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A Batch Arrival Retrial Queue with Two Phases of Service, Feedback and K Optional Vacations

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Abstract

We consider a batch arrival queueing system with two phases of service, feedback and K optional vacations under a classical retrial policy. At the arrival epoch, if the server is busy the whole batch joins the orbit. Whereas if the server is free, then one of the arriving customer starts its service immediately and the rest joins the orbit. For each customer, the server provides two phases of service. After the completion of two phases of service, the customer may rejoin the orbit as a feedback customer for receiving another regular service with probability p. If the system is empty, then the server become inactive and begins the first essential vacation. After the completion of first essential vacation, the server may either wait idle for a customer or may take one of K additional vacations. The steady state distribution of the server state and the number of customers in the orbit are obtained. Also the effects of various parameters on the system performance are analyzed numerically.

Mathematics Subject Classification: 60K25, 90B22, 68M20

Keywords: Retrial queue, two phases of heterogeneous service, Bernoulli feedback, optional vacations

1 Introduction

Queueing systems are powerful tool for modeling communication networks, transportation networks, production lines, operating systems, etc. In recent years, computer networks and data communication systems are the fastest growing technologies, which lead to glorious development in many applications.

For example, the swift advance in Internet, audio data traffic, video data traffic, etc.

Retrial queueing system are characterized by the fact that arriving customer who finds the server busy is to leave the service area and repeat his demand after some time called retrial time. Between trials, the blocked customer joins a pool of unsatisfied customers called orbit. For example, web access, telecommunication networks, computer systems, packet switching networks, collision avoidance star local area networks, etc.

Retrial queueing system operate under the classical retrial policy, where each block of jobs generate a stream of repeated attempts independently of the rest of the jobs in the orbit. For example, in call center, the customers may call again and again independently if their request are not completely fulfilled.

The feedback phenomenon are another important tool for communication systems. When the service of a customer is unsatisfied, the service can be retried again and again until the service is completed successfully. For example, in multiple access telecommunication systems, where messages turned out as errors are sent again can be modeled as retrial queues with feedback.

The server works continuously as long as there is at least one customer in the system. When the server finishes serving a customer and finds the system empty, it goes away for a length of time called a vacation. For example, maintenance activities, telecommunication networks, customized manufacturing, production systems, etc.

When no customers are found in the orbit, the server goes on a first essential vacation. After first essential vacation, the server may either wait idle for customers or the server may take one of Type k $(k = 1, 2, \dots K)$ vacations. At an optional vacation completion epoch, the server waits for the customers, if any in the orbit or for new customers to arrive.

Artalejo [1], Kulkarni [12] and Templeton [15] have given explicit survey on retrial queueing systems. Artalejo and Lopez Herrero [3] have investigated an information theoretic approach for the estimation of the main performance characteristics of the $M/G/1$ retrial queue. Gomez-Corral [7] widely discussed about a single server retrial queueing system with general retrial times. Artalejo and Gomez-Corral [2] have developed an $M/G/1$ retrial queue with finite capacity of the retrial group. Takacs [13] studied a single server queueing system with bernoulli feedback.

Krishnakumar and Arivudainambi [10] have analyzed a single server retrial queue with bernoulli vacation schedules and general retrial times. Krishnakumar et. al. [11] have introduced an $M/G/1$ retrial queueing systems with two phase service and preemptive resume. J. C. Ke et. al. [9] have analyzed the characteristics of an $M^{[X]}/G/1$ queueing system with N policy and almost J vacations.

Though lot of work has been done in retrial queueing systems, no one is done a batch arrival retrial queue with general retrial time, two phases of service, feedback and K optional vacations. To fill this gap, we have given a mathematical description in section 3, the practical justification for the model is discussed in section 2. Section 4 deals with the derivations of the steady state distribution of the server to be state. The mean number of jobs in the system and several performance measures are discussed in section 5. In section 6 some important special cases of this model are discussed briefly. Numerical results related to the effect of various parameters on the system performance measures are analyzed in section 7.

2 Practical Justification of our Model

2.1 Packet Switched Network

Two or more networks can be attached using an interconnection device called router and it is used to forward the packets within a network. Batch IP packets arrive at the router according to a Poisson process. When packets arrive at the router, one of the packet is selected for service and other packets will be kept in the buffer. In the buffer, each packet waits for some time and requires the service again. After all the packets are forwarded, the router will be in idle state and wait for the new packets to arrive. In the queueing terminology, router, buffer in the router, retransmission policy, maintenance activities in server idle time are considered to be the server, the orbit, the retrial discipline, essential vacation and optional vacation respectively.

2.2 Proxy of WWW server

The proxy of WWW server, HTTP requests arrive at the proxy according to a Poisson process. When the requests arrive at the proxy, one of the requests is selected for service and other requests will join in the buffer. In the buffer, each packet waits for some time and requires the service again. In web contents, proxy may do synchronize actions with WWW server when proxy is idle. Some maintenance activities are performed randomly when the buffer is empty (i.e) vacation. When the performed activity is finished, the proxy will enter the idle state again and wait for the new requests to arrive. In the queueing terminology, proxy, buffer in the proxy, retransmission policy, synchronize actions and maintenance activities in server idle time are corresponds to the server, the orbit, the retrial discipline, bernoulli vacation respectively.

2.3 Production Line

The raw materials arrive in batches of random size instead of single unit. The machine producing an item may require two phases of service such as prelimary check followed by usual processing of raw materials. After the completion of two phases of service the process either stopped for overhauling and maintenance of the system or may continue the further processing of the raw materials if no fault in the system.

3 Model Description

Customers arrive in batches according to a Poisson process with rate λ. Let X_j, $j = 1, 2 \cdots$ denote the number of customers with a probability distribution $P[X_j = n] = \chi_n$, $n = 1, 2, \cdots$ and probability generating function $X(z)$. One of the arriving customers begins his service immediately if the server is available and remaining customers leave the service area and join the orbit.

The customer access from the orbit to the server in governed by an arbitrary law with distribution function $R(t)$ and Laplace-Stieltjes Transform (LST) $R^*(\theta)$. In succession, a single server provides two phases of service to each customer. The first phase of service (FPS) is followed by the second phase of service (SPS). It is assumed that the service $S_i (i = 1, 2)$ of the i^{th} phases of service follows a random variable with distribution function $S_i(t)$ and Laplace-Stieltjes Transform $S_i^*(\theta)$. After completion of FPS and SPS, the service of a customer is unsatisfied, the customer may rejoin the orbit as a feedback customer with probability $p\,(0 \leq p \leq 1)$ or may leave the system with probability q.

When no customers are found in the system, the server deactivates and may decide to go for a first essential vacation of random variable V_0 with distribution function $V_0(t)$ and Laplace-Stieltjes Transform $V_0^*(\theta)$. After completion of first essential vacation, the server may either wait idle for customers with probability p_0 or may take one of Type $k\,(k = 1, 2, \ldots, K)$ vacations with probability p_k. The Type k vacation times are assumed to follow an arbitrary distributed random variable V_k with distribution function $V_k(t)$ and Laplace-Stieltjes Transform $V_k^*(\theta)$, where $k = 1, 2, \ldots, K$ and $\sum_{k=0}^{K} p_k = 1$. Note that when $p_0 = 1$, the server does not take any one of these optional vacations upon returning from the essential vacation and this case $K = 0$.

Let the random variable $Y(t)$ denotes the server state $0, 1, 2, 3, 4, \ldots k + 3, \ldots, K + 3$. If $Y(t) = 0$ the server being free at time t. If $Y(t) = 1$ and 2, the server is busy with FPS and SPS respectively at time t. If $Y(t) = 3, 4, k + 3$ and $K + 3$ the server is in essential vacation, type 1 vacation, type k vacation and type K vacation respectively at time t.

In addition, let $R^0(t)$, $S_i^0(t)$ and $V_k^0(t)$ be the elapsed retrial time, service time and vacation time respectively at time t. In order to obtain a bivariate Markov process $\{C(t),\ \xi(t)\}$, we introduce the supplementary variables $R^0(t)$, $S_i^0(t)$ and $V_k^0(t)$, where $C(t)$ is the number of the customers in the orbit and $\xi(t) = R^0(t)$ if $Y(t) = 0$ and $C(t) > 0$, $\xi(t) = S_i^0(t)$ if $Y(t) = i$ and $C(t) \geq 0$ where $i = 1, 2$, $\xi(t) = V_k^0(t)$ if $Y(t) = k + 3$ and $C(t) \geq 0$, $k = 0, 1, 2, \ldots, K$.

The functions $\theta(x)dx$, $\mu_i(x)dx$ and $\nu_k(x)dx$ are the conditional completion rates for repeated attempts, service and vacation respectively at time x. i.e., $\theta(x)dx = dR(x)/(1 - R(x))$, $\mu_i(x)dx = dS_i(x)/(1 - S_i(x))$, $\nu_k(x)dx = dV_k(x)/(1 - V_k(x))$.

Theorem 3.1 *Let $\{t_n; n = 1, 2, \cdots\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n = \{C(t_n+), \xi(t_n+)\}$ from a Markov chain which is embedded in the retrial queueing system. By similar arguments of Gomez-Corral [7], we show that the embedded Markov chain $\{Z_n; n = 1, 2, \cdots\}$ is ergodic if and only if $\rho/q < 1$, where $\rho = E(X)[1 - R^*(\lambda) + \lambda(E(S_1) + E(S_2))]$.*

4 Steady State Distribution of the Server State

For the process $\{X(t), t \geq 0\}$, the probability can be define as $P_0(t) = P\{C(t) = 0,\ Y(t) = 0\}$ and the probability densities $P_n(x, t)dx = P\{C(t) = n,\ \xi(t) = R^0(t);\ x < R^0(t) \leq x + dx\}$, $Q_{i,n}(x, t)dx = P\{C(t) = n,\ \xi(t) = S_i^0(t);\ x < S_i^0(t) \leq x + dx\}$ for $t \geq 0$, $x \geq 0$ and $n \geq 0$ where $i = 1, 2$ and $G_{k,n}(x, t)dx = P\{C(t) = n,\ \xi(t) = V_k^0(t);\ x < V_k^0(t) \leq x + dx\}$ for $x \geq 0$, $n \geq 1$ and $0 \leq k \leq K$.

We assume that the steady state condition $\rho/q < 1$ is fulfilled, so that we can set $P_0 = \lim_{t\to\infty} P_0(t), P_n(x) = \lim_{t\to\infty} P_n(t, x)$ for $x \geq 0$ and $n \geq 1$, $Q_n(x) = \lim_{t\to\infty} Q_n(t, x)$ for $x \geq 0$ and $n \geq 1$ and $G_n(x) = \lim_{t\to\infty} G_n(t, x)$ for $x \geq 0$ and $n \geq 1$. By the method of supplementary variables, we obtain

$$\lambda P_0 = p_0 \int_0^\infty G_{0,0}(x)\,\nu_0(x)dx + \int_0^\infty G_{1,0}(x)\,\nu_1(x)dx + \ldots + \int_0^\infty G_{K,0}(x)\,\nu_K(x)dx \quad (1)$$

$$\frac{d}{dx} P_n(x) + [\lambda + \theta(x)]\,P_n(x) = 0, x > 0, n \geq 1 \quad (2)$$

$$\frac{d}{dx} Q_{i,0}(x) + [\lambda + \mu_i(x)]\,Q_{i,0}(x) = 0, x > 0, i = 1, 2 \quad (3)$$

$$\frac{d}{dx} Q_{i,n}(x) + [\lambda + \mu_i(x)]\,Q_{i,n}(x) = \lambda \sum_{j=1}^{n} \chi_j Q_{i,n-j}(x), n \geq 1, i = 1, 2 \quad (4)$$

$$\frac{d}{dx}G_{k,0}(x)+[\lambda+\nu_k(x)]\,G_{k,0}(x) = 0, x>0, 0\le k\le K \quad (5)$$

$$\frac{d}{dx}G_{k,n}(x)+[\lambda+\nu_k(x)]\,G_{k,0}(x) = \lambda\sum_{j=1}^{n}\chi_j G_{k,n-j}(x), n\ge 1, 0\le k\le K \quad (6)$$

The above set of equations are to be solved using the steady state boundary conditions at $x=0$,

$$\begin{aligned} P_n(0) &= p_0\int_0^\infty G_{0,n}(x)\,\nu_0(x)dx+\int_0^\infty G_{1,n}(x)\,\nu_1(x)dx+\dots \\ &\quad +\int_0^\infty G_{K,n}(x)\,\nu_K(x)dx+q\int_0^\infty Q_{2,n}(x)\,\mu_2(x)dx \\ &\quad +p\int_0^\infty Q_{2,n-1}(x)\,\mu_2(x)dx, n\ge 1 \end{aligned} \quad (7)$$

$$Q_{1,0}(0) = \int_0^\infty P_1(x)\,\theta(x)dx+\lambda\chi_1 P_0 \quad (8)$$

$$Q_{1,n}(0) = \int_0^\infty P_{n+1}(x)\,\theta(x)dx+\lambda\int_0^\infty\sum_{j=1}^{n}\chi_j P_{n-j+1}(x)dx+\lambda\chi_{n+1}P_0 \quad (9)$$

$$Q_{2,n}(0) = \int_0^\infty Q_{1,n}(x)\,\mu_1(x)dx, n\ge 1 \quad (10)$$

$$\lambda P_0 = \int_0^\infty G_{0,0}(x)\nu_0(x)dx \quad (11)$$

$$G_{0,0}(0) = \begin{cases} q\int_0^\infty Q_{2,0}(x)\mu_2(x)dx, & n=0 \\ 0, & n\ge 1 \end{cases} \quad (12)$$

$$G_{k,n}(0) = p_k\int_0^\infty G_{0,n}(x)\,\nu_0(x)dx, n\ge 0, 0\le k\le K \quad (13)$$

The normalization condition is given by

$$P_0+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\sum_{i=1}^{2}\int_0^\infty Q_{i,n}(x)dx+\sum_{k=0}^{K}\sum_{n=0}^{\infty}\int_0^\infty G_{k,n}(x)dx=1 \quad (14)$$

Let us define the probability generating functions as $P(x,z)=\sum_{n=1}^{\infty}z^nP_n(x)$ for $|z|\le 1$ and $x>0$, $P(0,z)=\sum_{n=1}^{\infty}z^nP_n(0)$ for $|z|\le 1$, $Q_i(x,z)=\sum_{n=0}^{\infty}z^nQ_{i,n}(x)$ for $|z|\le 1$ and $x>0$, $Q_i(0,z)=\sum_{n=0}^{\infty}z^nQ_{i,n}(0)$ for $|z|\le 1$ and $G_k(x,z)=\sum_{n=0}^{\infty}z^nG_{k,n}(x)$ for $|z|\le 1$ and $x>0$.

Theorem 4.1 *Under the stability condition $\rho/q<1$, the stationary distributions of the number of customers in the system when the server is free, busy in FPS, busy in SPS and on vacations are given by*

$$P(z) = P_0\Big\{\frac{z[1-R^*(\lambda)][1-\frac{[(p_0+\sum_{k=1}^{K}p_kV_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))-1]}{V_0^*(\lambda)}]}{\gamma}$$

$$- \frac{[1-R^*(\lambda)][(pz+q)X(z)S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))]}{\gamma}\Big\} \quad (15)$$

$$Q_1(z) = P_0\Big\{\Big[\frac{z[1-\frac{[(p_0+\sum_{k=1}^{K}p_kV_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))-1]}{V_0^*(\lambda)}]}{\gamma} - \frac{(pz+q)X(z)S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))}{\gamma}\Big]\omega + \frac{X(z)}{z}\Big\} \times\Big\{\frac{1-S_1^*(\lambda-\lambda X(z))}{1-X(z)}\Big\} \quad (16)$$

$$Q_2(z) = P_0\Big\{\frac{[1-\frac{z[(p_0+\sum_{k=1}^{K}p_kV_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))-1]}{V_0^*(\lambda)}]\omega z - X(z)}{\gamma}\Big\} \times\Big\{\frac{S_1^*(\lambda-\lambda X(z))[1-S_2^*(\lambda-\lambda X(z))]}{1-X(z)}\Big\} \quad (17)$$

$$G_0(z) = \frac{p_0[1-V_0^*(\lambda-\lambda X(z))]}{V^*(\lambda)[1-X(z)]} \quad (18)$$

$$G_k(z) = \frac{p_0p_k\{V_0^*(\lambda-\lambda X(z))[1-V_k^*(\lambda-\lambda X(z))]\}}{V^*(\lambda)[1-X(z)]},\ k=1,2\cdots K \quad (19)$$

$$P_0 = \frac{q-\rho}{q[\frac{\lambda E(V_0)+\lambda\sum_{k=1}^{K}p_kE(V_k)}{V_0^*(\lambda)}+R^*(\lambda)]} \quad (20)$$

$$\text{where } \rho = E(X)[1-R^*(\lambda)+\lambda(E(S_1)+E(S_2))]$$

$$\omega = \frac{[R^*(\lambda)+X(z)(1-R^*(\lambda))]}{z}$$

$$\gamma = [(pz+q)(R^*(\lambda)+X(z)(1-R^*(\lambda)))S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))-z]$$

Proof

Multiplying equations (2) - (6) by suitable powers of z, summing over n and using generating functions, we obtain the partial differential equations

$$\frac{\partial P(x,z)}{\partial x}+[\lambda+\theta(x)]P(x,z) = 0,\ x>0 \quad (21)$$

$$\frac{\partial Q_i(x,z)}{\partial x}+[\lambda-\lambda X(z)+\mu_i(x)]Q_i(x,z) = 0,\ x>0,\ i=1,2 \quad (22)$$

$$\frac{\partial G_k(x,z)}{\partial x}+[\lambda-\lambda X(z)+\nu_k(x)]G_k(x,z) = 0,\ x>0, k=0,1,\cdots K \quad (23)$$

Solving the above partial differential equations (21) - (23), we get

$$P(x,z) = P(0,z)[1-R(x)]e^{-\lambda x},\ x>0 \quad (24)$$

$$Q_i(x,z) = Q_i(0,z)[1-S_i(x)]e^{-\lambda(1-X(z))x},\ x>0,\ i=1,2 \quad (25)$$

$$G_k(x,z) = G_k(0,z)[1-V_k(x)]e^{-\lambda(1-X(z))x},\ x>0,\ k=0,1,2,\cdots K \quad (26)$$

From equation(5), we obtain

$$G_{0,0}(x) \;=\; G_{0,0}(0)[1-V_0(x)]e^{-\lambda x},\ x>0 \tag{27}$$

Multiplying equation (27) by $\nu_0(x)$ on both sides and integrating with respect to x from $n=0$ *to* ∞ and using equation (11), we have

$$G_{0,0}(0) \;=\; \frac{\lambda P_0}{V_0^*(\lambda)} \tag{28}$$

From equations (12) and (28), we get the simplification

$$G_0(0,z) \;=\; \frac{\lambda P_0}{V_0^*(\lambda)} \tag{29}$$

Multiplying equation (7) by suitable powers of z, summing over n from 1 *to* ∞ and after some algebraic simplification we get,

$$\begin{aligned} P(0,z) \;=\; & p_0\int_0^\infty G_0(x,z)\nu_0(x)dx+\sum_{k=1}^{K}\int_0^\infty G_k(x,z)\nu_k(x)dx \\ & +(pz+q)\int_0^\infty Q_2(x,z)\mu_2(x)dx-\lambda P_0-G_{0,0}(0) \end{aligned} \tag{30}$$

Multiplying equations (8) - (10) and (13) by appropriate powers of z, summing over n from 0 *to* ∞ and after some algebraic manipulation we get,

$$Q_1(0,z) \;=\; \frac{1}{z}\int_0^\infty P(x,z)\,\theta(x)dx+\frac{\lambda X(z)}{z}\Big[\int_0^\infty P(x,z)dx+P_0\Big] \tag{31}$$

$$Q_2(0,z) \;=\; Q_1(0,z)S_1^*(\lambda-\lambda X(z)) \tag{32}$$

$$G_k(0,z) \;=\; p_k\frac{\lambda P_0}{V_0^*(\lambda)}V_0^*(\lambda-\lambda X(z)) \tag{33}$$

Further using equations (25) - (29) and (33) in equation (30), we get

$$\begin{aligned} P(0,z) \;=\; & \frac{\lambda P_0 p_0}{V_0^*(\lambda)}V_0^*(\lambda-\lambda X(z))\Big\{1+\sum_{k=1}^{K}p_kV_k^*(\lambda-\lambda X(z))\Big\} \\ & +(pz+q)Q_2(0,z)S_2^*(\lambda-\lambda X(z))-\lambda P_0-\frac{\lambda P_0}{V_0^*(\lambda)} \end{aligned} \tag{34}$$

Substituting equation (24) in (31), we obtain

$$Q_1(0,z) \;=\; P(0,z)\omega+\lambda P_0\frac{X(z)}{z} \tag{35}$$

Using equation (35) in equation (32), we get

$$Q_2(0,z) = \left[P(0,z)\omega + \lambda P_0 \frac{X(z)}{z}\right] S_1^*(\lambda - \lambda X(z)) \tag{36}$$

Substituting equation (36) in (34) and after some algebraic manipulation

$$\begin{aligned} P(0,z) &= \frac{\lambda z P_0 [1 - \frac{[(p_0 + \sum_{k=1}^{K} p_k V_k^*(\lambda - \lambda X(z))) V_0^*(\lambda - \lambda X(z)) - 1]}{V_0^*(\lambda)}]}{\gamma} \\ &\quad - \frac{\lambda P_0 [(pz+q) X(z) S_1^*(\lambda - \lambda X(z)) S_2^*(\lambda - \lambda X(z))]}{\gamma} \end{aligned} \tag{37}$$

Substituting equation (37) in (35), we get

$$\begin{aligned} Q_1(0,z) &= \lambda P_0 \left\{ \frac{\omega z [1 - \frac{[(p_0 + \sum_{k=1}^{K} p_k V_k^*(\lambda - \lambda X(z))) V_0^*(\lambda - \lambda X(z)) - 1]}{V_0^*(\lambda)}]}{\gamma} \right. \\ &\quad \left. - \frac{\omega [(pz+q) X(z) S_1^*(\lambda - \lambda X(z)) S_2^*(\lambda - \lambda X(z))]}{\gamma} + \frac{X(z)}{z} \right\} \end{aligned} \tag{38}$$

Utilizing equation (37) in (36) and simplifying we get

$$\begin{aligned} Q_2(0,z) &= \lambda P_0 S_1^*(\lambda - \lambda X(z)) \left\{ \frac{\omega z [1 - \frac{[(p_0 + \sum_{k=1}^{K} p_k V_k^*(\lambda - \lambda X(z))) V_0^*(\lambda - \lambda X(z)) - 1]}{V_0^*(\lambda)}]}{\gamma} \right. \\ &\quad \left. - \frac{\omega [(pz+q) X(z) S_1^*(\lambda - \lambda X(z)) S_2^*(\lambda - \lambda X(z))]}{\gamma} + \frac{X(z)}{z} \right\} \end{aligned} \tag{39}$$

Substituting equations (33), (37) - (39) in equations (24) - (26) and after some algebraic manipulation, we obtain

$$\begin{aligned} P(x,z) &= \lambda P_0 \left\{ \frac{z [1 - \frac{[(p_0 + \sum_{k=1}^{K} p_k V_k^*(\lambda - \lambda X(z))) V_0^*(\lambda - \lambda X(z)) - 1]}{V_0^*(\lambda)}]}{\gamma} \right. \\ &\quad \left. - \frac{[(pz+q) X(z) S_1^*(\lambda - \lambda X(z)) S_2^*(\lambda - \lambda X(z))]}{\gamma} \right\} [1 - R(x)]\, e^{-\lambda x} \\ Q_1(x,z) &= \lambda P_0 \left\{ \frac{\omega z [1 - \frac{[(p_0 + \sum_{k=1}^{K} p_k V_k^*(\lambda - \lambda X(z))) V_0^*(\lambda - \lambda X(z)) - 1]}{V_0^*(\lambda)}]}{\gamma} \right. \\ &\quad \left. - \frac{\omega [(pz+q) X(z) S_1^*(\lambda - \lambda X(z)) S_2^*(\lambda - \lambda X(z))]}{\gamma} + \frac{X(z)}{z} \right\} \\ &\quad \times \left\{ [1 - S_1(x)] e^{-\lambda(1 - X(z))x} \right\} \end{aligned}$$

$$\begin{aligned} Q_2(x,z) &= \lambda P_0 \Bigg\{ \frac{\omega z\left[1-\frac{[(p_0+\sum_{k=1}^{K} p_k V_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))-1]}{V_0^*(\lambda)}\right]}{\gamma} \\ &\quad -\frac{\omega[(pz+q)X(z)S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))]}{\gamma}+\frac{X(z)}{z}\Bigg\} \\ &\quad \times\Big\{[1-S_2(x)]e^{-\lambda(1-X(z))x}\Big\}S_1^*(\lambda-\lambda X(z)) \\ G_k(x,z) &= \frac{\lambda P_0 p_k}{V_0^*(\lambda)}V_0^*(\lambda-\lambda X(z))[1-V_k(x)]e^{-\lambda(1-X(z))x}, k=0,1,2,\cdots K \end{aligned}$$

Integrating the above equations with respect to x from 0 to ∞, we finally get the required results equations (15) - (19).

At this point, the only unknown is P_0, which can be determined by using the normalization condition $P_0+P(1)+Q_1(1)+Q_2(1)+G_0(1)+\sum_{k=1}^{K} G_k(1)=1$.

Let $K_s(z) = P_0 + P(z) + z[Q_1(z) + Q_2(z)] + G_0(z) + \sum_{k=1}^{K} G_k(z)$ and $K_0(z)=P_0+P(z)+[Q_1(z)+Q_2(z)]+G_0(z)+\sum_{k=1}^{K} G_k(z)$ be the probability generating function of the system and orbit size distribution at stationary point of time. Thus we have the following theorem.

Theorem 4.2 *Under the stability condition $\rho/q<1$, the probability generating function of the system and orbit size distribution at stationary point of time is given by*

$$\begin{aligned} K_s(z) &= P_0\Bigg\{q\Bigg[\frac{[\frac{1-(p_0+\sum_{k=1}^{K} p_k V_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))}{V_0^*(\lambda)}]}{\gamma[1-X(z)]} \\ &\quad \times[R^*(\lambda)+X(z)[1-R^*(\lambda)]+\frac{R^*(\lambda)}{\gamma}\Bigg] \\ &\quad \times[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))(1-z)]\Bigg\} \qquad (40) \\ K_0(z) &= P_0\Bigg\{q\Bigg[\frac{[\frac{1-(p_0+\sum_{k=1}^{K} p_k V_k^*(\lambda-\lambda X(z)))V_0^*(\lambda-\lambda X(z))}{V_0^*(\lambda)}]}{\gamma[1-X(z)]} \\ &\quad \times[R^*(\lambda)+X(z)[1-R^*(\lambda)]]-\frac{R^*(\lambda)}{\gamma}\Bigg] \\ &\quad \times[S_1^*(\lambda-\lambda X(z))S_2^*(\lambda-\lambda X(z))]\Bigg\}(1-z) \qquad (41) \end{aligned}$$

where P_0 is given (20).

5 Performance Measures

In this section, we analyze some system performance measures of the retrial queue. Differentiating equation (40) with respect to z and evaluating at $z = 1$, the mean number of customers in the system L_s is obtained as

$$L_s = \frac{Nr1}{Dr1} + \frac{Nr2}{Dr2} + \{\lambda E(X)[E(S_1) + E(S_2)]\} \tag{42}$$

$$\text{where } Dr1 = \{2qE(X)\left[\lambda E(V_0) + \lambda \sum_{k=1}^{K} p_k E(V_k) + V_0^*(\lambda)R^*(\lambda)\right]\}$$

$$\begin{aligned} Nr1 &= q\{[\lambda E(X)]^2[E(V_0^2) + 2\sum_{k=1}^{K} p_k E(V_k)E(V_0) + \sum_{k=1}^{K} p_k E(V_k^2)] \\ &\quad +2\lambda[E(X)]^2[1 - R^*(\lambda)][E(V_0) + \sum_{k=1}^{K} p_k E(V_k)]\} \\ Nr2 &= \{[\lambda E(X)]^2[E(S_1^2) + 2E(S_1)E(S_2) + E(S_2^2)] \\ &\quad +2\lambda[E(X)]^2[1 - R^*(\lambda)][E(S_1) + E(S_2)] \\ &\quad +E[X(X-1)][1 - R^*(\lambda) + \lambda(E(S_1) + E(S_2))] + 2p\rho\} \\ Dr2 &= [2(q - \rho)] \end{aligned}$$

Differentiating equation (41) with respect to z and evaluating at $z = 1$, we get the expected number of customers in the orbit L_q is obtained as

$$L_q = \frac{Nr1}{Dr1} + \frac{Nr2}{Dr2} - \frac{p\lambda[E(X)][E(S_1) + E(S_2)]}{q} \tag{43}$$

6 Special cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.
Case 1: If $P[X = 1] = 1$, $p_0 = 1$ and $p = 0$, the model reduces to the $M/G/1$ retrial queue with general retrial times and a single vacation. The probability generating function of the number of customers in the system $K_s(z)$, the idle probability P_0 and the mean queue size L_q can be simplified to the following expressions and which are in accordance with those of Krishnakumar et al [10].

$$\begin{aligned} P_0 &= \frac{[R^*(\lambda) - \lambda E(S)]V_0^*(\lambda)}{\{\lambda E(V_0) + V_0^*(\lambda)R^*(\lambda)\}} \\ K_s(z) &= P_0\left\{\frac{[1 - V_0^*(\lambda - \lambda z)][R^*(\lambda) + z(1 - R^*(\lambda))] - R^*(\lambda)V_0^*(\lambda)}{V_0^*(\lambda)[(z + (1-z)R^*(\lambda))S^*(\lambda - \lambda z) - z]}\right\} \\ &\quad \times\{S^*(\lambda - \lambda z)\} \end{aligned}$$

$$L_q = \Big\{\lambda^2[E(V_0^2)] + 2\lambda E(V_0)[1 - R^*(\lambda)]\Big\}\Big\{2[\lambda E(V_0) + R^*(\lambda)V_0^*(\lambda)]\Big\}^{-1} + \Big\{\lambda^2[E(S^2) + 2\lambda[1 - R^*(\lambda)]E(S)\Big\}\Big\{2[R^*(\lambda) - \lambda E(S)]\Big\}^{-1}$$

Case 2: If $P[X = 1] = 1$, $V_0^*(\lambda) = 1$ and $p = 0$, our model reduces to $M/G/1$ retrial queue with general retrial times and two phases of service. The probability generating function of the number of customers in the system $K_s(z)$, the idle probability P_0 and the mean queue size L_q can be simplified to the following expressions and this result is equivalent to the result obtained by Choudhury [4].

$$P_0 = \frac{[R^*(\lambda) - \lambda(E(S_1) + E(S_2))]}{R^*(\lambda)}$$

$$K_s(z) = P_0\Big\{\frac{R^*(\lambda)(1-z)S_1^*(\lambda - \lambda z)S_2^*(\lambda - \lambda z)}{[(z + (1-z)R^*(\lambda))S_1^*(\lambda - \lambda z)S_2^*(\lambda - \lambda z) - z]}\Big\}$$

$$L_q = \frac{\lambda^2[E(S_1^2) + 2E(S_1)E(S_2) + E(S_2^2)]}{2[R^*(\lambda) - \lambda[E(S_1) + E(S_2)]]} + \frac{2\lambda[1 - R^*(\lambda)][E(S_1) + E(S_2)]}{2[R^*(\lambda) - \lambda[E(S_1) + E(S_2)]]}$$

Case 3: If $V_0^*(\lambda) = 1$, $P[X = 1] = 1$, $p_0 = 1$ and $p = 0$, we get an $M/G/1$ retrial queue with general retrial times. In this case, the probability generating function of the number of customers in the system $K_s(z)$, the probability of no customers in the system P_0 and the mean queue size L_q can be rewritten in the following form and the results agree with Gomez-Corral [7].

$$P_0 = \frac{R^*(\lambda) - \lambda E(S)}{R^*(\lambda)}$$

$$K_s(z) = P_0\Big\{\frac{R^*(\lambda)(1-z)S^*(\lambda - \lambda z)}{[(z + (1-z)R^*(\lambda))S^*(\lambda - \lambda z) - z]}\Big\}$$

$$L_q = \frac{\lambda^2 E(S^2) + 2\lambda(1 - R^*(\lambda))E(S)}{2[R^*(\lambda) - \lambda E(S)]}$$

Case 4: If $V_0^*(\lambda) = 1$, $p_0 = 1$ and $p = 0$, we get the $M^{[X]}/G/1$ queue with classical retrial policy. In this case, the probability generating function of the number of customers in the system $K_s(z)$, the idle probability P_0 and the expected number of customers in the queue L_q can be simplified to the following forms and which is consistent with the results of Falin and Templeton [6].

$$K_s(z) = \frac{(1-z)(1-\rho)}{[R^*(\lambda) + X(z)R^*(\lambda)]}$$

$$L_q = \frac{(\lambda E(X))^2 E(S^2)}{2(1-\rho)} + \frac{\lambda(E(X))^2 E(S)(1 - R^*(\lambda))}{1-\rho} + \frac{E(X(X-1))[1 - R^*(\lambda) + \lambda E(S)]}{2(1-\rho)}$$

$$P_0 = \frac{1-\rho}{R^*(\lambda)}$$

Case 5: If $R^*(\lambda) \to 1$, $V_0^*(\lambda) = 1$, $P[X = 1] = 1$ and $p_0 = 1$, we obtain the $M/G/1$ queueing system with two phases of service and bernoulli feedback. In this case, the probability generating function of the number of customers in the system $K_s(z)$, the idle probability P_0 and the expected number of customers in the queue L_q can be simplified to the following expressions and which is equivalent the results obtained by Choudhury and Paul [5].

$$\begin{aligned} P_0 &= 1 - \frac{\rho}{q} \\ K_s(z) &= P_0\Big[\frac{q(1-z)S_1^*(\lambda-\lambda z)S_2^*(\lambda-\lambda z)}{(pz+q)S_1^*(\lambda-\lambda z)S_2^*(\lambda-\lambda z)-z}\Big] \\ L_q &= \rho + \frac{\lambda^2[E(S_1^2)+2E(S_1)E(S_2)+E(S_2^2)]}{2(q-\rho)} + \frac{\lambda p[E(S_1)+E(S_2)]}{q-\rho} \end{aligned}$$

Case 6: If $R^*(\lambda) \to 1$, $p_0 = 1$ and $p = 0$, we get the $M^{(X)}/G/1$ queueing system with a single vacation. In this case, the probability generating function of the number of customers in the system $K_s(z)$, the probability of no customers in the system P_0 and the expected number of customers in the queue L_q can be simplified to the following expressions and the equations coincides with equation of Takagi [14].

$$\begin{aligned} P_0 &= \frac{[1-\lambda E(X)E(S)]V_0^*(\lambda)}{\lambda E(V_0)+V_0^*(\lambda)} \\ K_s(z) &= P_0\Big\{\frac{[1-V_0^*(\lambda-\lambda X(z))+V_0^*(\lambda)[1-X(z)]](1-z)S^*(\lambda-\lambda X(z))}{V_0^*(\lambda)[S^*(\lambda-\lambda X(z))-z][1-X(z)]}\Big\} \\ L_q &= \Big\{(\lambda E(X))^2[E(V_0^2)]\Big\}\Big\{2E(X)[\lambda E(V_0)+V_0^*(\lambda)]\Big\}^{-1} \\ &\quad +\Big\{\lambda E(X))^2[E(S^2)+E(X(X-1))\lambda E(S)\Big\}\Big\{2[1-\lambda E(X)E(S)]\Big\}^{-1} \end{aligned}$$

Case 7: If $R^*(\lambda) \to 1$, $V_0^*(\lambda) = 1$, $P[X = 1] = 1$, $p_0 = 1$ and $p = 0$, our model can be reduced to the $M/G/1$ queueing system. In this case, the probability generating function of the number of customers in the system $K_s(z)$, the idle probability P_0 and the mean queue size L_q can be simplified to the following expressions and which are consistent with well known the P-K formula [8].

$$\begin{aligned} P_0 &= 1-\rho \\ K_s(z) &= \frac{(1-\rho)(1-z)S^*(\lambda-\lambda X(z))}{[S^*(\lambda-\lambda X(z))-z]} \\ L_q &= \frac{\lambda^2 E(S^2)}{2(1-\rho)} \end{aligned}$$

7 Numerical Illustration

We present some numerical results using Matlab in order to illustrate the effect of various parameters on the main performance of our system. For the effect of parameters λ, θ, p and k on the system performance measures, two dimensional graphs are drawn in Fig 1-3 such that the stability condition is satisfied. We assume that the service time distributions for chosen parametric values are Exponential, Erlangian and Hyper exponential distribution. Figure 1 shows that the mean orbit size L_q decreases for increasing the values of the feedback with probability p. Figure 2 shows that the mean orbit size L_q increases for increasing the values of the optional vacation k. Figure 3 shows that the mean orbit size L_q increases for increasing the values of the the retrial rate θ.

Three dimensional graphs are illustrated in figures 4-6. In figure 4, the surface displays a upward trend as expected for increasing value of the feedback probability p and the retrial rate θ against the mean orbit size L_q. The mean orbit size L_q increases for increasing value of the feedback probability p and the optional vacation k is shown in figure 5. In figure 6, the surface displays sharp fall trend as expected for increasing value of the retrial rate θ and the optional vacation k against the mean orbit size L_q.

8 Conclusion

In this paper, we introduced a single server retrial queueing system with general repeated attempts, batch arrival, two phases of service, feedback and K optional vacations. The probability generating function of the number of customers in the system is found using the supplementary variable technique. Various performance measures and special cases are analyzed. Some practical justification such as the packet-switched networks, proxy of WWW server and production line are given. The effect of various parameters on the performance measure are illustrated graphically. The result of this paper is useful for the network design engineers and software engineers to design various computer communication systems.

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Table 1: Mean number of customers for various values of p and θ.

	θ=0	θ=0.1	θ=0.2	θ=0.3	θ=0.4	θ=0.5	θ=0.6	θ=0.7	θ=0.8	θ=0.9
p=0	0.009	0.012	0.015	0.017	0.018	0.019	0.019	0.018	0.021	0.181
p=0.1	0.010	0.022	0.025	0.031	0.048	0.055	0.064	0.071	0.077	0.188
p=0.2	0.011	0.032	0.035	0.041	0.052	0.069	0.079	0.088	0.091	0.199
p=0.3	0.012	0.039	0.045	0.057	0.064	0.079	0.089	0.098	0.101	0.213
p=0.4	0.013	0.040	0.055	0.067	0.077	0.088	0.092	0.100	0.106	0.217
p=0.5	0.015	0.052	0.065	0.077	0.086	0.090	0.098	0.108	0.109	0.222
p=0.6	0.019	0.062	0.085	0.087	0.099	0.109	0.110	0.115	0.119	0.227
p=0.7	0.023	0.072	0.105	0.107	0.109	0.111	0.115	0.118	0.120	0.231
p=0.8	0.027	0.082	0.115	0.116	0.110	0.119	0.119	0.122	0.124	0.239
p=0.9	0.029	0.112	0.125	0.137	0.138	0.139	0.142	0.153	0.165	0.240

Table 2: Mean number of customers for various values of p and k.

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
p=0	0.034	0.035	0.037	0.039	0.042	0.046	0.052	0.064	0.095	0.341
p=0.1	0.034	0.035	0.037	0.038	0.040	0.044	0.049	0.060	0.088	0.325
p=0.2	0.034	0.035	0.036	0.037	0.039	0.042	0.047	0.056	0.081	0.308
p=0.3	0.034	0.035	0.036	0.037	0.038	0.040	0.044	0.051	0.073	0.291
p=0.4	0.034	0.035	0.035	0.036	0.037	0.038	0.041	0.047	0.066	0.275
p=0.5	0.034	0.034	0.034	0.035	0.035	0.036	0.038	0.042	0.058	0.258
p=0.6	0.033	0.033	0.033	0.033	0.033	0.033	0.034	0.037	0.049	0.240
p=0.7	0.032	0.031	0.031	0.031	0.030	0.030	0.030	0.031	0.041	0.222
p=0.8	0.030	0.030	0.029	0.029	0.028	0.027	0.026	0.026	0.032	0.204
p=0.9	0.029	0.028	0.028	0.027	0.025	0.024	0.022	0.020	0.024	0.186

Table 3: Mean number of customers for various values of θ and k.

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
θ=0	0.007	0.011	0.015	0.018	0.021	0.024	0.027	0.029	0.038	0.034
θ=0.1	0.008	0.012	0.016	0.019	0.022	0.025	0.028	0.031	0.034	0.036
θ=0.2	0.008	0.013	0.017	0.020	0.024	0.027	0.030	0.033	0.035	0.038
θ=0.3	0.009	0.014	0.018	0.021	0.025	0.028	0.031	0.034	0.037	0.040
θ=0.4	0.010	0.014	0.019	0.023	0.026	0.030	0.033	0.036	0.039	0.042
θ=0.5	0.010	0.015	0.020	0.024	0.027	0.031	0.034	0.037	0.040	0.043
θ=0.6	0.011	0.015	0.020	0.024	0.028	0.031	0.035	0.038	0.041	0.044
θ=0.7	0.011	0.016	0.020	0.024	0.028	0.032	0.035	0.039	0.042	0.045
θ=0.8	0.011	0.016	0.021	0.025	0.029	0.032	0.036	0.039	0.042	0.045
θ=0.9	0.011	0.016	0.021	0.025	0.029	0.033	0.036	0.040	0.043	0.046

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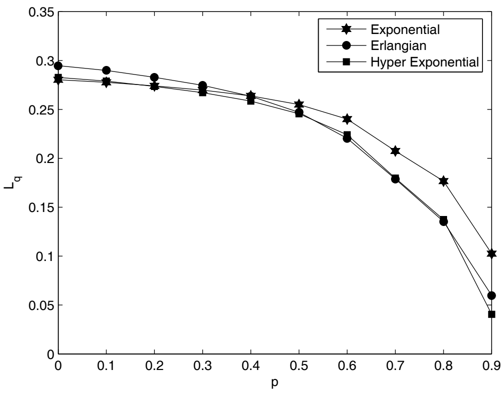

Figure 1: L_q versus p

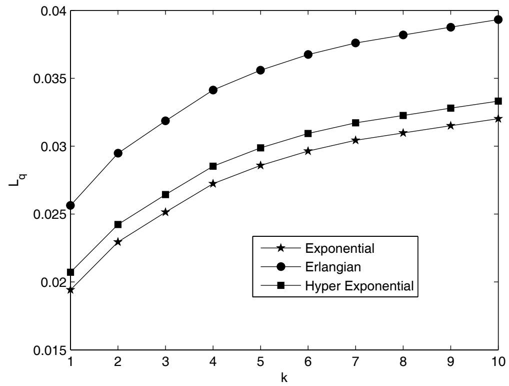

Figure 2: L_q versus k

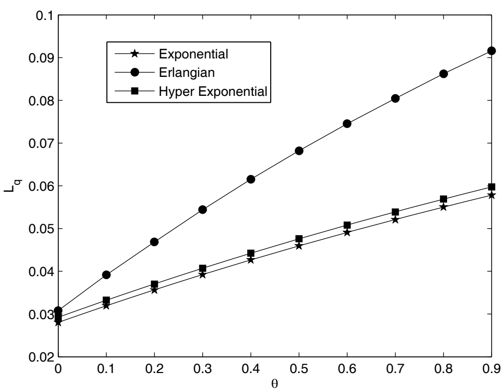

Figure 3: L_q versus θ

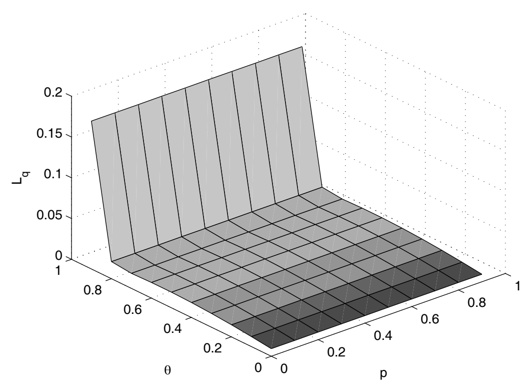

Figure 4: L_q versus p and θ

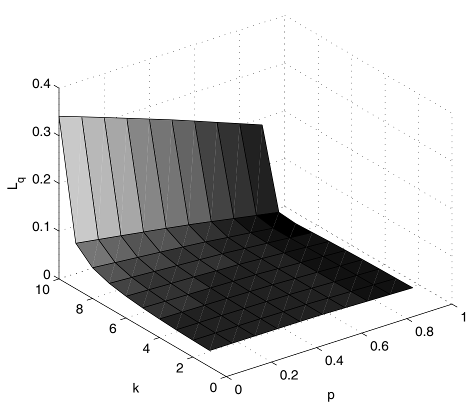

Figure 5: L_q versus p and k

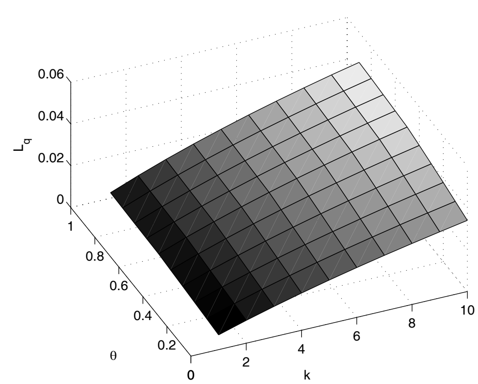

Figure 6: L_q versus θ and k