A BATCH ARRIVAL QUEUE WITH SECOND OPTIONAL SERVICE AND RENEGING DURING VACATION PERIODS

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ABSTRACT
We study a two phase queuing system model where arrivals come to the system in batches of variable size following a compound Poisson process. We consider that service is provided in two phases, the first service is essential and second service is optional. Service becomes unavailable when the server goes for vacation and customers may decide to renege. We treat reneging in this paper when service is unavailable as the server is on vacation. We obtain steady state results in terms of probability generating function. Some special cases are discussed and a numerical illustration is provided.

KEY WORDS: Second optional service, Reneging, Server vacation, Steady state queue size distribution

MSC: 90B22

RESUMEN
Estudiamos un sistema bifásico de colas donde los arribos llegan al sistema en lotes de tamaño variable que siguen un proceso compuesto de Poisson. Consideramos que el servicio esta provisto de dos fases, en el primero el servicio es esencial y el segundo el servicio es opcional. El servicio no está disponible si este sale de vacaciones y los clientes pueden optar por retirarse. Tratamos en este trabajo el retiro cuando el servicio no esta disponible por estar de vacaciones el servidor. Obtenemos un servicio estable en términos de la función generatriz de probabilidad. Algunos casos especiales son discutidos y se brinda una ilustración numérica.

1. INTRODUCTION

In real life, there are queuing situations when some customers are impatient and discouraged by a long wait in the queue. As such, the customers may decide not to join the queue (balking) or leave the queue after joining without receiving any service (reneging). We often witness such situations in real life like calls waiting in call centers, emergency patients in hospitals, programs on computer, banks etc.

Balking and reneging have attracted the attention of many authors and study of queues with behavior of impatient customers has significantly developed and we see an extensive amount of literature in this area. Daley [7] appears to be the first who studied queues with impatient customers. Since then queuing models with balking and reneging has been studied by many authors like Ancker et al [3], Altman and Yechiali[1], Choudhury and Medhi[6] to quote a few. In recent years, studies related to customers' impatience has been mainly concentrated on queuing models with single server. We refer to [2, 4] to readers for reference. Significant contribution by various authors on queues with server vacation has been seen in the last few years. Authors like Levy and Yechiali [11], Doshi[8], Ke [10], Wang et al[15]have studied queues under different vacation policies. Most of the literature on queues deals with one main server. However, in real life there are situations when there is a second server providing service to some customers demanding subsidiary services. Madan [12] was the first to introduce the concept of a second optional service.

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Such models with an optional service have been studied by many other authors mentioning a few are Medhi [13] Wang [11], Jain and Chauhan [9]

In this paper we have considered a batch arrival with two phases of services, one essential and the other as optional. Servers take vacation for a random length of time and customers renege during server vacation. The rest of the paper is structured as follows: The assumptions underlying the mathematical model are given in section2. Section 3 gives the definitions and notations used. In section 4 we give the equations governing the queuing system. In section 5 we derive the queue size distribution at a random epoch. The average queue size and average waiting time are obtained in section 6. Some special cases are discussed in section 7 and in section 8 we provide a numerical example to illustrate the feasibility of our results.

2. MODEL AND ASSUMPTIONS

The model has been defined under the following assumptions:

a) Customers or units arrive in batches following a compound Poisson process. Let $\lambda a_i dt$ be the first order probability of 'i' customers arriving at an instant of time *(t, t+dt]*, $0 \le a_i \le 1$, *i=1, 2,3...* The service to customers is based on a first come first served basis (FCFS); they receive the first essential service and may choose the second optional service (SOS) if needed. The first essential service (FES) is required by all customers. As soon as FES is completed by a customer then he may choose SOS with probability ϕ or leave the system with probability $1-\phi$. The service times of FES and SOS follow the general (arbitrary) distribution with distribution function $S_j(u)$ and density function $s_j(u)$. Let $\mu_j(x)dx,\ j=1,2$ be the conditional probability of service completion of FES and SOS respectively during the interval (x, x+dx] and is given by

$$\mu_j(x) = \frac{s_j(x)}{1-S_j(x)} \quad j=1,2 \text{ and thus } s_j(u) = \mu_j(u)e^{-\int_0^u \mu(x)dx} \qquad (1)$$

b) We assume that customers may renege (leave the system after joining the queue) when the server is on vacation and reneging is assumed to follow exponential distribution with parameter β Thus

$$f(t) = \beta e^{-\beta t}\, dt\ ,\ \beta > 0$$

Let βdt be the probability that a customer can renege during a short interval of time *(t, t+dt]*.

c) After each service the server goes on vacation with probability *p* or remains in the system with probability $1-p$. The vacation time is also assumed to follow general distribution with distribution function $F(v)$ and density function *f(v)*.Let $\gamma(x)dx$ be the conditional probability of a vacation period during the interval (x, x+dx] given that elapsed time is *x* such that

$$\gamma(x) = \frac{f(x)}{1-F(x)} \text{ and thus } f(v) = \gamma(v)e^{-\int_0^v \gamma(v)dx} \qquad (2)$$

3. DEFINITIONS AND NOTATIONS

Let $W_n^{(1)}(x,t)$ is the steady state probability that the server is providing first essential service working since elapsed time *x*, when there is *n* in the queue (*n*≥0) excluding one customer in first service.

Let $W_{n,}^{(2)}(x,t)$ = steady state probability that the server is providing second service since elapsed time x, when there is n in queue (*n ≥ 0)* excluding one customer in service.

Let $V_n(x,t)$ be the probability that there are *n c*ustomers in the queue *(n ≥0)* and the server is on vacation.

Let Q is the probability that the system is empty and the server is idle but available in the system.

We define the following probability generating functions

$$W^{(j)}(x,z)=\sum_{n=0}^{\infty}z^n W_n^{(j)}(x) \quad ; \quad W^{(j)}(z)=\sum_{n=0}^{\infty}z^n W_n^{(j)} \ ; \ j=1,2$$

$$V(x,z)=\sum_{n=0}^{\infty}z^n V_n(x) \quad ; \quad V(z)=\sum_{n=0}^{\infty}z^n V_n$$

$$A(z)=\sum_{i=1}^{\infty}a_i z^i$$

4. EQUATIONS GOVERNING THE SYSTEM

The steady state equations for our model are

$$\frac{d}{dx}W_n^{(1)}(x)+\left(\lambda+\mu_1(x)\right)W_n^{(1)}(x)=\lambda\sum_{i=1}^{n}a_i W_{n-i}^{(1)} \qquad n\geq 0 \tag{3}$$

$$\frac{d}{dx}W_0^{(1)}(x)+\left(\lambda+\mu_1(x)\right)W_0^{(1)}(x)=0 \tag{4}$$

$$\frac{d}{dx}W_n^{(2)}(x)+\left(\lambda+\mu_2(x)\right)W_n^{(2)}(x)=\lambda\sum_{i=1}^{n}a_i W_{n-i}^{(2)} \qquad n\geq 0 \tag{5}$$

$$\frac{d}{dx}W_0^{(2)}(x)+\left(\lambda+\mu_2(x)\right)W_0^{(2)}(x)=0 \tag{6}$$

$$\frac{d}{dx}V_n(x)+\left(\lambda+\gamma(x)+\beta\right)V_n(x)=\lambda\sum_{i=1}^{n}a_{n-i}V_{n-i}(x)+\beta V_{n+1}(x) \tag{7}$$

$$\frac{d}{dx}V_0(x)+\left(\lambda+\gamma(x)\right)V_0(x)=0 \tag{8}$$

$$\lambda Q=\int_0^{\infty}V_0(x)\gamma(x)dx+(1-p)(1-\phi)\int_0^{\infty}W_0^{(1)}(x)\mu_1(x)dx+(1-p)\int_0^{\infty}W_0^{(2)}(x)\mu_2(x)dx \tag{9}$$

The boundary conditions for solving the above differential equations at *x=0* are

$$W_n^{(1)}(0)=(1-p)(1-\phi)\int_0^{\infty}W_{n+1}^{(1)}(x)\mu_1(x)dx+(1-p)\int_0^{\infty}W_{n+1}^{(2)}(x)\mu_2 \tag{10}$$

$$+\int_0^{\infty}V_{n+1}(x)\gamma(x)dx+\lambda a_{n+1}Q \qquad n\geq 0$$

$$W_n^{(2)}(0)=\phi\int_0^{\infty}W_n^{(1)}(x)\mu_1(x)dx \qquad n\geq 0 \tag{11}$$

$$V_n(0) = p(1-\phi)\int_0^\infty W_n^{(1)}(x)\mu_1(x)dx + p\int_0^\infty W_n^{(2)}(x)\mu_2(x)dx \qquad n \geq 0 \tag{12}$$

The Normalizing condition is

$$Q + \sum_{j=1}^{2} \sum_{n=0}^{\infty}\int_0^\infty W_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty V_n(x)dx = 1$$

5. QUEUE SIZE DISTRIBUTION AT RANDOM EPOCH

We multiply equation (3) and (5) by z^n and taking summation over all possible values of n , we obtain

$$\frac{d}{dx}W^{(1)}(x,z) + \left(\lambda - \lambda A(z) + \mu_1(x)\right)W^{(1)}(x,z) = 0 \tag{13}$$

$$\frac{d}{dx}W^{(2)}(x,z) + \left(\lambda - \lambda A(z) + \mu_2(x)\right)W^{(2)}(x,z) = 0 \tag{14}$$

Similarly from (7) and (8)

$$\frac{d}{dx}V(x,z) + \left[\left(\lambda - \lambda A(z)\right) + \gamma(x) + \beta - \frac{\beta}{z}\right]V(x,z) = 0 \tag{15}$$

We now integrate equations (13), (14) and (15) between limits 0 and x and obtain,

$$W^{(1)}(x,z) = W^{(1)}(0,z)\exp\left[-\left(\lambda - \lambda A(z)\right)x - \int_0^x \mu_1(t)dt\right] \tag{16}$$

$$W^{(2)}(x,z) = W^{(2)}(0,z)\exp\left[-\left(\lambda - \lambda A(z)\right)x - \int_0^x \mu_2(t)dt\right] \tag{17}$$

$$V(x,z) = V(0,z)\exp\left[-\left(\lambda - \lambda A(z) + \beta - \frac{\beta}{z}\right)x - \int_0^x \gamma(t)dt\right] \tag{18}$$

Next we multiply equation (10) with appropriate powers of z and summing over suitable values of n, and utilizing (9) we get

$$\begin{aligned} zW^{(1)}(0,z) &= (1-p)(1-\phi)\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx + (1-p)\int_0^\infty W_{n+1}^{(2)}(x)\mu_2(x)dx + \int_0^\infty V(x,z)\gamma(x)dx \\ &\quad - \left[(1-p)(1-\phi)\int_0^\infty W_0^{(1)}(x)\mu_1(x)dx + (1-p)\int_0^\infty W_0^{(2)}(x)\mu_2(x)dx + \int_0^\infty V_0(x)\gamma(x)dx\right] + \lambda A(z)Q \end{aligned}$$

$$\begin{aligned} zW^{(1)}(0,z) &= (1-p)(1-\phi)\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx + (1-p)\int_0^\infty W^{(2)}(x)\mu_2(x)dx \\ &\quad + \int_0^\infty V(x,z)\gamma(x)dx + \left(\lambda A(z) - \lambda\right)Q \end{aligned} \tag{19}$$

Proceeding similarly with equations (11) and (12) we get

$$W^{(2)}(0,z)=\phi\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx \tag{20}$$

$$V(0,z)= p(1-\phi)\int_0^\infty W^{(1)}(x)\mu_1(x)dx+p\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx \tag{21}$$

Again we integrate equations (16), (17) and (18) with respect to *x* by parts and use (1) and (2). Thus it yields

$$W^{(1)}(z)=W^{(1)}(0,z)\left(\frac{1-S_1^*(\lambda-\lambda A(z))}{\lambda-\lambda A(z)}\right) \tag{22}$$

$$W^{(2)}(z)=W^{(2)}(0,z)\left(\frac{1-S_2^*(\lambda-\lambda A(z))}{\lambda-\lambda A(z)}\right) \tag{23}$$

$$V(z)=V(0,z)\left(\frac{1-F^*\left[\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right]}{\lambda-\lambda A(z)+\beta-\frac{\beta}{z}}\right) \tag{24}$$

where $S_j^*(\lambda-\lambda A(z))=\int_0^\infty e^{-(\lambda-\lambda A(z))x}dS_j(x);\ j=1,2$ and

$F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right)=\int_0^\infty e^{-\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right)x}dF(x)$ is the Laplace-Steiltjes transform of service and vacation time respectively.

To determine the integrals $\int_0^\infty W^{(i)}(x,z)\mu_i(x)dx, i=1,2$ and $\int_0^\infty V(x,z)\gamma(x)dx$ we multiply equations (16), (17) and (18) with $\mu_1(x),\mu_2(x)$ and $\gamma(x)$ respectively, integrate by parts with respect to *x* and using (1) and (2) obtain

$$\int_0^\infty W^{(1)}(x,z)\mu_1(x)dx=W^{(1)}(0,z)S_1^*[\lambda-\lambda A(z)] \tag{25}$$

$$\int_0^\infty W^{(2)}(x,z)\mu_2(x)dx=W^{(2)}(0,z)S_2^*[\lambda-\lambda A(z)] \tag{26}$$

$$\int_0^\infty V(x,z)\gamma(x)dx=V(0,z)F^*\left[\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right] \tag{27}$$

and $W^{(1)}(0,z),W^{(2)}(0,z),V(0,z)$ are given in (17), (18) and (19) respectively.

Now from (19) we have

$$zW^{(1)}(0,z) = (1-p)(1-\phi)W^{(1)}(0,z)S_1^*(\lambda-\lambda A(z)) + (1-p)W^{(2)}(0,z)S_2^*(\lambda-\lambda A(z)) + V(0,z)F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right) + (\lambda A(z)-\lambda)Q \tag{28}$$

$$W^{(2)}(0,z) = \phi W^{(1)}(0,z)S_1^*(\lambda-\lambda A(z)) \tag{29}$$

$$V(0,z) = p(1-\phi)W^{(1)}(0,z)S_1^*(\lambda-\lambda A(z)) + pW^{(2)}(0,z)S_2^*(\lambda-\lambda A(z)) \tag{30}$$

From equations (29) and (30) we have

$$V(0,z) = p(1-\phi)W^{(1)}(0,z)S_1^*(\lambda-\lambda A(z)) + p\phi W^{(1)}(0,z)S_1^*(\lambda-\lambda A(z))S_2^*(\lambda-\lambda A(z)) \tag{31}$$

Now using (29) and (31) in (28) we get

$$W^{(1)}(0,z) = \frac{(\lambda A(z)-\lambda)Q}{z-\begin{bmatrix}(1-p)(1-\phi)+(1-p)\phi S_2^*(\lambda-\lambda A(z)) \\ + p(1-\phi)F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right) \\ + p\phi S_2^*(\lambda-\lambda A(z))F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right)\end{bmatrix}S_1^*(\lambda-\lambda A(z))} \tag{32}$$

$$W^{(2)}(0,z) = \frac{\phi(\lambda A(z)-\lambda)S_1^*(\lambda-\lambda A(z))Q}{z-\begin{bmatrix}(1-p)(1-\phi)+(1-p)\phi S_2^*(\lambda-\lambda A(z)) \\ + p(1-\phi)F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right) \\ + p\phi S_2^*(\lambda-\lambda A(z))F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right)\end{bmatrix}S_1^*(\lambda-\lambda A(z))} \tag{33}$$

$$V(0,z) = \frac{(\lambda A(z)-\lambda)\left[p(1-\phi)S_1^*(\lambda-\lambda A(z)) + p\phi S_1^*(\lambda-\lambda A(z))S_2^*(\lambda-\lambda A(z))\right]Q}{z-\begin{bmatrix}(1-p)(1-\phi)+(1-p)\phi S_2^*(\lambda-\lambda A(z)) \\ + p(1-\phi)F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right) \\ + p\phi S_2^*(\lambda-\lambda A(z))F^*\left(\lambda-\lambda A(z)+\beta-\frac{\beta}{z}\right)\end{bmatrix}S_1^*(\lambda-\lambda A(z))} \tag{34}$$

Now using (32),(33) and (34) in(22), (23) and (24) we can obtain $W^{(1)}(z), W^{(2)}(z)$ and $V(z)$ respectively.

Now we use the normalizing condition $P_q(1)+Q$ =1 to determine the unknown probability Q.

Since $W_q(z) = W^{(1)}(z) + W^{(2)}(z) + V(z)$ is indeterminate of the 0/0 form at z =1, we use L'Hopital's rule. Thus

$$W^{(1)}(1) = \frac{\lambda E(I)E(U_1)Q}{1-\lambda E(I)E(U_1)-\phi\lambda E(I)E(U_2)-p(\lambda E(I)-\beta)E(V)} \quad (35)$$

$$W^{(2)}(1) = \frac{\phi\lambda E(I)E(U_2)Q}{1-\lambda E(I)E(U_1)-\phi\lambda E(I)E(U_2)-p(\lambda E(I)-\beta)E(V)} \quad (36)$$

$$V(1) = \frac{p\lambda E(I)E(V)Q}{1-\lambda E(I)E(U_1)-\phi\lambda E(I)E(U_2)-p(\lambda E(I)-\beta)E(V)} \quad (37)$$

Where *E(I)* is the mean size of batch of arriving customers, $S_1^{*/}(0) = -E(U_1), S_2^{*/}(0) = -E(U_2)$ is the mean of service time of FES and SOS time and $F^{*/}(0) = -E(V)$ is the mean of vacation time. Further $S_j^*(0) = 1, j = 1,2 \ \ F^*(0) = 1$.

Thus the unknown probability Q is derived as

$$Q = 1 - \frac{\lambda E(I)[\lambda E(U_1) + \phi E(U_2) + pE(V)]}{1 + p\beta E(V)} \quad (38)$$

Thus

$$\rho = \frac{\lambda E(I)[E(U_1) + \phi E(U_2) + pE(V)]}{1 + p\beta E(V)} < 1 \quad (39)$$

is the stability condition under which steady state exists.

Further using (39) in into equations into (22)-(24) yields

P (server is providing FES at random epoch) = $\lambda E(I)E(U_1)$ (40)

P (server is providing SOS at random epoch) = $\phi\lambda E(I)E(U_2)$ (41)

P (sever is on vacation at random epoch) = $\lambda pE(I)E(V)$ (42)

Let $W_s(z) = W^{(1)}(z) + W^{(2)}(z) + V(z)$ denote the probability generating function of queue size irrespective of the state of the system. Hence adding (22), (23), (24) we obtain

$$W_s(z) = \frac{Q\begin{bmatrix} n[S_1^*(m)-1] + n\phi S_1^*(m)(S_2^*(m)-1) - mp(1-\phi)S_1^*(m)[1-F^*(n)] \\ - mp\phi S_1^*(m)S_2^*(m)[1-F^*(n)] \end{bmatrix}}{n\begin{bmatrix} z-(1-p)(1-\phi)S_1^*(m)-(1-p)\phi S_1^*(m)S_2^*(m) \\ -p(1-\phi)S_1^*(m)F^*(n) - p\phi S_1^*(m)S_2^*(m)F^*(n) \end{bmatrix}} \quad (43)$$

Where we take $\lambda - \lambda A(z) = m, \ \lambda - \lambda A(z) + \beta - \dfrac{\beta}{z}$

Substituting Q from (38) into equation (43), we have completely and explicitly determined $W_s(z)$, the Probability Generating Function of the queue size.

6. THE AVERAGE QUEUE SIZE AND AVERAGE WAITING TIME

Let $L_q = \frac{d}{dz} W_q(z) \Big|_{z=1}$ denote the mean number of customers in the queue under the steady state .

Since the above relation is of *0/0* form at *z =1*, we use L'Hopital's rule twice to obtain L_q .

Let us write $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $W(z)$ are the numerator and denominator of RHS of (43), then

$$L_q = \frac{D^{/}(1)N^{//}(1) - N^{/}(1)D^{//}(1)}{2\left(D^{/}(1)\right)^2} \tag{44}$$

Where primes and double primes in (44) are first and second derivatives respectively at *z=1*

$$N^{/}(1) = Q\lambda E(I)\left[E(U_1) + \phi E(U_2) + pE(V)\right] \tag{45}$$

$$\begin{aligned} N^{//}(1) = {} & Q(\lambda E(I/I-1) + 2\beta)\left[\lambda E(I/I-1)\{E(U_1) + \phi E(U_2)\} + 2\phi(\lambda E(I))^2 E(U_1)E(U_2)\right] \\ & + Qp\lambda E(I/I-1)\begin{bmatrix} \lambda E(I/I-1)\{E(U_1) + \phi E(U_2)\} \\ + 2(\lambda E(I))^2\{E(U_1^2) + \phi E(U_2^2)\} \\ + 2\phi(\lambda E(I))^2 E(U_1)E(U_2) \end{bmatrix} \begin{bmatrix} \{\lambda E(I/I-1) + 2\beta\}E(V) \\ + (\lambda E(I) - \beta)^2 E(V^2) \end{bmatrix} \end{aligned} \tag{46}$$

$$D^{/}(1) = 1 - \lambda E(I)\left[E(U_1) + \phi E(U_2) + pE(V)\right] + p\beta E(V) \tag{47}$$

$$\begin{aligned} D^{//}(1) = {} & -\lambda E(I/I-1)\left[E(U_1) + \phi E(U_2) + pE(V)\right] \\ & -(\lambda E(I))^2\begin{bmatrix} E(U_1^2) + \phi E(U_2^2) \\ + 2\{\phi E(U_1)E(U_2) + pE(U_1)E(V) + \phi pE(U_2)E(V)\} \end{bmatrix} \\ & - 2p\beta\lambda E(I)\left[E(U_1)E(V) + \phi E(U_2)E(V)\right] - 2p\beta E(V) - p(\lambda E(I) - \beta)^2 E(V^2) \end{aligned} \tag{48}$$

where $E(I/I-1)$ is the second moment of batch of arriving customers, $E(U_1^2), E(U_2^2)$ and $E(V^2)$ is the second moment of FES, SOS and vacation time respectively. The value of *Q* has been obtained in (38). Substituting the values of $N^{/}(1), N^{//}(1), D^{/}(1), D^{//}(1)$ and Q from equations (45)-(48) and (38) we obtain L_q in a closed form. The mean waiting time of a customer can be obtained using the relation $W_q = \frac{L_q}{\lambda}$

7. SPECIAL CASES

Case1. No reneging during server vacation

In this situation customers do not renege when the server is on vacation. Then *β=0*. Thus $m = n$.Thus our probability generating function (43) reduces to

$$W_s(z)=\frac{Q\left[\left\{S_1^*(m)-1\right\}+\phi S_1^*(m)\left\{S_2^*(m)-1\right\}+\left\{\begin{array}{l}p(1-\phi)S_1^*(m)\\+p\phi S_1^*(m)S_2^*(m)\end{array}\right\}\left\{F^*(m)-1\right\}\right]}{\begin{array}{l}z-(1-p)(1-\phi)S_1^*(m)-(1-p)\phi S_1^*(m)S_2^*(m)\\-p(1-\phi)S_1^*(m)F^*(m)-p\phi S_1^*(m)S_2^*(m)F^*(m)\end{array}} \tag{49}$$

Where $m=\lambda-\lambda A(z)$

$$Q=1-\lambda E(I)\left[E(U_1)+\phi E(U_2)+pE(V)\right]$$

$$N^{/}(1)=Q\lambda E(I)\left[E(U_1)+\phi E(U_2)+pE(V)\right]$$

$$N^{//}(1)=Q\left[\begin{array}{l}\lambda E(I/I-1)\left\{E(U_1)+\phi E(U_2)\right\}\\+\left(\lambda E(I)\right)^2\left\{E(U_1^2)+\phi E(U_2^2)\right\}\\+2\phi\left(\lambda E(I)\right)^2E(U_1)E(U_2)\end{array}\right]\left[1+p\lambda E(I/I-1)\left\{\lambda E(I/I-1)E(V)-\left(\lambda E(I)\right)^2E(V^2)\right\}\right]$$

$$D^{/}(1)=1-\lambda E(I)\left[E(U_1)+E(U_2)+pE(V)\right]$$

$$D^{//}(1)=-\lambda E(I/I-1)\left\{E(U_1)+\phi(1-p)E(U_2)+pE(V)\right\}$$
$$-\left(\lambda E(I)\right)^2\left[\begin{array}{l}\left\{E(U_1^2)+\phi E(U_2^2)+pE(V^2)\right\}\\+2\left\{\phi E(U_1)E(U_2)+\phi E(U_2)E(V)+p\phi E(U_2)E(V)\right\}\end{array}\right]$$

Thus (49) is the queue size of a Batch Arrival Vacation Queue with second optional service.
Case 2. No second optional service. In this case, there is only one sever providing service, such thatϕ=*0*. Then the probability generating function in (43) reduces to

$$W_s(z)=\frac{Q\left[n\left\{S_1^*(m)-1\right\}-mpS_1^*(m)\left\{1-F^*(n)\right\}\right]}{n\left[z-(1-p)S_1^*(m)-pS_1^*(m)F^*(n)\right]} \tag{49a}$$

Where $m=\lambda-\lambda A(z),\ n=\lambda-\lambda A(z)+\beta-\dfrac{\beta}{z}$

$$Q=1-\frac{\lambda E(I)\left[E(U_1)+pE(V)\right]}{1+p\beta E(V)}$$

$$N^{/}(1)=Q\left[\lambda E(I)E(U_1)+p\lambda E(I)E(V)\right]$$

$$N^{//}(1)=Q\left[\begin{array}{l}\{\lambda E(I/I-1)+2\beta\}\{\lambda E(I/I-1)\}E(U_1)\\+p\left(\lambda E(I/I-1)\right)\left\{\begin{array}{l}\lambda E(I/I-1)E(U_1)\\+2\left(\lambda E(I)\right)^2E(U_1^2)\end{array}\right\}\left\{\begin{array}{l}\left(\lambda E(I/I-1)+2\beta\right)E(V)\\-\left(\lambda E(I)-\beta\right)^2E(V^2)\end{array}\right\}\end{array}\right]$$

$$D^{/}(1)=1-\lambda E(I)E(U_1)+p\big(-\lambda E(I)+\beta\big)E(V)$$

$$D^{//}(1)=-\lambda E(I/I-1)\{E(U_1)+pE(V)\}-\big(\lambda E(I)\big)^2\{E(U_1^2)+2pE(U_1)E(V)\}$$
$$-2p\beta\lambda E(I)\{E(U_1)E(V)\}-\big(\lambda E(I)-\beta\big)^2 pE(V^2)$$

Equation (49) gives the queue size distribution of a Batch Arrival with Reneging during vacation period.

Case 3. No server Vacation. If the server does not go for a vacation, in that case p = 0

Thus our P.G.F in (43) becomes

$$W_s(z)=\frac{\big[S_1^*\big(\lambda-\lambda A(z)\big)-1\big]+\phi\big[S_2^*\big(\lambda-\lambda A(z)\big)-1\big]}{z-(1-\phi)S_1^*\big(\lambda-\lambda A(z)\big)-\phi S_2^*\big(\lambda-\lambda A(z)\big)} \tag{50}$$

$$Q=1-\lambda E(I)\big[E(U_1)+\phi E(U_2)\big]$$

$$N^{/}(1)=Q\lambda E(I)\big[E(U_1)+\phi E(U_2)\big]$$

$$N^{//}(1)=\big(\lambda E(I/I-1)+2\beta\big)\begin{bmatrix}\big(\lambda E(I/I-1)\{E(U_1)+\phi E(U_2)\}\\ +2\phi\big(\lambda E(I)\big)^2E(U_1)E(U_2)\end{bmatrix}$$

$$D^{/}(1)=1-\lambda E(I)\big[E(U_1)+\phi E(U_2)\big]$$

$$D^{//}(1)=-\lambda E(I/I-1)\{E(U_1)+\phi E(U_2)\}-\big(\lambda E(I)\big)^2\big[E(U_1^2)+\phi E(U_2^2)+2\phi E(U_1)E(U_2)\big]$$

Equation (50) is the queue size of a Batch Arrival Queue with second optional service.

Case 4. No Reneging and no second optional service. Then *β=0, ϕ=0*.Thus from (43) we have

$$W_s(z)=\frac{Q\big[(1-p)S_1^*\big(\lambda-\lambda A(z)\big)-1+pS_1^*\big(\lambda-\lambda A(z)\big)F^*\big(\lambda-\lambda A(z)\big)\big]}{z-(1-p)S_1^*\big(\lambda-\lambda A(z)\big)-pS_1^*\big(\lambda-\lambda A(z)\big)F^*\big(\lambda-\lambda A(z)\big)} \tag{51}$$

$$Q=1-\lambda E(I)\big[E(U_1)+pE(V)\big]$$

$$N^{/}(1)=Q\lambda E(I)\big[E(U_1)+pE(V)\big]$$

$$N^{//}(1)=Q\big(\lambda E(I/I-1)\big]\big[\big(\lambda E(I/I-1)\big)E(U_1)\big]$$
$$+Qp\big(\lambda E(I/I-1)\big]\big[E(U_1)+2\big(\lambda E(I)\big)^2E(U_1^2)\big]\big[\big(\lambda E(I/I-1)\big)E(V)-\big(\lambda E(I)\big)^2E(V^2)\big]$$

$$D^{/}(1)=1-\lambda E(I)\big[E(U_1)+pE(V)\big]$$

$$D^{//}(1)=-\lambda E(I/I-1)\big[E(U_1)+pE(V)\big]-\big(\lambda E(I)\big)^2\big[E(U_1^2)+pE(V^2)+2E(U_1)E(V)\big]$$

The result obtained in (51) is the queue size for a Batch Arrival Vacation queue.

Case 5. No reneging, no second optional service and no sever vacation, then *β=0,ϕ=0, p=0*

Equation (43) reduces to

$$W_s(z) = \frac{S_1^*(\lambda - \lambda A(z)) - 1}{z - S_1^*(\lambda - \lambda A(z))} \tag{52}$$

$$Q = 1 - \lambda E(I)E(U_1)$$

$$N^{/}(1) = Q\lambda E(I)E(U_1)$$

$$N^{//}(1) = Q\left[\lambda E(I/I-1)E(U_1) + (\lambda E(I))^2 E(U_1^2)\right]$$

$$D^{/}(1) = 1 - \lambda E(I)E(U_1)$$

$$D^{//}(1) = -\left[\lambda E(I/I-1)E(U_1) + (\lambda E(I))^2 E(U_1^2)\right]$$

The result (52) tallies with the steady state queue size of a $M^X / G / 1$ queue.

Case 6. Exponential service time and vacation time.

The exponential distribution is the most common form of distribution for the service time and vacation time. For this distribution, the rate of service for first essential service is $\mu_1 > 0$ and rate of service for second optional service is $\mu_2 > 0$. The rate of vacation completion be $\eta > 0$. Then we have

$$S_1^*(\lambda - \lambda A(z)) = \frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}$$

$$S_2^*(\lambda - \lambda A(z)) = \frac{\mu_2}{\lambda - \lambda A(z) + \mu_2} \quad F^*\left(\lambda - \lambda A(z) + \beta - \frac{\beta}{z}\right) = \frac{\eta}{\lambda - \lambda A(z) + \beta - \frac{\beta}{z} + \eta}$$

Substituting the above relations in the expression for $W_s(z)$ in the main result (43), we get

$$W_s(z) = \frac{Q\left[\begin{array}{l} \left(\lambda - \lambda A(z) + \beta - \frac{\beta}{z}\right)\left\{\frac{\lambda A(z) - \lambda}{\lambda - \lambda A(z) + \mu_1} + \phi\left(\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\right)\frac{\lambda A(z) - \lambda}{\lambda A(z) - \lambda + \mu_2}\right\} \\ + (\lambda A(z) - \lambda)\left\{\begin{array}{l} p(1-\phi)\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\frac{(\lambda - \lambda A(z) + \beta - \frac{\beta}{z})}{\lambda - \lambda A(z) + \beta - \frac{\beta}{z} + \eta} \\ + p\phi\left(\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\right)\left(\frac{\mu_2}{\lambda - \lambda A(z) + \mu_2}\right)\left(\frac{\lambda - \lambda A(z) + \beta - \frac{\beta}{z}}{\lambda - \lambda A(z) + \beta - \frac{\beta}{z} + \eta}\right) \end{array}\right\} \end{array}\right]}{\left(\lambda - \lambda A(z) + \beta - \frac{\beta}{z}\right)\left[\begin{array}{l} z - (1-p)(1-\phi)\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1} - (1-p)\phi\left(\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\right)\left(\frac{\mu_2}{\lambda - \lambda A(z) + \mu_2}\right) \\ - p(1-\phi)\left(\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\right)\left(\frac{\mu_2}{\lambda - \lambda A(z) + \mu_2}\right) \\ - p\phi\left(\frac{\mu_1}{\lambda - \lambda A(z) + \mu_1}\right)\left(\frac{\mu_2}{\lambda - \lambda A(z) + \mu_2}\right)\left(\frac{\eta}{\lambda - \lambda A(z) + \beta - \frac{\beta}{z} + \eta}\right) \end{array}\right]}$$

$$W_s(z) = \frac{Q\left(\lambda A(z)-\lambda\right)\left[\frac{1}{k_1(z)} + \phi\frac{\mu_1}{k_1(z)k_2(z)} + p(1-\phi)\frac{\mu_1}{k_1(z)k_3(z)} + p\phi\frac{\mu_1\mu_2}{k_1(z)k_2(z)k_3(z)}\right]}{\begin{array}{l} z-(1-p)(1-\phi)\frac{\mu_1}{k_1(z)} - (1-p)\phi\frac{\mu_1\mu_2}{k_1(z)k_2(z)} \\ -p(1-\phi)\frac{\mu_1\eta}{k_1(z)k_3(z)} - p\phi\frac{\mu_1\mu_2\eta}{k_1(z)k_2(z)k_3(z)} \end{array}} \qquad (53)$$

Where $k_1(z) = \lambda - \lambda A(z) + \mu_1$, $k_2(z) = \lambda - \lambda A(z) + \mu_2$ and $k_3(z) = \lambda - \lambda A(z) + \beta - \frac{\beta}{z} + \eta$

$$Q = 1 - \frac{\lambda E(I)\left[\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right]}{1 + p\beta\frac{1}{\eta}}$$

$$\rho = \frac{\lambda E(I)\left[\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right]}{1 + p\beta\frac{1}{\eta}}$$

$$N^{/}(1) = Q\lambda E(I)\left[\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right]$$

$$\begin{aligned} N^{//}(1) = {} & Q\lambda E(I/I-1)\left\{\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right\} \\ & + Q2\left(\lambda E(I)\right)^2\left\{\frac{1}{\mu_1^2} + \frac{\phi}{\mu_2^2} + \frac{p}{\eta^2} + \frac{p}{\mu_1\eta} + \frac{\phi}{\mu_1\mu_2} + \frac{p\phi}{\mu_2\eta}\right\} \\ & - 2\lambda E(I)\beta\frac{p}{\eta^2}Q \end{aligned}$$

$$D^{/}(1) = 1 - \lambda E(I)\left[\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right] + p\beta\frac{1}{\eta}$$

$$\begin{aligned} D^{//}(1) = {} & -\lambda E(I/I-1)\left\{\frac{1}{\mu_1} + \frac{\phi}{\mu_2} + \frac{p}{\eta}\right\} - 2\left(\lambda E(I)\right)^2\left\{\frac{1}{\mu_1^2} + \frac{\phi}{\mu_2^2} + \frac{\phi}{\mu_1\mu_2} + \frac{p}{\mu_1\eta} + \frac{p\phi}{\mu_2\eta}\right\} \\ & + 2\lambda E(I)p\beta\left\{\frac{1}{\mu_1\eta} + \frac{\phi}{\mu_2\eta}\right\} - \frac{2\beta p}{\eta} \\ & - 2\left(\lambda E(I) - \beta\right)^2\left\{\frac{p}{\eta^2}\right\} \end{aligned}$$

The result (53) gives the PGF of a Batch Arrival with exponential second optional service and Reneging during Vacation periods.

8. A NUMERICAL ILLUSTRATION

We consider the special case of exponential service time and exponential vacation time as a numerical illustration for the validity of our results. All the values are arbitrarily chosen such that conditions of stability are satisfied. In this example we show the effect of the reneging parameter (β) on the server's idle time, utilization factor, mean queue size and mean waiting time.

Table (1): Computed values of some queue performance measures

p	ϕ	Q	ρ	L_q	L	W_q	W
$\beta = 7$							
0.50	0.25	0. 2206	0.7794	4.5055	5.2849	2.2528	2.6425
0.50	0.50	0.1471	0.8529	7.4528	8.3057	3.7264	4.1529
0.50	0.75	0.0735	0.9265	16.304	17.2305	8.152	8.615
0.60	0.25	0.2582	0.7418	3.7117	4.4535	1.85585	2.22675
0.60	0.50	0.1902	0.8098	5.5572	6.448	2.7786	3.224
0.60	0.75	0.8777	0.1223	67.4195	67.5418	33.70975	33.7709
0.75	0.25	0.3049	0.6951	2.9818	3.6769	1.4909	1.83845
0.75	0.50	0.2439	0.7561	4.016	4.7721	2.008	2.38605
0.75	0.75	0.1829	0.8171	5.8963	6.7134	2.94815	3.3567

We assume $\lambda = 2, \mu_1 = 2, \mu_2 = 4, \eta = 5, \phi = 0.25, E(I) = 1$ and $E(I/I-1) = 0$

We fixed the values of λ, μ_1, μ_2, and η, while β is assumed different varying values 7, 10 and 12. The above three tables shows the computed values of the proportion of idle time, utilization factor, the mean queue size and the mean waiting time. It clearly shows that as we increase the values of ϕ or p, the server idle time decreases while the utilization factor, mean queue size and mean waiting time increases for different values of the reneging parameter β .

Table (2): Computed Values of some Queue performance values

p	ϕ	Q	ρ	L_q	L	W_q	W
$\beta = 10$							
0.50	0.25	0.3375	0.6625	2.9444	3.6069	1.4722	1.8034
0.50	0.50	0.275	0.725	4.1182	4.8432	2.0591	2.4216
0.50	0.75	0.2125	0.7875	5.8068	6.5943	2.9034	3.2972
0.60	0.25	0.6205	0.3795	4.252	4.6315	2.126	2.31575
0.60	0.50	0.6773	0.3227	6.2857	6.6084	3.14285	3.3042
0.60	0.75	0.7341	0.2659	12.2121	12.478	6.10605	6.239
0.75	0.25	0.4300	0.5700	2.1451	2.7151	1.07255	1.35755
0.75	0.50	0.3800	0.6200	2.3348	2.9548	1.1674	1.4774
0.75	0.75	0.3300	0.6700	3.3046	3.9746	1.6523	1.9873

Table(3): Computed Values of some queue performance values

p	ϕ	Q	ρ	L_q	L	W_q	W
$\beta = 12$							
0.50	0.25	0.3977	0.6023	2.7057	3.308	1.3529	1.654
0.50	0.50	0.3409	0.6591	3.4415	4.1006	1.7208	2.0503
0.50	0.75	0.2841	0.7159	4.5028	5.2187	2.2514	2.6094
0.60	0.25	0.5594	0.4406	2.9386	3.3792	1.4693	1.6896
0.60	0.50	0.6107	0.3893	4.3323	4.7216	2.16615	2.3608
0.60	0.75	0.6619	0.3381	7.0271	7.3652	3.51355	3.6826
0.75	0.25	0.4911	0.5089	1.9219	2.4308	0.96095	1.2154
0.75	0.50	0.4464	0.5536	2.2849	2.8385	1.14245	1.41925
0.75	0.75	0.4018	0.5982	2.7778	3.376	1.389	1.688

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