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Title: A BAYESIAN MODEL FOR THE DETERMINATION OF OPTIMAL SAMPLING INTERVALS

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Dr. Edward D. McDowell

There are a nearly unlimited number of situations in which the status of time-varying processes must be updated. The monitoring of these processes usually occurs at periodic intervals. Whether the monitoring is performed by man or machine, a decision must be made regarding the frequency of these activities, that is, an optimal sampling interval must be determined.

This thesis presents two theoretical models, based on a Bayesian analysis, from which optimal sampling intervals can be determined. The necessary information for the use of these models includes the sampling costs, quadratic error costs, and a normally distributed measure of the uncertainty of the process as a function of the time since the last sample. This uncertainty can be measured either objectively, from historical data, or subjectively, from the decision maker's personal knowledge of the process. The first model assumes that immediately after sampling, the decision maker knows precisely the value of the process. That is, the variance at the time of sampling is zero. In the second model, this assumption is not made. A certain amount of uncertainty exists immediately after sampling.

This uncertainty can be reduced by taking a larger sample size.

With a knowledge of the value, or a distribution of the values of the process when a sample is taken, the decision maker "forecasts" values for the period until the next sample. Action will be taken on the basis of these forecast values. An error in these values will cause inappropriate actions to be taken. An error cost will be incurred on the squared difference between these two values. The extent of the difference will be dependent on the degree of uncertainty the decision maker has regarding the process. By sampling more frequently, he reduces the uncertainty and therefore the error cost, but increases the sampling cost. The sampling interval (and in the case of the second model, the sample size) that minimizes the sum of these costs determines the optimal sampling policy.

This thesis develops the necessary equations and suggest solution techniques from which these optimal intervals can be determined. A sensitivity analysis is also performed to show the effects of changes in cost parameters on the optimal sampling interval.

A Bayesian Model for the Determination of Optimal Sampling Intervals

by

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Assistant Professor of Industrial and General Engineering
in charge of major

Redacted for Privacy

Head of Department of Industrial and General Engineering

Redacted for Privacy

Dean of Graduate School

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From Senders, 1955 and 1964

$f_i(t)$	the sequence of pointer positions in time generated by instrument i.
$\phi_i(\omega)$	the power density spectrum of instrument i.
W_i	the cutoff frequency of instrument i.
A_i	the power of the message from instrument i.
E_i	the allowed RMS error between the original and the recovered messages of instrument i.
FF_i	the fixation frequency of instrument i.
H_i	the average amount of information from instrument i that the operator must assimulate at each sampling.
D_i	the duration of fixation on instrument i.
T_i	the proportion of total time spent on instrument i.
P_i	the probability of fixation on instrument i.

From Carbonell, 1966

M	the number of instruments.
t	the time of the observation.
$C_j(t)$	the total cost of looking at instrument j at time t.
C_i	the cost associated with exceeding the threshold of instrument i.
$P_i(t)$	the probability that instrument i will exceed the threshold at instant t.
$\hat{p}_i(t)$	the priority of instrument i at time t.
$\hat{a}_i$	a random number representing the last value read.
b_i	a constant of instrument i, relating the cost, variances, and divergencies
t_o	the time of the last reading.

From Sheridan, 1970

$\{x\}$	the probability density function of x.
$\langle V\|xy\rangle$	the expected value of V, given x and y
$\int_x$	the integral over x

Chapter III

μ_x	the mean of the prior distribution of x.
σ_x^2	the variance of the prior distribution of x.
τ	the time since the last sample.
T	the sampling interval.

$x(\tau)$ — the value of x at time τ.

$E[x(\tau)]$ — the expected value of $x(\tau)$.

$f(x|\tau)$ — the prior distribution of x given time τ.

$V_x(\tau)$ — the variance or uncertainty of x at time τ. This is assumed to be a known function of τ, continuous and monotonically increasing with time.

$V_x(0)$ — the posterior variance. This is the variance immediately after sampling, when $\tau = 0$. $V_x(0) = 0$.

$V_x(T)$ — the prior variance. This is the variance immediately prior to sampling when $\tau = T$.

T^* — the optimal sampling interval.

EVSI — the expected value of sample information.

ENGSI — the expected net gain of sampling information.

A — the limit of the uncertainty function.

b — the growth constant of the uncertainty function.

$\hat{C}$ — $(CA - bS)/CA$

$\hat{T}$ — bT^*

Chapter IV

$O_i(t)$ — the observed value on the ith observation at time t.

$e_i(t)$ — the error term associated with the ith observation at time t.

σ_e^2 — the variance of the error terms.

$\overline{O}(t)$ — the average of the observed values at time t.

n — the number of observations made per sample.

$V_x(n,T,\tau)$ — the uncertainty or variance function of x that is a function of n, T, and τ.

V_{pr} — the prior variance $V_{pr} = V_x(n,T,T)$.

V_{po} — the posterior variance $V_{po} = V_x(n,T,0)$.

T^* — the optimal sampling interval given a specific value for n.

n^* — the optimal sample size.

T^{**} — the optimal sampling interval given the optimal sample size, n^*.

$\sinh(bT^*)$ — the hyperbolic sine $= \dfrac{e^{bT^*} - e^{-bT^*}}{2}$.

$\cosh(bT^*)$ — the hyperbolic cosine $= \dfrac{e^{bT^*} + e^{-bT^*}}{2}$.

Chapter V

$\rho(\tau)$ — the autocorrelation function relating the variances of $x(t)$ and $x(t+\tau)$.

μ_x' — the mean of the posterior distribution.

$\sigma_x'^2$ — the variance of the posterior distribution.

A BAYESIAN MODEL FOR THE DETERMINATION OF OPTIMAL SAMPLING INTERVALS

I. INTRODUCTION

The world around us abounds with numerous situations in which one must periodically check the status of a certain process of interest. In these instances, the major question that arises is, "how often should this sample be taken?" This question becomes particularly important when the process parameters vary in some unpredictable manner and the decision maker's degree of uncertainty about the process increases with time. That is, the longer the time since the last sample, the more uncertain the decision maker is about the status of the process. This concept is illustrated graphically in Figure 1, shown below.

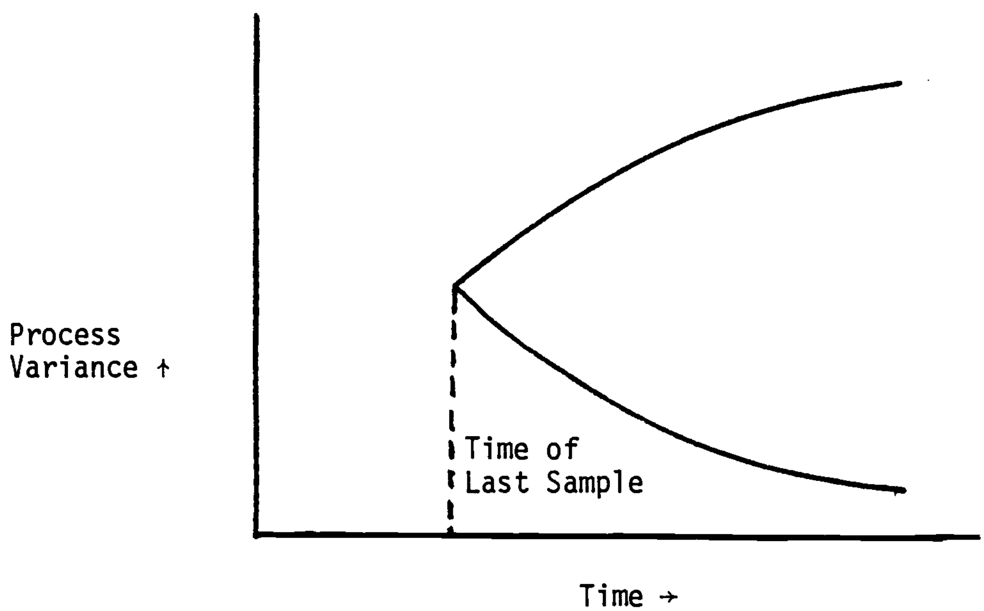

Figure 1. Range of Uncertainty at a Specific Confidence Interval, as a Function of Time Since the Last Sample.

As was mentioned earlier, there are many situations in which answering the sampling question is important. It will be helpful to now discuss a few examples.

The monitoring of continuous processes in industry requires samples to be taken periodically. Control of the process is based on the result of these samples. The refining of crude oil, liquefaction of natural gas, oxygen, or rare gases; the production of paper; and the processing of aluminum, steel, etc., all fall into this category.

Many so-called management functions are also subject to periodic sampling or review. A physical inventory must be taken at some point in time to determine an actual, as opposed to a perpetual inventory. This count is usually performed once a year, but it might be wiser to sample it more frequently. Financial auditing is another example of sampling that is performed on a process or processes. Audits are supposed to find any discrepancies that have developed since the last audit. In some cases, the period between audits may be too long, or on the other hand, it may be too short.

Many times, a company or a contractor will bid on a package of work or a job that will not be started for some period of time. At the time the bid is prepared, a certain amount of uncertainty exists concerning labor and material costs. The company may wish to update their estimate. This may or may not be necessary, however. If it is, it may be advantageous to update the bid more than once.

In a financial or investment situation, sampling problems also exist. An investment counselor must determine how often a portfolio needs to be reviewed. Another question that he is concerned with is how often he should check the price of a certain stock or commodity.

Finally, the transportation area also has some examples. The captain of a ship must evaluate his position in relation to some reference points. Some questions he may have are how often should this be done and does location of the ship, that is, the open seas or a shipping channel, have any bearing on this interval between "fixes." In a similar respect, a pilot of an airplane must make the same decisions.

The preceeding discussion has presented several examples of the sampling problem, as this type of research will be termed. Although this paper makes no pretense to solve all of these problems, it is hoped that the groundwork will be laid and with some additional research in each area, the specific problems may be solved.

The processes to be studied in this paper will, in a way, be similar. In all cases, the objective of the decision maker, supervisor or operator, will be to maximize the system's value function. This function is a function of the uncontrollable, independent input to the system, x; the control variable, y; and the output, Z. This general description of the processes of interest is shown on the following page in Figure 2.

In most cases, the output Z, is a function of the controlled and uncontrolled inputs. ($Z = Z(x,y)$). When this occurs, the variable Z becomes redundant. Therefore, Z can be treated as an intervening variable that may be of no direct interest to the operator. In any case, Z is not necessary in determining the value function. This relationship allows us to write $V = V(x,y)$ and reduces the system in Figure 2 to a simpler system, as shown in Figure 3.

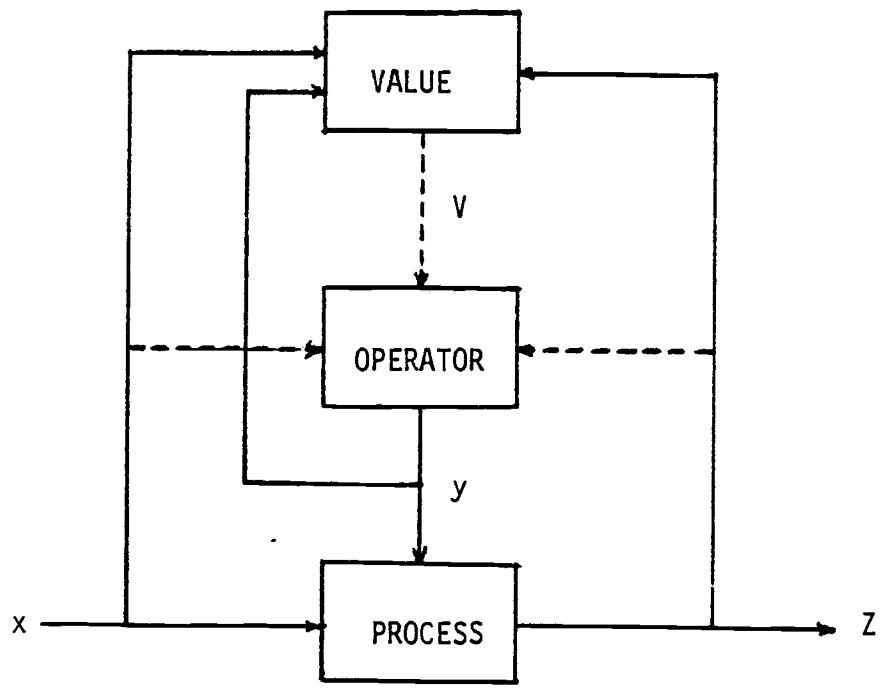

Figure 2. Schematic Diagram of Optimizing Control System
Sheridan [1970]

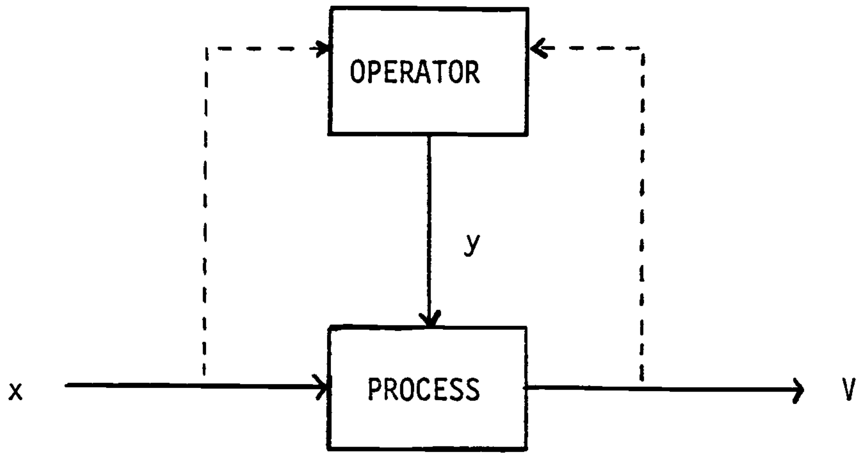

Figure 3. System of Interest to Operator
Sheridan [1970]

The controller may employ one of the three different strategies in his attempt to maximize the value function. At one extreme, he may continuously monitor x and modify y accordingly. That is, for a given x, he will set y so as to maximize V. On the other hand, he may refrain from sampling or monitoring x and set y on the basis of some distributional constraint of x and y. In essence, he is setting y so as to maximize the expected value of V. The third strategy is a mixed strategy in which he samples x periodically, at some time interval T. On the basis of this sample, he adjusts y in a manner similar to the second strategy.

There has been considerable research in continuous controls, where the process changes rapidly, with both automatic and manual operators. Research in areas where the process varies slowly has been limited. This is the area where continuous monitoring is not necessary.

The need for such research is illustrated in Crossman (1971).

> When designing displays, one tends to assume that the operator will attend it continuously or at frequent intervals because he needs all the information that they can provide and that the main problem is insuring that their data are conveyed as quickly and as accurately as possible. But in real life operators ignore displayed data for long periods without ill effect and they can often manage perfectly well with very poor instruments, as a brief visit to almost any factory will show.

Although Crossman is speaking specifically about displays, the same reasoning can be applied to all of the areas discussed previously.

This paper attempts to develop a model for the conditions discussed and proceeds to solve the sampling question for this model.

Why is such a model necessary? The answer is, of course, it will provide a means for answering the sampling question. This is important for many reasons. First, automatic controllers must be programmed. Someone must make the decision concerning the sampling interval. Since there is usually a cost associated with taking a sample, an "overkill," that is, sampling too often, may involve unnecessary expense.

By the same token, human operators must be programmed, that is, they must know how often an observation must be made if they are to operate at maximum efficiency.

A third use for the answers this model may give concerns operator work load. If an operator has to monitor many variables, he could become overloaded, having to sample faster than he is capable. This possibility has long been a concern to aircraft designers in the development of instrument panels used by pilots. The question of overload should also be of interest to managers, not only when determining operator work load, but also in determining managers' work load. How many areas should a supervisor or manager be responsible for? This question is related to how frequently he must look at each unit and the associated variables under his control.

In many situations, an operator will be assigned to monitor a certain number of variables. The monitoring process will not consume his entire time, the remainder will be spent performing other tasks. If a method is available for determining how often he must monitor these variables, standards can be set and the operator's time may be more efficiently used. This model is, "an attempt to devise a method of analysis and assessment (of the) mental loading of process operators." (Kitchin and Graham, 1974)

Scope of the Investigation

This research is of an exploratory nature, and consequently, many simplifying assumptions have been made. Although they are simplifying, they are not unreasonable and can be justified on the basis of the following:

1. The assumptions reasonably approximate real life situations. It is hoped that this research can be applied to actual situations. Therefore, there should be no objections to simplifying the problem.
2. Where exploratory research is involved, it is better to first

attack a relatively simple problem that can be solved rather than a complex problem that cannot be solved or has a solution that cannot be easily understood. It is hoped that the solution to the problems presented here will provide signigicant insight into the complex problem. With additional research, someone may carry this problem a step further. Afterall, the automobile had its beginning with a stone wheel. Therefore, the models presented in this paper may seem simple compared to the general family of problems, but they are neither unrealistic nor trivial.

Problem Formulation

The processes to be investigated have certain characteristics. The process must be normally distributed. This form is the normal or Gaussian distribution. This requirement is a very reasonable one. The normal distribution is very popular because many real-life processes do follow such a distribution. Although the calculations would be different, the approach to solving the case where another distribution is present would be similar.

Another important characteristic of the processes considered concerns the uncertainty. The decision maker's uncertainty about the process error (the difference between the actual value of the process and the forecast value of the process) increase with time and is symmetric and normally distributed. Figure 1, on page 1, shows how the uncertainty increases with time. A discussion of the distribution of this uncertainty will be delayed until later chapters.

Finally, the decision maker's uncertainty about the value of the process must be quantified. This may be done either objectively with historical data or subjectively with the decision maker's personal belief or "gut" feeling. A probability estimate of the process's uncertainty is needed.

The use of subjective estimates may be the most controversial part of this research. Although numerous references will support its use, a nearly equal number will dispute its validity. This disagreement is the fundamental difference between Bayesian and classical statisticians.

There is little disagreement about the probability measure itself or the method by which mathematical probabilities may be manipulated. There are, however, divergent opinions about "the nature of the relations between the mathematics of probability and the real thing and events to which it is applied." (Sheridan and Ferrell 1974, page 30)

> Classical statisticians contend that influences to be defensible, must be based only on the observation or measurement of appropriate data and must not be "biased" by investigator's prior information or beliefs. Bayesian statisticians, on the other hand, contend that the investigator's prior information and beliefs are themselves relevant data and should be considered along with more "objective" data, in making inference. (Weber, 1973, page 1)

Decision theorists and Bayesian statisticians admit a subjective element into the seemingly most objective procedures for determining qualitative probabilities. There does not have to be one correct value, unless the evidence logically entails it.

> The essence of the 'subjective' or 'personalist' view is that probability is intimately related to human decision making, reflecting a person's degree of belief that the event in question will actually occur. Degree of belief in this context is interpreted as the extent to which the belief would contribute to a disposition to act rather than as an intensity of feeling. (Sheridan and Ferrell, 1974, page 31-32)

There is a subjective element in the frequency view of probability. The act of hypothesizing a limiting relative frequency and the reassessment of the probability if the evidence indicate it, introduces some human judgement.

The use of prior distributions and probabilities as well as personal beliefs are important in this research. Therefore, the subjective view of probability will not be discounted.

The costs used in the models developed in this paper are very significant. Anytime that a sample is taken, a sampling cost is incurred. This cost is equal to a constant (set-up cost) plus a variable cost which is linearly related to the number of samples taken. The general relationship is;

$$C_s = S + U \cdot n$$

where C_s = sample cost

S = fixed sampling cost

U = cost per observation

n = number of observations per sample

As was mentioned before, there is also a cost of an error in a forecast. This cost will be proportional to the squared error. The algebraic relationship is;

$$C_e = C\,[x(t) - \hat{x}(t)]^2$$

where C_e = cost of an error

C = quadratic error coefficient

$x(t)$ = process value at time t

$\hat{x}(t)$ = forecast value at time t

The forecast value may be either an "objective" forecast or a subjective estimate by the decision maker. The value of this forecast will cause certain actions to be taken, therefore, the importance of this value.

There are two other important cost functions in addition to the quadratic costs, that could be used, linear and step functions. Quadratic costs will be used in this paper since they not only simplify some calculations later on, but also approximate the other costs reasonably well. Figure 4, on the following page, illustrates this approximation. Although, as the error becomes large, the quadratic costs deviate significantly, the approximation is quite good within the region of interest.

The value function, V, can now be thought of as a cost, or negative value. As the sampling interval is increased (fewer samples per unit time) the sampling cost is decreased, but the cost of an error increases because the uncertainty about the process increases with time. Similarly, as the sampling interval decreases, there is a higher sampling cost but a smaller error cost. Since V is the total of these two costs, the optimal value of T is such that V is minimized, that is, the smallest cost. Restated, the value of the optimal sampling interval T minimizes the total cost of sampling and error costs.

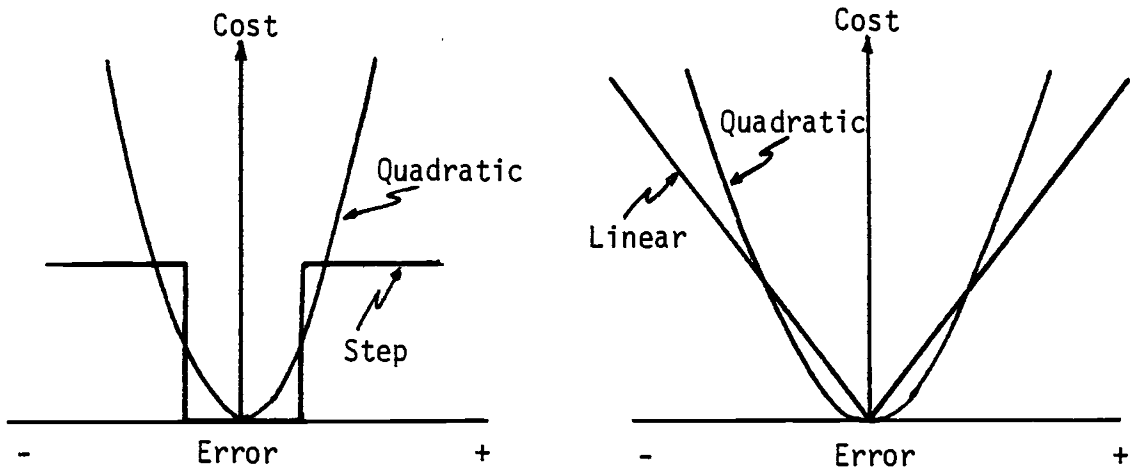

Figure 4. Quadratic Approximation of Step and Linear Costs

II. LITERATURE REVIEW

The problem and situations noted in Chapter I are by no means new. The question being asked, namely, "how often should a sample be taken?" has been asked many times before. Many times, the answers have been determined out of convenience rather than reason. That is to say, a manager decides to look at his reports once a week, the process is checked every hour, etc. In other situations, however, the operator or supervisor will evaluate the process in his own mind, based on his knowledge of the process in question and perform the sampling accordingly. This is an attempt to systematically determine the sampling interval. Only in a few cases is the process actually analyzed, taking into consideration the effects of sampling versus the effects of not sampling.

The reason for this lack of analysis in situations where sampling must be performed could stem from the lack of any widely accepted technique for determining the sampling intervals. Literature on this subject is limited. That which is available is usually limited to a specific area, such as control charts, or multi-instrument sampling associated with aircraft pilots.

Control Charts

Acheson Duncan, a leader in the field of statistical quality control, published one of the first papers in which economic factors were used in determining the design of control charts ($\bar{X}$ charts) (Duncan, 1956). The design is based on the determination of the sample size, the upper and lower control limits, and the time interval between samples. The specific application is for control charts that were designed to maintain control of a process, as opposed to those used to bring a process under control.

The primary objective in designing the charts was to maximize the long run average net income per unit time, assuming there is information or knowledge available concerning the risk of occurrence of an assignable cause that will cause a shift in the process average. Cost and income parameters must also be available. The maximum average income criterion was chosen because it is of most interest to business. Initially, statistical considerations were used to design control charts, but more recent work have used this income criteria (Montgomery and Heikes, 1976).

Two important assumptions are made concerning the process in Duncan's model. First, the process is not shut down while the search is conducted and second, the cost to bring the process back into control after the assignable cause is found is not included in the income function. It is not charged to the control chart program.

Assuming that the probability of a non-occurrence of an assignable cause before time t, when the process begins from a state of control, is $e^{-\lambda t}$ (consistant with waiting analysis), Duncan proceeds to develop implicit equations for the sample size and control limits, and an explicit equation for the sampling interval. The solutions are, however, only approximations. The sample size and the control limits are determined using a graphical technique. The sampling interval, h, is then found using the following equation,

$$h = \sqrt{\frac{\alpha T + b + cn}{\lambda M \ (1/P-1/2)}} \ .$$

A description of the various terms can be found in the Glossary. The sampling interval is a function of the various costs, the sample size, and the various terms that relate to the probability of an occurrence and detection of an assignable cause.

Certain assumptions and limitations are inherent in any model. The fact that such limitations will be mentioned in this paper is not an attempt to criticize the work of the original paper's author. It will, however, point out the differences between the work upon which this paper is based and that work which is being reviewed.

Duncan's article focuses only on control chart processes and in doing so makes its application somewhat limited. Control charts are assumed to be either in control or out of control. It is hard to adjust the control chart model to situations in which this is not the case.

Also, the process is considered Markovian or memoryless in nature. This Markovian model, which has been used extensively, was first applied to quality control by Girshick and Rubin (1952). More recent studies (Baker, 1971 and Heikes, et.al., 1974) have shown that control charts are sensitive to the Markov assumption. While both of these studies develop non-Markovian models, they do not solve the sampling interval problem. The time intervals are referred to as time periods, with no attempt to determine the length of this time period.

One final criticism of Duncan's model is the fact that the values for the design parameters are only approximations. However, an algorithm for determining the exact values has been developed in a subsequent article by Goel, Jain, and Wu (1968).

Multi-Instrument Sampling

In addition to the area of control charts, another field in which the work done in this paper may be applied is the area of multi-instrument sampling, usually associated with aircraft pilots.

Visual system overload is of great importance to aircraft designers. Therefore, there must be a method of determining how often an instrument must be sampled (looked at) or how much time is spent looking at all the instruments.

John W. Senders (1955) proposed the first theoretical model dealing with man's capacity to use information generated from several instruments on a control panel. In a later article (1964), he compares the theoretical calculations to experimental results.

The underlying idea of these articles is that the amount of time spent observing a given instrument will be related to the amount of information presented.

An instrument, i, generates a time function of indicated values $f_i(t)$. From this function, the power density spectrum $\phi_i(\omega)$, can be calculated. The cutoff frequency of ϕ is W_i. If W_i is given, and if the original $f_i(t)$ is to be specifiable from the sample, then according to the information theory, the minimum sampling rate for periodically taken samples must be equal to $2W_i$. Furthermore, if $f_i(t)$ is assumed to be a white noise source, then instrument i is generating information at the rate of $W_i \log_2 \frac{A_i^2}{E_i^2}$ bits/sec., where A_i is the power of the message (RMS amplitude) of instrument i and E_i is the allowed RMS error between the original and the recovered messages of instrument i.

The operator wishes to be able to reconstruct the original time function from remembered or recorded readings. He must, therefore, sample with a fixation frequency (FF_i) at least equal to $2W_i$. If FF_i is exactly equal to $2W_i$, then the average amount of information which the operator must assimulate at each sampling is,

$$H_i = \frac{1}{2W_i} W_i \log_2 \frac{A_i^2}{E_i^2} = \log_2 \frac{A_i}{E_i} \text{ bits}$$

The duration of fixation (D_i), that is the time spent observing the instrument, is linearly related to the amount of information presented in a manner similar to the relation of reaction time and information, $D_i = k \log_2 \frac{A_i}{E_i} + C$ sec. where k has dimensions of time per bit and C, a constant to account for movement time and minimum fixation time has dimensions of seconds per fixation.

The proportion of the total time spent on instrument i (T_i), is related to the amount of information generated.

$$T_i = FF_i \times D_i = 2kW_i \log_2 \frac{A_i}{E_i} + 2W_i C \quad \text{where } FF_i = 2W_i.$$

Senders proceeds to show that the total time spent on an instrument is minimized when $FF_i = 2W_i$.

If the operator has many instruments that he must observe, his total workload will be the summation of the individual workloads. The workload will be minimized when $FF_i = 2W_i$ for all the various instruments. Given m different instruments,

$$\text{Min } T_m = 2 \sum_{i=1}^{m} W_i[k \log_2 \frac{A_i}{E_i} + C].$$

If T_o is the duration of the duty cycle during which all instruments must be observed and $T_o > \text{Min } T_m$, then another instrument may be added if $T_o > \text{Min } T_{m+1}$. In this way, a designer can determine when he must stop adding new instruments.

In the second article by Senders (1964), the theoretical base is extended to include fixation sequences and then tested with actual experiences. Fixation sequence refer to the frequency distributions of transitions from one instrument to another. The sequence of transitions is assumed to be a random series, constrained only by the relative frequencies of fixation of the instrument involved in the transition. Over a sufficiently long interval the relative number of fixations on each instrument will be an estimate of the probability of fixation on that instrument. If P_i equals the probability of fixation on instrument i, then,

$$P_i = \frac{T_i \times FF_i}{T_i \times \sum_{i=1}^{m} FF_i} = \frac{FF_i}{\sum_{i=1}^{m} FF_i}$$

The probability of a transition between two instruments, a and b, will be the product of their individual probabilities of fixation. The probability of going from a to b is $P_{ab} = P_a P_b$ and the probability of transition in both directions, $P_{\overline{ab}}$ is $2P_a P_b$.

The freedom of path through the set of instruments increases as the transition probabilities approach one another and is maximal when the probabilities are equal. There are greater opportunities for logical scanning to occur when the constraints of relative frequency diminish.

In addition to transition probabilities between a pair of instruments, there is also a finite probability of observing the same instrument on two consecutive sampling instances. This fact effects empirical data obtained from measurements from multi-instrument tasks in two ways. First, the measured frequency of observation of instrument a will fall short of the value predicted by $P_a \times FF_a$, samples per second. The observed frequency of a, FF_a, must be corrected,

$$FF_a = FF_a\ (1-P_a) = 2W_a\ (1-P_a) \text{ if } FF_a = 2W_a.$$

Secondly, the probability of an unobservable transition between instruments a and a will equal $P_a^{\,2}$. The observable transition probability between instruments a and b, in either direction, must also be corrected,

$$P_{\overline{oab}} = \frac{2P_aP_b}{1-\sum_{i=1}^{m} (P_i)^2} .$$

Likewise, the observable mean duration of fixation needs correction,

$$D_{oa} = \frac{1}{1-P_a} \left(k \log_2 \frac{A_a}{E_a} + C\right).$$

An experimental test using four instruments with different band widths and five subjects produced results showing the observed transition probabilities very close to the calculated probabilities.

Sender's model does have some limitations, but, these limitations "do not detract from its merit, especially taking its simplicity into account" (Carbonell, 1966, page 158). One such limitation is the fact that no control is assumed on the part of the pilot. He is, in essence, a passive observer who cannot modify the signals. Secondly, there is no cost structure in this model. While E_i can be adjusted to show the need for tight control, it does not show the critical nature of an error, that is, there is no error cost. There is also no provisions for sampling costs, however, in the situation investigated, sampling costs could be assumed to be zero.

This model may give an aircraft designer help in determining the number of instruments to include on a panel, but it does not give the pilot much help. It will predict the number of times an instrument will be looked at and thus gives the pilot an indication of the relative importance of the various instruments, but it does not specify and sequence.

Jaime R. Carbonell (1966) also investigated the muli-instrument sampling problem. However, he tries to predict what instrument should be observed . As with Senders, he is concerned with pilots of aircraft.

The pilot has two types of instruments to observe; desired value instruments, such as air speed, altitude, etc., and threshold instruments, such as indicator lights. All of these instruments compete for his attention. When he looks at one, he postpones looking at the others. Some instruments are "more important" than others. This urgency is measured by the risk of an instrument exceeding a certain threshold while not being observed. The risks vary somewhat, depending on the type of maneuver the pilot is executing. For example, speed and altitude have a greater importance while landing than during level flight.

Carbonell's model assumes that the pilot evaluates the risks involved in not looking at an instrument and tries to minimize this risk. At each sampling instant, he compares the instruments noting the probability and cost of exceeding each threshold.

The variables used in this model are defined below:

M = number of instruments
t = observation time
$C_j(t)$ = the total cost of looking at instrument j at time t
C_i = the cost associated with exceeding the threshold of instrument i
$P_i(t)$ = the probability that instrument i will exceed the threshold at instant t

Given the variables

$$C_j(t) = \sum_{i=1}^{m} \frac{C_i P_i(t)}{1-P_i(t)} - C_j P_j(t)$$

The pilot theoretically will look at the instrument j that makes $C_j(t)$ a minimum. In other words, he will choose the instrument with the largest value $C_j P_j(t)$.

The instruments can now be considered to queue for the observer's attention. Arrivals can be considered to be coincident with service completion. The queue discipline will be based on three priority factors. The first factor is a conditional random number, representing the last known value of the instrument. The probability of an instrument exceeding a threshold will be less if the last reading was "right on" a desired value as opposed to being near the threshold limit. The second factor is based on waiting time. The final factor in the queue discipline is the risk factor. As time increases, the observer's uncertainty about a particular instrument also increases. The risk of exceeding the threshold will also increase and approach one if a sufficiently long period of time has elapsed. Figure 5 shows this growth of uncertainty or risk.

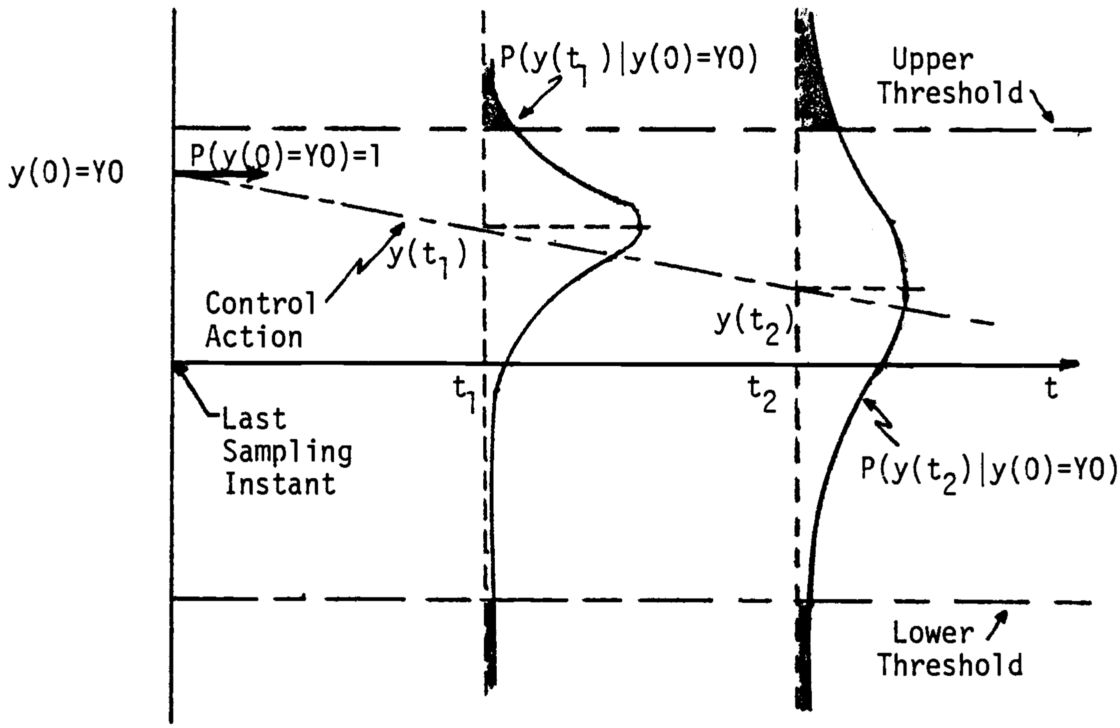

Figure 5. Risk Growth With Time
Carbonell [1966]

The model represents a priority queueing system with complex priorities. The priority function developed is,

$$\hat{p}_i(t) = \hat{a}_i + b_i\ (t-t_o)$$

where $\hat{a}_i$ is a random number representing the last value read, b_i is a constant particular to each instrument (related to the costs, variances, and divergencies), t_o is the time of the last reading, and t is the sampling instant.

In order to solve this problem, Carbonell developed a simulation program to determine the instrument to be sampled at each sampling instant. Two examples were given, one in which 110 decisions were used and in the other, only 70 decisions were used. The various parameters were changed with the different runs.

Tests were to be conducted to see if changing the parameters could match actual test results. Carbonell's model may be useful in explaining actions of pilots, but it is not helpful in determining how often a pilot should sample. Some of the other limitations of the model include the fact that the pilot is assumed to be a first order observer, that is, he observes only the position of the pointer on any given instrument. No consideration is given to the rate of change of the pointer. The cost structure is similar to that of Sender's, a go-no go type situation. Only deviations outside of the threshold are considered significant.

However, Carbonell's discussion concerning the increase of uncertainty as time increases are an integral part of the models being discussed in this thesis.

Supervisory Sampling

The term supervisory sampling, as used in this paper, will refer to the situations in which someone must determine when to sample a process input and what values to specify for the controlled input in order to maximize the system's value function output.

The uncertainty concepts used by Carbonell have been extended by Sheridan (1970) to questions that are generally the same as those in this thesis.

The relationships shown in Figure 3 are repeated in Figure 6. The variables are defined in a manner consistant with Chapter I, that is, x is the independent input, y is the controlled input, and V is the value of the output.

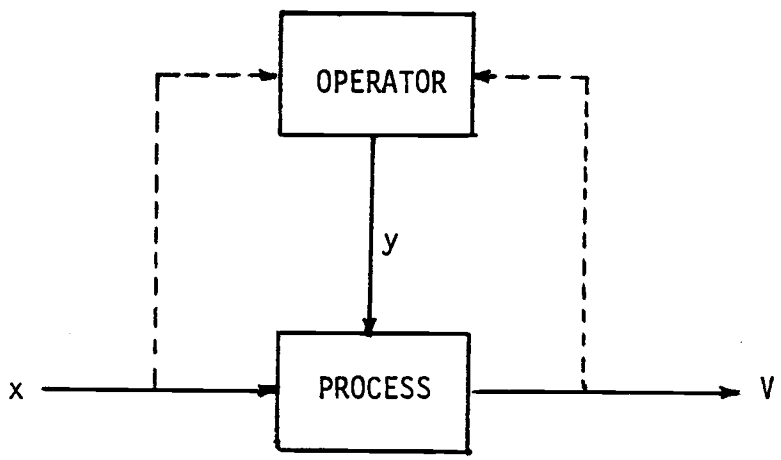

Figure 6. System of Interest to Operator
Sheridan [1970]

It is the controller's objective to adjust the input y in order to maximize V. Three alternatives are open to him. At one extreme, he may set y at a value based on his prior knowledge of the process and not monitor the process at all. At the other extreme, he may monitor x and V continuously and constantly modify y to maximize V. Finally, he may adopt a mixed strategy in which he samples x at regular intervals.

In Sheridan's notation,

x is a random event with known probability density $\{x\}$

V(x,y) is the value gained per unit time and has a known expected value $\langle V|xy\rangle$ when x and y are specified.

The expected value of V(y) is

$$\langle V|y\rangle = \int_x \langle V|xy\rangle \{x\} .$$

If the controller makes a decision based only on the prior distribution $\{x\}$, then the best strategy will be to determine $\langle V|y\rangle$ and adjust y to maximize this function

$$\langle V_1\rangle = \max_y \langle V|y\rangle = \max_y \int_x \langle V|xy\rangle \{x\} .$$

The situation in which the controller samples continuously corresponds to perfect information. In this case, the best strategy will be to find a value for y that maximizes V for all x.

$$\langle V_2|x\rangle = \max_y \langle V|xy\rangle$$

The expected value of V before x is known will be,

$$\langle V_2\rangle = \int_x \langle V_2|x\rangle \{x\} = \int_x \max_y \langle V|xy\rangle \{x\} .$$

In many cases, knowledge of x can be updated periodically. Let x' represent the controller's state of knowledge about x at some intermediate time, and $x = x_0$ at $t = 0$. The distribution $\{x'|x_0 t\}_{t=\infty} = \{x\}$. The best strategy at each t is to optimize y based on the distribution of x' given x_0 and t.

$$<V_3|x_0 yt> = \int_{x'} <V|x'y> \{x'|x_0 t\}$$

where $<V|x'y> = <V|xy>$.

$$<V_3|t> = \int_{x_0} <V_3|x_0 t>\{x\} \text{ since } \{x\} = \{x\}$$

$$= \int_{x_0}[\max_{y} \int_{x'} <V|x'y> \{x'|x_0 t\} \quad \{x_0\}.$$

The expected value per unit time averaged over a sampling interval T is,

$$<V_3>_T = \frac{1}{T} \int_{t=0}^{T} <V_3|t>.$$

If the cost per sample is C,

$$<V_{net}>_T = <V_3>_t - C/T.$$

The optimal strategy is to pick T such that $<V_{net}>_T$ is maximized. Figure 7 illustrates this graphically.

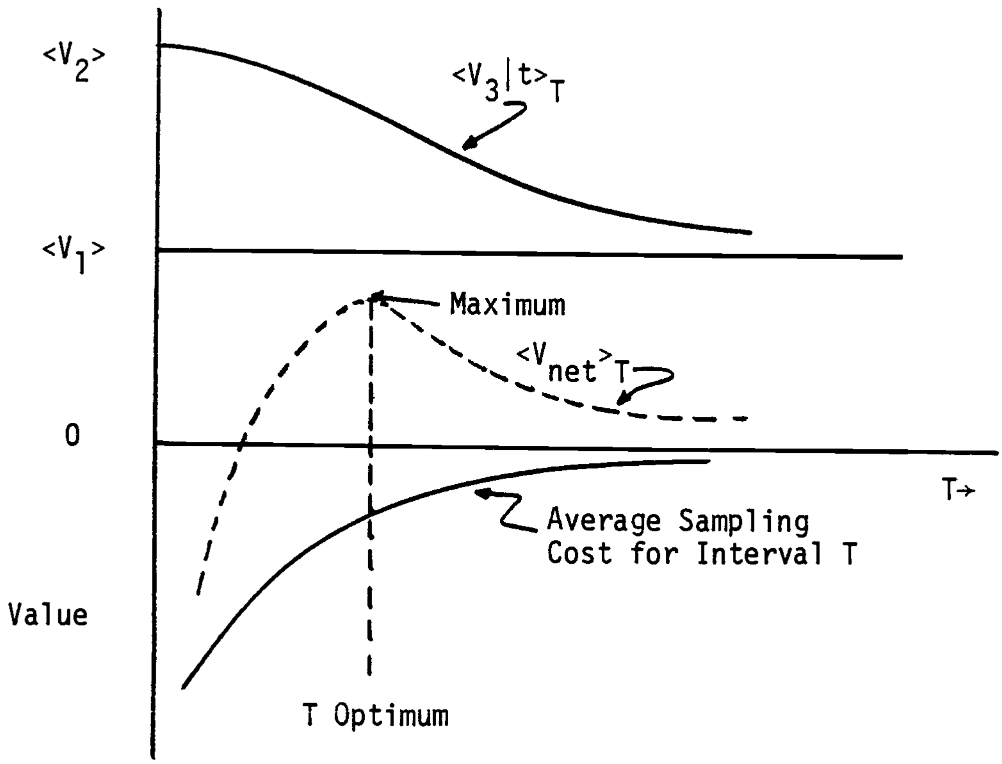

Figure 7. Determination of Optimum T
Sheridan [1970]

Sheridan and Rouse (1971) used this theoretical model in comparing the behavior of actual subjects in predicting values of a time series, a second-order digital filter driven by white noise. The subjects had to "pay" to sample the process, but were rewarded for correct predictions.

This study revealed that the subjects tended to choose shorter sampling intervals than the optimal intervals. Two reasons were given for this sub-optimal behavior. First, the subjects appeared unable to predict over longer periods of time without sampling. Second, there appeared to be a tendency toward risk aversion. There was a certain amount of the "fear-of-the-unknown." The possibility existed that a major error may occur during a longer sampling period.

The work of Sheridan is an appealing approach to determining optimal sampling intervals for time-varying processes, and has formed as a significant reference in the work of this thesis. Sheridan mentions one area for future work should include partially random processes. This thesis does investigate this area.

Summary

In this chapter, some of the more relevant works in the area of optimal sampling intervals have been reviewed. The volume of such work is small. The models developed in this paper use ideas from some of the articles reviewed, and new ideas to build what is intended to be, a more refined technique for determining optimal sampling intervals. Of course, it is not a panacea and will have its limitations. Some of these will be discussed in Chapter VI.

III. THE LOOK-SEE PROBLEM

Bayesian statistics form an integral part of the models developed in this paper. A discussion of the concepts involved in this type of analysis will not be included here, however. The reader who is unfamiliar with these concepts is referred to any elementary book on Bayesian statistics. An excellent discussion can be found in Morris (1968).

The processes to be considered in this paper consist of two inputs, the independent input x, and the controlled input y, as shown in Figure 3. The controlled input is used to adjust the value of x so that the total process input will be equal to the desired value. This desired value is known beforehand. The system's output will be a negative value function, that is, a cost or loss function. The value of this function will depend not only on how successful the controller is at adjusting the input to equal the desired input, but also on how often he samples the process. That is, a cost is incurred every time the controller observes the value of x. The controller must determine the time interval between samples so as to minimize the sum of the error cost (assessed on the squared difference between the actual value of x and the forecast value) and the sampling cost.

As stated in the introduction, the approach to solving the sampling problem will be to take a simplified approach to the problem and then progress to the more sophisticated problem.

Model Description

The first model of the general sampling problem to be investigated will be referred to as the "look-see" problem. In this case, in order to determine the value of x at any given time, the decision maker needs only to look and see the value. The value of x is precisely known, that is, the variance of x at the time of sampling is zero (x is a mathematical variable). There is, however, a cost associated with looking.

Since the value of x changes with time, it is reasonable to assume that the decision maker's uncertainty about x will increase with time. Figure 8 illustrates this change in uncertainty. In this case, the prior distribution, that is, the decision maker's belief about the distribution of x immediately prior to sampling is normally distributed with mean equal to μ_X and variance equal to σ_X^2.

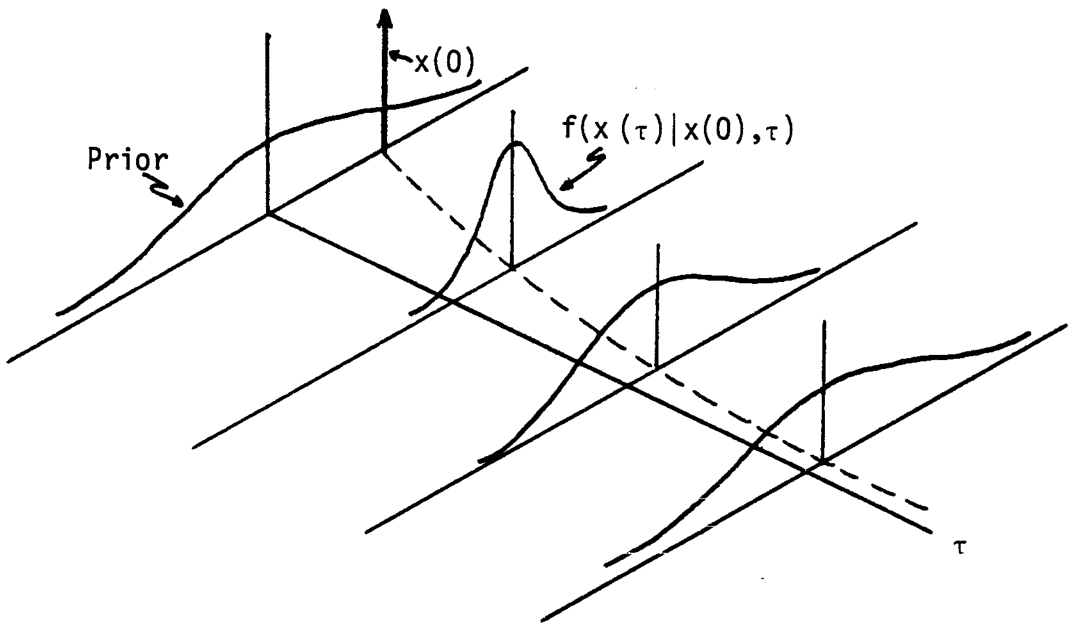

Figure 8. Uncertainty Change
Sheridan [1970]

If the decision maker were to proceed with this problem based only on the prior information, his best action would be to assume a forecast value of x that would be equal to the prior expected value of x. That is, $\hat{x}(\tau) = E[x(\tau)]$, where τ is the time since the last sample. If the error costs are symmetric about zero and quadratic costs are used, the instantaneous expected error cost of such a policy is,

$$\text{Instantaneous Expected Error Cost} = \int_{-\infty}^{+\infty} C[x(\tau) - \hat{x}(\tau)]^2 f(x|\tau)\, dx$$

$$= C \int_{-\infty}^{+\infty} (x(\tau)-E[x(\tau)])^2 f(x|\tau)\, dx$$

$$= C\, V_x(\tau) \qquad (3\text{-}1)$$

where $f(x|\tau)$ is the prior distribution of x given time τ, and $V_x(\tau)$ is the variance or uncertainty of x at time τ. *

This result is an interesting and important result. It implies the instantaneous expected error cost is only dependent on the unit cost of a squared error and the variance or uncertainty of the process. While the prior distribution must be normal, the expected error cost is independent of the parameters of this distribution.

The expected error cost per unit time given prior information for a sampling interval of length T, will be,

$$E[\text{Error Cost Per Unit Time}|\text{Prior Info.}] = \frac{1}{T}\int_0^T C\, V_x(\tau)\, d\tau \qquad (3\text{-}2)$$

If no sampling is performed, the expected error cost per unit time given this policy is found by letting the sampling interval

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* If the error costs are one-sided, that is, assessed only when the deviation is positive (negative) and zero when the deviation is negative (positive), equation (3-1) becomes,
Instantaneous Expected Error Cost = $1/2\ C\, V_x(\tau)$.

approach infinity,

$$E[\text{Error Cost Per Unit Time}|\text{No Sampling}] = \lim_{T\to\infty} E[\text{Error Cost}|\text{Prior Info}]$$

$$= \lim_{T\to\infty} [\frac{1}{T} \int_0^T C\, V_x(\tau)\, d\tau] = \lim_{T\to\infty} [\frac{\int_0^T C\, V_x(\tau)\, d\tau}{T}] . \quad (3\text{-}3)$$

Since $V_x(\tau)$ is assumed to be monotonically increasing and both the numerator and demoninator in equation (3-3) approach infinity as T approaches infinity, it can be evaluated by l'Hospital's rule to yield the following relationship.

$$E[\text{Error Cost Per Unit Time}|\text{No Sampling}] = C \lim_{T\to\infty} V_x(T). \quad (3\text{-}4)$$

Equation (3-4) shows the expected error cost per unit time given a no sampling policy increases as the limit of the variance or uncertainty increases. This cost will be finite if the decision maker's uncertainty about the process is bounded above by some finite value. If the uncertainty approaches a value of A, the expected error cost per unit time would be CA. If, on the other hand, the variance increases without bound as time increases, the expected error cost will be infinite. A further discussion about bounded and unbounded uncertainty functions is included in a later part of this chapter.

The decision maker will be able to determine the value of x exactly if he samples. This is a characteristic of the "look-see" problem. Therefore, if he decides to sample he will need only one sample, that is, one look. His problem, as stated above, is to determine how often to sample.

Since only one sample is taken at fixed time intervals equal to T, the sampling cost per unit time will be S/T. The total expected cost per unit time for periodic sampling policy will be the sum of S/T and the expected error per unit time, equation (3-2).

$$E[\text{Cost Per Unit Time} \mid \text{Sample of } 1] = E[C \mid 1] = \frac{1}{T} \int_0^T C\, V_X(\tau)\, d\tau + \frac{S}{T}\,. \qquad (3\text{-}5)$$

The decision maker's objective, as stated before, is to minimize the expected cost per unit time. To do this, equation (3-5) can be differentiated with respect to T and set equal to zero. Using the chain rule of differentiation, this yields,

$$\frac{\delta\, E\,[C \mid 1]}{\delta\, T} = -\frac{C}{T^2} \int_0^T V_X(\tau)\, d\tau + \frac{C}{T} V_X(T) - \frac{S}{T^2} = 0. \qquad (3\text{-}6)$$

The value of T that satisfies equation (3-6) will be the optimal sampling interval, referred to as T*. Because of the complicated form of this expression, it is not convenient to solve it explicitly for T* unless the form of the uncertainty function is known. This will be done in the next section.

However, if an optimal value for T is assumed to exist, equation (3-6) may be written

$$\frac{C}{T^*} \int_0^{T^*} V_X(\tau)\, d\tau + \frac{S}{T^*} = C\, V_X(T^*). \qquad (3\text{-}7)$$

The left hand side of equation (3-7) is equal to the right hand side of equation (3-5), when T=T*. Therefore, the expected cost per unit time given an optimal sampling policy will be,

$$E[C|T^*] = C\ V_x(T^*). \quad (3\text{-}8)$$

In situations in which sampling is performed, the sample information has a value to the decision maker. This value, defined as the expected value per unit time of sample information (EVSI) is the difference between the expected cost per unit time given no sampling and the expected cost per unit time given sample information.

$$EVSI = C \lim_{T\to\infty} V_x(T) - C\ V_x(T^*) \quad (3\text{-}9)$$

where the two expected costs are defined in equations (3-4) and (3-8).

Since the sample information will cost the decision maker a certain value to obtain, the expected net gain, per unit time, of sample information (ENGSI) is the difference between the EVSI and the cost per unit time of obtaining this information.

$$ENGSI = EVSI - S/T^* \quad (3\text{-}10)$$

The expressions for EVSI and ENGSI are valid only for situations in which periodic sampling is optimal.

The next step in the analysis of the "look-see" model will be to introduce two general classes of uncertainty functions; the unbounded case, and the bounded case.

Unbounded Uncertainty

There may be some instances when the decision maker's uncertainty about the process becomes very large as time increases and increases without bound. A random walk process would be an example. Depending on the circumstances, there could be a number of equations that would express this growth of uncertainty. One such equation would be,

$$V_x(\tau) = k\tau \qquad (3\text{-}11)$$

where k is a constant greater than zero. Solving equation (3-6) for this example (when C > 0) results in the following:

$$\frac{-C}{(T^*)^2} \int_0^{T^*} (k\tau)d\tau + \frac{C}{T^*}(kT^*) - \frac{S}{(T^*)^2} = 0$$

$$\frac{-C}{(T^*)^2} \left[\frac{k(T^*)^2}{2} - 0\right] + \frac{C}{T^*}(kT^*) - \frac{S}{(T^*)^2} = 0$$

$$\frac{-Ck}{2} + Ck = \frac{S}{(T^*)^2}$$

$$T^* = \sqrt{\frac{2S}{kC}}\,. \qquad (3\text{-}12)$$

This is an interesting and intuitively appealing result. It shows that if the cost of sampling increases, the optimal sampling interval will increase. That is, the more expensive it is to take a sample, the less often the decision maker will sample. Also, the more costly an error, the more often the sampling will be performed. Finally, the faster the rate of growth of the decision maker's uncertainty, the more often he should sample.

If equation (3-11) is substituted into equation (3-8), the expected cost per unit time of this policy can be found.

$$E[C|1] = CkT^*$$

$$= Ck\sqrt{\frac{2S}{kC}}$$

$$= \sqrt{2CSk}. \qquad (3\text{-}13)$$

Again, this appears to be a logical result. When either the error cost, the sampling cost, or the uncertainty constant increases, the expected cost per unit time will also increase.

As mentioned in the beginning of this section, $V_x(\tau) = k\tau$, is just one of the many ways the unbounded variance can be expressed, but solutions to different functions would be handled in a similar manner.

Another alternative form of the uncertainty to be studied in this section is the bounded case.

Bounded Uncertainty

In many instances, it is reasonable to assume that the decision maker's uncertainty about the process will be bounded above by some value. This uncertainty will not increase without bound, but will approach a limiting value. Without sampling, the uncertainty function may resemble the function in Figure 9a. However, with sampling, the picture would change to resemble Figure 9b.

A reasonable example of such a function (again, this is just one of many possible examples) is

$$V_x(\tau) = A\,(1 - e^{-b\tau}) \qquad (3\text{-}14)$$

where A and b are constants, greater than zero. The expected cost per unit time given no sampling is found by substituting the function in equation (3-14) into equation (3-4).

$$E[\text{Cost Per Unit Time}|\text{No Sampling}] = C \lim_{T\to\infty} A(1-e^{-bT})$$

$$= CA \qquad (3\text{-}15)$$

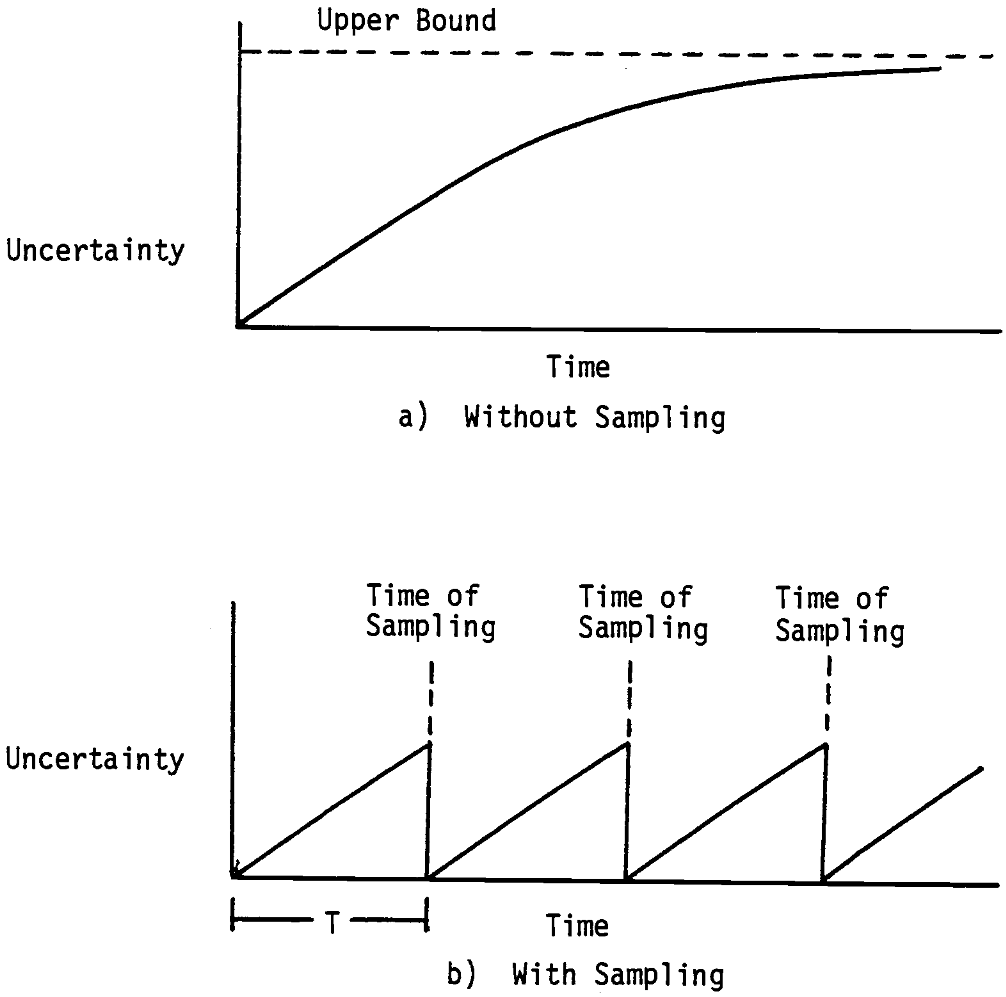


Figure 9. Uncertainty Function

In order to determine the optimal sampling interval T*, the uncertainty function must be substituted into equation (3-6) and solved. Using elementary calculus, this will reduce to,

$$e^{-bT^*}(1 + bT^*) = \frac{CA-bS}{CA} \qquad (3\text{-}16)^*$$

* See Appendix 1 for mathematical derivation.

where C is again assumed to be greater than zero. Three statements can be discerned from this equation, relating the optimal sampling interval to the process parameters.

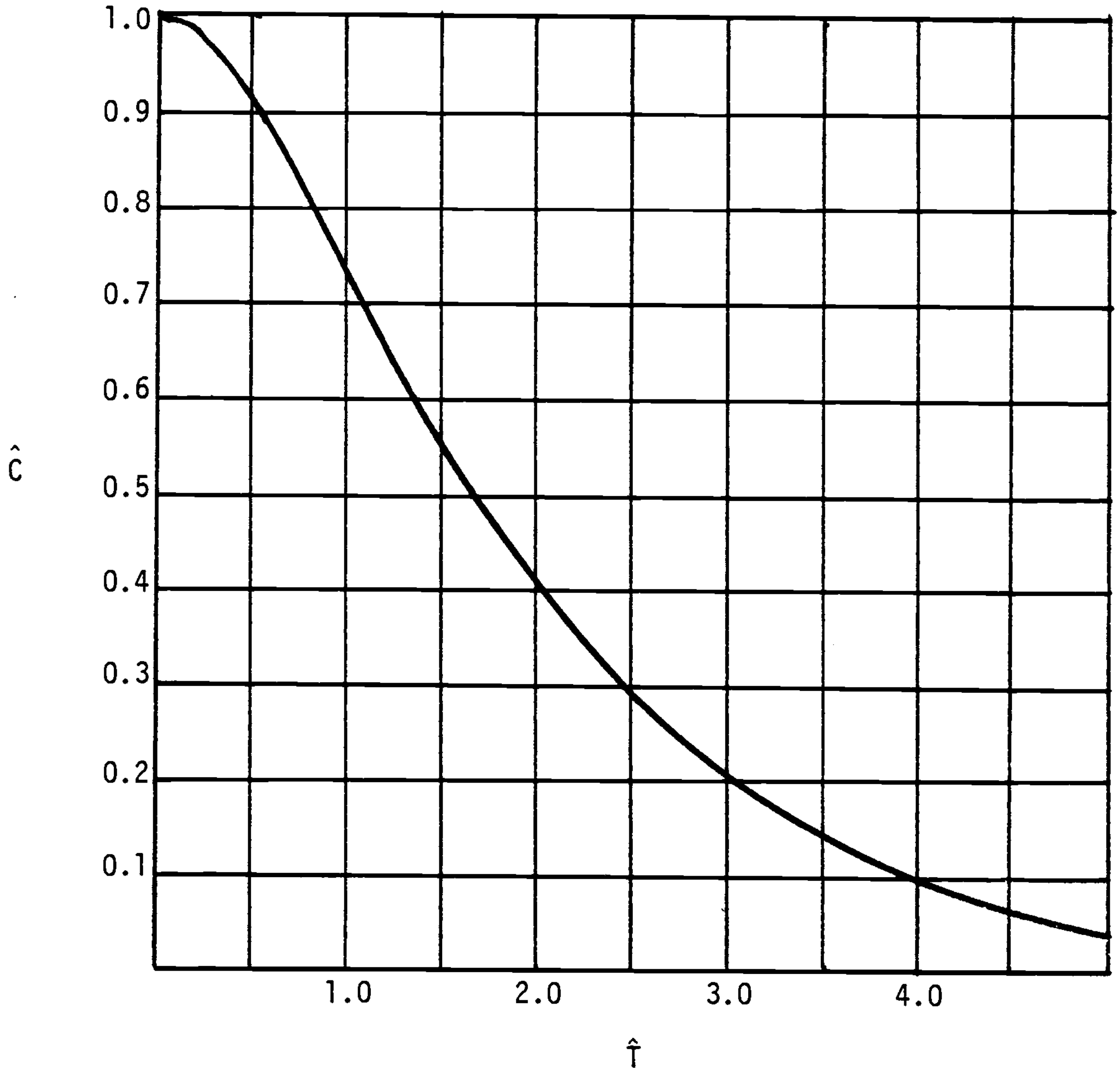
1. $T^* = 0$ if and only if $S = 0$. This situation corresponds to continuous monitoring.
2. $T^* = \infty$ if and only if $CA < bS$. This situation corresponds to no sampling.
3. T^* is contained on the open interval $(0, \infty)$ in all other cases. Periodic sampling will be optimal.

Given numerical values for the process parameters, several methods can be used to solve for the optimal sampling interval. A trial and error approach or the Newton-Raphson method are two examples. Appendix 2 discusses some of these solution techniques. Figure 10 presents a third approach, a graphical solution.

In order to use this graph, the value $\hat{C} = (CA - bS)/CA$ must be calculated. The value for $\hat{T}$ corresponding to $\hat{C}$ is then read from the graph. The optimal sampling interval, T^*, is found by dividing $\hat{T}$ by b. The expected cost for this optimal policy, the EVSI, and the ENGSI, then can be determined by substituting the appropriate information into equations (3-8), (3-9), and (3-10) respectively.

Table I presents a numerical example of a bounded uncertainty function.

An analysis of this example, and in fact any example in which $V_X(\tau) = A(1-e^{-b\tau})$ reveals the fact that as the cost of sampling increases, sampling is performed less often. As the cost of an error increases, or the limit of the uncertainty is increased, sampling is performed more often. Finally, as the rate of growth of uncertainty increases, the sampling interval will decrease. These results are



$$V_x(\tau) = A\,(1 - e^{-b\tau})$$

$$\hat{C} = \frac{CA-bS}{CA}$$

$$T^* = \hat{T}/b$$

Figure 10. Graphical Solution Technique

TABLE I. LOOK-SEE PROBLEM: BOUNDED UNCERTAINTY EXAMPLE

$V_X(\tau) = A(1-e^{-b\tau})$

$A = 2.0 \text{ (feet)}^2$ $\qquad$ $C = \$10/(\text{feet})^2\text{-hour}$

$b = .25 \text{ (hours)}^{-1}$ $\qquad$ $S = \$20$

E[Cost Per Unit Time|No Sampling] = CA = \$20/hour

$$\hat{C} = \frac{CA-bS}{CA} = \frac{20-5}{20} = .75$$

From Figure 8, T = .96

Therefore, T* = 3.84 hours

$$E[\text{Cost Per Unit Time}|\text{No Sampling}] = 10[2.0(1-e^{-.25(3.84)})]$$

$$= \$12.35/\text{hour}$$

$$EVSI = \$20.00/\text{hour} - \$12.35/\text{hour}$$

$$= \$7.65/\text{hour}$$

$$ENGSI = \$7.65/\text{hour} - \$20/3.84 \text{ hours}$$

$$= \$2.44/\text{hour}$$

consistant with those obtained in the unbounded uncertainty case. Again, the results are intuitively appealing.

Summary

A model has now been developed to determine optimal sampling intervals in which $V_X(0) = 0$. That is, the value of the process is known precisely whenever a sample is taken. Because it is known precisely, the sample size needs never to be greater than one.

In order to determine the optimal sampling interval, three pieces of information must be known. First, the cost of taking a sample, the second item is the quadratic error cost, and finally, the functional form of the decision maker's uncertainty about the process, whether bounded or unbounded, and the values of the associated parameters. The optimal sampling interval can then be found by substituting this information into the appropriate equation contained in this section.

The next step in the logical development of a general model will be to consider the case where $V_x(0) \neq 0$. The following chapter presents such a model.

IV. THE SAMPLING ERROR PROBLEM

The previous chapter laid the ground work and developed the concepts necessary to solve a more general class of sampling interval problems. The "look-see" problem was a simplified, but by no means trivial example of the general problem.

Model Description

A more difficult and in some instances a more realistic problem occurs when the process input, $x(\tau)$ is not directly observable to the controller. In this case, the value the controller observes is not the exact value of the process input as it was in the "look-see" problem but contains a noise or error component. In many instances, this error may be the variance of the instrumentation used to record the input.

$$O_i(\tau) = x(\tau) + e_i(\tau).$$

Since more than one observation may be taken at the time of sampling, the subscripts of $O_i(\tau)$ and $e_i(\tau)$ refer to specific observations. Thus,

$$\overline{O}\,(\tau) = \frac{1}{n} \sum_{i=1}^{n} O_i(\tau)$$

where n is the number of observations per sample.

If the variance of $e(\tau)$ is σ_e^2, then the variance of $\overline{O}\,(\tau)$ will be σ_e^2/n. Unlike the "look-see," performing more observations at the time of sampling will reduce the uncertainty. Therefore, the optimal sample size will also need to be determined. This problem will be dealt with later.

The value of $x(\tau)$ is again considered to be a realization of a random process. However, since the value of $x(\tau)$ immediately after sampling is not precisely known, Figure 8 must be revised. Figure 11 shows this revision.

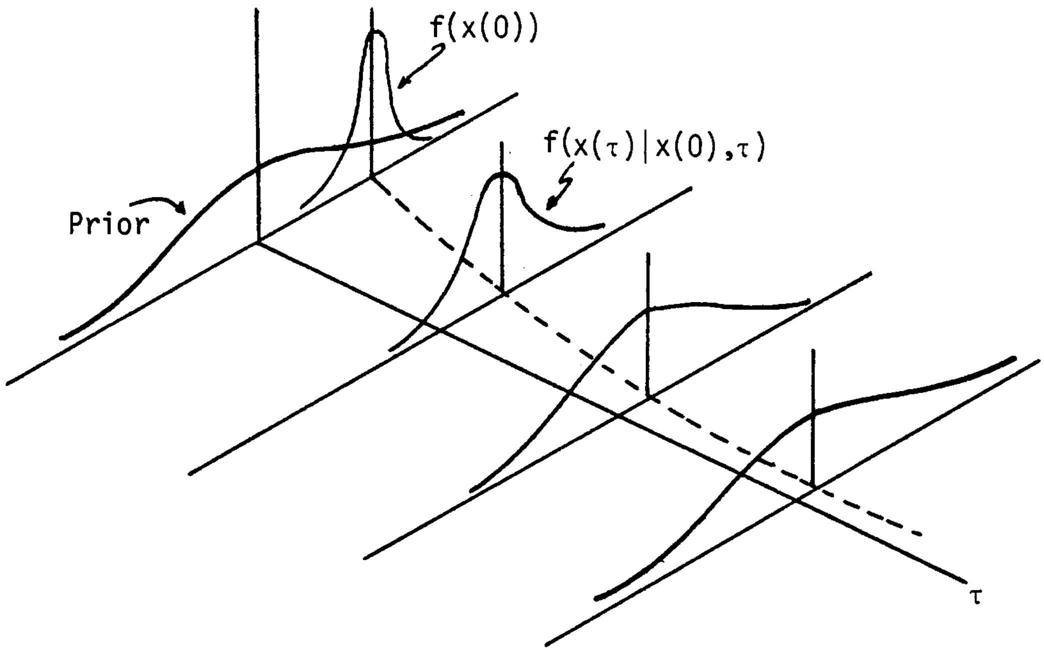


Figure 11. Uncertainty Change With Time

Variance Restriction

The purpose of sampling is to reduce the decision maker's uncertainty about the state of the process. In the "sampling error" problem, sampling will not, however, reduce the uncertainty at the time of sampling to zero. The variance at the time of sampling (when $\tau = 0$) referred to as the posterior variance (V_{po}), will be a function of the sample size n, and the sampling interval T. The decision maker's uncertainty about the process an instant before sampling (when $\tau = T$) will be referred to as the prior variance (V_{pr}). The uncertainty or variance function itself will be a function of the two variables previously mentioned, and the time since the last sample, τ. Figure 12 shows this relationship.

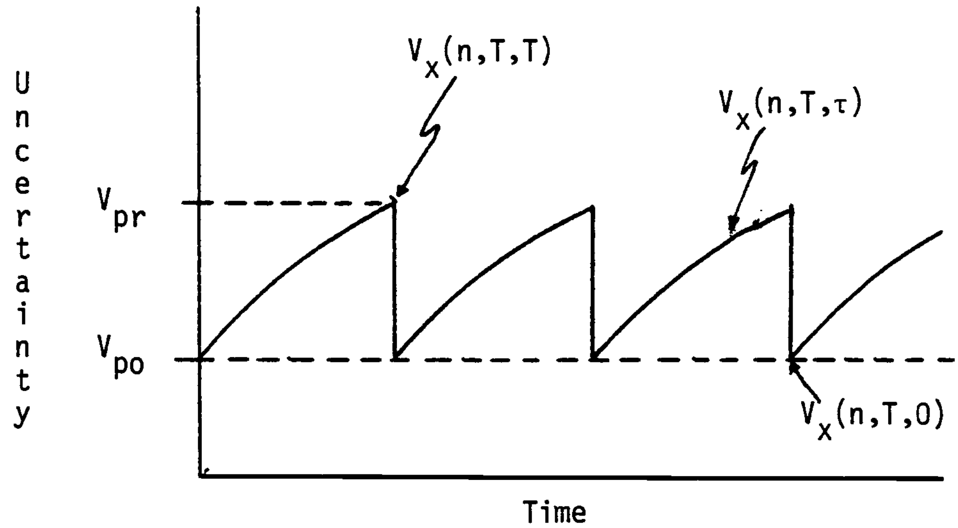


Figure 12. Prior and Posterior Variance Relationship

Because the prior distribution of $x(\tau)$ has been assumed to be normally distributed, and the samples that are taken are also normally distributed, then as shown in Morris (1968), the posterior distribution will also be normally distributed. The prior and posterior variances will be related as follows:

$$V_{po} = \frac{V \cdot V_{pr}}{V + n \cdot V_{pr}}, \text{ where } V = \sigma_e^2.$$

With this in mind, the development of the necessary equations to describe this model will follow.

Solution Development

Given the quadratic cost assumption as previously discussed and the decision rule to adjust the controllable unit $y(\tau)$ on the basis of a forecast value of $x(\tau)$, the expected error cost at any time τ, after a sample given a sampling interval of T, and a sample size of n, is,

$$\text{Instantaneous Expected Error Cost} = CV_x(n,T,\tau). \qquad (4\text{-}1)$$

This solution is derived in the same manner as equation (3-1). The logic used to develop the remaining equations is similar to the logic used in Chapter III.

If no sampling is performed, $y(\tau)$ is adjusted on the basis of prior information about the process. In this case, the sampling interval will approach infinity. Using l'Hospital's rule, the expected cost per unit time given this situation will be,

$$E[\text{Cost Per Unit Time}|\text{No Sampling}] = \lim_{T\to\infty} [C \cdot \frac{1}{T} \int_0^T V_x(n,T,\tau)\, d\tau]$$

$$= C \lim_{T\to\infty} V_x(n,T,\tau). \qquad (4\text{-}2)$$

If the uncertainty function is unbounded as T approaches infinity, the expected cost will also approach infinity. If, on the other hand, the uncertainty is bounded above, the expected cost per unit time will be the quadratic error cost times this limiting value. The implication is that sampling will be optimal if the uncertainty is unbounded.

If there is a sampling cost, whether a fixed or setup cost, S, or a variable or unit cost, U, the expected cost of continuous monitoring will be infinite. This would imply a continuous sampling policy will never be optimal.

The extreme cases of no sampling and continuous sampling have now been investigated for the "sampling error" problem. The results are very similar to those obtained in the "look-see" problem. The next step is to look at the case of periodic sampling. Here too, the development and results will be similar to those in Chapter III.

The expression for the expected cost given periodic sampling is relatively easy to obtain but, as will be seen later, may be difficult to explicitly evaluate. Following the same steps as in the previous chapter;

$$E[\text{Cost Per Unit Time} \mid n \text{ observations}] = \frac{1}{T}\int_0^T C\, V_X(n,T,\tau)\,d\tau + \frac{S+U\cdot n}{T} \; . \qquad (4\text{-}3)$$

It is the decision maker's objective to minimize this expression. Therefore, the value of T that will minimize this cost, given a fixed number of observations per sample, will be the optimal sampling interval T* for that sample size.

In order to attempt to derive an explicit solution for T*, equation (4-3) must be differentiated with respect to T and set equal to zero.

$$-\frac{C}{T^{*2}} \int_0^{T^*} V_X(n,T^*,\tau)d\tau + \frac{C}{T^*} \left[\int_0^{T^*} \frac{\delta V_X(n,T^*,\tau)}{\delta T^*} d\tau + V_X(n,T^*,T^*)\right] - \frac{S+U\cdot n}{T^{*2}} = 0 \qquad (4\text{-}4)$$

This equation cannot be solved explicitly until the functional form of the uncertainty is known. This will be done in a later section.

Equation (4-4) can, however, be rewritten in the following form,

$$\frac{C}{T^*} \int_0^{T^*} V_X(n,T^*,\tau)d\tau + \frac{S+U\cdot n}{T^*} = C\left[\int_0^{T^*} \frac{\delta V_X(n,T^*,\tau)}{\delta T^*} d\tau + V_X(n,T^*,T^*)\right] \qquad (4\text{-}5)$$

The right hand side of equation (4-5) is equal to the left hand side of equation (4-3) when T = T*. Therefore, once the optimal sampling interval is known for a given sample size, the expected cost per unit time can be found using equation (4-6).

$$E[\text{Cost Per Unit Time}|T^*,n] = C\left[\int_0^{T^*} \frac{\delta V_X(n,T^*,\tau)}{\delta T} d\tau + V_X(n,T^*,T^*)\right] \qquad (4\text{-}6)$$

It should be noted that equation (4-6) gives the expected cost per unit time given a sample size of n only when the optimal sampling interval has been determined. As n increases, so does the optimal sampling interval, but the expected cost will decrease to a minimum and then begin to increase. The optimal sampling policy will be to choose the sample size with the lowest expected cost.

$$E[Cost|T^*_{n-1}]_{n-1} > E[Cost|T^*_n]_n \leq E[Cost|T^*_{n+1}]_{n+1} \qquad (4\text{-}7)$$

The value of n that satisfies equation (4-7) will be the optimal sample size, referred to as n*. The optimal sampling interval, T** will be equal to T^*_{n*}.

In order to solve equation (4-7) for n*, the sampling interval and expected cost must be calculated for a sample size of one and for two. If a sample size of one is optimal, $E[Cost|T^*_1]$ will be less than $E[Cost|T^*_2]$. If it is not, n must be increased by one until the minimum cost is found.

The optimal policy is actually determined using a two stage dynamic programming process. The first step is the calculation of T* while fixing the value of n. Then, in the second stage, the value for n* is determined on the basis of minimum cost.

Since the functional form of the uncertainty affects of optimal sampling policy, the next step will be to examine the special cases where the uncertainty function is either unbounded or bounded. This concludes the development of the general equations for the "sampling error" problem.

Unbounded Uncertainty

The first uncertainty function case to be looked at is the one in which the uncertainty increases without bound as time increases. There are many ways to express such a function. A discussion on choosing an appropriate one is contained in Chapter V. One such example is,

$$V_x(n,T,\tau) = (V/n)(1-e^{-bT})\,(e^{b\tau}). \qquad (4\text{-}8)$$

Notice that $V_x(n,T,\tau)$ increases to infinity as τ increases and V_{po} ($V_x(n,T,0)$) and V_{pr} ($V_x(n,T,T)$) satisfy the prior, posterior relationship presented earlier.

If equation (4-8) represents the uncertainty function, the optimal sampling interval can be found by substituting this expression into equation (4-4). With some algebraic manipulation, this reduces to,

$$bT^* \sinh(bT^*) - \cosh(bT^*) = \frac{(S+Un)bn-2CV}{2CV} \qquad (4\text{-}9)^*$$

where $\sinh(bT^*) = \dfrac{e^{bT^*} - e^{-bT^*}}{2}$ and $\cosh(bT^*) = \dfrac{e^{bT^*} + e^{-bT^*}}{2}$.

The expected cost per unit time given a sample size of n and the corresponding optimal sampling interval is found using equation (4-6).

$$E[\text{Cost Per Unit Time} \mid T^*,n] = \frac{2CV}{n} \sinh(bT^*). \qquad (4\text{-}10)$$

There are numerical methods available to solve the preceding equations. Appendix 2 mentions some of the more established techniques. Table II presents a numerical example of an unbounded variance function in which the uncertainty function was substituted into equation (4-3) and minimized using a Golden Section Search procedure. The solutions were verified using the Newton-Raphson numerical approximation method to solve equation (4-9). The computer programs used for this example are contained in Appendix 2. In this example, the optimal policy would be to take six observations every seven hours and twenty minutes. This policy would result in an expected cost of

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* See Appendix 3 for mathematical derivation.

\$10.15 per hour.

TABLE II. SAMPLING ERROR PROBLEM: UNBOUNDED UNCERTAINTY EXAMPLE

$V_x(n,T,\tau) = (V/n)(1-e^{-bT})\,(e^{-b\tau})$

$b = .25\ (\text{hour})^{-1}$ $V = 1\ (\text{foot})^2$ $S = \$15$

$U = \$5/\text{observation/sample}$ $C = \$10/(\text{feet})^2\text{-hour}$

n(obs/sample)	T*(hours)	$E[C\mid T^*]_n$ (\$/hour)
1	2.68	\$14.42
2	3.98	11.68
3	5.02	10.74
4	5.89	10.34
5	6.66	10.18
6	7.33	10.15 *
7	7.94	10.20

ENGSI and ENSI are infinite since $V_x(n,T,\tau)$ is unbounded.

* Optimal Policy

Bounded Uncertainty

In many situations, there is an upper limit on the decision maker's uncertainty. The uncertainty approaches, but does not exceed a certain upper limit. An example of such a function is,

$$V_x(n,T,\tau) = \frac{nB^2(1-e^{-b\tau})^2}{V - nB(1-e^{-b\tau})} + B(1-e^{-bT}) \qquad (4\text{-}11)$$

$$\text{where } B = \frac{VA}{V + nA}$$

The limit of $V_x(n,\tau,\tau)$ as τ approaches infinity is A.

The solution to this example, is somewhat more complicated than the previous examples. The major difficulty occurs when the variance is substituted into equation (4-3) or (4-4). The term $\int_0^T V_x\ (n,T,\tau)d\tau$ is difficult to explicitly evaluate. Tables of indefinite integrals can provide a solution, however, the solution will be correct only for certain values of the parameters. The values of the parameter in this example make some of the restrictions impossible to satisfy. For example, one term in the solution involves LN (1/(1-(V/nB))). Obviously 1-(V/nB) must be greater than zero. This means B must be greater than V/n. Since B is defined as VA/(V+nA) it becomes clear after some algebraic manipulation that B will never be greater than V/n since V,n, and A are all greater than zero. Therefore, the solution by indefinite integration is inappropriate for this example. A numerical approximation of the integral must be used. Using the trapezoidal method of integral approximation and the Golden Section Search procedure to find the minimum of equation (4-3) an answer can be found when the parameter values are specified. Table III shows the results of such an example. Again, the computer programs are shown in Appendix 2.

In the example presented in Table III, the optimal policy would be to take five observations every seven hours and thirty-six minutes. The expected cost of such a policy will be $9.13 per hour.

TABLE III. SAMPLING ERROR PROBLEM: BOUNDED UNCERTAINTY EXAMPLE

$$V_x\ (n,T,\tau) = \frac{nB^2(1-e^{-b\tau})^2}{V - nB(1-e^{-b\tau})} + B\ (1-e^{-bT})$$

where $B = \dfrac{VA}{V+nA}$

$A = 4.0\ (\text{feet})^2$ $\quad C = \$10/(\text{feet})^2\text{-hr}$ $\quad U = \$5/\text{observation/sample}$

$S = \$15$ $\quad b = .25(\text{hours})^{-1}$ $\quad V = 1\ (\text{foot})^2$

n(observations/sample)	T*(hours)	E[C|T*]n ($/hour)
1	3.55	11.81
2	4.91	10.06
3	5.97	9.43
4	6.86	9.19
5	7.63	9.13 *
6	8.31	9.16

$$EVSI_{n^*=5} = 4(10) - 9.13 = \$30.87/\text{hour}$$

$$ENGSI_{n^*=5} = 30.87 - \frac{15 + 5(5)}{7.63} = \$25.63/\text{hour}$$

* Optimal Policy

Sensitivity Analysis

A sensitivity analysis performed on the two examples presented in this chapter resulted in some interesting observations concerning the changes in the optimal sampling interval and the optimal sample size as one of the three cost parameters (C, S, or U) was varied while holding the other values constant. The general results were the same in both the unbounded and bounded uncertainty examples. The following discussion therefore refers to both examples and should apply to the general "sampling error" problem.

As was mentioned in a previous section, for a given set of parameter values, as the number of observations per sample is increased, the optimal sampling interval increases. That is, the time between samples will be longer if the sample size is larger.

The first results of the sensitivity analysis concerns the change in T*, given a fixed sample size, while varying one of the cost parameters and holding the remaining two constant. The analysis shows;

1) as the quadratic error cost C, increases, the optimal sampling interval T*, decreases;
2) as the fixed sampling cost S, increases, T* increases,
3) as the variable or unit sampling cost U, increases, T* also increases.

In other words, given a fixed sample size, the more costly an error, the more often the decision maker should sample; the most costly the setup charge for taking a sample, the less often he should sample; and finally, the more costly it is to make an observation, the less often he should sample. These three results are intuitively appealing.

Also,

1) if the increase of C is great enough, n* increases;
2) if the increase of S is great enough, n* increases;
3) if the increase of U is great enough, n* decreases.

The meaning of "great enough" will be explained in more detail later in this section. For the present, assume this means doubling the value of the parameter. Then, according to the previous results: when the increase in the cost of an error is great enough, the sample size should be increased in order to reduce some of the uncertainty; when the setup charge is increased enough, more observations should be made each time a sample is taken; and, if the cost of making these observations is increased enough, the number of observations per sample should be decreased. Once again, these results are to be expected.

If n is thought of as a continuous variable rather than the discrete variable it actually is, the preceding statements could be made without reference to a "great enough" increase. For instance, an increase in C, would cause an increase in the optimal sample size. However, since n is a discrete variable, there are intervals of values for C (or S or U) for which a specific value of n is optimal. The length of these intervals is a function of the sample size. While holding the other parameters constant:

1) when C increases, the length of the interval for which n is optimal is less than the interval for which n+1 is optimal;
2) when S increases, the length of the interval for which n is optimal is less than the interval for which n+1 is optimal;
3) when U increases, the length of the interval for which n is optimal is greater than the interval for which n+1 is optimal.

Thus, when either C or S becomes large, the optimal sample size will be large and the interval for which this sample size is optimal is also large. When U becomes large a sample size of one will be optimal. The interval for which one is optimal will be very large.

Figure 13 presents a graphical display of the previous results.

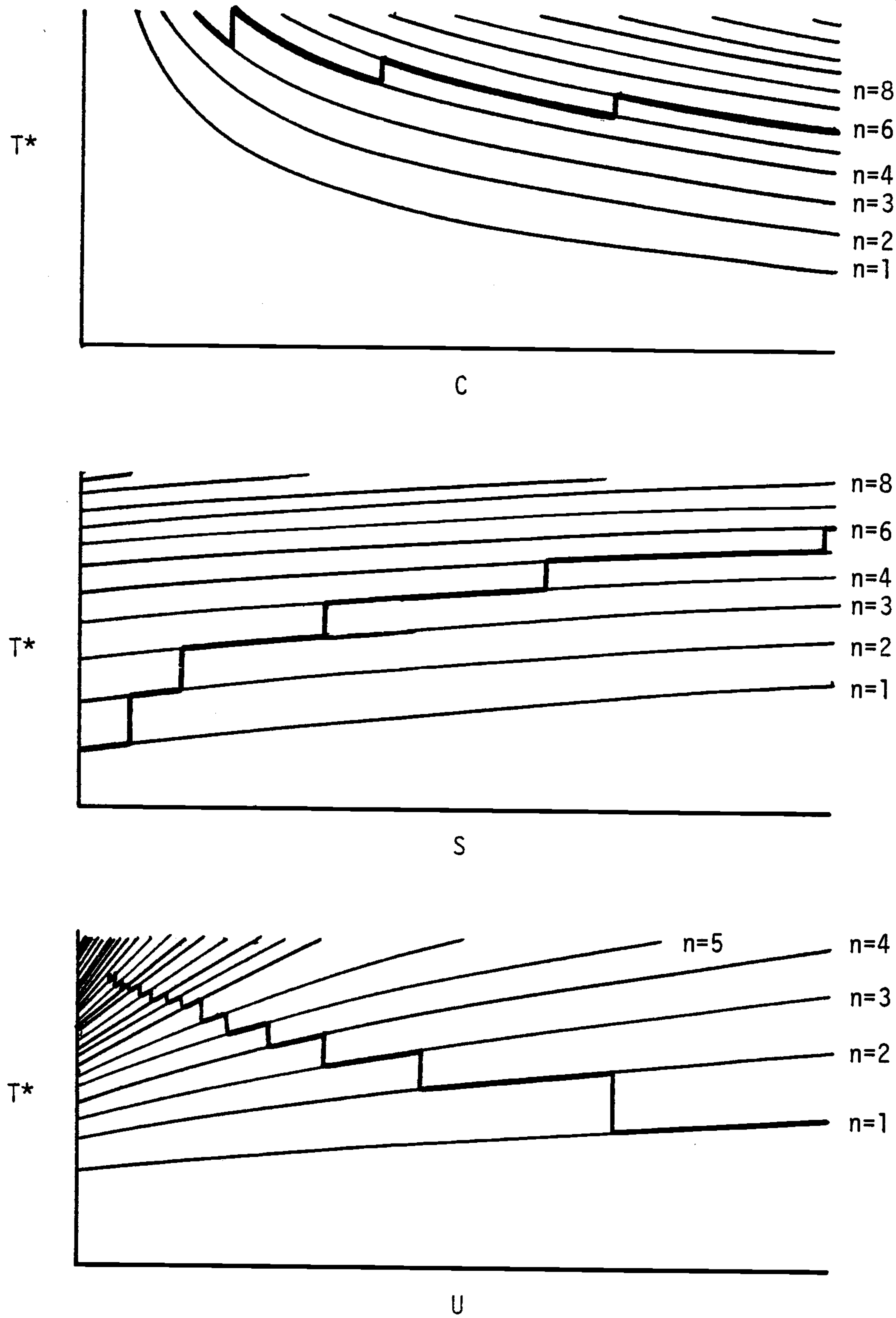

Figure 13. Sensitivity Analysis

In each graph, there is a family of curves representing optimal sampling intervals for given sample sizes. The bold lines represent the optimal sampling policies. The optimal policies were determined on the basis of minimum expected cost per unit time.

Notice that as C increases, T* decreases on a given interval. However, there exists a critical value where taking an additional observation with a longer sampling interval results in the same expected cost per unit time. A further increase in C will cause an increase in the optimal sample size and a jump in the optimal sampling interval. Still further increases cause a decrease in T* until another critical value is reached.

The opposite situation occurs when U increases. Within a given interval, T* increases until a critical value is reached. A value of C to the right of this point, causes a change in the optimal policy. The new policy will be to take one less observation at the value of T* less than the previous value.

When S increases, the value of T* increases within a given interval. When a critical value of S is reached, the optimal sampling interval increases as does the optimal sample size.

Summary

Once the groundwork was laid in Chapter III for determining optimal sampling intervals, the solution to the "sampling error" problem was fairly straightforward.

Chapter IV developed the equations needed to determine an optimal sampling policy, when the variance at the time of sampling is not equal to zero, once the functional form of the uncertainty is known. The optimal policy will consist of the optimal sampling interval and the optimal sample size.

Since the solutions are heavily dependent on the functional form of the decision maker's uncertainty, two forms were introduced, unbounded and bounded uncertainty. Numerical examples were given for both situations. Although solutions to these examples are difficult and time consuming to obtain using hand calculations, a computer will provide speedy, accurate answers.

A sensitivity analysis performed on both examples yielded the same general results. These results are assumed to be applicable to the general class of "sampling error" problems.

Since the uncertainty function is a major part of the analysis of this problem, the following chapter discusses the various interpretations of the uncertainty and methods for determining it.

V. PHYSICAL INTERPRETATION OF THE UNCERTAINTY FUNCTION

In the previous chapters, discussions concerning the uncertainty function, $V_x(n,T,\tau)$, assumed that this function was known. It was mentioned that the functional form of the uncertainty could be determined by objective data or subjective estimation, but the method of determination was not discussed. The purpose of this chapter is to discuss these methods.

Objective Uncertainty

In situations in which there are historic records of the process to be sampled, it may be possible to develop an objective estimate for the uncertainty function.

In one situation, the values of x(t) and x(t+τ) are considered to be multi-variate, normally distributed random variables, both with mean μ_x and variance $\rho(\tau)\sigma_x^2$. The term $\rho(\tau)$ is the autocorrelation function and σ_x^2 is the process variance. If the probability distribution of x(t) equals the probability distribution of x(t+τ), the process is strongly stationary and the conditional probability of x(t+τ) given x(t) will be as follows;

$$f(x(t+\tau)|x(t)) = N(\mu_x+\rho(\tau)(x(t)-\mu_x),\ \sigma_x^2(1-\rho^2(\tau))).$$

When a sample of size n is taken at time t

where $O_i(t) = x(t) + e_i(t)$

and $\overline{O}(t) = \frac{1}{n}\sum_{i=1}^{n} O_i(t)$

then $\overline{O}(t) = N(x(t),\ \sigma_e^2/n)$.

If the prior distribution is normally distributed and the sample observations are also normally distributed, then, as discussed in Chapter IV, the posterior distribution of x(t) given $\overline{O}(t)$ and prior information will also be normally distributed when mean μ_x' and variance $\sigma_x'^2$.

$$Po(x(t)|\overline{O}(t), Pr) = N(\mu_x', \sigma_x'^2) \qquad (5\text{-}1)$$

$$f(x(t+\tau)|\overline{O}(t), Pr) = \int_{-\infty}^{+\infty} f(x(t+\tau)|x(t))\ f(x(t)|\overline{O}(t), Pr)\ dx(t)$$

$$= N(\mu_x + \rho(\tau)\ (\mu_x' - \mu_x),\ \sigma_x^2 - \rho^2(\tau)\ (\sigma_x^2 - \sigma_x'^2)) \qquad (5\text{-}2)$$

(Morris, 1968)

Therefore,

$$V_x(n,T,\tau) = \sigma_x^2 - \rho^2(\tau)\ (\sigma_x^2 - \sigma_x'^2) \qquad (5\text{-}3)$$

and $V_{po} = V_x(n,T,0) = \sigma_x^2 - \sigma_x^2 + \sigma_x'^2$ because $\rho^2(0) = 1$

$= \sigma_x'^2$ (5-4); as stated in (5-1)

$$V_{pr} = V_x(n,T,T) = \sigma_x^2 - \rho^2(T)\ (\sigma_x^2 - \sigma_x'^2). \qquad (5\text{-}5)$$

In order to determine the value of $\sigma_x'^2$, the equation relating the prior variance to the posterior variance that was presented in Chapter IV must be used.

$$V_{po} = \frac{V \cdot V_{pr}}{V + nV_{pr}}$$

therefore, $$\sigma_x'^2 = \frac{V(\sigma_x^2 - \rho^2(T)\ (\sigma_x^2 - \sigma_x'^2))}{V+n(\sigma_x^2 - \rho^2(T)(\sigma_x^2 - \sigma_x'^2))} \qquad (5\text{-}6)$$

With some basic algebraic manipulation, this equation will reduce to a quadratic equation from with $\sigma_x'^2$ can be determined.

$$\sigma_x'^2 = \sqrt{\frac{-b + b^2 - 4ac}{2a}} \qquad \text{(5-7) since } b \geq 0 \text{ and } ac < 0$$

where $a = n\,\rho^2(T)$

$b = (\sigma_e^2 + n\sigma_x^2)\ (1-\rho^2(T))$

$c = -\sigma_x^2\ \sigma_e^2\ (1-\rho^2(T))$

In order to illustrate the use of the objective data, two examples will be given.

Look-See Example

In the "look-see" problem, the uncertainty function is only a function of τ. Also, the prior variance, $\sigma_x'^2$ is zero, since at the time of sampling the process value is known precisely. Therefore, equation (5-3) will reduce to,

$$V_x(\tau) = \sigma_x^2\ (1-\rho^2(\tau)). \qquad (5\text{-}8)$$

If the autocorrelation function of the process is $e^{-k\tau}$, then

$$V_x(\tau) = \sigma_x^2\ (1-e^{2k\tau}). \qquad (5\text{-}9)$$

Furthermore, if σ_x^2 is referred to as the constant A,

$$V_x(\tau) = A\ (1-e^{-2k\tau}). \qquad (5\text{-}10)$$

Equation (5-10) is identical to equation (3-12), the examples used in the section in Chapter III concerned with bounded uncertainty functions, when $k = b/2$.

This is, of course, just one example of an autocorrelation function. The determination of the uncertainty function for different functions would be handled in a similar manner.

Sampling Error Example

In using objective data in the "sampling error" problem, two pieces of information are needed in addition to the autocorrelation function. These two items are the process variance, σ_x^2 and the error variance, σ_e^2. With this information and equations (5-3) and (5-7), $V_x(n,T,\tau)$ can be determined and substituted into equation (4-3). A Golden Section Search or similar procedure can be used to minimize the function and determine T* for a given sample size. The same method as presented in Chapter IV can be used to determine n*. Table IV presents the results of an example. These results were obtained using a Golden Section Search method. The computer program used to solve this example is contained in Appendix 2.

TABLE IV. SAMPLING ERROR PROBLEM: OBJECTIVE UNCERTAINTY EXAMPLE

$\rho^2(\tau) = e^{-b\tau}$ $\sigma_x^2 = 4.0\ (\text{feet})^2$ $\sigma_e^2 = 1.0\ (\text{feet})^2$

$b = .25(\text{hr})^{-1}$ $C = \$10/(\text{feet})^2\text{-hour}$

$S = \$15$ $U = \$5/\text{observation/sample}$

n(obs/sample)	T*(hours)	$E[C\|T^*]_n$($/hour)
1	2.56	23.39
2	2.94	23.08 *
3	3.29	23.84

$$\text{EVSI} = 4.0(10)-23.08 = \$16.92/\text{hour}$$

$$\text{ENGSI} = 16.92 - \frac{(15 + 5(2))}{2.94} = \$8.42/\text{hour}$$

* Optimal Policy

A method has now been developed from which objective data can be used in determining the uncertainty function. The next step in this analysis is to investigate the use of subjective data.

Subjective Uncertainty

If records are not available to the decision maker concerning the autocorrelation function, he must rely on another method for determining uncertainty function. Two options available to him include a subjective estimate of the autocorrelation function, or a subjective estimate of the entire uncertainty function. A brief discussion of these two alternatives will follow.

Autocorrelation Function Estimation

Since it has been assumed that the decision maker is relatively familiar with the process, he may have an estimate of the autocorrelation function. He may be able to sketch the function directly, from which the equation can be derived, or a subjective estimation procedure may be used. One such procedure is a paired comparison method in which several possible shapes of the autocorrelation function are shown to the decision maker, one pair at a time. He is then asked to decide which curve better expresses his belief about the process. The various combinations of pairs are presented to him and he is asked to make a decision for each pair. After this process is finished, the results for each test can be used to determine the curve that best fits the decision maker's belief about the autocorrelation function.

Once this function has been determined it can be considered "objective" and substituted into the appropriate equations in the first section of this chapter.

Uncertainty Function Estimation

If the decision maker is unsure about the shape of the autocorrelation function or is unfamiliar with the concept of autocorrelation, it may be advisable to subjectively estimate the uncertainty function itself.

The first step in developing this estimate is to obtain a functional relationship between the sampling interval T, and the prior variance V_{pr} ($V_{pr} = V_x(n,T,T)$). The prior variance should increase as the sampling interval increases. This increase may be unbounded or bounded. Once the decision maker has determined whether the uncertainty is unbounded or bounded, a paired comparison test can be performed to determine the functional form of the prior variance.

Once V_{pr} is known, the posterior variance, V_{po}, can be found using the equation relating V_{po} to V_{pr}.

$$V_{po} = \frac{V \cdot V_{pr}}{V + n\, V_{pr}}$$

When the prior and posterior variance have been determined, the uncertainty function $V_x(n,T,\tau)$ can be developed. For simplicity, it will be assumed that $V_x(n,T,\tau)$ will be in one of the two following forms.

$$V_x(n,T,\tau) = V_{po} + X. \qquad (5\text{-}11)$$

or $$V_x(n,T,\tau) = Y\, V_{po}. \qquad (5\text{-}12)$$

There are admittedly many forms the uncertainty can take, however, these forms would be very difficult to obtain. The amount of effort needed could not be justified since any form is only a subjective approximation. At this point, the uncertainty function cannot be drawn unless the sampling interval is known. Therefore, it is questionable whether the decision maker could make a valid distinction between a function in the form of equation (5-11) or (5-12) and a function of another form.

The functions X and Y will be a function of the time since the last sample, τ. Since $V_x(n,T,T) = V_{po}$,

$$V_{po} + X' = V_{pr} \qquad (5\text{-}13)$$

or $$Y' V_{po} = V_{pr} \qquad (5\text{-}14) \quad \text{when } \tau = T.$$

Therefore, given equations (5-13) and (5-14), X' and Y' can be found. The solutions show X' and Y' as functions of T. To obtain the final estimate for the uncertainty function, T must be set equal to τ for the X' and Y' terms ($X = X'$ when $\tau = T$). Finally, the decision maker must determine which of the final forms of equations (5-11) and (5-12) best fits his process.

To illustrate the preceding procedure, suppose it has been determined that,

$$V_{pr} = V_x(n,T,T) = (V/n)\ (e^{bT} - 1)$$

then, $$V_{po} = \frac{V\ V_{pr}}{V + nV_{pr}} = (V/n)\ (1 - e^{-bT})$$

so, $$(V/n)\ (e^{bT} - 1) = (V/n)\ (1-e^{-bT}) + X'$$

or $$(V/n)\ (b^{bT} - 1) = (V/n)\ (1-e^{-bT})\ Y'.$$

Solving these equations yields,

$$X' = \frac{(V/n)\ (1 - e^{-bT})^2}{e^{-bT}} \quad \text{and} \quad X = \frac{(V/n)\ (1 - e^{-b\tau})^2}{e^{-b\tau}}$$

$$Y' = e^{bT} \text{ and } Y = e^{b\tau}.$$

Therefore,

$$V_x(n,T,\tau) = (V/n)\ (1 - e^{-bT}) + \frac{(V/n)\ (1 - e^{b\tau})^2}{e^{-b\tau}} \tag{5-15}$$

or $$V_x(n,T,\tau) = (V/n)\ (1 - e^{-bT})\ (e^{b\tau}) \tag{5-16}$$

Notice that equation (5-16) is the function used in the example in the unbounded section of Chapter IV.

Summary

This chapter has attempted to give a physical interpretation to the uncertainty function $V_x(n,T,\tau)$. A determination of this function can be made using objective or subjective data.

The use of objective data involves determining the autocorrelation function between $x(t)$ and $x(t+\tau)$. The uncertainty function can be determined from the autocorrelation function.

If the decision maker does not have this objective data available to him, he may attempt to estimate the autocorrelation function or the prior variance function, V_{pr}.

There may be some discussion concerning the methods used in developing these functions from subjective data. The argument will be that these functions may be something other than one of the functions shown the decision maker. However, since the functions will be subjective estimates, the decision maker will not be able to distinguish between subtle differences between two different functions. Therefore, only common functions, such as the autocorrelation function $e^{-b\tau}$, need be shown him. Refinements can be made only if there is an abundance of actual data available.

The uncertainty function is one of the major variables in the determination of an optimal sampling policy. The ability to use objective or subjective data to determine this function should allow for a much broader application of the procedures developed in this thesis.

VI. SUMMARY AND CONCLUSIONS

The sampling interval problem, as defined in this thesis, refers to any situation in which someone must determine how often a time-varying process should be sampled. Sampling refers to the actions that must be taken in order to monitor the process. There are many examples where the sampling interval problem exists. The monitoring of continuous manufacturing processes or the publication of business reports are just two examples.

In this thesis, a theoretical model, based on a Bayesian analysis, is developed, from which the optimal sampling interval can be determined. The information needed to use this model includes sampling costs, a quadratic error cost, and a normally distributed measure of the growth of the uncertainty of the process with time. The sampling interval with the least expected cost per unit time is considered optimal.

The first step in developing a general model to solve this problem was to look at a specific, but simplified application. This is the "look-see" problem. In this problem, the value of the process is known exactly whenever the decision maker samples or looks at the process. His uncertainty immediately after sampling will be equal to zero. The sample size needs never be greater than one, since nothing can be gained by additional sampling. Equations were developed from which the optimal sampling interval can be determined once the functional form of the uncertainty is known.

The next step was to examine the case where the uncertainty immediately after sampling was not equal to zero. In this case, an error term is introduced into the reading. This is the "sampling error" problem. Additional samples will tend to decrease the amount of uncertainty that exists concerning the state of the process. Therefore, in addition to determining an optimal sampling interval, an optimal sampling size must also be found. The solution to this problem involves determining the optimal sampling interval in a manner similar to the "look-see" problem for different sample sizes and choosing the sample size with the lowest expected cost per unit time.

To illustrate the use of the solution techniques that were developed, two examples were presented for each problem. In one case, the uncertainty was unbounded, that is, it increased without bound as time increased. In the other case, there was a limit that the uncertainty would never exceed. The uncertainty was considered bounded.

A sensitivity analysis was performed to show the effect on the optimal sampling policy when one of the cost parameters is changed.

Although several examples were used, they did not represent "real world" examples. One of the obvious areas for future research is in the application of the models developed here to a practical example. The purpose of this paper was to develop a general model that could be adopted to specific situations and from which more sophisticated models could evolve.

There are, of course, some significant areas for future research.

Some of the items to be mentioned may be more important than others and may have a larger impact on the sampling intervals. The order in which they are presented is not in order of importance.

One area that can and should be explored concerns the effect of different error cost functions on the optimal sampling intervals. While quadratic costs can approximate linear and step cost functions, it may be found that the approximation is not sufficiently accurate.

A second area for future research involves the correction action. The time needed for correction should be investigated. In this paper the assumption has been made that an instantaneous correction is possible. In some instances, performing the necessary adjustments may take a significant amount of time during which the process will remain "out of control" or have to be shut down. The cost of correction may also be significant.

An area that may prove very interesting is in the area of parametric analysis, showing the effects of simultaneous changes in the costs.

A final area in which additional research may prove useful involves multi-process sampling. One method of determining sampling intervals when several processes are involved is to determine the interval for each process separately. This method will be adequate if the processes are independent, but a different approach must be used if the processes are related in some manner. In this situation, the uncertainty functions would have to be revised to reflect this dependence.

There are undoubtedly other extensions to the research presented in this paper. Solutions involving the items discussed will not solve the entire sampling problem, but would provide solutions to a wider range of processes.

In conclusion, the models in this paper provide a general approach to the sampling interval problem. While not claiming to be a panacea, it should provide insight into the problems of this nature and set the stage for "scientific" methods for determining optimal sampling intervals.

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APPENDICES

APPENDIX 1

DERIVATION OF EQUATION (3-16)

$$V_x(t) = A(1 - e^{-bt}), \qquad (3\text{-}14)$$

from equation (3-6),

$$\frac{-C}{(T^*)^2} \int_0^{T^*} A(1 - e^{-b\tau})d\tau + \frac{C}{T^*}A(1 - e^{-bT^*}) - \frac{S}{(T^*)^2} = 0$$

$$\int_0^{T^*} (1 - e^{-b\tau})d\tau - T^*(1 - e^{-bT^*}) = \frac{-S}{CA}$$

$$T^* + \frac{1}{b} e^{-bT^*} - \frac{1}{b} - T^* + T^*e^{-bT^*} = \frac{-S}{CA}$$

$$bT^* + e^{-bT^*} - bT^* + bT^*e^{-bT^*} = 1 - \frac{bS}{CA}$$

$$e^{-bT^*}(1 + bT^*) = \frac{CA - bS}{CA} . \qquad (3\text{-}16)$$

APPENDIX 2

SOLUTION TECHNIQUES

Given the various solution equations presented in this thesis, there are several ways in which the optimal sampling intervals can be determined. The purpose of this section is to explain some of the techniques that can be used and illustrate how the solutions were obtained to the examples presented in this paper.

The first example presented was the unbounded uncertainty case in the "look-see" problem. Solving the appropriate equation when $V_X(\tau) = k\tau$, yields an explicit equation for T*, as shown in equation (3-12). Therefore, no special solution techniques were needed. If the uncertainty function was something other than the one presented, one of the methods to be discussed may be needed.

In the second example of Chapter III, the bounded uncertainty case, an explicit equation is not given, what is given is,

$$e^{-bT^*}(1 + bT^*) = \frac{CA - bS}{CA}. \qquad (3\text{-}16)$$

One way to solve this equation is a trial-and-error method. Once the right hand side of the equation is calculated, various values for T* can be tried until the correct value is found. Another technique is to use the graph in Figure 10. In order to draw this graph, the values for bT* were calculated for various values of (CA-bS)/CA using the Newton-Raphson method. Given a function, f(x), the solution is found using an iterative process with the following relationship

$$X_{i+1} = X_i - \frac{f(X_i)}{f'(X_i)} .$$

Since equation (3-16) was easily differentiated, a computer program was written to solve the equation for values of $\hat{C}$ ranging from 0.01 to 1.00 in increments of 0.01. The listing of this program is at the end of this Appendix and is labeled Program A.

In the unbounded example contained in Chapter IV, equation (4-9) can be solved by trial-and-error or by the Newton-Raphson method. Program B is a Newton-Raphson solution to this example. Although it was not done, Program B could be easily modified to compute values for bT* for any number of values for the right side of this equation. This would provide a graphical solution procedure.

Another method that can be used to solve this example is to use a Golden Section Search method on equation (4-3) to minimize this equation. This eliminates the need to calculate the integral of a partial derivative. If differentiation is difficult, this procedure is very helpful. Since the objective is to minimize the expected cost per unit time, the expected cost for the optimal policy is calculated without having to make use of a separate equation. Program C is a listing of a program from which the optimal sampling policy can be determined.

Due to the complex nature of equation (4-11), a Golden Section Search method is suggested for solving this example. Program D is a listing which will calculate this solution. This program uses the trapezoidal method of numerical integration to evaluate the function since, as discussed in Chapter IV, indefinite integration will not yield satisfactory results.

Finally Program E is the program used to solve the objective uncertainty example in Chapter V.

Program B, C, D, and E have been written so that only the data file must be changed in order to solve an example with different parameter values. Also, the Golden Section Search routine in Programs C, D, and E is identical. A functional subroutine is used to solve the individual problems.

These problems do not represent the only way and perhaps not the best way to solve the examples presented. They did, however, when run on an IBM 370-168, provide accurate and speedy answers at a reasonable cost.

Program A

```
      DIMENSION T(100)
      C=0.01
      WRITE(6,10)
10     FORMAT(1H0,3X,'C*',6X,'T*')
      ERROR=0.00001
20    I=1
      T(I)=1.0
30    A1=EXP(-T(I))*(1+T(I))-C
      A2=EXP(-T(I))*T(I)
      T(I+1)=T(I)+A1/A2
      DIFF=ABS(T(I+1)-T(I))
      IF(DIFF.LT.ERROR) GOTO 40
      I=I+1
      GOTO 30
40    X=T(I+1)
      WRITE(6,50) C,X
50     FORMAT(1H ,F6.2,F9.4)
      C=C+0.01
      IF(C.GT.1.0) GOTO 60
      GOTO 20
60    CALL EXIT
      END

R;

C>
```

Program B

```
      DIMENSION T(100)
      READ(9,5) N,B,S,U,V,C,ERROR
 5     FORMAT(I2,5F8.4,F7.5)
      WRITE(6,10)
10     FORMAT(1H0,1X,'I',4X,'T(I)')
      I=1
      T(I)=5
      WRITE(6,20) I,T(I)
20     FORMAT(1H ,I2,F13.9)
30    A1=(EXP(B*T(I))-EXP(-B*T(I)))/2
      A2=(EXP(B*T(I))+EXP(-B*T(I)))/2
      A3=(S+(U*N))*B*N
      A4=2*C*V
      A5=(A3-A4)/A4
      A6=B*T(I)*A1
      A7=B**2*T(I)*A2
      T(I+1)=T(I)-((A6-A2-A5)/A7)
      J=I+1
      X=T(I+1)
      WRITE(6,40) J,X
40     FORMAT(1H0,I2,F10.5)
      DIFF=ABS(T(I+1)-T(I))
      IF(DIFF.LT.ERROR) GOTO 50
      I=I+1
      GOTO 30
50    CALL EXIT
      END

R;

C>T
```

Program C

```
      DIMENSION EC(50)
      REAL L,LOWER
      INTEGER D,N
      N=1
      EC(1)=999.
      WRITE(6,10)
10    FORMAT(1H1,3(/),2X,'N',8X,'T*',8X,'E(C)')
11    READ(9,15) C,V,B,S,U,LOWER,UPPER,ERROR
15    FORMAT(8F8.4)
       D=N+1
       L=UPPER-LOWER
       R=(SQRT(5.)-1)/2
       X=LOWER+(L*(1-R))
       Y=LOWER+(L*R)
       FOFX=CALC(N,C,V,B,S,U,X)
       FOFY=CALC(N,C,V,B,S,U,Y)
25    IF(L.LE.ERROR) GOTO 80
       L=L*R
       IF(FOFX.LE.FOFY) GOTO 50
       LOWER=X
       X=Y
       Y=LOWER+(R*L)
       FOFX=FOFY
       FOFY=CALC(N,C,V,B,S,U,Y)
       GOTO 25
50     UPPER=Y
       Y=X
       X=LOWER+((1-R)*L)
       FOFY=FOFX
       FOFX=CALC(N,C,V,B,S,U,X)
       GOTO 25
80    TOPT=LOWER+(L/2)
      EC(D)=FOFX+((FOFY-FOFX)/2)
      WRITE(6,85) N,TOPT,EC(D)
85    FORMAT(/1H ,I2,2F12.4)
      IF(EC(D).GT.EC(D-1))GOTO 87
      N=N+1
      REWIND 9
      GOTO 11
87    CALL EXIT
       END
       FUNCTION CALC(N,C,V,B,S,U,T)
       A1=(C*V)/(B*N*T)
       A2=EXP(B*T)+EXP(-B*T)-2.0
       A3=(S+(U*N))/T
       CALC=(A1*A2)+A3
       RETURN
       END

R;

C>
```

Program D

```
      DIMENSION EC(50)
      REAL L, LOWER, FOFX, FOFY,M,FPART,FSUB,FUPP,INT,FVAL
      INTEGER D,Z
      N=1
      EC(1)=999
      WRITE(6,10)
10    FORMAT(1H1,3(/),2X,'N',8X,'T*',8X,'E(C)')
11    READ(9,15) A,B,S,C,U,V,LOWER,UPPER,ERROR
15    FORMAT(9F8.4)
       D=N+1
       L=UPPER-LOWER
       R=(SQRT(5.)-1.)/2.
       X=LOWER+(L*(1.-R))
       Y=LOWER+(L*R)
       FOFX=RINTEG(A,B,S,C,U,N,V,X)
       FOFY=RINTEG(A,B,S,C,U,N,V,Y)
25    IF(L.LE.ERROR) GOTO 80
       L=R*L
      IF(FOFX.LE.FOFY) GOTO 50
       LOWER=X
       X=Y
       Y=LOWER+(R*L)
       FOFX=FOFY
       FOFY=RINTEG(A,B,S,C,U,N,V,Y)
       GOTO 25
50     UPPER=Y
       Y=X
       X=LOWER+((1.-R)*L)
       FOFY=FOFX
       FOFX=RINTEG(A,B,S,C,U,N,V,X)
       GOTO 25
80    TOPT=LOWER+(L/2.)
      EC(D)=FOFX+((FOFY-FOFX)/2)
      WRITE(6,85)N,TOPT,EC(D)
85    FORMAT(/1H ,I2,2F12.4)
      IF(EC(D).GT.EC(D-1)) GOTO 87
      N=N+1
      REWIND 9
      GOTO 11
87    CALL EXIT
      E N D
      FUNCTION RINTEG(A,B,S,C,U,N,V,T)
       INTEGER Z,N
       REAL FOFX,FOFY,M,FPART,FSUB,FUPP,INT,FVAL
       M=(V*A)/(V+(N*A))
       Z=IFIX(50*T)
       FSUB=0.
       N1=Z-1
      DO 90 I2=1,N1
       TI=I2*T/Z
       FPART=N*M**2.*(1-EXP(-B*TI))**2./(V-N*M*(1-EXP(-B*TI)))
       FSUB=FSUB+FPART
90    CONTINUE
       FLOW=0.
       FUPP=N*M**2.*(1-EXP(-B*T))**2./(V-N*M*(1-EXP(-B*T)))
       INT=(T/Z)*(((FLOW+FUPP)/2)+FSUB)
       RINTEG=((C/T)*INT)+(C*M*(1-EXP(-B*T)))+((S+(U*N))/T)
      RETURN
      E N D

R;

C>
```

Program E

```
      DIMENSION EC(50)
      REAL L, LOWER, FOFX, FOFY,M,FPART,FSUB,FUPP,INT,FVAL
      INTEGER D,Z
      N=1
      EC(1)=999
      WRITE(6,10)
10    FORMAT(1H1,3(/),2X,'N',8X,'T*',8X,'E(C)')
11     READ(9,15) VARX,VARE,C,S,U,B,LOWER,UPPER,ERROR
15    FORMAT(9F8.4)
       D=N+1
       L=UPPER-LOWER
       R=(SQRT(5.)-1.)/2.
       X=LOWER+(L*(1.-R))
       Y=LOWER+(L*R)
       FOFX=CALC(N,VARX,VARE,C,S,U,B,X)
       FOFY=CALC(N,VARX,VARE,C,S,U,B,Y)
25    IF(L.LE.ERROR) GOTO 80
       L=R*L
      IF(FOFX.LE.FOFY) GOTO 50
       LOWER=X
       X=Y
       Y=LOWER+(R*L)
       FOFX=FOFY
       FOFY=CALC(N,VARX,VARE,C,S,U,B,Y)
       GOTO 25
50     UPPER=Y
       Y=X
       X=LOWER+((1.-R)*L)
       FOFY=FOFX
       FOFX=CALC(N,VARX,VARE,C,S,U,B,X)
       GOTO 25
80    TOPT=LOWER+(L/2.)
      EC(D)=FOFX+((FOFY-FOFX)/2)
      WRITE(6,85)N,TOPT,EC(D)
85    FORMAT(/1H ,I2,2F12.4)
      IF(EC(D).GT.EC(D-1)) GOTO 87
      N=N+1
      REWIND 9
      GOTO 11
87    CALL EXIT
      E N D
      FUNCTION CALC(N,VARX,VARE,C,S,U,B,T)
      RHO2=EXP(-B*T)
      AQ=N*RHO2
      BQ=(VARE+(N*VARX))*(1-RHO2)
      CQ=VARX*VARE*(1-RHO2)
      VPRIM=(-BQ+SQRT((BQ**2)+(4*AQ*CQ)))/(2*AQ)
      S1=T*VARX
      S2=VARX-VPRIM
      S3=1/B
      S4=1-EXP(-B*T)
      PART=S1-(S2*S3*S4)
      CALC=((C/T)*PART)+((S+(U*N))/T)
      RETURN
      E N D

R;
```

APPENDIX 3

DERVIATION OF EQUATION (4-9)

$$V_x(n,T,\tau) = (V/n)\ (1 - e^{bT})\ (e^{-b\tau}) \qquad (4\text{-}8)$$

$$\int_0^T V_x(n,T,\tau)d\tau = (V/n)\ (1 - b^{-bT})\ (1/b)\ (e^{bT} - 1)$$

$$\frac{\delta}{\delta T}\ V_x(n,T,\tau) = (V/n)\ (e^{b\tau})\ (-be^{-bT})$$

$$\int_0^T \frac{\delta}{\delta T}\ V_x(n,T,\tau)d\tau = (V/n)\ (be^{-bT})\ (1/b)\ (e^{bT} - 1)$$

$$V_n(n,T,T) = (V/n)\ (e^{bT} - 1)$$

from equation (4-4)

$$- \frac{C}{T^{*2}}\ (V/n)\ (1 - e^{-bT^*})\ (1/b)\ (e^{bT^*} - 1)$$

$$+ \frac{C}{T^*}\ [(V/n)\ (be^{-bT^*})\ (1/b)\ (e^{bT^*} - 1) + (V/n)\ (e^{bT^*} - 1)]$$

$$- \frac{S + Un}{T^{*2}} = 0$$

then

$$-(e^{bT^*} - 2 + e^{-bT^*}) + bT^*(1 - e^{-bT^*} + e^{bT^*} - 1) = \frac{(S+Un)bn}{CV}$$

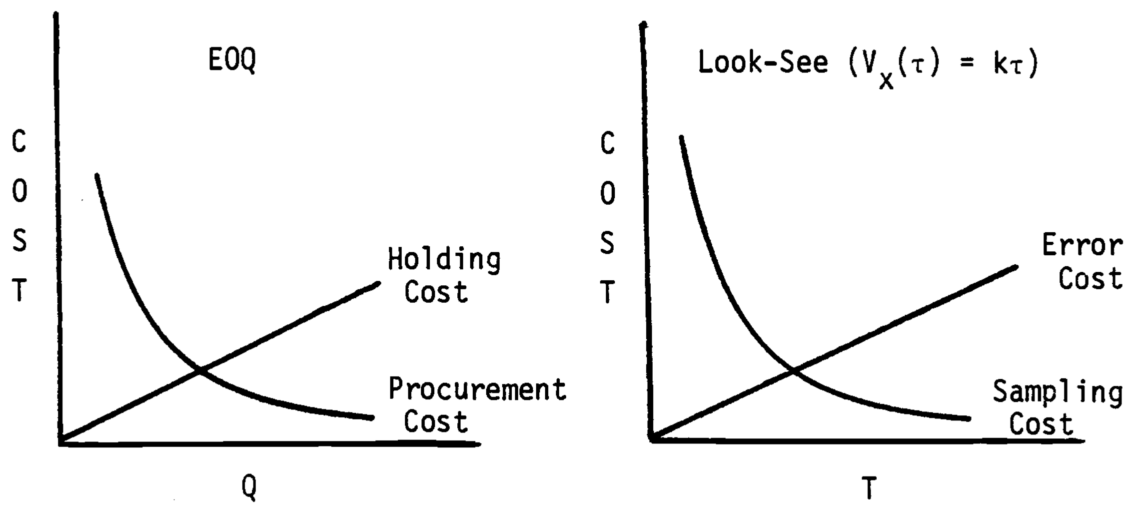
$$bT^*\ (e^{bT^*} - e^{-bT^*}) - (e^{bT^*} + e^{-bT^*}) = \frac{(S+Un)bn}{CV} - 2$$

$$bT^*\ \sinh(bT^*) - \cosh(BT^*) = \frac{(S+Un)bn - 2CV}{2CV}\ . \qquad (4\text{-}9)$$

APPENDIX 4

MISCELLANEOUS OBSERVATIONS ABOUT SELECTED RESULTS AND ASSUMPTIONS

The solution to the unbounded example in the "look-see" problem bears a striking resemblance to this classic EOQ model. It will be worthwhile to draw a parallel between these two models. The two cost components in the EOQ model are the carrying costs and the procurement costs. The two costs in the "look-see" model are the error cost and the sampling cost. Figure 14 graphically portrays both situations.



Holding Cost = $\frac{HQ}{2}$

Procurement Cost = $\frac{OD}{Q}$

$$Q^* = \sqrt{\frac{2OD}{H}}$$

Error Cost = $\frac{CkT}{2}$

Sampling Cost = $\frac{S}{T}$

$$T^* = \sqrt{\frac{2S}{Ck}}$$

Figure 14. EOQ, Look-See Model Comparisons

If holding costs in the EOQ model are based on a maximum inventory as opposed to the average inventory, the holding cost expression becomes HQ, and the solution for Q becomes,

$$Q^* = \sqrt{\frac{OD}{H}} \quad .$$

Likewise, if the error costs in the "look-see" model are assessed on the maximum error, the error cost will become CkT, and the solution for T will be,

$$T^* = \sqrt{\frac{S}{Ck}} \quad .$$

At several points during the development of the models used in this paper, the term infinite costs has been used. Some may argue that there is no such thing as an infinite cost. However, when one of the costs is three or four orders of magnitude greater than the other cost, the first cost would be essentially infinite. For example, a process that has a small sampling cost and a very narrow tolerance limit outside of which a catastrophic situation occurrs, could be considered as having an infinite error cost.

In situations in which the variance function is determined on the basis of a subjective estimate, the resulting determination of optimal sampling intervals may be different than if the variance was objectively determined. As Sheridan and Rouse (1971) showed, the subjectively determined variance will result in a smaller sampling interval. The important consideration is that the decision maker must be convinced that he is providing a reasonable estimate of the actual process variance. This is similar to the use of the Markov Assumption. In this situation, the process in question must at least "appear" to be Markovian. Likewise, the subjective estimate of the process variance must "appear" to be a reasonable estimate of the actual process variance.

APPENDIX 5

SELECTED RESULTS FROM NUMERICAL EXAMPLES

Sampling Error Problem

$$V_x(n,T,\tau) = V/n\ (1-e^{-bT})\ (e^{b\tau})$$

where $b = .25\ (\text{hours})^{-1}$ and $V = 1\ (\text{foot})^2$

S = $ U = $/obs/sample T** = hours

C = $/(feet)^2-hr n = observations/sample E[C] = $/hours

S	C	U	n*	T**	E[C]
15	5	5	5	8.25	11.54
15	10	5	6	7.33	10.15
15	20	5	7	6.36	13.45
15	5	2.5	8	9.09	6.0
15	10	2.5	9	7.85	7.75
15	20	2.5	12	7.33	10.15
15	10	2.5	9	7.85	7.75
15	10	5	6	7.33	10.15
15	10	10	4	6.87	13.45
15	20	2.5	12	7.33	10.15
15	20	5	7	6.36	13.45
15	20	10	4	5.40	18.02
5	10	5	4	5.22	8.54
15	10	5	6	7.33	10.15
25	10	5	7	8.37	11.42
5	5	10	2	5.22	8.54
15	5	10	3	7.33	10.15
25	5	10	4	8.90	11.44

Sampling Error Problem

$$V_x(N,T,\tau) = \frac{nB^2\ (1-e^{-b\tau})^2}{V-nB\ (1-e^{-b\tau})} + B\ (1-e^{-bT})$$

where $B = \frac{VA}{V + nA}$ and $A = 4\ (\text{feet})^2$

$b = .25\ (\text{hours})^{-1}$

$V = 1\ (\text{feet})^2$

S = \$ U = \$/obs/sample T** = hours

C = $\$/(\text{feet})^2$-hr n* = observations/sample E[C] = \$/hour

S	C	U	n*	T**	E[C]
15	5	5	4	8.88	6.81
15	10	5	5	7.63	9.13
15	20	5	6	6.52	12.29
15	5	10	2	7.74	8.56
15	10	10	3	7.03	11.73
15	20	10	4	6.25	16.09
15	10	2.5	8	8.21	7.13
15	10	5	5	7.63	9.13
15	10	10	3	7.03	11.73
15	20	2.5	11	7.58	9.43
15	20	5	6	6.51	12.29
15	20	10	4	6.25	16.09
5	10	5	3	5.05	7.62
15	10	5	5	7.63	9.13
25	10	5	6	8.90	10.32
5	10	10	2	4.91	10.06
15	10	10	3	7.03	11.73
25	10	10	4	8.65	13.04