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Nicholas M. Kiefer
Princeton University

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Nicholas M. Kiefer
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The patterns and determinants of consumers' demand for commodities are studied so that consequences of changes in income and prices can be predicted and our understanding of consumer's behavior can be increased. The pure theory of demand, in particular the hypothesis of utility maximization, facilitates empirical analysis by providing restrictions on the parameters of systems of demand equations; these restrictions may be used to reduce the number of parameters to be estimated. Houthakker (1960) used the assumption of additivity in the utility function combined with the theory of utility maximization to provide an estimable system of demand equations. Barten (1964) relaxed Houthakker's additivity assumption and imposed the theoretical restrictions stochastically using the Theil-Goldberger (1961) technique. In these and other studies the theory of demand was used as a tool to make possible or improve the explanation of expenditure variation by price and income variation.

In 1967 A.P. Barten published the paper "Evidence on the Slutsky Conditions for Demand Equations," which provided the first formal test of the classical restrictions. Since then much of the work on demand analysis has centered on improving the estimation techniques for systems of demand equations and on testing the restrictions.^{1/} This line of research led to

^{1/}See Barten (1969), Byron (1970), and Deaton (1972).

the 1975 paper by Christensen, Jorgenson and Lau in which the authors provide a system of demand equations appropriate for their tests, perform the statistical tests, and reject the theory of demand.

Recently however some authors, for example Abbott and Ashenfelter (forth/coming) Kiefer and MacKinnon (1975) and Philips (1975), have expressed concern about the meaning or appropriateness of tests of the restrictions arising from utility theory. These restrictions are on the parameters of the exact demand functions of a single individual; the demand functions actually estimated are often approximations and are estimated with aggregate data. Can the classical restrictions still be useful even if they do not hold exactly? This question is treated in section two from a Bayesian point of view and is answered affirmatively.

The model we estimate, discussed in detail in section one, attempts to explain the joint decision to supply labor and demand commodities. Empirical work on this joint decision has been done by Abbott and Ashenfelter, whose purpose is to provide a framework in which to make empirical judgements about the efficiency or "optimality" of income or commodity taxation,^{2/} and by Barnett (1974).

In section one the restrictions arising from the hypothesis of utility maximization are outlined and the functional form of the system we estimate is presented. Section two includes a description of the Bayesian approach to the estimation of the system discussed in section one and a derivation of a Bayesian estimator. This estimator is compared with more conventional estimators. An interpretation of the conventional tests of the theoretical ^{2/}As treated theoretically by Diamond and Mirlees (1971), for example.

restriction is provided; this interpretation answers questions previously raised about the appropriateness of such tests. In section three the Bayesian estimates are presented (conventional estimates are also presented for reference purposes), while the paper concludes with general comments on the Bayesian vs. the conventional approach.

1. The Results from Utility Theory

The consumer is assumed to maximize a differentiable utility function

$$(1-1) \quad u = u(q) = u(q_0, \dots, q_n)$$

where q_0 is leisure and q_1, \dots, q_n are n goods, subject to a budget constraint

$$(1-2) \quad p'q = m$$

where p_1 is the price of q_1 and m is non-labor income. Solving the necessary conditions for the demand equations yields

$$(1-3) \quad q = q(p, m).$$

Using the Slutsky decomposition of the effect of a price change into an income and a substitution effect

$$(1-4) \quad S = \partial q / \partial p + (\partial q / \partial m) q'$$

and using the transformation $dz = z dnz$, Theil (1967) writes a first-order approximation to (1-3) as^{3/}

$$(1-5) \quad \hat{w} d \ln q = K d \ln p + b d \ln y$$

where \hat{w} denotes the diagonal matrix whose i th element is w_i , the ratio of the expenditure on the i th good to nonlabor income. K is the matrix $\hat{p} S \hat{p} / m$ and $b = \hat{p} \partial q / \partial m$. The quantity y is an index of the change in nonlabor income $d \ln y = d \ln m - w' d \ln p = w' d \ln q$.

^{3/}We follow Deaton's (1974) elegant presentation of this model.

The classical restrictions are restrictions on the partial derivatives of the demand functions (Goldberger (1967)). The Engel aggregation conditions

$$(1-6) \quad b'1 = 1$$

and the Cournot aggregation conditions

$$(1-7) \quad K'1 = 0$$

are direct results of differentiating (1-2) with respect to m and P .

The other restrictions are the result of utility maximization. They are

$$(1-8) \quad \text{Homogeneity} \quad K1 = 0,$$

$$(1-9) \quad \text{Symmetry} \quad K = K',$$

and negativity; K must be negative semidefinite. In practice only Engel aggregation, Cournot aggregation, homogeneity and symmetry are imposed (but see Barten and Geyskens (1975) where negativity is imposed).

Since hours of labor supplied is equal to total available time minus hours of leisure q_0 the labor supply function has derivatives which are equal in magnitude but opposite in sign to those of the leisure demand function. Consequently we can and will interpret the first equation in (1-5) as the negative of a labor supply function. For purposes of estimation we use finite differences of the logarithms of prices and quantities and define the share of a good to be the average of the shares at the endpoints of the first difference. A constant term is added to each equation of (1-5). The system of equations to be estimated is known as the Rotterdam system and was used by Theil (1967) and Barten (1967, 1969) and others.^{4/}

^{4/} This parameterization is slightly different from that used by Abbott and Ashenfelter, who define y in (1-5) as full income rather than nonlabor income.

The system will be estimated for seven aggregates of commodities; 1) durables, 2) food, 3) clothing, 4) other nondurables, 5) housing services, 6) transportation services, 7) other services, and labor supply. The data are from the National Income and Product Accounts and are described in detail in Abbott and Ashenfelter (1974). The time series run from 1929 to 1967 and the observations are annual.

II. A Bayesian Estimation Procedure

A normally distributed error term will be added to equations (1-5). This error will be assumed to have, in each time period, mean zero and constant covariance matrix Σ . This covariance matrix can be shown to be singular (using the Cournot aggregation conditions) so, for purposes of estimation, one equation will be dropped. Its parameters are uniquely determined by the parameters of the other equations. The labor supply equation is dropped and the remaining equations are written

$$(2-1) \quad y = X\beta + \epsilon$$

where y is a $nT \times 1$ vector (T is the number of periods) whose first T elements are the observations on the first equation and so on, X is a matrix equal to $I \otimes Z$ where Z is the $T \times (n+3)$ matrix of independent variables, β is an m -vector of parameters to be estimated and ϵ is normally distributed error with mean zero and variance $\Sigma \otimes I$. Thus

$$(2-2) \quad y \sim N(X\beta, \Sigma \otimes I).$$

is the likelihood function. The restrictions on β may be written

$$(2-3) \quad \beta = Q\gamma;$$

using the Bayesian framework we will take this as prior information with some uncertainty. Indeed our assumption is related to the Lindley-Smith (1972) assumption of exchangeability. We take as a prior density on β

$$(2-4) \quad \beta \sim N(Q\gamma, \sigma^2 I).$$

This prior reflects a feeling that the coefficients β_i satisfy the classical restrictions outlined in section one and (loosely) that no one β_i is more likely to violate the restriction than any other.

Assume for the moment that Φ and σ^2 are known. Then, taking a diffuse prior for γ the posterior distribution for β may be found by combining the prior (2-4) with the likelihood (2-2). (Following Lindley and Smith (1972)). The posterior distribution of β is normal with mean, used here as an estimator,

$$(2-5) \quad \beta^* = [X'(\Phi^{-1} \otimes I)X + \frac{1}{\sigma^2}(I - Q(Q'Q)^{-1}Q')]^{-1} X'(\Phi^{-1} \otimes I)y$$

and variance equal to the inverse of the term in brackets. This estimator is a weighted sum of the unrestricted Gauss-Markov estimator, which will be recognized as the first term in the brackets, and the prior information, with weights equal to the inverses of the respective variances.

The value of the prior restrictions is clearly shown by (2-5). In situations in which there are a lot of commodities and not many data points the restrictions conventionally become identifying restrictions; this is true in the case of (2-5) since the term $X'(\Phi^{-1} \otimes I)X$ need not be invertible. On the other hand, when the sample is very large relative to the number of commodities the restrictions are not necessary for estimation; indeed they have a diminishing effect as the size of the sample increases. The value of the restrictions, measured as their effect on the parameter estimates,

is lower when the sample size is larger. This entirely desirable property is the cause of Abbott and Ashenfelter's concern about their tests of the restrictions.

When testing the classical restrictions (2-3) Abbott and Ashenfelter comment that "...with a large enough sample we are bound to ... reject these null hypotheses if they are even slightly amiss. ..." In consequence they use a conventional estimator but weaker-than-conventional tests proposed by Toro-Vizcarrondo and Wallace (1968) and Wallace (1972). Thus Abbott and Ashenfelter try to find a way to use the classical restrictions but cannot because they are faced with the choice of either accepting or rejecting them.

The estimator (2-5) is not plagued with problems in imposing restrictions when the data set is large. Indeed, the conventional tests of the restrictions can now be seen as tests of the hypothesis that the variance in the restrictions σ^2 is zero against the hypothesis that the precision $1/\sigma^2$ is zero. Clearly (2-5) covers a large middle ground.

Unfortunately the variances ξ and σ^2 are unknown in practice. We assume vague prior knowledge about both; that is, the prior densities are^{5/}

$$(2-6) \quad p(\sigma^2) \propto 1/\sigma^2$$

and

$$(2-7) \quad p(\xi) \propto |\xi|^{-\frac{(n+1)}{2}}.$$

The likelihood function (2-2) may be combined with the prior densities (2-4), (2-6), (2-7) and a uniform density for γ to form a joint density over β , γ ,

^{5/}The problem of estimating the variance-covariance matrix of a normal distribution has not yet been completely resolved. See Lindley (1971) pp. 68-69 for a discussion of the problems. Our prior (2-7) is adopted by Zellner (1971) on the basis of invariance, a criteria due to Jeffreys (1961).

γ , ξ , and σ^2 . Upon integrating out γ the remaining density is proportional to

$$(2-8) \quad |\xi|^{-\frac{(T+n+1)}{2}} \exp\{-\frac{1}{2}(Y-X\beta)'(\xi^{-1}\otimes I)(Y-X\beta)\} \\ \times (\sigma^2)^{-\frac{m}{2}} \exp\{-\frac{1}{2}(\beta-Q(Q'Q)^{-1}Q'\beta)'(\beta-Q(Q'Q)^{-1}Q'\beta)\}.$$

The nuisance parameters ξ and σ^2 can be integrated out of (2-8)^{6/} and the mean of the β -margin can in principle be used as a point estimate for β . In fact the mean of the marginal distribution would have to be found by numerical integration in m -space^{7/}; we choose instead to use a modal approximation suggested by Lindley and Smith (1972), which leads to tractable estimating equations. Setting the derivatives of (2-8) with respect to ξ and σ^2 equal to zero and solving yields the modal estimators

$$(2-9) \quad \beta^* = \sum_{t=1}^T (Y_t - X_t\beta)(Y_t - X_t\beta)' / (T+n-1)$$

and

$$(2-10) \quad s^2 = (\beta-Q(Q'Q)^{-1}Q'\beta)'(\beta-Q(Q'Q)^{-1}Q'\beta)/m.$$

The estimates can be inserted into equation (2-5) to give a value of β^* .

In practice we set ξ equal to the identity matrix, $1/\sigma^2$ equal to zero, use (2-5) to estimate β^* and iterate until the process converges. Each iteration is linear.

^{6/}Box and Tiao (1973) warn against the practice of casually integrating out nuisance parameters, although their reasons are not the same as ours.

^{7/} $m = 70$ in our case.

III. Results

Bayes estimates of the parameters of the equation system (1-5) are presented in Table 1.^{8/} For comparison the conventional ordinary least squares estimates are reported in Table 3. A comparison indicates that the prior information arising from demand theory had an effect on the estimates of the parameters of the system, but not a large effect. This is as can be expected from the Bayesian approach; as the sample size gets large the effect of the restrictions is negligible,^{9/} for smaller samples the restrictions are necessary for estimability.

Elasticities are calculated to give an idea of what the estimates mean. These are presented in Table 2 and are calculated at the 1967 values of the data. The labor supply elasticities are of particular interest as they are rarely estimated jointly with commodity demand elasticities. Killingsworth (1975) suggests that uncompensated labor supply elasticities calculated from annual time series data are likely to be negative, or small if positive, because annual wage changes are likely to be permanent and the income effect will be strong. Our estimate, $-.18$, is in accordance with Killingsworth's prior beliefs and with estimates using other models and other data sets, (Ashenfelter and Heckman (1973) estimate an elasticity of $-.15$, using cross-section data). The estimates imply that the supply curve of labor is backward bending. Our estimated compensated elasticity of labor supply is zero

^{8/}These were computed on the IBM 360/91 computer at the Princeton University Computing Center. Convergence, defined as the maximum absolute change in β being less than $.00001$, was achieved in five iterations each of which took approximately twenty seconds.

^{9/}This is, of course, true of Bayesian estimation in general - as the sample size increases the likelihood function dominates the prior density.

Table 1

Bayes Estimates of the Rotterdam Model^{1/}

Commodity	Const.	b ₁	K _{1j} for j =							
			1	2	3	4	5	6	7	8
1. Durables	-.027 (.008)	.218 (.021)	-.301 (.226)	.232 (.173)	-.556 (.239)	.060 (.193)	-.701 (.226)	-.074 (.158)	.239 (.323)	.761 (.217)
2. Food	.011 (.005)	.254 (.015)	.239 (.156)	-.223 (.119)	.615 (.165)	-.523 (.133)	.551 (.156)	-.267 (.109)	-.231 (.221)	-.273 (.148)
3. Clothing	.000 (.004)	.071 (.011)	-.325 (.115)	-.058 (.088)	-.289 (.121)	.291 (.098)	.128 (.115)	.355 (.080)	.171 (.165)	-.093 (.111)
4. Other Non- durables	.001 (.000)	.031 (.001)	.002 (.012)	-.004 (.009)	.063 (.013)	-.043 (.011)	.051 (.012)	.014 (.009)	-.049 (.018)	-.013 (.012)
5. Housing Services	-.002 (.002)	.073 (.005)	.049 (.052)	.005 (.039)	.094 (.054)	-.058 (.044)	-.081 (.051)	.009 (.036)	.025 (.073)	.030 (.049)
6. Transporta- tion Services	.015 (.003)	.082 (.008)	.087 (.087)	-.092 (.067)	.214 (.092)	.182 (.074)	.197 (.087)	-.387 (.061)	.073 (.125)	-.209 (.084)
7. Other Services	.007 (.003)	.090 (.007)	-.134 (.076)	-.055 (.058)	.354 (.080)	.030 (.065)	.292 (.076)	.017 (.053)	-.185 (.108)	-.203 (.073)
8. Leisure	-.005	.181	.383	.195	-.495	.061	-.437	.333	-.043	.000

^{1/} Numbers in parentheses are the square roots of the variances in the posterior density.

Table 2

Estimated Elasticities

	Uncompensated Own-Price Elasticities		Compensated Own-Price Elasticities		Non-Labor Income Elasticities ^{1/}	
	Bayes	OLS	Bayes	OLS	Bayes	OLS
1. Durables	-1.77	-1.82	-1.55	-1.60	1.12	
2. Food	-.540	-.549	-.286	-.295	.326	
3. Clothing	-.472	-.468	-.401	-.396	.099	
4. Other Non-Durables	-.448	-.449	-.417	-.408	.301	
5. Housing Services	-.343	-.350	-.270	-.277	.243	
6. Transportation Services	-.927	-.923	-.845	-.841	.179	
7. Other Services	-.383	-.380	-.293	-.290	.142	
8. Labor Supply	-.181	-.188	0.000	-.007	-.082	

^{1/} These estimates are equal to the accuracy reported, there is no reason to expect them always to be equal in the Bayes and OLS cases.

(Ashenfelter and Heckman estimate a substitution elasticity of .12). The nonlabor income elasticities of demand for commodities all have the expected signs; an increase in nonlabor income will increase the demand for commodities and decrease the supply of labor.

IV. Conclusions

The theoretical advantages of the Bayesian approach to estimation are discussed in a number of places, including Lindley (1971) and Jeffreys (1961). The estimation of the parameters of a system of demand equations has been found to be an attractive practical application of Bayesian ideas and methods. In particular the implications of classical utility theory for the behavior of a single consumer can be used as prior information in the study of aggregate expenditure patterns. This seems to be an appealing way to use these theoretical restrictions on aggregate data in view of the weak connection between restrictions derived for an individual consumer and the properties of aggregate demand and supply functions. The Bayes estimator (2-5) uses the restrictions more or less according to how much information is supplied by the data: this is a departure from the conventional practice of testing the restrictions whenever there are enough data to do so.

The Bayes estimator (2-5) has been applied to a practical problem and was found to give sensible results and to be easily computed. In view of its desirable properties it is a useful alternative to conventional estimators of the parameters of the Rotterdam model.

Table 3

OLS Estimates of the Rotterdam Model^{1/}

Commodity	Const.	b _i	i	K _{ij} for j =							
				2	3	4	5	6	7	8	
1. Durables	-.026 (.008)	.218 (.076)	-.310 (.228)	-.243 (.173)	.545 (.241)	.063 (.194)	-.689 (.228)	-.071 (.159)	.231 (.325)	.743 (.218)	
2. Food	.011 (.005)	.254 (.015)	.245 (.157)	-.230 (.120)	.604 (.166)	-.514 (.134)	.548 (.157)	-.255 (.109)	-.223 (.224)	-.277 (.150)	
3. Clothing	.000 (.004)	.071 (.011)	-.326 (.115)	-.060 (.088)	-.285 (.122)	.286 (.098)	.126 (.115)	.347 (.081)	.164 (.165)	-.079 (.110)	
4. Other Non- durables	.001 (.000)	.031 (.001)	.002 (.012)	-.005 (.009)	.063 (.013)	-.042 (.011)	.051 (.012)	.015 (.009)	-.049 (.018)	-.013 (.012)	
5. Housing Services	-.002 (.002)	.073 (.005)	.051 (.051)	.003 (.039)	.091 (.054)	-.057 (.044)	-.083 (.051)	.010 (.036)	.025 (.073)	.032 (.049)	
6. Transporta- tion Services	.015 (.003)	.082 (.008)	.086 (.087)	-.093 (.066)	.215 (.092)	.183 (.075)	.199 (.087)	-.385 (.061)	.075 (.125)	-.215 (.084)	
7. Other Services	.007 (.003)	.090 (.007)	-.135 (.076)	-.053 (.058)	.353 (.080)	.030 (.065)	.292 (.075)	.018 (.053)	-.183 (.108)	-.206 (.072)	
8. Leisure	-.006	.181	.387	.195	-.496	.051	-.393	.321	-.040	.015	

^{1/} Standard errors are in parentheses.

References

- Abbott, M. and O. Ashenfelter, "Labor Supply, Commodity Demand, and the Allocation of Time," Review of Economic Studies, forthcoming.
- Ashenfelter, O. and J. Heckman, "Estimating Labor-Supply Functions," in G. Cain and H. Watts (eds.), Income Maintenance and Labor Supply, Chicago: Markham, 1973.
- Barnett, W.A., "Labor Supply and the Allocation of Consumption Expenditure," unpublished paper, August, 1974.
- Barten, A.P., "Consumer Demand Functions Under Conditions of Almost Additive Preferences," Econometrica 32:1 (1964) pp. 1-38.
- _____, "Evidence on the Slutsky Conditions for Demand Equations," The Review of Economics and Statistics, 49:1 (1967).
- _____, "Maximum Likelihood Estimation of a Complete System of Demand Equations," European Economic Review 1:1 (1969) pp. 7-73.
- _____, and E. Geyskens, "The Negativity Condition in Consumer Demand," European Economic Review, 6:3 (1975) pp. 227-260.
- Box, G.E.P. and G.C. Tiao, Bayesian Inference in Statistical Analysis. Reading: Addison-Wesley, 1973.
- Byron, R.P. "The Restricted Aitken Estimation of Sets of Demand Relations," Econometrica, 38:6 (1970) pp. 816-830.
- Christensen, L.R., D.W. Jorgenson, and L.J. Lau, "Transcendental Logarithmic Utility Functions," American Economic Review 65:3 (1975), pp.367-383.
- Deaton, A.S., "The Estimation and Testing of Systems of Demand Equations: A Note," European Economic Review 3:4 (1972), pp. 399-401.
- _____, "The Analysis of Consumer Demand in the United Kingdom, 1900-1970," Econometrica 42:2 (1974), pp. 341-367.
- Diamond, P.A. and J.A. Mirrlees, "Optimal Taxation and Public Production II: Tax Rules," American Economic Review, 64:2 (1974), pp. 175-180.
- Goldberger, A.S., "Functional Form and Utility: A Review of Consumer Demand Theory," unpublished paper, October 1967.
- Houthakker, H.S., "Additive Preferences," Econometrica, 28:2 (1960), pp. 244-257.
- Jeffreys, H., Theory of Probability (3rd ed.). Oxford: Clarendon, 1961.

Kiefer, N.M., and J.G. MacKinnon, "Transcendental Logarithmic Utility Functions: Comment," unpublished paper, August 1975.

Killingsworth, M., "Neoclassical Labor Supply Models: A Survey of Recent Literature on Determinants of the Supply of Time to the Labor Market," unpublished paper, September 1975.

Lindley, D.V., Bayesian Statistics, A Review. Philadelphia: Society for Industrial and Applied Mathematics, 1971.

_____, and A.F.M. Smith, "Bayes Estimates for the Linear Model," Journal of the Royal Statistical Society, Series B, 34:1 (1972), pp. 1-18.

Phlips, L. "Comment," Annals of Economic and Social Measurement, 4:1 (1975), pp. 45-48.

Theil, H. and A.S. Goldberger, "On Pure and Mixed Statistical Estimation in Economics," International Economic Review, 2:1 (1961), pp. 65-78.

_____, Economics and Information Theory. Amsterdam: North-Holland, 1967.

Toro-Vizcarrondo, C. and T.D. Wallace, "A Test of the Mean Square Error Restrictions in Linear Regression," Journal of the American Statistical Association, 63:322 (1968), pp. 558-572.

Wallace, T.D., "Weaker Conditions and Tests for Linear Restrictions in Regression," Econometrica, 40:4 (1972), pp. 689-698.

Zellner, A., An Introduction to Bayesian Inference in Econometrics. New York: Wiley, 1971.