TI 2012-012/1
Tinbergen Institute Discussion Paper

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A bankruptcy approach to the core cover*

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Abstract

In this paper we establish a relationship between the core cover of a compromise admissible game and the core of a particular bankruptcy game: the core cover of a compromise admissible game is, indeed, a translation of the set of coalitional stable allocations captured by an associated bankruptcy game. Moreover, we analyze the combinatorial complexity of the core cover and, consequently, of the core of a compromise stable game.

Keywords: Cooperative game theory, compromise admissible games, bankruptcy, core cover, complexity.

1 Introduction

In the theory of cooperative TU games, the investigation of relations among different set valued solutions is crucial for a better understanding of these solutions. The core (Gillies (1953)) of a TU game is the set of all efficient allocations that are coalitional stable. In other words, all the core allocations are coalitional stable in the sense that there is no coalition S with incentives to split off. The core cover (Tijs and Lipperts, 1982) is the set of all efficient allocations satisfying that every player receives neither more than his utopia payoff, nor less than his minimal right. Both set valued solutions are convex polytopes and therefore can be described by the convex hull of their extreme points. Besides, when the game is convex, the set of extreme points of the core coincides with the set of marginal vectors (Shapley, 1953; Ichiishi, 1981). Quant et al. (2005) showed that the extreme points of the core cover of admissible games are the larginal vectors. Recently, Platz et al. (2011) characterize sets of larginal vectors satisfying that the game is compromise stable if, and only if, every larginal vector of the set is in the core.

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*This paper was started while Arantza Estévez-Fernández was visiting the University of Vigo. Authors acknowledge the financial support of *Ministerio de Ciencia e Innovación* through projects ECO2008-03484-C02-02/ECO and MTM2011-27731-C03-03, and of *Xunta de Galicia* through project INCITE09-207-064-PR.

As the name implies, the core cover is a core catcher. The games with a non-empty core cover satisfying that all core cover allocations are coalitional stable are called compromise stable games, that is, for this subclass of games, the core and the core cover coincide. The subclass of compromise stable games contains both convex and not convex games. Quant et al. (2005) showed that convex compromise stable games are strategically equivalent to bankruptcy games (O'Neill, 1982 and Aumann and Maschler, 1985).

Our aim is to investigate new relations between the core cover of compromise admissible games and the core of bankruptcy games. Our main contribution here is Theorem 3.2, where we show that the core cover of a compromise admissible game is a translation of the core of a particular associated bankruptcy game. Therefore, the core cover of a compromise admissible game is, up to a translation, the set of coalitional stable allocations captured by the associated bankruptcy game.

Shapley (1971) studied in detail the core of convex games. Recently, González-Díaz and Sánchez-Rodríguez (2008) further analyzed the core of convex games by introducing face games. Given a game (N, v) with a non-empty core and a coalition $T \subset N$, a T-face game is defined in such a way that the core of this T-face game coincides with the core allocations of the game (N, v) that provide the best payoff for coalition T and the worst payoff for its complementary coalition $N \setminus T$. González-Díaz and Sánchez-Rodríguez (2008) showed that the core of convex games can be rebuilt with the cores of the face games. Any face game is related to a specific coalition T, and there are so many face games as coalitions. In this paper, we establish that all bankruptcy face games are new bankruptcy games. Combining the results of González-Díaz and Sánchez-Rodríguez (2008) and Theorem 3.2, we obtain that the core cover of a compromise admissible game can also be rebuilt with the core covers of some specific bankruptcy games.

Several rules for bankruptcy problems have been redefined in the context of compromise admissible games: the adjusted proportional rule (τ value) in González-Díaz et al. (2005), the Talmud rule (nucleolus) in Quant et al. (2005), and the run to the bank rule (the Shapley value) in Quant et al. (2006). Here, we consider a general formula, which is already used in the papers previously mentioned, for extending bankruptcy rules to the class of compromise admissible games. It turns out that, if the bankruptcy rule is invariant under claims truncation, then, the corresponding value always belongs to the core cover. Particularly, we consider the constrained equal awards rule (CEA) and show that its associated value for compromise admissible games belongs to the core cover of a specific T-face game of an associated bankruptcy game.

Another goal of this paper is to show the complexity of the core cover with regard to the maximal number of extreme points. It is well known that for an n-player game, $n!$ is the maximal number of extreme core allocations. With the exception of 3-player games, the maximal number of extreme core cover vertices is strictly less than $n!$. As an example, for a 7-player game, the upper bound is 140, much less than $7! = 5040$ (maximal number of marginal vectors). In this paper, we derive the precise upper bound of the number of extreme points of the core cover.

The paper is structured as follows. In Section 2, we present the basic definitions and notations. We analyze, in Section 3, the relation between the core cover and the core of the associated bankruptcy game, and introduce and study the CEA value for compromise admissible games. Section 4 is devoted to the study of the complexity of the core cover. Finally, in Section 5, we conclude with a summary of the major contributions of this paper.

2 Preliminaries

A *cooperative n-player game with transferable utility*, shortly a TU game, is an ordered pair (N, v) where N is a finite set (the set of players) with $|N| = n$ and $v : 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \subseteq N$, a payoff $v(S)$; by convention, $v(\emptyset) = 0$. Let G^n be the set of n-player TU games. Given $S \subseteq N$, let $|S|$ be the number of players in S.

A TU game $(N, v) \in G^n$ is said to be *additive* if there exists a vector $a \in \mathbb{R}^n$ such that $v(S) = \sum_{i \in S} a_i$ for all $S \subseteq N$. The game (N, v) is then denoted by (N, a). A TU game $(N, v) \in G^n$ is *strategically equivalent* to another TU game $(N, w) \in G^n$ if there exists a scalar $k > 0$ and an additive game $(N, a) \in G^n$ such that $w = a + kv$. A *value* is a function $\varphi : G^n \longrightarrow \mathbb{R}^n$ that assigns to each TU game $(N, v) \in G^n$ a vector $\varphi(N, v) \in \mathbb{R}^n$.

The *core* (Gillies, 1953) of a cooperative TU-game (N, v) is defined as

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^n \ : \ \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N \right\},$$

that is, the core is the set of efficient allocations of $v(N)$ such that there is no coalition with an incentive to split off. A game is said to be *balanced* (see Bondareva, 1963; Shapley, 1967) if the core is nonempty. Let BG^n be the set of n-player balanced TU games.

An important subclass of balanced games is the class of convex games (see Shapley, 1971). A game (N, v) is said to be *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. Let CG^n be the set of n-player convex TU games.

Given $S \subseteq N$, an order of the players in S is a bijection $\sigma_S : \{1, \ldots, |S|\} \to S$, where $\sigma_S(k)$ is the player in S that is in position k. We denote by $\Pi(S)$ the set of all orders of the players in S. For $S = N$, we denote σ instead of σ_S. Given $\sigma \in \Pi(N)$ and a player $\sigma(k) \in N$ we denote $P_\sigma(\sigma(k)) = \{\sigma(1), \ldots, \sigma(k-1)\}$. For $\sigma \in \Pi(N)$, we define the *inverse order of* σ, $\sigma^{in} \in \Pi(N)$, as the order satisfying $\sigma^{in}(k) = \sigma(n - k + 1)$, for every $k \in \{1, \ldots, n\}$. Let $(N, v) \in G^n$ and $\sigma \in \Pi(N)$. The *marginal vector* associated with (N, v) and σ, $m^\sigma(v)$, is defined, for each $k \in \{1, \ldots, n\}$, by $m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$. It is known that convexity of a game is equivalent to every marginal vector being a core element and, moreover, $\mathcal{C}(v) = \text{con}\{m^\sigma(v) \ : \ \sigma \in \Pi(N)\}$[1] (see Shapley, 1953; Ichiishi, 1981).

Next, we recall the terminology used in Shapley (1971) and in González-Díaz and Sánchez-Rodríguez (2008). Let $(N, v) \in BG^n$. For each $\emptyset \neq T \subseteq N$, let H_T be the hyperplane $H_T = \{x \in \mathbb{R}^n \ : \ \sum_{i \in T} x_i = v(T)\}$. Next, for $T \subset N$, let $F_T = \mathcal{C}(v) \cap H_{N \setminus T}$. Clearly, $F_\emptyset = \mathcal{C}(v)$.

[1]Given a finite set $A \subset \mathbb{R}^n$, $\text{con}(A)$ represents the convex hull of A.

For convenience, we define $F_N = \mathcal{C}(v)$. For convex games, each F_T is a nonempty face of $\mathcal{C}(v)$ and we refer to F_T as the T-*face* of $\mathcal{C}(v)$. By definition, for each allocation in F_T, coalition T receives $v(N) - v(N \setminus T)$. Clearly, for each $\emptyset \neq T \subset N$, since both F_T and $F_{N\setminus T}$ lie in H_N, they are parallel to each other. Subsequently, we recall, for each coalition $T \subseteq N$, the T-face game (N, v_{F_T}) that is closely related to F_T. The *T-face game* (González-Díaz and Sánchez-Rodríguez, 2008) (N, v_{F_T}) is defined, for each $S \subseteq N$, as

$$v_{F_T}(S) = v\Big((S \cap T) \cup (N \setminus T)\Big) - v\Big(N \setminus T\Big) + v\Big(S \cap (N \setminus T)\Big).$$

Face games were introduced in González-Díaz and Sánchez-Rodríguez (2008) in order to analyze the core of convex and strictly convex games. Hence, the idea behind the T-face game of a convex game (N, v) is the following. F_T are the best core allocations for coalition T and the worst ones for coalition $N \setminus T$ since coalition T always receives $v(N) - v(N \setminus T)$ and coalition $N \setminus T$ gets exactly $v(N \setminus T)$. Moreover, note that there is still freedom for $v(N) - v(N \setminus T)$ to be shared among the players in T and for $v(N \setminus T)$ to be shared among the players in $N \setminus T$.

Proposition 2.1 (González-Díaz and Sánchez-Rodríguez, 2008). *Let $(N, v) \in CG^n$ and let $T \subseteq N$. Then, $\mathcal{C}(v_{F_T}) = F_T$. Therefore, $\mathcal{C}(v) = con\{\mathcal{C}(v_{F_T}) \,:\, \emptyset \neq T \subset N\}$.*

Following Tijs and Lipperts (1982), the *utopia vector* of a TU game (N, v), $M(v) \in \mathbb{R}^n$, is defined by $M_i(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$. The *minimum right vector* $m(v) \in \mathbb{R}^n$ is defined, for all $i \in N$, by

$$m_i(v) = \max_{S \subseteq N, i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right\}.$$

The *core cover* (Tijs and Lipperts, 1982) of a TU game (N, v) consists of all allocations of $v(N)$ giving each player at least his minimum right, but no more than his utopia payoff:

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^n \,:\, \sum_{i \in N} x_i = v(N),\ m(v) \leq x \leq M(v) \right\}.$$

A TU game is called *compromise admissible* if it has a nonempty core cover. Let CA^n be the set of n-player compromise admissible TU games. Let us note that $CG^n \subset CA^n$. Mathematically, a TU game (N, v) is compromise admissible if

$$m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v).$$

The extreme points of the core cover are called *larginal vectors* or *larginals* (see Quant et al., 2005). Let $(N, v) \in CA^n$ and $\sigma \in \Pi(N)$. The larginal vector $\ell^\sigma(v)$ is the allocation of $v(N)$ that gives the utopia payoffs to the first players with respect to σ as long as it is still possible to assign the remaining players their minimum rights:

$$\ell^{\sigma}_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)}(v) & \text{if } \sum_{r=1}^{k} M_{\sigma(r)}(v) + \sum_{r=k+1}^{n} m_{\sigma(r)}(v) \leq v(N), \\ m_{\sigma(k)}(v) & \text{if } \sum_{r=1}^{k-1} M_{\sigma(r)}(v) + \sum_{r=k}^{n} m_{\sigma(r)}(v) \geq v(N), \\ v(N) - \sum_{r=1}^{k-1} M_{\sigma(r)}(v) - \sum_{r=k+1}^{n} m_{\sigma(r)}(v) & \text{otherwise,} \end{cases}$$

for every $k = 1, \ldots, n$.

It is well-known that, for all compromise admissible game (N, v), $\mathcal{C}(v) \subseteq \mathcal{CC}(v)$ (Tijs and Lipperts, 1982). A TU game $(N, v) \in CA^n$ is said to be *compromise stable* (see Quant et al., 2005) if $\mathcal{C}(v) = \mathcal{CC}(v)$.

Theorem 2.1 (Quant et al., 2005). *A TU game $(N, v) \in CA^n$ is compromise stable if, and only if, for all $\emptyset \neq S \subseteq N$,*

$$v(S) \leq \max \left\{ \sum_{i \in S} m_i(v),\ v(N) - \sum_{i \in N \setminus S} M_i(v) \right\}.$$

A *bankruptcy problem* (cf. O'Neill, 1982; Aumann and Maschler, 1985) is a triple (N, E, d), where $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}^n_+$ is the vector of claims satisfying $\sum_{i \in N} d_i \geq E$. The corresponding *bankruptcy game* (N, v) is defined, for each $S \subseteq N$, by

$$v(S) = \max \left\{ 0,\ E - \sum_{j \in N \setminus S} d_j \right\}.$$

We denote the class of bankruptcy problems with n players by BR^n. The class of bankruptcy games is a proper subclass of CG^n. A *bankruptcy rule* is a function $f : BR^n \longrightarrow \mathbb{R}^n_+$ assigning to each bankruptcy problem $(N, E, d) \in BR^n$ a payoff vector $f(N, E, d) \in \mathbb{R}^n_+$ such that $\sum_{i \in N} f_i(N, E, d) = E$ and $f_i(N, E, d) \leq d_i$ for every $i \in N$.

Let us note that there exist compromise stable TU games that are not convex and that there exist convex TU games which are not compromise stable. The next theorem, due to Quant et al. (2005), relates bankruptcy games with convex and compromise stable games.

Theorem 2.2 (Quant et al., 2005). *A TU game is both convex and compromise stable if, and only if, it is strategically equivalent to a bankruptcy game.*

3 The structure of the core cover via bankruptcy games

The main result of this section is the relation between the core cover of a compromise admissible game and the core of an associated bankruptcy game. Based on this relation, we define the family of values on the class of compromise admissible games that arise from bankruptcy rules. It is shown that, if the bankruptcy rule is invariant under claims truncation, then, the corresponding value on the class of compromise admissible games belongs to the core cover of the game. Particularly, we study the constrained equal awards (CEA) value and show that it always belongs to the core of a specific face game of the associated bankruptcy game.

The next lemma describes the contribution of a player to any coalition in a bankruptcy game.

Lemma 3.1. *Let (N, E, d) be a bankruptcy problem and (N, v) be the associated TU game. For each $i \in N$ and each $S \subseteq N \setminus \{i\}$,*

$$v(S \cup \{i\}) - v(S) = \min \{d_i, v(S \cup \{i\})\}.$$

Proof. Let $i \in N$ and $S \subseteq N \setminus \{i\}$. If $v(S) = 0$, then, $E - \sum_{j \in N \setminus S} d_j \leq 0$ and, consequently, $E - \sum_{j \in N \setminus (S \cup \{i\})} d_j \leq d_i$. Since $v(S \cup \{i\}) = \max\{0, E - \sum_{j \in N \setminus (S \cup \{i\})} d_j\}$, it follows

$$v(S \cup \{i\}) - v(S) = v(S \cup \{i\}) = \min \{d_i, v(S \cup \{i\})\}.$$

If $v(S) > 0$, then, $v(S) = E - \sum_{j \in N \setminus S} d_j > 0$ and, consequently, $0 \leq d_i < E - \sum_{j \in N \setminus (S \cup \{i\})} d_j$. Since $v(S \cup \{i\}) = \max\{0, E - \sum_{j \in N \setminus (S \cup \{i\})} d_j\}$, it follows

$$v(S \cup \{i\}) - v(S) = E - \sum_{j \in N \setminus (S \cup \{i\})} d_j - (E - \sum_{j \in N \setminus S} d_j) = d_i = \min \{d_i, v(S \cup \{i\})\}. \qquad \square$$

Given a compromise admissible TU game (N, v), we can associate a bankruptcy problem and the corresponding bankruptcy game to (N, v) as follows.

Definition 3.1. *Let $(N, v) \in CA^n$. We define the* associated bankruptcy problem, *(N, E, d), as*

$$E = v(N) - \sum_{j \in N} m_j(v) \text{ and } d = M(v) - m(v).$$

We denote by $(N, \bar{v})$ the corresponding bankruptcy game.

Note that the game $(N, \bar{v})$ in Definition 3.1 is, indeed, a bankruptcy game since $m(v) \leq M(v)$ and $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. Besides, Theorem 2.1 can be rewritten as follows. A TU game $(N, v) \in CA^n$ is compromise stable if, and only if, for all $\emptyset \neq S \subseteq N$, $v(S) \leq \sum_{i \in S} m_i(v) + \bar{v}(S)$, where $(N, \bar{v})$ is the bankruptcy game of Definition 3.1.

Theorem 3.1. *Let $(N, v) \in CA^n$, $(N, \bar{v})$ be its associated bankruptcy game, and $\sigma \in \Pi(N)$. Then, $\ell^\sigma(v) = m(v) + m^{\sigma^{in}}(\bar{v})$.*

Proof. Let $k \in \{1, \ldots, n\}$, it follows, by definition of σ^{in}, that $\sigma^{in}(n-k+1) = \sigma(k)$ and $P_{\sigma^{in}}(\sigma^{in}(n-k+1)) = \{\sigma^{in}(1), \sigma^{in}(2), \ldots, \sigma^{in}(n-k)\} = \{\sigma(n), \sigma(n-1), \ldots, \sigma(k+1)\}$. Therefore,

$$m^{\sigma^{in}}_{\sigma(k)}(\bar{v}) = m^{\sigma^{in}}_{\sigma^{in}(n-k+1)}(\bar{v})$$

$$
\begin{aligned}
&= \bar{v}(P_{\sigma^{in}}(\sigma^{in}(n-k+1)) \cup \{\sigma^{in}(n-k+1)\}) - \bar{v}(P_{\sigma^{in}}(\sigma^{in}(n-k+1))) \\
&= \bar{v}(\{\sigma(n), \sigma(n-1), \ldots, \sigma(k)\}) - \bar{v}(\{\sigma(n), \sigma(n-1), \ldots, \sigma(k+1)\}) \qquad (1) \\
&= \min\{d_{\sigma(k)}, \bar{v}(\{\sigma(n), \sigma(n-1), \ldots, \sigma(k)\})\} \\
&= \min\{d_{\sigma(k)}, \max\{0, E - \sum_{l=1}^{k-1} d_{\sigma(l)}\}\} \\
&= \min\{M_{\sigma(k)} - m_{\sigma(k)}, \max\{0, v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k}^{n} m_{\sigma(l)}(v)\}\}
\end{aligned}
$$

where the fourth equality follows by Lemma 3.1.

Following the three cases in the definition of larginals, we distinguish between three possible situations:

1. $v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k}^{n} m_{\sigma(l)}(v) \leq 0$.
   In this case $\ell^{\sigma}_{\sigma(k)}(v) = m_{\sigma(k)}(v)$. Moreover, we obtain $m^{\sigma^{in}}_{\sigma(k)}(\bar{v}) = 0$ by Expression (1).

2. $0 < v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k}^{n} m_{\sigma(l)}(v) < M_{\sigma(k)}(v) - m_{\sigma(k)}(v)$.
   In this case, $\ell^{\sigma}_{\sigma(k)}(v) = v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k+1}^{n} m_{\sigma(l)}(v)$. Moreover, we obtain $m^{\sigma^{in}}_{\sigma(k)}(\bar{v}) = v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k}^{n} m_{\sigma(l)}(v)$ by Expression (1).

3. $M_{\sigma(k)}(v) - m_{\sigma(k)}(v) \leq v(N) - \sum_{l=1}^{k-1} M_{\sigma(l)}(v) - \sum_{l=k}^{n} m_{\sigma(l)}(v)$.
   In this case, $\ell^{\sigma}_{\sigma(k)}(v) = M_{\sigma(k)}(v)$. Moreover, we obtain $m^{\sigma^{in}}_{\sigma(k)}(\bar{v}) = M_{\sigma(k)}(v) - m_{\sigma(k)}(v)$ by Expression (1).

Therefore, $\ell^{\sigma}(v) = m(v) + m^{\sigma^{in}}(\bar{v})$. □

Theorem 3.2. *Let $(N, v) \in CA^n$ and $(N, \bar{v})$ be its associated bankruptcy game. Then,*

$$\mathcal{CC}(v) = m(v) + \mathcal{C}(\bar{v}) = m(v) + \mathcal{CC}(\bar{v}).$$

Moreover, (N, v) is compromise stable if, and only if, $\mathcal{C}(v) = m(v) + \mathcal{C}(\bar{v})$.

Proof. The first part of the theorem follows from Theorem 3.1 and because $(N, \bar{v})$ is a bankruptcy game. The second part of the theorem straightforwardly follows from the definition of compromise stability. □

Note that from Theorem 3.2 we also derive Theorem 2.1 and Theorem 2.2. Namely, the alternative proof for Theorem 2.1 is the following. If for all $\emptyset \neq S \subseteq N$, $v(S) \leq \sum_{i \in S} m_i(v) + \bar{v}(S)$, then, $m(v) + \mathcal{C}(\bar{v}) \subseteq \mathcal{C}(v)$. By Theorem 3.2, $\mathcal{CC}(v) = m(v) + \mathcal{C}(\bar{v}) \subseteq \mathcal{C}(v)$, and then, $\mathcal{C}(v) = \mathcal{CC}(v)$ and the game v is compromise stable. Reciprocally, if v is compromise stable, then, $\mathcal{C}(v) = \mathcal{CC}(v) = m(v) + \mathcal{C}(\bar{v})$ where the last equality follows from Theorem 3.2. Since the game $m(v) + \bar{v}$ is convex, for every $S \subset N$ there is $x^S \in m(v) + \mathcal{C}(\bar{v}) = \mathcal{C}(v)$ such that $\sum_{i \in S} x_i^S = \sum_{i \in S} m_i(v) + \bar{v}(S)$. Then, $\sum_{i \in S} m_i(v) + \bar{v}(S) = \sum_{i \in S} x_i^S \geq v(S)$.

The alternative proof for Theorem 2.2 is the following. If (N,v) is convex and compromise stable, then, the equality $\mathcal{C}(v) = m(v) + \mathcal{C}(\bar{v})$ implies that $v = m(v) + \bar{v}$, and, therefore, game v is strategically equivalent to a bankruptcy game. Reciprocally, if v is strategically equivalent to a bankruptcy game, then, clearly, v is convex and compromise stable.

Note that the second part of Theorem 3.2 can also be derived from the proof of Theorem 4.2 in Quant et al. (2005).

By Proposition 2.1, the core of a bankruptcy game can be rebuilt through the cores of the face games. The following proposition shows that each face game of a bankruptcy game is strategically equivalent to a bankruptcy game.

Proposition 3.1. *Let (N,E,d) be a bankruptcy problem, (N,v) be the associated TU game, and $T \subseteq N$. Then, the T-face game is strategically equivalent to a bankruptcy game in the following way: $v_{F_T} = \tilde{v} + a$, where*

1. *$a_i = 0$, for every $i \in N$, and $\tilde{v}$ is the bankruptcy game associated to the bankruptcy problem $(N,E,\tilde{d})$ where $\tilde{d}_i = d_i$, for every $i \in T$, and $\tilde{d}_i = 0$, for every $i \in N \setminus T$ if $\sum_{i\in T} d_i \geq E$.*
2. *$a_i = d_i$, for every $i \in T$, and $a_i = 0$, for every $i \in N \setminus T$, and $\tilde{v}$ is the bankruptcy game associated to the bankruptcy problem $(N, E - \sum_{i\in T} d_i, \tilde{d})$ with $\tilde{d} = d - a$ if $\sum_{i\in T} d_i < E$.*

Proof. Let $T \subseteq N$. The T-face game (N, v_{F_T}) associated with the bankruptcy game (N,v) is defined, for each $S \subseteq N$, by

$$\begin{aligned} v_{F_T}(S) &= v\big((S\cap T)\cup(N\setminus T)\big) - v\big(N\setminus T\big) + v\big(S\cap(N\setminus T)\big) \\ &= \max\{0, E - \textstyle\sum_{j\in T\setminus S} d_j\} - \max\{0, E - \sum_{j\in T} d_j\} + \max\{0, E - \sum_{j\in (N\setminus S)\cup(S\cap T)} d_j\}. \end{aligned}$$

1. $\sum_{i\in T} d_i \geq E$. In this case, it follows that $E - \sum_{j\in(N\setminus S)\cup(S\cap T)} d_j \leq 0$ since $(N\setminus S)\cup(S\cap T) = T \cup (N\setminus(S\cup T))$. Therefore, $v(N\setminus T) = v(S\cap(N\setminus T)) = 0$ and

$$v_{F_T}(S) = \max\{0, E - \sum_{j\in T\setminus S} d_j\}.$$

Hence,

$$v_{F_T}(S) = \begin{cases} 0 & \text{if } S \subseteq N\setminus T \\ \max\{0, E - \sum_{j\in T\setminus S} d_j\} & \text{if } S\cap T \neq \emptyset \end{cases}$$

It is clear that v_{F_T} is the bankruptcy game associated to the bankruptcy problem $(N,E,\tilde{d})$, where $\tilde{d}_i = d_i$, for every $i \in T$, and $\tilde{d}_i = 0$, for every $i \in N\setminus T$.

2. $\sum_{i\in T} d_i < E$. Then,

$$\begin{aligned} v_{F_T}(S) &= v((S\cap T)\cup(N\setminus T)) - v(N\setminus T) + v(S\cap(N\setminus T)) \\ &= E - \textstyle\sum_{j\in T\setminus S} d_j - E + \sum_{j\in T} d_j + \max\{0, E - \sum_{j\in(N\setminus S)\cup(S\cap T)} d_j\} \\ &= \textstyle\sum_{j\in T\cap S} d_j + \max\{0, E - \sum_{j\in T} d_j - \sum_{j\in N\setminus(S\cup T)} d_j\} \end{aligned}$$

It is clear that $\max\{0, E - \sum_{j\in T} d_j - \sum_{j\in N\setminus(S\cup T)} d_j\}$ is the bankruptcy game associated to the bankruptcy problem $(N, \tilde{E}, \tilde{d})$, where $\tilde{E} = E - \sum_{i\in T} d_i$, and $\tilde{d_i} = 0$, for every $i \in T$, and $\tilde{d_i} = d_i$, for every $i \in N \setminus T$. $\square$

The following theorem states that the core cover of a compromise admissible game can be rebuilt with the core covers of the bankruptcy face games.

Theorem 3.3. *Let $(N, v) \in CA^n$ and $(N, \bar{v})$ be its associated bankruptcy game. Then,*

$$\mathcal{CC}(v) = m(v) + con\{\mathcal{CC}(\bar{v}_{F_T}) : T \subset N\}.$$

Proof. Combining the results in Theorem 3.2 and in Proposition 2.1, we rewrite the core cover of a compromise admissible game (N, v) as

$$\mathcal{CC}(v) = m(v) + \text{con}\{\mathcal{C}(\bar{v}_{F_T}) : T \subset N\}.$$

By Proposition 3.1, for each $T \subset N$, it follows that $\bar{v}_{F_T}$ is a new bankruptcy game, and then, $\mathcal{CC}(\bar{v}_{F_T}) = \mathcal{C}(\bar{v}_{F_T})$, which establishes the result. $\square$

Theorem 3.2 suggests that we may allocate the value $v(N)$ of a compromise admissible game among the players in a reasonable way using bankruptcy rules. This has been done in González-Díaz et al. (2005) using the adjusted proportional rule (the τ value) and in Quant et al. (2005) using the Talmud rule (the nucleolus for compromise stable games). Following these two papers, Quant et al. (2006) describe a method to apply bankruptcy rules to compromise admissible games and study the run to the bank rule and the Tal-family rules. A common property of all these bankruptcy rules is the property of *invariance under claims truncation*. A bankruptcy rule f satisfies *invariance under claims truncation* if for every bankruptcy problem (N, E, d), $f(N, E, d) = f(N, E, d')$, where $d'_i = \min\{E, d_i\}$ for every $i \in N$. Curiel et al. (1987) show that a bankruptcy rule satisfies invariance under claims truncation if, and only if, the allocation provided by the rule is always in the core of the associated bankruptcy game. We show that if a bankruptcy rule satisfies invariance under claims truncation, then, the corresponding value always provides allocations belonging to the core cover of compromise admissible games. Moreover, we investigate the CEA-value and show that the corresponding allocations always belong to a specific face of the core cover of compromise admissible games.

Definition 3.2. *Let $(N, v) \in CA^n$, let (N, E, d) be its associated bankruptcy problem, and let f be a bankruptcy rule. We define the f-*value*, $\varphi^f(N, v)$, as*

$$\varphi^f(N, v) = m(v) + f(N, E, d).$$

The following result is an immediate consequence of Theorem 5 in Curiel et al. (1987) and Theorem 3.2 and, therefore, the proof is omitted.

Theorem 3.4. *Let $(N,v) \in CA^n$ and let f be a bankruptcy rule satisfying invariance under claims truncation. Then, $\varphi^f(N,v) \in \mathcal{CC}(v)$.*

Subsequently, we consider the CEA-value which arises from the Constrained Equal Awards (CEA) rule for bankruptcy problems and show that it always belongs to a face of the core cover. From now on, given (N,v) a compromise admissible TU-game we assume, without loss of generality and for easiness of exposition, that $N = \{1,\ldots,n\}$ and $M_1(v) - m_1(v) \leq M_2(v) - m_2(v) \leq \ldots \leq M_n(v) - m_n(v)$. Given a bankruptcy problem (N,E,d), the *constrained equal awards* rule, CEA, provides an allocation $CEA(N,E,d) \in \mathbb{R}^n$ defined as $CEA_i(N,E,d) = \min\{d_i,\lambda\}$ for every $i \in N$, with λ chosen such that $\sum_{i\in N}\min\{d_i,\lambda\} = E$. Given a compromise admissible TU-game (N,v) and (N,E,d) the associated bankruptcy problem, we denote by $k(v) \in N$ the player satisfying $CEA_i(N,E,d) = d_i$ for every $i < k(v)$ and $CEA_i(N,E,d) < d_i$ for every $i \geq k(v)$. By definition of the CEA-value, we have that $\varphi_i^{CEA}(N,v) = M_i(v)$ for every $i < k(v)$ and $\varphi_i^{CEA}(N,v) < M_i(v)$ for every $i \geq k(v)$. Note that the players in $\{1,\ldots,k(v)-1\}$ are receiving their utopia values and, therefore, they are allocated their maximum obtainable payoffs.

Proposition 3.2. *Let $(N,v) \in CA^n$ and let $(N,\bar{v})$ be its associated bankruptcy game. Then, $\varphi^{CEA}(N,v) \in m(v) + F_{\{1,\ldots,k(v)-1\}}(\bar{v})$.*

Proof. Recall that $F_{\{1,\ldots,k(v)-1\}}(\bar{v}) = \mathcal{C}(\bar{v}) \cap H_{N\setminus\{1,\ldots,k(v)-1\}}(\bar{v}) = \mathcal{C}(\bar{v}) \cap H_{\{k(v),\ldots,n\}}(\bar{v})$. By Theorem 3.4 and Theorem 3.2, $\varphi^{CEA}(N,v) \in \mathcal{CC}(v) = m(v) + \mathcal{C}(\bar{v})$.

By definition of φ^{CEA} and of $k(v)$,

$$\begin{aligned}\sum_{j=k(v)}^{n} \varphi_j^{CEA}(N,v) &= \sum_{j=k(v)}^{n} m_j(v) + \sum_{j=k(v)}^{n} CEA_j(N,E,d) \\ &= \sum_{j=k(v)}^{n} m_j(v) + v(N) - \sum_{j\in N} m_j(v) - \sum_{j=1}^{k(v)-1} (M_j(v) - m_j(v)) \\ &= \sum_{j=k(v)}^{n} m_j(v) + \bar{v}(\{k(v),\ldots,n\})\end{aligned}$$

where the last equality follows by definition of $k(v)$. Therefore, $\varphi^{CEA} \in m(v) + H_{\{k(v),\ldots,n\}}(\bar{v})$. As a result,

$$\varphi^{CEA}(N,v) \in (m(v) + H_{\{k(v),\ldots,n\}}(\bar{v})) \cap (m(v) + \mathcal{C}(\bar{v})) = m(v) + F_{\{1,\ldots,k(v)-1\}}(\bar{v}). \qquad \square$$

Note that if $k(v) > 1$, then, the CEA value belongs to a specific face of the core cover polytope. Nevertheless, Proposition 3.2 applied to the case $k(v) = 1$ indicates that $\varphi^{CEA}(N,v) \in m(v) + \mathcal{C}(\bar{v})$, which is the face of the empty set. In this case, the constrained equal awards value coincides with the egalitarian value that assigns $\frac{v(N)}{n}$ to every agent.

4 Core cover complexity

In this section, we describe the complexity of the core-cover of a compromise admissible game by looking at the maximal number of its extreme points. Consequently, the complexity of the core of a compromise stable game is also analyzed and, particularly, the complexity of the core of bankrupcty games.

We now focus on the maximal number of extreme points of the core cover. Since the core cover is the convex hull of the larginal vectors, it has at most $n!$ extreme points. However, this number is never achieved for games with at least four players. Notice that the larginal vector $l^{\sigma}(v)$ is the efficient payoff vector giving the first players in σ their utopia payoffs as long as it is possible to assign the remaining players their minimum rights. Therefore, for each $\sigma \in \Pi(N)$, players can be divided into three groups: those receiving their utopia payoffs (group G_1), those receiving their minimum rights (group G_2), and the player which is between both groups (the pivot player). Clearly, given a game $(N, v) \in CA^n$, for each order σ, the position of the pivot player, l, varies between $1, \ldots, n$ and depends on the vector of minimal rights and the utopia vector.

To have some regularity inside the class of compromise admissible games, consider now the subclasses of compromise admissible games where the position of the pivot player is fixed. That is, take $k \in \{1, \ldots, n\}$, a TU game $(N, v) \in CA^n_k$ if $(N, v) \in CA^n$ and, for each $\sigma \in \Pi(N)$,

$$\sum_{r=1}^{k-1} M_{\sigma(r)}(v) + \sum_{r=k}^{n} m_{\sigma(r)}(v) < v(N) < \sum_{r=1}^{k} M_{\sigma(r)}(v) + \sum_{r=k+1}^{n} m_{\sigma(r)}(v).$$

Given a game $(N, v) \in CA^n_k$, the pivot player is always in position k for all the larginal vectors. There are compromise admissible games with such property as we illustrate in the following example

Example 4.1. *Let (N, v) be a symmetric*[2] *4-player compromise admissible game such that, for all $i \in N$, $v(S) = 0$ for all $S \subset N \backslash \{i\}$, $v(N) = 10$, and, for all $i \in N$,*

1. $v(N \backslash \{i\}) = 0$ *(example of a game $(N, v) \in CA^4_1$).*

2. $v(N \backslash \{i\}) = 4$ *(example of a game $(N, v) \in CA^4_2$).*

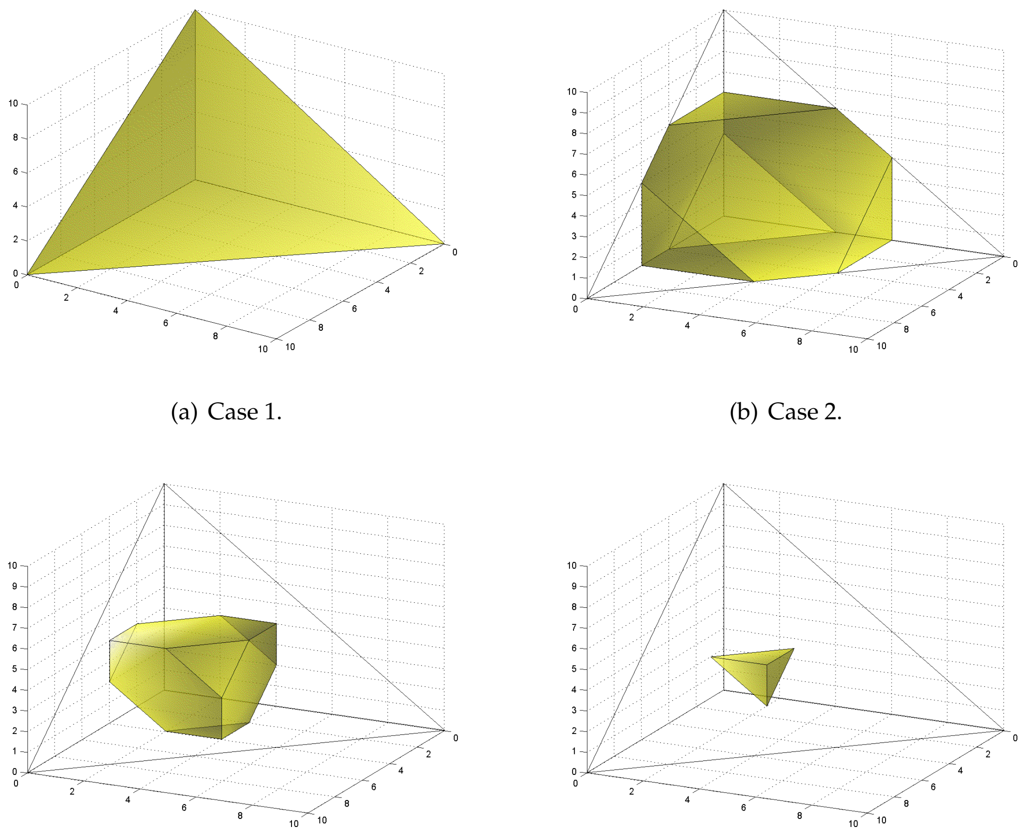
3. $v(N \backslash \{i\}) = 6$ *(example of a game $(N, v) \in CA^4_3$).*

4. $v(N \backslash \{i\}) = 7$ *(example of a game $(N, v) \in CA^4_4$).*

In the four cases presented in Figure 1 we have $\mathcal{C}(v) = \mathcal{CC}(v)$.[3] *However, there are compromise admissible games in the classes CA^n_k where the core is a strict subset of the core cover. Consider, for*

[2]A TU game (N, v) is symmetric if the value of a coalition only depends on its cardinality. Although the subclasses CA^n_k ($k \in \{1, \ldots, n\}$) certainly contain games that are not symmetric, we restrict to symmetric games for easiness of exposition.

[3]The graphics in Figure 1 and Figure 2 were built with the toolbox TUGlab of MATLAB® (Mirás-Calvo and Sánchez-Rodríguez, 2008). The web page of TUGlab can be found in `http://eio.usc.es/pub/io/xogos/index.php`



(a) Case 1.

(b) Case 2.

(c) Case 3.

(d) Case 4.

Figure 1: $n = 4$, core covers of cases 1, 2, 3, and 4.

instance, the 4*-player games* (N, v) *and* (N, w) *such that*

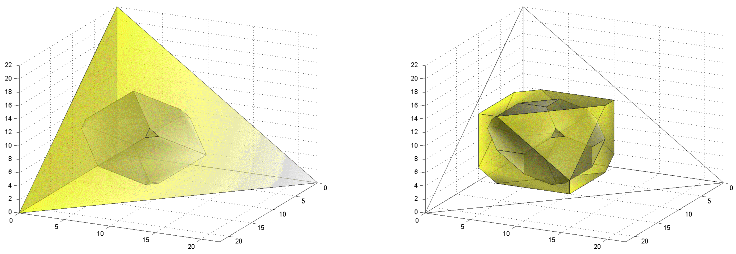
$$v(S) = \begin{cases} 0 & \textit{if } |S| = 1, \\ 7 & \textit{if } |S| = 2, \\ 0 & \textit{if } |S| = 3, \\ 22 & \textit{if } S = N \end{cases} \quad \textit{and} \quad w(S) = \begin{cases} 0 & \textit{if } |S| = 1, \\ 7 & \textit{if } |S| = 2, \\ 12 & \textit{if } |S| = 3, \\ 22 & \textit{if } S = N \end{cases}$$

Their cores and core covers are represented in Figure 2. Clearly, $(N, v) \in CA_1^4$ *and* $(N, w) \in CA_3^4$*. Besides, the game* (N, w) *has the maximal number of core vertices* (24) *and, as we will show in Theorem 4.1, the maximal number of core cover vertices* (12)*.*

Next, for each $k \in \{1, \ldots, n\}$, we investigate the computational complexity of a game $(N, v) \in CA_k^n$ attending to the maximal number of extreme points of its core cover.

Lemma 4.1. *Let* $k \in \{1, \ldots, n\}$*.*

1. *Let* $(N, v) \in CA_k^n$*. The maximal number of extreme points is given by the number* $n\binom{n-1}{k-1}$*.*



(a) Core cover and core of (N, v). (b) Core cover and core of (N, w).

Figure 2: Core cover contains the core.

2. *The classes CA_k^n and CA_{n-k+1}^n have the same combinatorial complexity. The maximal complexity is obtained at $k = \frac{n+1}{2}$ if n is odd, and $k = \frac{n}{2}$ (or $k = \frac{n}{2} + 1$) if n is even.*

Proof. 1. If $k = 1$ (respectively, $k = n$), the pivot player is in the first (last) position for each order $\sigma \in \Pi(N)$. Take a pivot player $j \in N$, then, all the orders $\sigma \in \Pi(N)$ where j is in first (last) position give rise to the same larginal vector (an extreme point of the core-cover). Hence, there are at most n different larginal vectors, one for each possible pivot player.

Take $1 < k < n$ and fix $j \in N$ as the pivot player. Then, any group of $k-1$ players out of the players in $N \setminus \{j\}$ receive their utopia payoffs if they are located ahead of player j according to an order $\sigma \in \Pi(N)$. As a result, there are at most $\binom{n-1}{k-1}$ different larginal vectors associated with orders where the pivot player j is in position k. By changing the pivot player, it is proved that there are at most $n\binom{n-1}{k-1}$ larginal vectors.

2. The first part is clear since $n\binom{n-1}{k-1} = n\binom{n-1}{n-k}$. Moreover, $n\binom{n-1}{k-1} = \frac{n!}{(k-1)!(n-k)!} = \binom{n}{k}k$ and therefore it can be seen that $\binom{n}{k}k \leq \binom{n}{k+1}(k+1)$ whenever $k \leq \frac{n}{2}$. If n is even, $n\binom{n-1}{\frac{n}{2}-1} = n\binom{n-1}{\frac{n}{2}}$ and the result follows. If n is odd, we have just shown the result for $k \leq \frac{n-1}{2}$. Then, we need to compare the case $k = \frac{n-1}{2}$ with the case $k' = \frac{n+1}{2}$. Note that

$$\binom{n}{\frac{n-1}{2}}\frac{n-1}{2} = \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!}\frac{n-1}{2} < \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!}\frac{n+1}{2} = \binom{n}{\frac{n+1}{2}}\frac{n+1}{2}$$

and the result follows. □

From point 2 in Lemma 4.1, it is clear that the maximal complexity of the class of compromise admissible games CA_k^n depends on the parity of n. For easiness of exposition, we define $k_0 = \frac{n+1}{2}$ if n is odd and $k_0 = \frac{n}{2}$ if n is even.

Theorem 4.1. *The computational complexity of the class of admissible games CA^n is $n\binom{n-1}{\frac{n-1}{2}}$ when n is odd and $n\binom{n-1}{\frac{n-2}{2}}$ when n is even.*

Proof. Take an order $\sigma \in \Pi(N)$, and let k be the position of the pivot player (where $k \in \{1, \ldots, n\}$). Clearly, the number of larginal vectors that give rise to the same vector is given by $(n-k)!(k-1)!$. By the proof of Lemma 4.1, this number is minimized whenever the pivot player is in position k_0.

Subsequently, we show that the core cover of a compromise admissible game has the maximal number of vertices when the pivot player is in position k_0 (a middle position) for every order $\sigma \in \Pi(N)$. In other words, we show that the core cover has maximal number of vertices when the game belongs to the class $CA^n_{k_0}$. Let $(N, v) \in CA^n$. For each $k \in \{1, \ldots, n\}$, let p^v_k be the number of different core cover vertices of the game (N, v) that arise from orders $\sigma \in \Pi(N)$ such that the pivot player is in position k. Then, the number of core cover vertices of the game (N, v) is given by $\sum_{k=1}^n p^v_k$. Moreover, since each core cover vertex corresponds to some larginal vector, the number of different orders of players in N can be written as

$$n! = \sum_{k=1}^{n} p^v_k (n-k)!(k-1)! \tag{2}$$

Let $(N, w) \in CA^n_{k_0}$. The number of its core cover vertices is $p^w_{k_0}$ and

$$n! = p^w_{k_0}(n-k_0)!(k_0-1)! \tag{3}$$

Combining (2) and (3), it is obtained that

$$p^w_{k_0} = \sum_{k=1}^{n} p^v_k \frac{(n-k)!(k-1)!}{(n-k_0)!(k_0-1)!} \geq \sum_{k=1}^{n} p^v_k$$

where the inequality follows because $(n-k)!(k-1)!$ is minimized at k_0, as mentioned above. Therefore, it follows that $p^w_{k_0} = n\binom{n-1}{k_0-1}$ gives the computational complexity of CA^n. □

As an illustration, we compare some of these maximal numbers in Table 1.

n	$n!$	k_0	$n\binom{n-1}{k_0-1}$
3	6	2	6
4	24	2	12
5	120	3	30
6	720	3	60
7	5040	4	140
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Core and core cover complexity.

Easy computations show that the ratio between the complexity of the core cover of an $(n+1)$-player game and of an n-player game is 2 when n is odd and $\frac{2(n+1)}{n}$ when n is even. Observe that this ratio is always $n+1$ for the extreme points of the core. Besides, given an n-player game, the ratio between the complexity of the core and the complexity of the core cover is $(k_0-1)!(n-k_0)!$. This shows that the convex structure of the core cover is much simpler than the convex structure of the core.

5 Concluding Remarks

In this paper we analyze the relationship between compromise admissible games and bankruptcy games by establishing an identification, up to a translation, of the core cover of a compromise admissible and the core of a particular bankruptcy game. In fact, given a compromise admissible game, first, we assign each player his minimum right and, then, players are involved in a particular bankruptcy problem. This bankruptcy problem has as demand vector the remainder of the utopia vector and as the estate the remainder of $v(N)$ once the minimum right vector is allocated.

As a consequence of this, on the one hand, the study of the core cover of a compromise admissible game, from a geometric point of view, is equivalent to the analysis of the core of a bankruptcy game. Thus, we study this core by means of its faces. All the face games are strategically equivalent to new bankruptcy games, but with specific interpretations: a coalition of agents has priority over its complementary coalition. Since the face games are new bankruptcy games, the same procedure can be applied several times to define a complete hierarchical structure between coalitions. We show that the core cover of the original game can be recovered with the core covers of the associated hierarchical bankruptcy games. On the other hand, we can define values or allocations of $v(N)$ among the players based on those allocation rules which have been proposed in the context of bankruptcy problems and that satisfy invariance under claims truncation. These allocations always propose core cover elements. Here, we consider the CEA rule and corresponding CEA value. The CEA valeu gives some players their utopia values whenever the value of the grand coalition, $v(N)$, is high enough to ensure all players their minimal right plus an extra amount, which is obtained as the smallest coordinate of the difference between the utopia vector and the minimal rights vector. We show that the CEA value belongs to the face of the core cover given by the coalition of players that get their utopia values. It remains to analyze the natural hierarchical structure described above to propose new rules and to find new axiomatic characterizations.

Finally, we emphasize that the core cover polytope is much simpler than the core polytope. Their convex structures are determined by their extreme points, and therefore, the maximal number of extreme points is a measure of their computational complexity. For a game with a large number of players, the core cover has considerably less extreme points than the core.

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