A BAYES APPROACH TO A QUALITY CONTROL MODEL[1]

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Summary. This paper deals with a class of statistical quality control procedures and continuous inspection procedures which are optimum for a specified income function and a production model which can only be in one of four states, two of which are states of repair, with known transition probabilities. The Markov process, generated by the model and the class of decision procedures, approaches a limiting distribution and the integral equations from which the optimum procedures can be derived are given.

1. Introduction. A machine which is producing items possessing a measurable quality characteristic x can be in one of four states. In state $i = 1, 2$ the machine is in production and is characterized by a probability density $f_i(x)$ of the quality characteristic x. In state $j = 3, 4$ the machine is being repaired, having previously been in state $j - 2$. The machine remains in the repair shop for n_j time units, where a time unit is taken as the length of time required to produce one item. Repair puts the machine in state 1 which is assumed to be the desirable state. When the machine is in state 1 there is a constant probability g that in the next time unit it will go into state 2. This probability is inherent in the production process and is assumed to be known. Once the machine enters state 2 it remains in this state until it is brought to repair (i.e., state 4). The machine is brought from production to repair by a statistical quality control rule R based on observations on x.

Two cases are considered. In case 1 it is assumed that 100% inspection of the items is based upon grounds other than inspection costs. In this case the rule R specifies only when to terminate production and put the machine in the repair shop. In case 2 inspection costs are taken into account or alternatively 100% inspection is precluded by the destructiveness of the tests so that the rule R, in addition to being a "stop" rule, also specifies which items in the production sequence are to be inspected. In both cases, the aim is to maximize the long run average income. Case 1 will be discussed first.

2. Optimum quality control rule for the case of 100% inspection. The economic considerations involved in the production model in the case of 100% inspection are (a) a function $V(x)$ which gives the income per item of quality x produced, and (b) two positive constants c_j, $j = 3, 4$, which represent the cost of repair per unit of time the machine is in state j.

For the given production model, income function, and repair costs, the expected income per unit of time the production process is in operation in any specified length of time depends on the particular quality control rule employed.

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For any rule R, let $I_N(R)$ stand for the sample average income per unit of time if the production process has been in operation for N time units and initially the machine is in state 1. Furthermore, let r_{kN} ($k = 1, 2, 3, 4$) be the number of time units in the N time units that the machine is in state k. Then

$$I_N(R) = \frac{1}{N} \sum_{\text{state } 1} V(x) + \frac{1}{N} \sum_{\text{state } 2} V(x) - \frac{r_{3N}}{N} c_3 - \frac{r_{4N}}{N} c_4 . \tag{2.01}$$

Let $E[V(x) \mid f_i]$ stand for the expected value of $V(x)$ given that the machine is in state $i = 1, 2$, and let $\pi_{kN} = E(r_{kN}/N)$. Then from (2.01)

$$E[I_N(R)] = \pi_{1N} E[V(x) \mid f_1] + \pi_{2N} E[V(x) \mid f_2] - \pi_{3N} c_3 - \pi_{4N} c_4 . \tag{2.02}$$

A rule R^* will be called optimum or Bayes if it yields $\max_R \lim_{N \to \infty} E[I_N(R)]$. That is, letting $I(R) = \lim_{N \to \infty} E[I_N(R)]$, then R^* is defined by

$$I(R^*) = \max_R I(R). \tag{2.03}$$

In spite of the apparent complexity of the problem, the characterization of R^* turns out to be fairly simple.

Let $x_1, x_2, \ldots$ be the quality of the items produced in sequence from the time the machine comes out of the repair shop. Define

$$y_k = \frac{f_2(x_k)}{(1 - g) f_1(x_k)}, \qquad Z_0 = 0, Z_k = y_k(1 + Z_{k-1}). \tag{2.04}$$

For any positive constant a let $R(a)$ be the rule which states that inspection is to continue as long as $Z_k < a$, and inspection is to terminate and the machine is to be put in the repair shop as soon as for some k, $Z_k \geq a$. Furthermore, let a^* be such that

$$I(R(a^*)) = \max_a I(R(a)). \tag{2.05}$$

The optimum quality control rule R^* is completely characterized by

THEOREM 1. *$R^* = R(a^*)$ if there exists a constant a such that*

$$E(V(x) \mid f_2) < I(R(a)).$$

The following definitions and lemmas are required to prove this theorem.

For any rule R in use, the time period during which the machine, having left the repair shop, stays in production until it is placed back in the repair shop and stays there for the specified length of time, will be called a cycle. Let n be the number of time units the machine is in production under the rule R during a cycle and let $m = n + n_j$ where $j = 3$ if the machine entered the repair shop from state 1 and $j = 4$ if it entered the repair shop from state 2. Thus m is a random variable and represents the length of the cycle. Let $u = V(x)$ in any time unit the machine is in production and $u = -c_j$ ($j = 3, 4$) in any time unit the machine is in the repair shop. Let $J_m(R)$ be the total income per cycle. Then $J_m(R) = \sum_{i=1}^{m} u_i$. A rule R_1^* will be called Bayes if it yields $\max_R EJ_m(R)$. That is, setting $J(R) = EJ_m(R)$, R_1^* is defined by

$$J(R_1^*) = \max_R J(R). \tag{2.06}$$

Let $R(a)$ be defined as above and let $\bar{a}$ be such that

$$J(R(\bar{a})) = \max_a J(R(a)). \tag{2.07}$$

Lemma 1. *$R_1^* = R(\bar{a})$ if $E(V(x) \mid f_2) < 0$.*

Proof. The proof of Lemma 1 follows directly from the general characterizations of Bayes solutions given by Arrow, Blackwell, and Girshick [1] and only a brief sketch of the argument will be presented here.

If at any time that the machine is in production it were known that it is in state 2, by the conditions of the lemma it would pay to place it in the repair shop. Thus the only relevant information obtainable from the observations is the *a posteriori probability* that the machine is in state 2. Whether or not for a given a posteriori probability the expected income per cycle is maximized by placing the machine in the repair shop depends on the existence or nonexistence of a continuation rule which from this stage on would guarantee an expected income exceeding the expected cost of repair. It is proved in the paper cited above that the set of a posteriori probabilities for which the best procedure is to take a given action is an interval. In the case under consideration, the set of a posteriori probabilities for which the best procedure is to put the machine in the repair shop is an interval from g^* to 1, where g^* is a nonnegative fraction and its value depends on $V(x)$, c_3, c_4, and g. The optimum procedure R_1^* can therefore be described as follows. At each stage of inspection compute the a posteriori probability that the next item will be produced in state 2. As long as this a posteriori probability is less than g^* continue inspection. However, as soon as this a posteriori probability equals or exceeds g^*, terminate inspection and place the machine in the repair shop. That this procedure is equivalent to $R(\bar{a})$ can be seen from the following.

At the kth stage of inspection, let q_k be the a posteriori probability that item $k + 1$ will be produced while the machine is in state 1. Then

$$q_k = \frac{(1 - g)q_{k-1}f_1(x_k)}{q_{k-1}f_1(x_k) + (1 - q_{k-1})f_2(x_k)}, \qquad q_0 = 1 - g. \tag{2.08}$$

Let y_k be defined as in (2.04). Then

$$\frac{1}{q_k} = \frac{1}{1 - g} + y_k\left(\frac{1}{q_{k-1}} - 1\right), \tag{2.09}$$

$$\frac{1}{q_k} - \frac{1}{1 - g} = y_k\left(\frac{1}{q_{k-1}} - \frac{1}{1 - g} + \frac{g}{1 - g}\right). \tag{2.10}$$

Let

$$(2.11) \qquad Z_k = \frac{1-g}{g}\left(\frac{1}{q_k} - \frac{1}{1-g}\right).$$

Then from (2.10)

$$(2.12) \qquad Z_k = y_k(1 + Z_{k-1}), \qquad Z_0 = 0.$$

Let now $p_k = 1 - q_k$. Then from (2.11),

$$(2.13) \qquad p_k = \frac{g(Z_k + 1)}{gZ_k + 1}.$$

Consequently the relationship $p_k \lesseqgtr g^*$ is equivalent to the relationship

$$(2.14) \qquad Z_k \lesseqgtr \frac{g^* - g}{g(1 - g^*)} = \bar{a}.$$

Thus, R_1^* is equivalent to the rule, continue inspection as long as $Z_k < \bar{a}$, and terminate inspection as soon as $Z_k \geq \bar{a}$.

LEMMA 2. *For any positive constant a*

$$(2.15) \qquad P(m \geq m_0 \mid R(a)) \to 0 \text{ as } m_0 \to \infty,$$

where $m = n + n_j$ *is the length of a cycle.*

PROOF. It suffices to show that the lemma holds for n, i.e., that $R(a)$ terminated production with probability 1. It is clear that $P(y > 1 \mid f_2) = r > 0$. Let n_0 be a large positive integer and let $i < n_0$ be any integer such that $n_0 - i > [a + 1] = d$, where the symbol $[t]$ stands for the smallest integer greater than or equal to t. Furthermore let $P(i, 2)$ stand for the probability that the machine is in state 2 after i time units of production and let $P(i, n_0 \mid 2)$ be the probability that there has been a run of at least d y's each greater than 1 between the time period i and the time period n_0 given that the machine is in state 2. Then

$$(2.16) \quad P(n < n_0 \mid R(a)) \geq P(i, 2)\, P(i, n_0 \mid 2) \geq (1 - (1 - g)^i)\,(1 - (1 - r^d)^k),$$

where $k = [(n_0 - i)/d]$. Setting $i = \alpha n_0$, equation (2.16) becomes

$$(2.17) \qquad P(n < n_0 \mid R(a)) \geq (1 - \delta_1^{\alpha n_0})\,(1 - \delta_2^{\beta n_0}) \geq 1 - A\delta^{n_0},$$

where $0 < \delta < 1$. Thus

$$(2.18) \qquad P(n \geq n_0) < A\delta^{n_0},$$

which proves the lemma.

LEMMA 3. *For any positive constant a,* $E[m \mid R(a)] < \infty$, *where m is the length of a cycle.*

PROOF. Again it suffices to show that $E[n \mid R(a)] < \infty$. In view of (2.18) the series $\sum_{k=1}^{\infty} P(n \geq k)$ is convergent. But

$$(2.19) \qquad \sum_{k=1}^{\infty} P(n \geq k) = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} P(n = k) = \sum_{j=0}^{\infty} jP(n = j),$$

which proves the lemma.

LEMMA 4. *For any rule R with* $E(m \mid R) < \infty$, $I(R) = E[\sum_{i=1}^{m} u_i/E(m \mid R)]$, *where* $I(R) = \lim_{N\to\infty} EI_N(R)$.

PROOF. Let W_N be the number of completed cycles in the time period of length N and L_N be their total length. A repeated application of the Strong Law of Large Numbers shows that with probability 1 the following sequences approach the indicated limits as $N \to \infty$:

$$
\begin{aligned}
&\text{(a)}\ \frac{N - L_N}{W_N} \to 0; \qquad\qquad \text{(b)}\ \frac{L_N}{W_N} \to E(m);\\
&\text{(c)}\ \frac{N}{W_N} \to E(m); \qquad\qquad \text{(d)}\ \left[\sum_{i=L_N+1}^{N} u_i/W_N\right] \to 0;\\
(2.20)\quad &\text{(e)}\ E\left[I_N(R) - \sum_{i=1}^{L_N} u_i/(W_N E(m))\right] \to 0;\\
&\text{(f)}\ E\left[\sum_{i=1}^{L_N} u_i/(W_N E(m))\right] \to E\left[\sum_{i=1}^{m} u_i/E(m)\right].
\end{aligned}
$$

Therefore,

$$EI_N(R) - E\left[\sum_{i=1}^{m} u_i/E(m)\right] \to 0 \tag{2.21}$$

as $N \to \infty$, which completes the proof.

LEMMA 5. *If for any rule R,* $E(m \mid R) = \infty$, *then* $I(R) = E(V(x) \mid f_2)$.

PROOF. To prove this lemma it will suffice to show that the proportion of time units in which the machine is in state 2 in N time units approaches 1 as $N \to \infty$. If as $N \to \infty$ there are only a finite number of cycles, then with probability 1 the machine will enter state 2 and remain in state 2 so that the lemma is established. Assume that this is not the case.

Let s_i $(i = 1, 2, \cdots)$ be the number of time units required for the machine to enter state 2 in the ith cycle if no stop rule were employed plus the number of time units it stays in the repair shop. Then $s_1, s_2, \cdots$ are identically and independently distributed variates with $Es_i \le (1/g) + \max(n_3, n_4)$. Let t_i $(i = 1, 2, \cdots)$ be the length of the ith cycle. Then $t_1, t_2, \cdots$ are identically and independently distributed variates with t_i independent of $s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots$. Define

$$
\begin{aligned}
z_i &= t_i - s_i \quad \text{if} \quad t_i > s_i,\\
&= 0 \qquad\quad\ \ \text{if} \quad t_i \le s_i.
\end{aligned} \tag{2.22}
$$

Then z_i = number of time units in the ith cycle that the machine is in state 2. The fraction of time units that the machine is not in state 2 in the first n cycles is given by

$$Q_n = \frac{w_1 + w_2 + \cdots + w_n}{t_1 + t_2 + \cdots + t_n}, \tag{2.23}$$

where

$$(2.24)\qquad \begin{aligned} w_i &= t_i \text{ if } t_i \le s_i\,, \\ {}_i &= s_i \text{ if } t_i > s_i\,. \end{aligned}$$

Combining (2.22) and (2.24) yields

$$(2.25)\qquad z_i + w_i = t_i\,.$$

Now by (2.24) $Ew_i \le Es_i \le (1/g) + \max(n_3, n_4)$, and since by assumption $Ez_i = \infty$, it follows from (2.25) that $Et_i = \infty$. By (2.23) and (2.25),

$$(2.26)\qquad \begin{aligned} Q_n &= \frac{w_1 + w_2 + \cdots + w_n}{z_1 + \cdots + z_n + w_1 + \cdots + w_n} \\ &= \frac{\dfrac{w_1 + \cdots + w_n}{n}}{\dfrac{z_1 + \cdots + z_n}{n} + \dfrac{w_1 + \cdots + w_n}{n}}, \end{aligned}$$

so that by the Strong Law of Large Numbers, $\lim_{n\to\infty} Q_n = 0$ with probability 1. Let

$$(2.27)\qquad \sum_{i=1}^{n} (w_i + z_i) < N \le \sum_{i=1}^{n+1} (w_i + z_i).$$

Then the relative length of time that the machine is not in state 2 in N time units is given by

$$(2.28)\qquad \left\{\frac{1}{N}\sum_{i=1}^{n} w_i + \text{length of time that it is not in state 2 between } \sum_{i=1}^{n}(w_i + z_i) \text{ and } N\right\} \le \frac{\sum_{i=1}^{n+1} w_i}{\sum_{i=1}^{n}(w_i + z_i)} \to 0$$

with probability 1. This proves the lemma.

The above lemmas will now be used to prove Theorem 1.

By Lemma 5 the only rules R that need to be considered are those for which $E(m \mid R) < \infty$. Let

$$(2.29)\qquad K(R) = E\left[\sum_{i=1}^{m} (u_i - I(R(a^*))\right].$$

The income function $u - I(R(a^*))$ satisfies the conditions of Lemma 1. Hence there exists a constant $\bar{a}$ such that

$$(2.30)\qquad K(R(\bar{a})) = \max_{a} K(R(a)).$$

Now by Lemma 4,

$$K(R) = E(m \mid R)[I(R) - I(R(a^*))]. \tag{2.31}$$

Since by Lemma 3 $E(m \mid R(\bar{a})) < \infty$, then for all R

$$K(R) \leq K(R(\bar{a})) = E(m \mid R(\bar{a}))\,[I(R(\bar{a})) - I(R(a^*))]. \tag{2.32}$$

But by definition $I(R(a)) \leq I(R(a^*))$. It follows therefore that $K(R) \leq 0$ so that by (2.31) $I(R) - I(R(a^*)) \leq 0$, which proves the theorem.

By Lemma 3, a sequential procedure defined by the rule $R(a)$ terminates with probability 1. It is of interest to investigate under what conditions $R(a)$ is a truncated sequential procedure. The answer to this is given by

THEOREM 2. *Let r_0 be the least upper bound of numbers r such that $P(y < r \mid f_1) = 0$. A necessary and sufficient condition that $R(a)$ be a nontruncated sequential procedure is that $r_0 < a(1 - r_0)$.*

PROOF. Since $y = f_2(x)/((1 - g)f_1(x)) \geq 0$, it follows that $r_0 \geq 0$, so that if the condition of the theorem is satisfied r_0 must be less than 1. Assume that $r_0 < a(1 - r_0)$. Let $r_1 > r_0$ but still satisfying the condition $r_1 < a(1 - r_1)$. Then $P(y < r_1 \mid f_1) > 0$. Thus for any finite n whatever, there is a positive probability that (a) the machine is in state 1 during the n time units and (b) there exists a sequence $y_1, \cdots, y_n$ such that $y_i < r_1$ for all i. But for such a sequence of y's inspection cannot terminate since $Z_j < \sum_{i=1}^{j} r_1^i < a$ for $j = 1, \cdots, n$. Conversely if $r_0 > a(1 - r_0)$, the series $\sum_{i=1}^{j} r_0^i > a$. Thus there exists an n_0 for which $\sum_{i=1}^{n_0} r_0^i > a$, which implies that $P(n > n_0) = 0$. This completes the proof of Theorem 2.

3. Integral equations for the Markov process in the case of 100% inspection. In the previous section it was shown that the optimum quality control rule is given by $R(a)$ for $a = a^*$. The problem is to find a^* for a given $V(x)$, c_3 and c_4. Since a^* is that value of a for which $I(R(a)) = \lim_{n\to\infty} EI_N(R(a))$ is a maximum, this problem will be solvable if $I(R(a))$ is determined for an arbitrary a. But by (2.02) this is equivalent to finding for any a $\lim_{N\to\infty} \pi_{kN}$ $(k = 1, 2, 3, 4)$. The solution to the latter problem will be given in this section.

The production model under consideration together with the stop rule $R(a)$ creates a Markov process with states and transitions which can be represented schematically as follows:

$(1, Z_n)$: If $Z_n < a$, take another observation; if $Z_n \geq a$, go into [3, 1].
$(2, Z_n)$: If $Z_n < a$, take another observation; if $Z_n \geq a$, go into [4, 1].
$[3, k] \to [3, k + 1]$ $(k = 1, \cdots, n_3 - 1)$.
$[3, n_3]$: Take an observation.
$[4, k] \to [4, k + 1]$ $(k = 1, \cdots, n_4 - 1)$.
$[4, n_4]$: Take an observation.

Here the symbol (i, E) $i = 1, 2$, stands for the joint event, the machine is in state i and the event E occurred. The symbol $[j, k]$, $(j = 3, 4, k = 1, \cdots, n_j)$, stands for the event, the machine is in the kth time unit of repair and prior to repair has been in state $j - 2$. The arrows indicate transitions from one state to another.

From the results of a paper by Erdös, Feller, and Pollard [2], it follows that the probability that at time m the machine is either in state $[3, n_3]$ or $[4, n_4]$ approaches a limit, and hence the probability of the event $(i, Z_n \lesseqgtr a)$ or the event $(i, Z_n \varepsilon S)$ where S is any Borel set, approaches a limit.

While in the Markov process under consideration it was assumed that when the machine leaves the repair shop $Z_0 = 0$ and $p_k = g$, where p_k (see (2.08)) is the a posteriori probability that item $k + 1$ will be produced in state 2, it is found just as convenient to derive the integral equations for the limiting distribution for any arbitrary value for Z_0 and a corresponding a posteriori probability h.

Write Z for Z_k, w for Z_{k-1} and y for y_k. Then $Z = y(w + 1)$. For any $x > 0$ let $P(i, Z < x)$ stand for the probability of the event $(i, Z < x)$.

Lemma 6. *If* $P(y = \infty \mid f_2) = 0$, *then*

$$P(2, Z < x) = \frac{g}{1 - g} \int_0^x (1 + t)\, dP(1, Z < t). \tag{3.01}$$

Proof. The truth of this lemma can readily be seen from the fact that if t is written for Z_k and p for p_k then

$$\frac{p}{q} = \frac{g}{1 - g}(1 + t),$$

as can be verified from equation (2.13).

In what follows, it will be assumed that $P(y = \infty \mid f_2) = 0$. This will reduce the problem to that of finding an integral equation only for $P(1, Z < x)$ as will be evident from the following equations:

$$\begin{aligned} P(1, Z < x) &= (1 - g)\, P(1, w < a, y(w + 1) < x) \\ &\quad + (1 - g)\,(1 - h)\,(\mu + \nu)\, P(y(Z_0 + 1) < x \mid 1), \end{aligned} \tag{3.02}$$

$$P[3, k] = \mu = P(1, Z < \infty) - P(1, Z < a), \qquad (k = 1, \cdots, n_3), \tag{3.03}$$

$$P[4, k] = \nu = P(2, Z < \infty) - P(2, Z < a), \qquad (k = 1, \cdots, n_4), \tag{3.04}$$

$$P(1, Z < \infty) + P(2, Z < \infty) + n_3\mu + n_4\nu = 1. \tag{3.05}$$

Let $G_1(y)$ be the cumulative distribution of y given state 1. Then from (3.02)

$$\begin{aligned} P(1, Z < x) &= (1 - g) \int_0^a G_1\left(\frac{x}{1 + t}\right) dP(1, Z < t) \\ &\quad + (1 - g)(1 - h)(\mu + \nu) G_1\left(\frac{x}{1 + Z_0}\right). \end{aligned} \tag{3.06}$$

Interchanging order of integration yields

$$P(1, Z < x) = (1 - g)\left[\int_0^{x/(1+a)} P(1, w < a)\, dG_1(y) + \int_{x/(1+a)}^{x} P(1, w < (x/y) - 1)\, dG_1(y)\right] + (1-g)(1-h)(\mu+\nu)G_1\left(\frac{x}{1+Z_0}\right) \tag{3.07}$$

$$= (1-g)\left[P(1, w < a)G_1\left(\frac{x}{1+a}\right) + (1-h)(\mu+\nu)G_1\left(\frac{x}{1+Z_0}\right)\right] + (1-g)\int_{x/(1+a)}^{x} P(1, w < (x/y) - 1)\, dG_1(y).$$

In terms of the quantities defined in (3.01) to (3.05) $\lim_{N\to\infty}\pi_{kN}$ with $h = g$ is given by

$$\lim_{N\to\infty} \pi_{1N} = P(1, Z < a) + (1-g)(\mu+\nu), \tag{3.08}$$

$$\lim_{N\to\infty} \pi_{2N} = P(2, Z < a) + g(\mu+\nu), \tag{3.09}$$

$$\lim_{N\to\infty} \pi_{3N} = n_3\mu, \tag{3.10}$$

$$\lim_{N\to\infty} \pi_{4N} = n_4\nu. \tag{3.11}$$

The computation of the quantities involved in (3.08) to (3.11) can be simplified by the following device.

Let $\rho = \mu + \nu$. Assign a value to ρ, say ρ'. Solve for $P(1, Z < x)$ from either (3.06) or (3.07). Call the solution $P'(1, Z < x)$. Compute $P'(2, Z < x)$ from (3.01). Compute μ' from (3.03) and ν' from (3.04). Compute

$$D = P'(1, Z < \infty) + P'(2, Z < \infty) + n_3\mu' + n_4\nu'. \tag{3.12}$$

Then

$$P(1, Z < x) = \frac{P'(1, Z < x)}{D}, \tag{3.13}$$

$$P(2, Z < x) = \frac{P'(2, Z < x)}{D}, \tag{3.14}$$

$$\mu = \frac{\mu'}{D}, \qquad \nu = \frac{\nu'}{D}. \tag{3.15}$$

4. Optimum quality control rule when inspection costs are considered or when tests are destructive. As was previously pointed out, in case inspection costs are taken into consideration, or when inspection is destructive, a quality control rule R must not only specify when to stop inspection and put the machine in the repair shop, but it must also specify which items are to be inspected. That is, R must be a *continuous inspection plan* as well as a stop rule.

The income considerations involved in the present situation are somewhat different from those in case 1. To begin with, the cost of inspecting an item has to be specified. In addition, the income from an item may depend not only on its quality but also on whether or not it has been inspected. This is obvious if inspection destroys the item. But even if the tests are not destructive, throwing away or repairing a defective item, for example, may involve a different cost consideration from that of selling a possible defective item with a resulting loss of good will, etc.

To distinguish between the two types of income functions, let $V_0(x)$ be the income of an uninspected item of quality x and $V(x)$ be the income of an inspected item of quality x. It may be assumed that inspection costs have already been reflected in $V(x)$. In addition let c_j $(j = 3, 4)$ be, as above, the cost of repair when the machine is in state j. Again as above let $I_N(R)$ represent the average income per unit of time if the production process has been in operation for N time units and the rule R is employed. Then

$$\begin{aligned} EI_N(R) &= \pi_{01N}\, E[V_0(x) \mid f_1] + \pi_{02N}\, E[V_0(x) \mid f_2] + \pi_{1N}\, E[V(x) \mid f_1] \\ &\quad + \pi_{2N}\, E[V(x) \mid f_2] - \pi_{3N}c_3 - \pi_{4N}c_4 , \end{aligned} \tag{4.01}$$

where in the N time units π_{0iN} $(i = 1, 2)$ is the expected proportion of time units in which items are not inspected and the machine is in state i, π_{iN} $(i = 1, 2)$ is the expected proportion of time units in which items are inspected and the machine is in state i and π_{jN} $(j = 3, 4)$ is the expected proportion of time units the machine is in state j (i.e., repair).

A rule R^* will be called Bayes if it yields $\max_R \lim_{N\to\infty} EI_N(R)$.

Without going through the details of the argument, which are similar to the case previously considered, the optimum rule R^* is characterized as follows: Let

$$y_n = \frac{f_2(x_n)}{(1 - g)f_1(x_n)} \tag{4.02}$$

if in the nth stage of production the nth item is inspected, and let

$$y_n = \frac{1}{1 - g} \tag{4.03}$$

if in the nth stage of production the nth item is not inspected. Let

$$Z_n = y_n\,(1 + Z_{n-1}), \qquad Z_0 = 0. \tag{4.04}$$

Assume that when the machine leaves the repair shop the first item is not inspected. Then for suitably chosen positive constants a^* and b^* with $b^* < a^*$, R^* is the rule which states that items are not inspected as long as $Z_n < b^*$. Inspection begins as soon as $Z_n \geq b^*$, and inspection continues until either $Z_n < b^*$ or $Z_n \geq a^*$. In the former case production continues but inspection terminates, in the latter case inspection terminates and the machine is put in the repair shop.

It is to be noted that whenever for some n_0, $Z_{n_0} < b^*$, the number of items to be skipped is completely determined. For if k is the number of items to be skipped, then k must satisfy the equation

$$Z_{n_0+k} = \sum_{j=1}^{k} \left(\frac{1}{1-g}\right)^j + \left(\frac{1}{1-g}\right)^k Z_{n_0} \geq b^*. \tag{4.05}$$

Summing the above equation and solving for k yields

$$k = \left[\log\left(\frac{gb^* + 1}{gZ_{n_0} + 1}\right) \Big/ -\log(1-g)\right], \tag{4.06}$$

where the symbol $[t]$ stands for the smallest integer greater than or equal to t. The interesting fact is that R^* prescribes that inspection or noninspection shall occur in batches of items.

5. Integral equations for the Markov process in case inspection costs are considered. The integral equations for the limiting distribution of the present Markov process are understandably more complicated. They are obtained as follows:

As in the previous case, let Z_0 be arbitrary and let h be the corresponding a posteriori probability. Let $Z = y(1 + w)$, where y is defined by (4.02) and (4.03). For any arbitrary a and b (a^* and b^* are obtainable by a maximization process), let the symbols $P(1, Z < x)$, $P(2, Z < x)$, and $P[i, k]$ have the same meaning as in the previous case. Then

$$P(2, Z < x) = \frac{g}{1-g} \int_0^x (1 + t)\, dP(1, Z < t), \tag{5.01}$$

$$\begin{aligned} P(1, Z < x) = (1-g) \Big\{ &P\left(1, w < b, \frac{1+w}{1-g} < x\right) \\ &+ P(1, b \leq w < a, y(w+1) < x) + (1-h)(\mu + \nu) P\left(\frac{Z_0 + 1}{1-g} < x\right)\Big\}, \end{aligned} \tag{5.02}$$

$$P[3, k] = \mu = P(1, Z < \infty) - P(1, Z < a), \quad (k = 1, 2, \cdots, n_3), \tag{5.03}$$

$$P[4, k] = \nu = P(2, Z < \infty) - P(2, Z < a), \quad (k = 1, 2, \cdots, n_4), \tag{5.04}$$

$$P(1, Z < \infty) + P(2, Z < \infty) + n_3\mu + n_4\nu = 1. \tag{5.05}$$

Let $G_1(y)$ be the cumulative distribution of y for an inspected item given that the machine is in state 1. Then the integral equation for $P(1, Z < x)$ is given by

$$P(1, Z < x) = (1 - g) \min [P(1, w < b)\, P(1, w < (1 - g)x - 1)]$$

$$+ (1 - g)\left[\int_0^a G_1\left(\frac{x}{1 + t}\right) dP(1, w < t) + (1 - h)(\mu + \nu)\eta \left(x - \frac{Z_0 + 1}{1 - g}\right)\right], \tag{5.06}$$

where $\eta = 1$ if argument is positive, $\eta = 0$ otherwise.

Equation (5.06) can also be written as

$$\begin{aligned} P(1, Z < x) = (1 - g)\Big[&\min (P(1, w < b), P(1, w < (1 - g)x - 1)) \\ &+ (1 - h)(\mu + \nu)\eta\left(x - \frac{Z_0 + 1}{1 - g}\right)\Big] \\ + (1 - g)\Big[&G_1\left(\frac{x}{1 + a}\right) (P(1, w < a) - P(1, w < b)) \\ &+ \int_{x/(1+a)}^{x/(1+b)} \left\{P\left(1, w < \frac{x}{y} - 1\right) - P(1, w < b)\right\} dG_1(y)\Big]. \end{aligned} \tag{5.07}$$

The limiting probabilities involved in (4.01) for this process are

$$\lim_{N\to\infty} \pi_{01N} = P(1, Z < b) + (1 - g)(\mu + \nu), \tag{5.08}$$

$$\lim_{N\to\infty} \pi_{02N} = P(2, Z < b) + g(\mu + \nu), \tag{5.09}$$

$$\lim_{N\to\infty} \pi_{1N} = P(1, Z < a) - P(1, Z < b), \tag{5.10}$$

$$\lim_{N\to\infty} \pi_{2N} = P(2, Z < a) - P(2, Z < b), \tag{5.11}$$

$$\lim_{N\to\infty} \pi_{3N} = n_3\mu, \qquad \lim_{N\to\infty} \pi_{4N} = n_4\nu. \tag{5.12}$$

The previous remarks about computing the integral equation apply to this case also.

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