=f \\ [a,f]=g \end{array}$$

$1,2,3,4,5,7_B$

$$\begin{array}{ll} [a,b]=c & [b,c]=g \\ [a,c]=d & \\ [a,d]=e & \\ [a,e]=f & \\ [a,f]=g & \end{array}$$

$1,2,3,4,5,7_C$

$$\begin{array}{lll} [a,b]=c & [b,e]=g & [c,d]=-g \\ [a,c]=d & & \\ [a,d]=e & & \\ [a,e]=f & & \\ [a,f]=g & & \end{array}$$

$1,2,3,4,5,7_D$

$$\begin{array}{ll} [a,b]=c & [b,c]=f \\ [a,c]=d & [b,d]=g \\ [a,d]=e & \\ [a,e]=f & \\ [a,f]=g & \end{array}$$

$1,2,3,4,5,7_E$

$$\begin{array}{ll} [a,b]=c & [b,c]=f+g \\ [a,c]=d & [b,d]=g \\ [a,d]=e & \\ [a,e]=f & \\ [a,f]=g & \end{array}$$

$1,2,3,4,5,7_F$

$$\begin{array}{lll} [a,b]=c & [b,c]=f & [c,d]=-g \\ [a,c]=d & [b,d]=g & \\ [a,d]=e & [b,e]=g & \\ [a,e]=f & & \\ [a,f]=g & & \end{array}$$

$1,2,3,4,5,7_G$

$$\begin{array}{ll} [a,b]=c & [b,c]=e \\ [a,c]=d & [b,d]=f \\ [a,d]=e & [b,e]=g \\ [a,e]=f & \\ [a,f]=g & \end{array}$$

$1,2,3,4,5,7_H$

$$\begin{array}{ll} [a,b]=c & [b,c]=e+g \\ [a,c]=d & [b,d]=f \\ [a,d]=e & [b,e]=g \\ [a,e]=f & \\ [a,f]=g & \end{array}$$

$1,2,3,4,5,7_I : \xi,\ \ \xi \neq 1$

$$\begin{array}{lll} [a,b]=c & [b,c]=e & [c,d]=(1-\xi)g \\ [a,c]=d & [b,d]=f & \\ [a,d]=e & [b,e]=\xi g & \\ [a,e]=f & & \\ [a,f]=g & & \end{array}$$

These algebras were classified by Umlauf [28]. The following notes are included for the sake of completeness.

For ABC, $L/Z \cong 1,2,3,4,6_A$; in A, $\langle b, c, d, e, f, g\rangle$ is a 6-dimensional abelian ideal; and in B, $[Z^5, Z^4] = 0$.

For DEF, $L/Z \cong 1, 2, 3, 4, 6_B$; in DE, $[Z^5, Z^4] = 0$; and in D, there are linearly independent vectors $a, b \pmod{Z^5}$ such that $[b, Z^5] \subset Z^2$ and $ad_a^4(b) = -ad_b^2(a)$.

For GHI, $L/Z \cong 1, 2, 3, 4, 6_D$; in GH, $[Z^5, Z^4] = 0$; and in G, there are linearly independent vectors a, b (mod Z^5) such that $ad_a^3(b) = -ad_b^2(a)$. The proof that the $I : \xi$ algebras are pairwise nonisomorphic is omitted.

Acknowledgments

As mentioned in the Introduction, this is one of three recent attempts to solve the classification problem in dimension 7. The others are due to Romdhani [18] and Ancochea-Bermudez and Goze [1]. Roger Carles has scrutinized these three works prior to publication, and has identified several mistakes in all of them. Thus it should be noted that the results of the other two papers have led to the discovery of some of the omissions which once appeared in this paper due to carelessness. Robert Lee Wilson and Duncan Melville also corrected a few of the mistakes in earlier drafts of this paper. Finally, I wish to thank Boris Kunin and especially my thesis advisor Stephen Yau for their encouragement while I was lost in the details of this work.

This paper is a revised and abbreviated version of my thesis which was submitted as partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate College of the University of Illinois at Chicago, 1988.

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