Journal of Statistical and Econometric Methods, vol.1, no.3, 2012, 39-52
ISSN: 2241-0384 (print), 2241-0376 (online)
Scienpress Ltd, 2012

A Bayesian Estimation of Stable Distributions

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Abstract

Stable distributions are a rich class of probability distributions that are widely used to model leptokurtic data. Since the probability density and distribution functions are not known in closed form, stable distributions are often specified by their characteristic functions. This paper reviews both the techniques used to compute the density functions and the methods used to estimate parameters of the stable distributions. A new Bayesian approach using Metropolis random walk chain and direct numerical integration is proposed. The performance of the method is examined by a simulation study.

Mathematics Subject Classification: 46N30, 60E07, 62F15, 97K80
Keywords: Characteristic exponent, Distribution functions, Metropolis-Hastings algorithm, Parameter estimation, Posterior distribution, Stable distribution

1 Introduction

Stable distributions allow skewness and heavy tails. Therefore, they are widely used in modeling heavy tailed data. Stable distributions, also called "Levy-Pareto distributions" are used to describe complex systems in physics,

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The views expressed are those of the authors and should not be attributed to the Central Bank of the Republic of Turkey.

Article Info: *Received* : September 22, 2012. *Revised* : October 24, 2012
Published online : November 30, 2012

biology, sociology and economics [1]. Recent studies show that stable distributions have been used for modeling stock returns, foreign exchange rate changes, commodity-price movements, and real estate returns [2].

A stable distribution does not have an analytic closed form but can be expressed by its characteristic function,

$$\phi_X(t) = E\left(e^{itx}\right) = \exp\left(i\delta t - |ct|^{\alpha}\left[1 + i\beta\,\mathrm{sgn}(t)\,w(t,\alpha)\right]\right) \tag{1}$$

where $w(t,\alpha) = \begin{cases} -\tan\left(\dfrac{\pi\alpha}{2}\right) & , \quad \alpha \neq 1 \\ (2/\pi)\ln|t| & , \quad \alpha = 1 \end{cases}$

$-\infty<t<\infty$, $0<\alpha\leq 2$, $|\beta|\leq\min(\alpha, 1-\alpha)$, $c>0$, $-\infty<\delta<\infty$.

A stable distribution has four parameters α, β, δ and γ ($\gamma=c^{\alpha}$). The parameter α is called the characteristic exponent (or index of stability) and it is interpreted as a shape parameter. The normal distribution is stable with $\alpha=2$ and is the only stable distribution for which second and higher absolute moments exist. When $\alpha<2$, absolute moments of order equal to and greater than α do not exist while those of order less than α do. δ and c are the location and scale parameters respectively. β is the skewness parameter. When β is positive, the distribution is skewed to the right. When β is negative, the distribution is skewed to the left. When β is zero, the distribution is symmetric about location parameter, δ. As α approaches 2, β loses its effect and the distribution approaches the normal distribution regardless of β [3].

2 Computation of density functions

It is a widely known fact that stable distributions are appropriate for modeling extreme events because of allowing heavy tails. However, their usage has been limited due to lack of closed form of probability density and distribution functions.

Researchers have worked on different techniques in order to compute probability density functions. Zolotarev [1,4,5] and Skorohod [6] approximately calculated the probability density and distribution function of stable distribution by using series expansion and proper integral representations. Nolan [7], improved the integration approach of Zolotarev [1]. Doganoglu and Mittnik [8] showed another algorithm for calculating the stable probability density function by using Fast Fourier Transform (FFT).

Zolotarev [1], introduced five different parameterizations for the characteristic function of a stable distribution. Nolan [9], defined another parameterization based on Zolotarev’s (M) parameterization that is jointly continuous in all four parameters. Joint continuity makes parameters more meaningful and parameter estimation well behaved over the entire parameter space.

The direct numerical integration method will be introduced in the following way:
The new parameterization that Nolan [7] used in integration is a slight variation of (1) and can be given as:

$$\ln\varphi_{X^0}(t)=\begin{cases}-c^{\alpha}|t|^{\alpha}[1+i\beta\tan(\pi\alpha/2)\operatorname{sgn}(t)(c|t|^{1-\alpha}-1)]+i\delta^0 t & ,\ \alpha\neq 1\\ -c|t|[1+i\beta(2/\pi)\operatorname{sgn}(t)(\ln|t|+\ln c)]+i\delta^0 t & ,\ \alpha=1\end{cases}\tag{2}$$

Some definitions of Nolan [7] are

$$\zeta=\zeta(\alpha,\beta)=\begin{cases}-\beta\tan\dfrac{\pi\alpha}{2}, & \alpha\neq 1\\ 0, & \alpha=1\end{cases}\tag{3}$$

$$\theta_0=\theta_0(\alpha,\beta)=\begin{cases}\dfrac{1}{\alpha}\tan^{-1}(\beta\tan\dfrac{\pi\alpha}{2}), & \alpha\neq 1\\ \dfrac{\pi}{2}, & \alpha=1\end{cases}\tag{4}$$

$$c_1(\alpha,\beta)=\begin{cases}\dfrac{1}{\pi}(\dfrac{\pi}{2}-\theta_0), & \alpha<1\\ 0, & \alpha=1\\ 1, & \alpha>1\end{cases}\tag{5}$$

$$V(\theta;\alpha,\beta)=\begin{cases}(\cos\alpha\theta_0)^{1/(\alpha-1)}\left(\dfrac{\cos\theta}{\sin\alpha(\theta_0+\theta)}\right)^{\alpha/(\alpha-1)}\left(\dfrac{\cos\theta_0+(\alpha-1)\theta}{\cos\theta}\right), & \alpha\neq 1\\ \dfrac{2}{\pi}\left(\dfrac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right)\exp\left(\dfrac{1}{\beta}\left(\dfrac{\pi}{2}+\beta\theta\right)\tan\theta\right), & \alpha=1,\ \beta\neq 0\end{cases}\tag{6}$$

Then, the probability density function for different values of α can be given by

$\alpha\neq 1$ & $x>\zeta$:

$$f(x;\alpha,\beta)=\frac{\alpha(x-\zeta)^{1/(\alpha-1)}}{\pi|\alpha-1|}\int_{-\theta_0}^{\pi/2}V(\theta;\alpha,\beta)\exp\left(-(x-\zeta)^{\alpha/(\alpha-1)}V(\theta;\alpha,\beta)\right)d\theta \tag{7}$$

$\alpha \neq 1$ & $x = \zeta$:

$$f(\zeta\ ;\alpha,\beta\ =\frac{\Gamma\left(1+(1/\alpha)\right)\cos\theta_0}{\pi(1+\zeta^2)^{(1/2\alpha)}} \tag{8}$$

$\alpha \neq 1$ & $x<\zeta$:

$$f(x;\alpha,\beta) = f(-x;\alpha,-\beta) \tag{9}$$

$\alpha = 1$:

$$f(x;1,\beta)=\begin{cases}\dfrac{1}{2|\beta|}e^{-(\pi x/2\beta)}\displaystyle\int_{-\pi/2}^{\pi/2}V(\theta;1,\beta)\exp\left(-e^{-(\pi x/2\beta)}V(\theta;1,\beta)\right)d\theta, & \beta\neq 0\\ \dfrac{1}{\pi(1+x^2)}\quad, & \beta=0\end{cases} \tag{10}$$

3 Parameter estimation methods

3.1 General Methods

Quantile, sample characteristic function, maximum likelihood and Bayesian methods are different estimation techniques for parameters of stable distributions. Fama and Roll [3,10] provided estimates for parameters of standardized symmetric stable distributions. They used series expansions to approximate probability density and distribution functions. They found parameter estimates using quantiles. McCulloch [11] generalized and improved the quantile method for skewness parameter, β. Press [12] presented his method via method of moments using sample characteristic function. Koutrouvelis [13] also used sample characteristic function but the method that he showed was a regression-type method for the estimation. DuMouchel [14] tried to get approximate maximum likelihood estimates of α. Later, maximum likelihood estimation was enhanced [15,16]. Hill proposed Hill estimator in 1975 [17]. Georgiou and Tsakalides [18] introduced Sinc Function and wavelet transform methods [19].

3.2 Bayesian Inference

Buckle [20] proposed a Bayesian inference by using MCMC (Markov Chain Monte Carlo) method, especially the Gibbs sampler.

Posterior density in Bayesian inference can be given as,

$$\pi(\theta|x) \propto f(x|\theta)\, \pi(\theta)$$

If the probability density function of x is not obtainable in closed form whereas the joint probability density function of x and y exists, then the posterior density is found by taking integration,

$$\pi(\theta|x) \propto \int f(x,y|\theta)\, \pi(\theta)dy$$

Let f(z,y) be given by

$$f(z,y\,|\,\alpha,\beta) = \frac{\alpha}{|\alpha-1|}\exp\left\{-\left|\frac{z}{t_{\alpha,\beta}(y)}\right|^{\alpha/(\alpha-1)}\right\}\left|\frac{z}{t_{\alpha,\beta}(y)}\right|^{\alpha/(\alpha-1)}\frac{1}{|z|} \tag{11}$$

where

$f: (-\infty,0)*(-1/2, l_{\alpha,\beta}) \cup (0,\infty)*(l_{\alpha,\beta}, 1/2) \rightarrow (0,\infty)$

$$t_{\alpha,\beta}(y) = \left(\frac{\sin\left[\pi\alpha y + \eta_{\alpha,\beta}\right]}{\cos \pi y}\right)\left(\frac{\cos \pi y}{\cos\left[\pi(\alpha-1)y + \eta_{\alpha,\beta}\right]}\right)^{(\alpha-1)/\alpha}$$

and $\alpha \in (0,1) \cup (1,2]$, $\beta \in [-1,1]$, $z \in (-\infty,\infty)$, $y \in (-1/2,1/2)$, $\eta_{\alpha,\beta} = \beta\min(\alpha,2-\alpha)\pi/2$, $l_{\alpha,\beta} = -\eta_{\alpha,\beta}/\pi\alpha$.

Then, Buckle [20] explained that the function in (11) is a bivariate probability density function for the distribution of (Z,Y) and the marginal distribution of Z is a stable distribution with parameters α, β, $\delta=0$, and $c=1$:

$$f(z\,|\,\alpha,\beta) = \frac{\alpha|z|^{1/(\alpha-1)}}{|\alpha-1|}\int_{-1/2}^{1/2}\exp\left\{-\left|\frac{z}{t_{\alpha,\beta}(y)}\right|^{\alpha/(\alpha-1)}\right\}\left|\frac{1}{t_{\alpha,\beta}(y)}\right|^{\alpha/(\alpha-1)} dy$$

The bivariate density in (11) is used in order to have a representation of the posterior density of the stable parameters:

$$\pi(\alpha,\beta,\delta,c\,|\,\underset{\sim}{x}) \propto \int\left(\frac{\alpha}{|\alpha-1|c}\right)^{n}\exp\left\{-\sum_{i=1}^{n}\left|\frac{z_i}{t_{\alpha,\beta}(y_i)}\right|^{\alpha/(\alpha-1)}\right\}\prod_{i=1}^{n}\left|\frac{z_i}{t_{\alpha,\beta}(y_i)}\right|^{\alpha/(\alpha-1)}\frac{1}{|z_i|}\pi(\alpha,\beta,\delta,c)\,dy$$

where $z_i = (x_i-\delta)/c \neq 0$, $i=1,\ldots,n$, $\alpha \in (0,1) \cup (1,2]$, $\beta \in [-1,1]$, $\delta \in (-\infty,\infty)$, $c \in (0,\infty)$.

The Gibbs sampler can be implemented by the following procedure. At first, for each observation x_i, y_i is generated from $f(y_i\,|\,\alpha,\beta,\delta,c,x_i)$. After generating vector $\underset{\sim}{y}$, $\pi(\alpha\,|\,\beta,\delta,c,\underset{\sim}{x},\underset{\sim}{y})$, $\pi(\beta\,|\,\alpha,\delta,c,\underset{\sim}{x},\underset{\sim}{y})$, $\pi(\delta\,|\,\alpha,\beta,c,\underset{\sim}{x},\underset{\sim}{y})$ and $\pi(c\,|\,\alpha,\beta,\delta,\underset{\sim}{x},\underset{\sim}{y})$ are generated.

This approach presents some difficulties. Due to lack of information about the shape of the conditional distributions in the generation process, sequential Metropolis steps is resorted. If there are n observations in a model, there have to be n augmentation variables whose movement around the parameter space will be highly serially correlated when they are updated on a sequential basis and they result in a decline in efficiency. Tsionas [21] considered using Metropolis sampler that updates all components of the parameter vector at the same time in order to have simpler and more efficient method. He computed the likelihood by means of FFT and used a Metropolis random walk chain to explore the parameter space. Not using data augmentation makes the computations very fast and nearly independent of the sample size "n". Although Metropolis chains do induce serial correlation, their performance is expected to be better when compared to augmentation methods since they update all components of the parameter vector at the same time. Tsionas [21] used a regression model given by:

$$y_i=g(x_i;\phi)+\sigma u_i \text{ , } i=1,\ldots,n$$

(g is a given function, ϕ a parameter vector, σ an unknown scale parameter, u_is are independent and identically distributed random variables from a standard stable distribution with parameters α and β.

The steps for the procedure can be given as:

Let $\theta=[\phi, \sigma, \alpha, \beta]'$ be parameter vector and $\hat{\theta}$ maximum likelihood estimator.

$\sqrt{n}\,(\hat{\theta}-\theta) \to N(0, V(\theta))$ in distribution where $V(\theta)=-E\left[\partial^2 L(\theta; x, y)/\partial\theta\partial\theta'\right]^{-1}$.

Let $h(\theta \mid \theta^*, d^2 V(\hat{\theta}))$ represent a fixed density with location vector θ^*, scale matrix $d^2 V(\hat{\theta})$ (d>0). Then, Metropolis-Hastings chain produces a sequence $\theta^{(i)}$ that converges in distribution to $p(\theta|x,y)/\int p(\theta|x,y)d\theta$. Given $\theta^{(i)}$, $\theta^{(i+1)}$ is produced as:

$\tilde{\theta}$ is a draw from $h(\theta \mid \theta^{(i)}, d^2 V(\hat{\theta}))$ which is multivariate normal $(N(\theta^{(i)}, d^2 V(\hat{\theta})))$.

With probability:

$$\pi(\theta, \tilde{\theta})=\min[1,p(\tilde{\theta}|x,y)/p(\theta^{(i)}|x,y)], \text{ set } \theta^{(i+1)}=\tilde{\theta}$$

else with probability $1-\pi(\theta, \tilde{\theta})$, set $\theta^{(i+1)}=\theta^{(i)}$.

3.3 Proposed Bayesian Approach

Our proposed Bayesian approach, which does not use data augmentation, consists of Metropolis random walk chain and therefore increases the efficiency. In addition to this, the direct numerical integration method as an alternative to the FFT method is used to compute the likelihood function required for Bayesian estimation. The advantage of using the direct numerical integration method is that, the probability density function value for a particular value can be computed via direct numerical integration method, whereas the FFT method basically needs calculation of a set of probability density function values [8]. As a result, the aim of this approach is to find a better technique used for estimation of the stable distribution parameters.

4 Simulation Experiment

The experiment consists of generating stable random variables and using Metropolis random walk chain in order to find parameter estimates. First of all, standard stable random variables of size n with parameter vector $\theta=[\alpha,\beta]'$ having the characteristic function in (2) are generated. Then, the likelihood function is as follows

$$L(\theta; y) = \prod_{i=1}^{n} f\left(y_i \mid \alpha, \beta\right)$$

and posterior density can be given as

$$f(\theta|y) \propto L(\theta;y)\pi(\theta).$$

The random sample of different sizes (25, 50, 75, 100, 500 and 1000) is generated from a standard stable distribution with parameters $\theta=[1.3, 0.5]'$. The maximum likelihood estimator of θ and its variance-covariance matrix is calculated for each sample. $\theta_0=[0.5, 0.8]'$, $\theta_0=[1.5, 0.2]'$ and maximum likelihood estimates are taken as starting values. 50 independent sequences of length 1000, 3000, 5000 and 10000 are produced. After that, the final values of each sequence are used in order to obtain an approximate independent and identically distributed sample from posterior density. The comparisons according to different starting values, sample sizes and iteration numbers can be made by using Table 1.

The analysis of variance tables are constructed in order to test the effects of different sample sizes, starting values and iteration numbers on the estimates of α and β parameters (tables 2-3). It can be concluded that starting values and iteration numbers don't have any effect on the estimates while the effect of sample size exists with 99 percent confidence. When sample size gets larger, the estimated values get closer to real values. It can also be seen from Figures 1 and 2.

As a result of having no effect on estimates, any starting value and iteration number can be chosen. Therefore, initial value is chosen as $\theta_0=(\alpha_0=0.5, \beta_0=0.8)'$ and independent sequences of length 1000 are produced in order to test the performance of the proposed method. Performance test is done by comparing Bayesian estimates with the maximum likelihood, quantile and sample characteristic function estimates at different sample sizes. 50 different random samples of sizes 25, 50, 75, 100, 500 and 1000 are generated from a standard stable distribution with parameters $\theta=[1.3, 0.5]'$. 50 independent sequences of length 1000 are produced for each 50 sample of different size group. The results show that the mean square errors of Bayesian estimates are a bit less than the mean square errors of the maximum likelihood estimates whereas much less than those of the other two estimates. Moreover, all estimates' mean square errors are getting smaller when the sample size is getting larger (Table 4).

5 Conclusion

Different parameter estimation techniques like quantile, sample characteristic function and maximum likelihood can be used for the stable distributions. In addition to these methods, Bayesian inference can also be used for the distributions and it is introduced in details. Buckle's Bayesian method puts in use the data augmentation that can cause serial correlation. Tsionas [21] introduced a more efficient method which doesn't use data augmentation. He calculated the likelihood by means of FFT and used a Metropolis random walk chain to explore the parameter space. In this paper, the likelihood is computed with the usage of the direct numerical integration method instead of FFT with the purpose of having a more effective Bayesian inference. Simulation experiment is done for testing the effect of starting values, iteration numbers and sample sizes. It is found that only the sample size has an effect on estimates. Bayesian estimates are compared with the maximum likelihood, quantile and sample characteristic function estimates at different sample sizes. According to the results, the mean square error of Bayesian estimates is less than the mean square error of all the other estimates.

Table 1: Simulation Results

		Iteration Numbers			
		1000		3000	
Sample Size	Starting Values	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
25	α_0=0.5, β_0=0.8	1.4050	0.0495	1.3185	-0.0451
	α_0=1.5, β_0=0.2	1.4096	0.1439	1.3823	0.1326
	α_0=1.36, β_0=0.10 (mle)	1.4192	0.0791	1.4503	0.0475
50	α_0=0.5, β_0=0.8	1.1310	0.2157	1.1561	0.2405
	α_0=1.5, β_0=0.2	1.0962	0.2092	1.1244	0.2103
	α_0=1.12, β_0=0.25 (mle)	1.1165	0.1914	1.1429	0.2262
75	α_0=0.5, β_0=0.8	1.2185	0.5833	1.1934	0.5508
	α_0=1.5, β_0=0.2	1.2247	0.5754	1.2174	0.5700
	α_0=1.22, β_0=0.61 (mle)	1.2506	0.5864	1.2265	0.5449
100	α_0=0.5, β_0=0.8	1.2416	0.4064	1.2241	0.3730
	α_0=1.5, β_0=0.2	1.2093	0.3837	1.1895	0.3568

	α_0=1.21, β_0=0.39 (mle)	1.2319	0.3544	1.2494	0.3598
500	α_0=0.5, β_0=0.8	1.3223	0.4303	1.3163	0.4334
	α_0=1.5, β_0=0.2	1.3205	0.4204	1.3124	0.4253
	α_0=1.31, β_0=0.42 (mle)	1.3095	0.4272	1.3287	0.4216
1000	α_0=0.5, β_0=0.8	1.2807	0.5160	1.3031	0.5474
	α_0=1.5, β_0=0.2	1.2913	0.5390	1.2899	0.5551
	α_0=1.28, β_0=0.55 (mle)	1.2851	0.5443	1.2813	0.5422

Table 1: Simulation Results (continued)

		Iteration Numbers			
		5000		10000	
Sample Size	Starting Values	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
25	α_0=0.5, β_0=0.8	1.4001	0.0632	1.3827	0.0240
	α_0=1.5, β_0=0.2	1.4158	0.0950	1.3451	0.0993
	α_0=1.36, β_0=0.10 (mle)	1.4537	0.0128	1.3451	-0.0073
50	α_0=0.5, β_0=0.8	1.1756	0.1948	1.1174	0.2164
	α_0=1.5, β_0=0.2	1.1802	0.2664	1.1324	0.1998
	α_0=1.12, β_0=0.25 (mle)	1.1420	0.1690	1.1202	0.1400
75	α_0=0.5, β_0=0.8	1.2248	0.5557	1.2465	0.5612
	α_0=1.5, β_0=0.2	1.2476	0.5756	1.2654	0.5934
	α_0=1.22, β_0=0.61 (mle)	1.2354	0.5583	1.2431	0.6173
100	α_0=0.5, β_0=0.8	1.2076	0.3845	1.2142	0.3856
	α_0=1.5, β_0=0.2	1.2164	0.3557	1.2357	0.4106
	α_0=1.21, β_0=0.39 (mle)	1.2305	0.3828	1.2085	0.3368
500	α_0=0.5, β_0=0.8	1.3108	0.4289	1.3083	0.4136
	α_0=1.5, β_0=0.2	1.3094	0.4175	1.3211	0.4060
	α_0=1.31, β_0=0.42 (mle)	1.3099	0.4346	1.3153	0.4231
1000	α_0=0.5, β_0=0.8	1.2832	0.5462	1.2862	0.5406
	α_0=1.5, β_0=0.2	1.2913	0.5305	1.2824	0.5516
	α_0=1.28, β_0=0.55 (mle)	1.2828	0.5470	1.2827	0.5331

Table 2: Analysis of Variance for α

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	Significance Level
Sample Size	0.4700	5	0.09393	185.704*	0.000
Starting Values	0.0008	2	0.00043	0.848	0.433
Iteration Numbers	0.0022	3	0.00072	1.431	0.243
Error	0.0309	61	0.00051		
Total	0.5040	71			

* Significant at % 5 level.

Table 3: Analysis of Variance for β

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F	Significance Level
Sample Size	2.3640	5	0.47300	574.555*	0.000
Starting Values	0.0068	2	0.00340	4.131*	0.021
Iteration Numbers	0.0014	3	0.00045	0.551	0.649
Error	0.0521	61	0.00082		
Total	2.4230	71			

* Significant at % 5 level.

Table 4: Comparisons of Estimation Methods

Sample Size		Bayesian Estimate		Maximum Likelihood Estimate	
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
25	Mean	1.1670	0.1903	1.1447	0.2506
	Bias	-0.1330	-0.3097	-0.1553	-0.2494
	MSE	0.0464	0.1715	0.0630	0.1997
50	Mean	1.2780	0.4008	1.2665	0.4683
	Bias	-0.0220	-0.0992	-0.0335	-0.0317
	MSE	0.0280	0.0737	0.0335	0.1001
75	Mean	1.2798	0.3882	1.2711	0.4313
	Bias	-0.0202	-0.1118	-0.0289	-0.0687
	MSE	0.0192	0.0696	0.0226	0.0745
100	Mean	1.2987	0.4179	1.2938	0.4649
	Bias	-0.0013	-0.0821	-0.0062	-0.0351
	MSE	0.0211	0.0684	0.0244	0.0791
500	Mean	1.3139	0.5005	1.3115	0.5092
	Bias	0.0139	0.0005	0.0115	0.0092
	MSE	0.0035	0.0109	0.0033	0.0108
1000	Mean	1.3042	0.4861	1.3035	0.4929
	Bias	0.0042	-0.0140	0.0035	-0.0071
	MSE	0.0024	0.0047	0.0025	0.0044

MSE: Mean Square Error

Table 4: Comparisons of Estimation Methods (continued)

Sample Size		Quantile Estimate		Sample Characteristic Function Estimate	
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
25	Mean	1.0136	0.2456	1.2206	0.2283
	Bias	-0.2864	-0.2544	-0.0794	-0.2717
	MSE	0.1585	0.1772	0.0676	0.3267
50	Mean	1.3297	0.5327	1.2928	0.4205
	Bias	0.0297	0.0327	-0.0072	-0.0795
	MSE	0.0998	0.1455	0.0451	0.1564
75	Mean	1.2617	0.4548	1.3123	0.3565
	Bias	-0.0383	-0.0452	0.0123	-0.1435
	MSE	0.0583	0.0765	0.0287	0.1937
100	Mean	1.2713	0.4493	1.3130	0.4648
	Bias	-0.0287	-0.0507	0.0130	-0.0352
	MSE	0.0544	0.0965	0.0262	0.1003
500	Mean	1.3178	0.5244	1.3024	0.5095
	Bias	0.0178	0.0244	0.0024	0.0095
	MSE	0.0091	0.0180	0.0039	0.0129
1000	Mean	1.3128	0.5077	1.3014	0.4968
	Bias	0.0128	0.0077	0.0014	-0.0032
	MSE	0.0050	0.0066	0.0036	0.0106

MSE: Mean Square Error

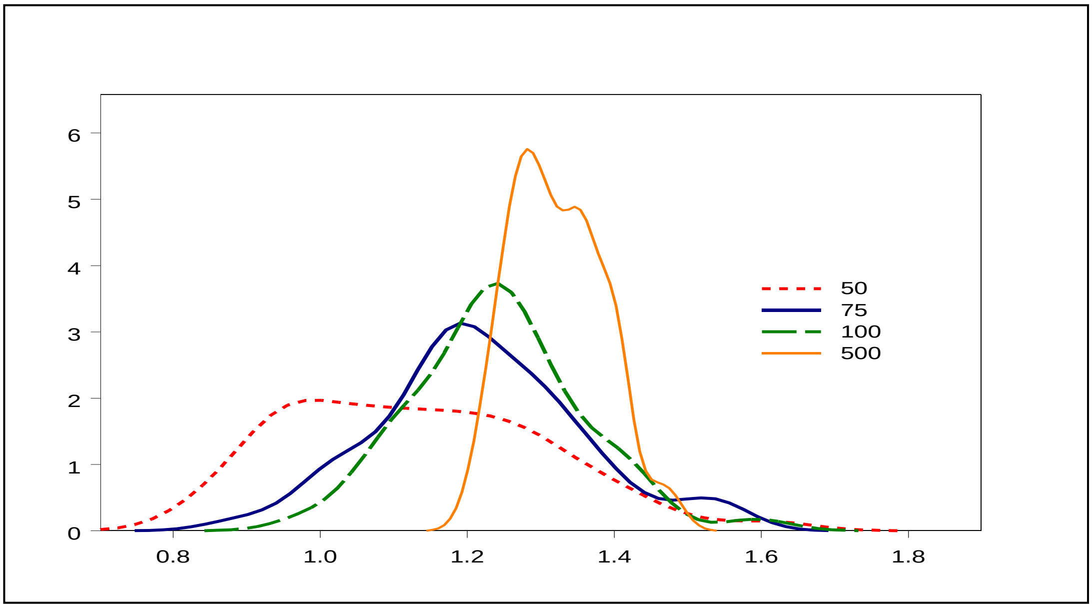

Figure 1: Bayes estimates of α at different sample sizes

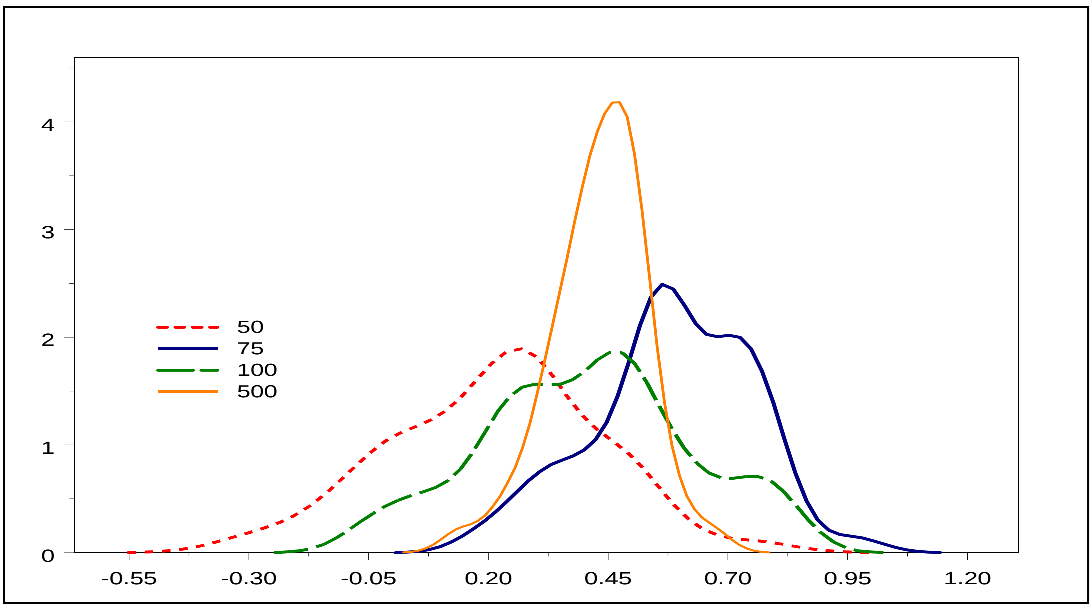

Figure 2: Bayes estimates of β at different sample sizes

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