

- [4] J. J. Charatonik, *Ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 227-252.
- [5] — *On decompositions of 2-dendroids*, Fund. Math. 67 (1970), pp. 15-30.
- [6] M. L. Curtis, *The covering homotopy theorem*, Proc. Amer. Math. Soc. 7 (1956), pp. 682-684.
- [7] W. T. Ingram, *Decomposable circle-like continua*, Fund. Math. 63 (1968), pp. 193-198.
- [8] J. B. Fugate, *A sufficient condition that a compact metric continuum be chainable*, to appear.
- [9] — *Retracting dendroids onto trees*, Notices Amer. Math. Soc. 15 (1968), pp. 773.
- [10] K. Kuratowski, *Topologie II*, Warszawa 1961.
- [11] S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math. 25 (1935), pp. 327-328.
- [12] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ., vol. 13, New York, 1962.

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## A 2-complex is collapsible if and only if it admits a strongly convex metric

by

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**§ 1. Introduction.** A metric  $d$  on a compact space  $X$  is *strongly convex* if, for any two points  $x, y \in X$ , there is a unique point  $m \in X$  such that  $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$ . In the last few years, there has been considerable interest in characterizing the spaces which admit convex metrics. Lelek and Nitka [3] and Rolfsen [4] have shown that cells are the only compact 2 and 3-dimensional spaces which admit strongly convex metrics with the property that no midpoint of  $x$  and  $y$  is a midpoint of  $x$  and  $y'$  unless  $y = y'$ . Rolfsen [4] has further shown that the only compact  $n$ -manifold,  $n \leq 3$ , admitting a strongly convex metric is the cell.

It is well known (see [2]) that any compact space which admits a strongly convex metric is contractible, but Sieklucki [5] has demonstrated a contractible 2-complex which admits no strongly convex metric. Joseph Martin conjectured in 1966 that the stronger condition of collapsibility does characterize the 2-complexes which admit strongly convex metrics, and a proof of this is the object of this note. It is interesting to note that this theorem also provides, conversely, a topological characterization of collapsibility in 2-complexes, and thus cannot be directly extended to higher dimensions, for a 3-cell can have a non-collapsible triangulation [1].

### § 2. A collapsible 2-complex admits a strongly convex metric.

**DEFINITIONS.** All simplices are closed simplices. If  $a_1, a_2, \dots, a_k$  are points in a simplex  $\sigma$ , then  $a_1 a_2 \dots a_k$  is their convex hull in the linear structure of  $\sigma$ . A *triangle* is a 2-simplex in  $E^2$  with the regular euclidean metric  $\|x - y\|$ .

All maps are continuous; if  $X$  and  $Y$  are spaces, the notation  $f: X \rightarrow Y$  denotes a map from  $X$  onto  $Y$ . If  $K$  is a complex, then  $K^{(k)}$  denotes the  $k$ -skeleton of  $K$ .

Let  $X$  be a compact space with a strongly convex metric  $d$ . Any two points  $x, y$  of  $X$  are joined in  $X$  by a unique arc, the *segment*  $\widehat{xy}$ , which is isometric to a closed interval of the real line ([2]). A *concave collection* for  $d$  is a finite collection  $T$  of segments in  $X$  satisfying:

(2.1) If  $\varrho, \tau \in T$  and  $x_1, x_2 \in \varrho, y_1, y_2 \in \tau$ , then  $d(x_m, y_m) \leq \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)]$ , where  $x_m, y_m$  are the midpoints of  $\overline{x_1 x_2}$  and  $\overline{y_1 y_2}$ .

(2.2) If  $\tau \in T$  and  $x_1, x_2 \in \tau$  then, for any point  $y \in X$ ,  $d(y, x_m) \leq \frac{1}{2}[d(y, x_1) + d(y, x_2)]$ , where  $x_m$  is the midpoint of  $\overline{x_1 x_2}$ .

LEMMA 2. Suppose that  $X \cup \sigma$  is a metric space and  $X \cap \sigma = \tau$  is an arc. Let  $d$  be a strongly convex metric for  $X$  and let  $T$  be a concave collection for  $d$ ,

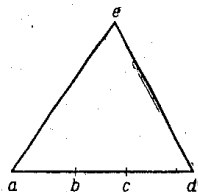


Fig. 1

an element of which contains  $\tau$ . Suppose  $abcde$  (Figure 1) is a triangle with vertices  $a, d$ , and  $e$ , and let  $\varphi: abcde \rightarrow \sigma$  be a homeomorphism such that  $\varphi(bc) = \tau$  and  $d(\varphi(x), \varphi(y)) = \|x - y\|$  for every  $x, y \in bc$ .

Then there is a strongly convex metric  $d'$  for  $X \cup \sigma$  such that:

$$(2.3) \quad d'(x, y) = d(x, y) \quad \text{for all } x, y \in X,$$

$$(2.4) \quad d'(x, y) = \|\varphi^{-1}(x) - \varphi^{-1}(y)\| \quad \text{for all } x, y \in \sigma,$$

(2.5)  $T \cup \{\varphi(ab), \varphi(cd), \varphi(de), \varphi(ea)\}$  is a concave collection for  $d'$ .

Proof. Define  $d'$  by:

$$d'(x, y) = \begin{cases} d(x, y), & x, y \in X, \\ \|\varphi^{-1}(x) - \varphi^{-1}(y)\|, & x, y \in \sigma, \\ \min_{p \in \tau} \{d'(x, p) + d'(p, y)\}, & x \in \sigma, y \in X \text{ or } x \in X, y \in \sigma. \end{cases}$$

Checking that  $d'$  satisfies 2.3-2.5 is then a straightforward process. ■

THEOREM 2. Let  $L$  be a 2-complex with a strongly convex metric  $d$  on  $|L|$ , and let  $T$  be a concave collection for  $d$  which covers  $|L^{(1)}|$ . Suppose that  $\sigma$  and  $\tau$  are a simplex and face, respectively, such that  $L' = L \cup \{\sigma, \tau\}$  is a 2-complex and  $L' \rightarrow L$  is an elementary collapse.

Then there is a strongly convex metric  $d'$  for  $|L'|$  and a concave collection  $T'$  for  $d'$  satisfying:

(2.6)  $T'$  covers  $|L^{(1)}|$ ,

$$(2.7) \quad d'(x, y) = d(x, y) \quad \text{for } x, y \in |L|.$$

Proof. The case when  $\sigma$  is a 1-simplex is left to the reader to check. Suppose  $\sigma$  is a 2-simplex; then  $\sigma$  meets  $L$  in two of its faces,  $\tau_1$  and  $\tau_2$  (Figure 2).  $T$  covers the one-skeleton of  $L$ , so we can find a subtriangulation  $u_0 u_1, u_1 u_2, \dots, u_{k-1} u_k$  of  $\{\tau_1, \tau_2\}$  such that each element  $u_{i-1} u_i$

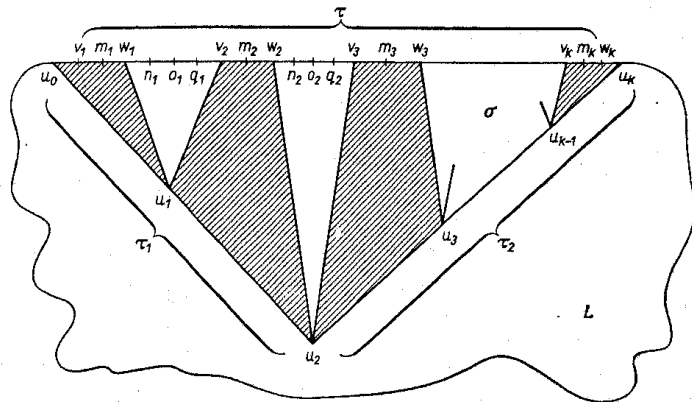


Fig. 2

belongs to  $T$ . Choose points  $v_i, w_i, m_i, i = 1, \dots, k$ , and  $n_i, o_i, q_i, i = 1, \dots, k-1$ , so that the order of points along  $\tau$  is  $u_0, v_1, m_1, w_1, n_1, o_1, q_1, v_2, m_2, w_2, n_2, o_2, q_2, \dots, v_{k-1}, m_{k-1}, w_{k-1}, n_{k-1}, o_{k-1}, q_{k-1}, v_k, m_k, w_k, u_k$ , as illustrated in Figure 2.

PROPOSITION 2A. There is a strongly convex metric  $d_1$  for  $L_1 = |L| \cup \cup u_0 u_1 u_1 v_1 \cup \dots \cup u_{k-1} u_k w_k v_k$  (shaded in Figure 2) such that:

(1)  $T_1 = T \cup \{v_1 u_0, u_1 w_1, w_1 m_1, m_1 v_1, \dots, v_k u_{k-1}, u_k w_k, w_k m_k, m_k v_k\}$  is a concave collection for  $d_1$ .

(2)  $v_j u_{j-1} \cup u_{j-1} u_j \cup u_j w_j = \widehat{v_j w_j}$ , for each  $j = 1, \dots, k$ .

Proof. The quadrilaterals  $u_{j-1} u_j w_j v_j, j = 1, \dots, k$ , meet each other only in  $|L|$ , so we can apply Lemma 2 repeatedly to obtain  $d_1$ . ■

We now extend  $d_1$  to the rest of  $|L \cup \sigma|$  by induction. Let

$$L_j = L_1 \cup w_1 u_1 v_2 \cup \dots \cup w_{j-1} u_{j-1} v_j$$

and

$$T_j = T_1 \cup \{w_1 n_1, n_1 o_1, o_1 q_1, q_1 v_2, \dots, w_{j-1} n_{j-1}, n_{j-1} o_{j-1}, o_{j-1} q_{j-1}, q_{j-1} v_j\}$$

for  $j = 2, \dots, k$ , and suppose that, for some  $i \in \{1, \dots, k\}$ , we have a strongly convex metric  $d_i$  for  $L_i$  such that:

(i)  $\bar{d}_i(x, y) = \bar{d}_1(x, y)$  for all  $x, y$  in  $L_i$ .

(ii)  $T_i$  is a concave collection for  $\bar{d}_i$ .

If  $i < k$ , consider  $w_i u_i \cup u_i v_{i+1}$ .

PROPOSITION 2B.  $w_i u_i \cup u_i v_{i+1} = \widehat{w_i v_{i+1}}$ .

Proof. This is certainly true if  $u_i \in \widehat{w_i v_{i+1}}$ , since  $\widehat{w_i u_i} = w_i u_i$  and  $\widehat{u_i v_{i+1}} = u_i v_{i+1}$ . But  $u_i$  must lie on  $\widehat{w_i v_{i+1}}$ , since  $\widehat{w_i v_{i+1}}$  has to hit  $u_i u_{i+1}$  and  $v_{i+1} u_i \cup u_i u_{i+1}$  is a segment by Proposition 2A(2) and (i). ■

PROPOSITION 2C. If  $p \in \bar{L}_i \setminus u_{i-1} u_i w_i v_i$  (or  $p \in \bar{L}_i \setminus u_i u_{i+1} w_{i+1} v_{i+1}$ ) and  $x \in w_i u_i$  ( $x \in u_i v_{i+1}$ ), then  $\bar{d}_i(p, x) = \bar{d}_i(p, u_i) + \bar{d}_i(u_i, x)$ .

Proof.  $\widehat{p x}$  must hit  $v_i u_{i-1} \cup u_{i-1} u_i$  ( $u_i u_{i+1} \cup u_{i+1} w_{i+1}$ ), but  $v_i u_{i-1} \cup u_{i-1} u_i \cup u_i w_i$  ( $v_{i+1} u_i \cup u_i u_{i+1} \cup u_{i+1} w_{i+1}$ ) is a segment, so  $u_i \in \widehat{p x}$ . ■

PROPOSITION 2D.  $T_i \cup \{\widehat{w_i v_{i+1}}\}$  is a concave collection for  $\bar{d}_i$ .

Proof. We will show that  $T_i \cup \{\widehat{w_i v_{i+1}}\}$  and  $\bar{d}_i$  satisfy (2.1). Suppose that  $\tau \in T_i \setminus \{u_i w_i, w_i m_i, m_i v_i\}$ , and let  $x_1, x_2 \in \widehat{w_i v_{i+1}}$ ,  $y_1, y_2 \in \tau$  (Figure 3).

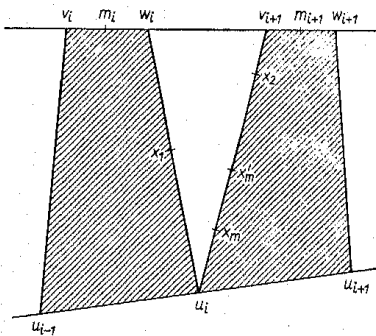


Fig. 3

If  $x_1, x_2 \in u_i w_i$  or  $x_1, x_2 \in u_i v_{i+1}$  then we are through, since  $u_i w_i$  and  $u_i v_{i+1} \in T_i$ , so assume that  $x_1 \in u_i w_i$  and  $x_2 \in u_i v_{i+1}$ . Let  $x'_m, y'_m$ , and  $x''_m$  be the midpoints of  $x_1 x_2, y_1 y_2$ , and  $u_i x_2$  respectively. Now, we know that  $\bar{d}_i(x'_m, y'_m) \leq \frac{1}{2}[\bar{d}_i(u_i, y_1) + \bar{d}_i(x_2, y_2)]$ . By Proposition 2C,  $\bar{d}_i(x_1, y_1) = \bar{d}_i(x_1, u_i) + \bar{d}_i(u_i, y_1)$ .  $\bar{d}_i(x_m, x''_m) = \frac{1}{2} \bar{d}_i(x_1, u_i)$ , so

$$\begin{aligned} \bar{d}_i(x_m, y_m) &\leq \bar{d}_i(x_m, x'_m) + \bar{d}_i(x'_m, y'_m) = \frac{1}{2} \bar{d}_i(x_1, u_i) + \bar{d}_i(x'_m, y'_m) \\ &\leq \frac{1}{2}[\bar{d}_i(x_1, u_i) + \bar{d}_i(u_i, y_1) + \bar{d}_i(x_2, y_2)] = \frac{1}{2}[\bar{d}_i(x_1, y_1) + \bar{d}_i(x_2, y_2)]. \end{aligned}$$

A symmetric argument works if  $\tau$  is  $u_i w_i, w_i m_i$ , or  $m_i v_i$ , and the proof that  $T_i \cup \{\widehat{w_i v_{i+1}}\}$  and  $\bar{d}_i$  satisfy (2.2) is similar. ■

PROPOSITION 2E. There is a strongly convex metric  $\bar{d}_{i+1}$  for  $L_{i+1}$  which satisfies conditions (i) and (ii) with  $i+1$  substituted for  $i$ .

Proof. Propositions 2B and 2D allow us to apply Lemma 2. ■

By induction, there is a strongly convex metric  $\bar{d}_k$  for  $L_k = |L'|$  which satisfies conditions (i) and (ii) with  $k$  substituted for  $i$ . It is easy to check that  $\bar{d}' = \bar{d}_k$  and  $T' = T_k$  then satisfy the conclusions of the theorem. ■

COROLLARY 2. A collapsible 2-complex admits a strongly convex metric.

### § 3. A 2-complex which admits a strongly convex metric is collapsible.

DEFINITIONS. We will consider all complexes to be embedded linearly in some euclidean space, although not, of course, with the inherited metric. If  $C$  is a complex and  $x$  a point of  $|C|$ , we define

$$\text{St}(x, C) = \{\sigma \in C: x \in \sigma\},$$

$$\text{Lk}(x, C) = \{\tau \in C: \tau \subset |\text{St}(x, C)|, x \notin \tau\}.$$

$|\text{St}(x, C)| = x | \text{Lk}(x, C)|$ , and we can use this cone structure of  $\text{St}(x, C)$  to define the natural projection  $\pi(x, C): |\text{St}(x, C)| \setminus \{x\} \rightarrow |\text{Lk}(x, C)|$ . Similarly, if  $S^\varepsilon(x) = \{y \in |C|: \|y - x\| = \varepsilon\}$  is contained in  $|\text{St}(x, C)|$ , we can define a natural homeomorphism  $\pi^\varepsilon(x, C): |\text{Lk}(x, C)| \rightarrow S^\varepsilon(x)$  such that  $\pi(x, C)\pi^\varepsilon(x, C) = I$ , the identity. A subcomplex  $C'$  of  $C$  is a *spine* of  $C$  if  $C$  collapses to  $C'$ .

LEMMA 3. Let  $K$  be an  $n$ -complex,  $n = 2$  or  $3$ , with a strongly convex metric  $d$ , and let  $L \subset K$  be a subcomplex consisting of  $n$ -simplices and their faces. Then there is an  $n$ -simplex of  $L$  with a face free in  $L$ .

Proof. Fix a point  $p$  in the interior of an  $n$ -simplex of  $L$ . The metric  $d$  induces a contraction  $H: |K| \times [0, 1] \rightarrow |K|$  such that (see [2]):

- (i)  $H(x, 0) = x$  for all  $x \in |K|$ ,
- (ii)  $H(x, 1) = p$  for all  $x \in |K|$ ,
- (iii)  $H(x, t) \in \widehat{p x}$  for all  $x \in |K|, t \in [0, 1]$ .

PROPOSITION 3A. There is a point  $x_0$  of  $|L|$  such that:

- (1)  $x_0$  is not a vertex of  $K$ ,
- (2) for any  $y \in |L|, x_0 \in \widehat{p y} \Rightarrow x_0 = y$ .

Proof.  $S^\varepsilon(p)$  separates the  $n$ -cell containing  $p$  if  $\varepsilon$  is small enough, and  $S \cap H(K^{(0)} \times [0, 1])$  is a finite set. Choose  $x \in S \setminus H(K^{(0)} \times [0, 1])$ , and let  $F = \{y \in |L|: x \in \widehat{p y}\}$ .  $F$  is closed and hence compact, and so contains a point  $x_0$  such that  $d(p, x) \leq d(p, x_0)$  for all  $x \in F$ . It is easy to check that  $x_0$  satisfies Proposition 3A (1) and Proposition 3A (2). ■

PROPOSITION 3B.  $|\text{Lk}(x_0, L)|$  can be shrunk to a point in  $|\text{Lk}(x_0, K)|$ .

Proof. Pick  $t_1 \in [0, 1]$  so that  $H(x_0, t_1) \neq x_0$  and  $H(x_0, t) \in \text{int}|\text{St}(x_0, K)|$  for all  $t \in [0, t_1]$ . Let  $\pi = \pi(x_0, K)$ . We can find a neighborhood  $N_0$  of  $x_0$  in  $|K|$  such that:

- (iv)  $H(N_0, t) \subset \text{int}|\text{St}(x_0, K)|$  for all  $t \in [0, t_1]$ ,
- (v)  $\pi[H(N_0, t_1)]$  can be shrunk to a point in  $|\text{Lk}(x_0, K)|$ .

The second condition is possible because  $|\text{Lk}(x_0, K)|$ , being a polyhedron, is locally contractible. Let  $\pi_1 = \pi^s(x_0, K)$ , where  $s$  is small enough that  $S^s(x_0) \subset N_0$ , and consider the function  $\pi H(\cdot, t)\pi_1$  on  $|\text{Lk}(x_0, L)|$  for  $t \in [0, t_1]$ . Proposition 3A (2) and (iv) show that it is well-defined, since  $\pi_1[|\text{Lk}(x_0, L)|] \subset |L|$ . It is a continuous family of mappings from  $|\text{Lk}(x_0, L)|$  into  $|\text{Lk}(x_0, K)|$ , and  $\pi H(\cdot, 0)\pi_1 = I$ . By (v),  $\pi H(\cdot, t_1)\pi_1$  of  $[|\text{Lk}(x_0, L)|]$  can be shrunk to a point in  $|\text{Lk}(x_0, K)|$ , and we are through. ■

PROPOSITION 3C.  $x_0$  lies on a  $(n-1)$ -simplex which is the face of exactly one  $n$ -simplex of  $L$ .

Proof. We will consider the case when  $n = 3$ , leaving the case  $n = 2$  for the reader to check. Let  $\tau$  be the simplex of  $L$  containing  $x_0$  in its interior. It follows from Proposition 3B that  $|\text{Lk}(x_0, L)|$  contains no 2-sphere, so  $\tau$  is not a 3-simplex and, if  $\tau$  is a 2-simplex, it is the face of exactly one 3-simplex in  $L$ .

If  $\tau$  is a 1-simplex, then  $\text{Lk}(x_0, L)$  is the suspension of  $\text{Lk}(\tau, L)$ , where  $\text{Lk}(\tau, L) = \{\mu: \tau\mu \in L\}$ .  $\text{Lk}(\tau, L)$  contains no simple closed curve, because  $\text{Lk}(x_0, L)$  contains no 2-sphere.  $\text{Lk}(\tau, L)$  does contain a 1-simplex, since  $\tau$  is the face of a 3-simplex in  $L$ .  $\text{Lk}(\tau, L)$  therefore contains a 1-simplex  $\mu$  with a vertex  $v$  free in  $\text{Lk}(\tau, L)$ .  $\mu\tau$  is then a 3-simplex of  $\text{St}(x_0, L)$  with a face  $v\tau$  free in  $L$ . ■

THEOREM 3. If  $K$  is an  $n$ -complex,  $n = 2$  or  $3$ , with a strongly convex metric, then  $K$  has an  $(n-1)$ -dimensional spine.

Proof. Suppose we have already collapsed  $K$  down to a subcomplex  $L$ . If  $L$  contains  $n$ -simplices, let  $L'$  be the collection of  $n$ -simplices of  $L$  and their faces. Applying Lemma 3 to  $L'$ , we get an  $n$ -simplex of  $L$  with a face which must be free in  $L$ , since  $L-L'$  is  $(n-1)$ -dimensional. ■

COROLLARY 3. If  $K$  is a 2-complex with a strongly convex metric, then  $K$  is collapsible.

Proof.  $K$  is contractible and, by Theorem 3, collapses to a 1-complex, which must also be contractible. Any contractible 1-complex is collapsible. ■

#### References

- [1] R. H. Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*, Lectures on Modern Mathematics II edited by T. L. Saaty, New York, 1964.

- [2] K. Borsuk, *On a metrization of polytopes*, Fund. Math. 47 (1959), pp. 325-341.  
 [3] A. Lelek and W. Nitka, *On convex metric spaces I*, Fund. Math. 49 (1961), pp. 183-204.  
 [4] D. Rolfsen, *Strongly convex metrics in cells*, Bull. Amer. Math. Soc. 74 (1968), 171-175.  
 [5] K. Sieklucki, *On a contractible polytope which cannot be metrized in the strong convex manner*, Bull. Acad. Polon. Sci. 6 (1958), pp. 361-364.

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