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1 Introduction

The purpose of this paper is to present a general framework for the study of control problems using the behavioral formalism and to specialize such a setting to the ℓ_1 optimal control problem. Especially in the control community, dynamical systems are dominantly viewed as operators acting on inputs and producing output signals. It has been argued in [9, 10] that for many applications in modeling, control and simulation, the traditional input-output framework may not be a natural starting point. Also, the causality structure which is often assumed in feedback configurations imposes constraints on the design of control systems which may not be necessary or which may not correspond to a physical structure. It is a distinguishing feature of the behavioral theory that systems are described in terms of equations rather than input-output operators. Behavioral equations define relationships among system variables in which input and output signals are not necessarily distinguished. System variables are therefore treated in a symmetric way which may have major conceptual advantages for theoretical and practical considerations in control.

Control of dynamical systems concerns the manipulation of a selected set of variables so as to achieve some kind of desirable behavior. Here, by 'selected variables' we will mean a distinguished set of system variables which can be interconnected with a control system. Desirable behavior will be expressed in terms of qualitative or quantitative properties of the system which is obtained by interconnecting plant and controller. For such an interconnection, we will make a crucial distinction between interconnection and external variables. *Interconnection variables* describe the interaction between plant and controller. These variables have been selected a priori and can be used for control purposes. *External variables* are the variables by means of which the controlled plant interacts with its environment. Typically, actuator inputs and measured process outputs are interconnection variables; external variables include reference signals, disturbances and to-be-controlled system variables. A *plant* is viewed as a dynamical system that imposes constraints on both external and interconnection variables. We will view a *controller* as a set of laws which imposes constraints on the interconnection variables only.

This paper is motivated by earlier work on control in a behavioral context [5, 6, 11, 14], and by papers on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control in this formalism [8, 12, 13]. The papers [8, 12, 13] concentrate on the full-information case, while in this paper we study the partial-information case since we explicitly specify the variables on which the controller can impose constraints. The ℓ_1 optimal control problem has not been considered in the behavioral formalism before. For the main ideas of ℓ_1 optimal control in the more common input-output framework we refer to [2].

2 System interconnections

The analysis of system interconnections is the core of many problems in modeling, simulation and control. In this section we concentrate on the interconnection of a *plant* with a *controller*. We will consider dynamical systems $\Sigma = (T, W, \mathcal{B})$ with discrete time set $T = \mathbb{Z}_+$ or $T = \mathbb{Z}$, finite dimensional real valued signal spaces $W = \mathbb{R}^q$, $q > 0$ and behaviors $\mathcal{B}$ which are linear shift-invariant and complete subsets of W^T. This class of linear systems will be denoted by $\mathcal{L}^q$.

It has been shown in [9, 10] that $\mathcal{L}^q$ admits a parametrization by means of real polynomial matrices with q columns. Precisely, for every system $\Sigma \in \mathcal{L}^q$ there exists a polynomial $R \in \mathbb{R}^{\cdot \times q}[z]$, i.e. a polynomial of the form $R(z) = \sum_{i=0}^{L} R_i z^i$ with R_i real matrices with q columns, such that the behavior $\mathcal{B}$ of Σ can

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be written as

$$\mathcal{B} = \mathcal{B}_{\text{ker}}(R) := \{w : \mathbb{Z}_+ \to \mathbb{R}^q \mid R(\sigma)w = 0\}.$$

Here, $R(\sigma)$ is to be interpreted as a polynomial operator in the left-shift σ (i.e. $[\sigma w](t) = w(t+1)$) acting on the signal space $(\mathbb{R}^q)^T$.

Let $\Sigma_p \doteq (T, W, \mathcal{B}_p)$ be a dynamical system and suppose that its signal space $W = \mathbb{R}^q$ is partitioned as

$$W = W_e \times W_i$$

where $W_e = \mathbb{R}^{q_e}$ is the *external signal space* and $W_i = \mathbb{R}^{q_i}$ is a non-empty set called the *interconnection space*. Here, $q = q_e + q_i$ and $q_i > 0$. We refer to Σ_p as the *plant*. A *controller* for Σ_p is a dynamical system $\Sigma_c = (T, W_i, \mathcal{B}_c)$ which, when interconnected with Σ_p imposes constraints on the interconnection variables only. We formalize this as follows.

Definition 2.1 *The* interconnection *of the systems* $\Sigma_p = (T, W_e \times W_i, \mathcal{B}_p)$ *and* $\Sigma_c = (T, W_i, \mathcal{B}_c)$ *is the system*

$$\Sigma_p \sqcap \Sigma_c := (T, W_e \times W_i, \mathcal{B}_p \sqcap \mathcal{B}_c)$$

where

$$\mathcal{B}_p \sqcap \mathcal{B}_c := \{(w_e, w_i) \mid (w_e, w_i) \in \mathcal{B}_p \text{ and } w_i \in \mathcal{B}_c\}. \tag{2.1}$$

If W_e *is void then* $\Sigma_p \sqcap \Sigma_c$ *is called a* full interconnection.

Note that in a full interconnection $\mathcal{B}_p \sqcap \mathcal{B}_c = \mathcal{B}_p \cap \mathcal{B}_c$. Further, it is easily seen that the interconnection $\Sigma_p \sqcap \Sigma_c \in \mathcal{L}^q$ if $\Sigma_p \in \mathcal{L}^q$ and $\Sigma_c \in \mathcal{L}^{q_i}$. In what follows, we will view Σ_c as a dynamical system that is (or needs to be) designed to be interconnected with the plant Σ_p. The interconnection $\Sigma_{cp} := \Sigma_p \sqcap \Sigma_c$ is refered to as the *controlled plant.*

Not all system interconnections will qualify for the purpose of control. A well-posed interconnection is defined as follows.

Definition 2.2 *Let* $\Sigma_p = (T, W_e \times W_i, \mathcal{B}_p)$ *and* $\Sigma_c = (T, W_i, \mathcal{B}_c)$ *be two dynamical systems. Their interconnection* $\Sigma_p \sqcap \Sigma_c$ *is said to be* well-posed *if there exists* $t_0 \in T$ *such that*

$$\{(w_e, w_i'), (w_e, w_i'') \in \mathcal{B}_p \sqcap \mathcal{B}_c,\ w_i'(t) = w_i''(t) \text{ for } t \leqslant t_0\} \implies \{w_i' = w_i''\}.$$

This means that once the external trajectories w_e in an interconnected system are specified, the set of all interconnection variables w_i for which $(w_e, w_i) \in \mathcal{B}_p \sqcap \mathcal{B}_c$ define an autonomous behavior. In other words, a well-posed interconnection does not allow inputs in the interconnection variables for the interconnected system.

Note that the interconnection variables w_i do not need to be partitioned in inputs (actuators) and outputs (measurements). This is one of the reasons that we avoid the usage of more classical terminology like 'feedback' and 'closed-loop' as the causality structure of the interconnection variables is irrelevant in this setting.

Let $\Sigma_p \in \mathcal{L}^q$ be a dynamical system and suppose that its behavior is represented in polynomial form by

$$\mathcal{B}_p = \mathcal{B}_{\text{ker}}\left(\begin{bmatrix} R_e & R_i \end{bmatrix}\right)$$

where $R_e \in \mathbb{R}^{\cdot \times q_e}[z]$ and $R_i \in \mathbb{R}^{\cdot \times q_i}[z]$. If $\Sigma_c \in \mathcal{L}^{q_i}$ is a controller, then its behavior can be represented as

$$\mathcal{B}_c = \mathcal{B}_{\text{ker}}(R_c)$$

where $R_c \in \mathbb{R}^{\cdot \times q_e}[z]$ and the resulting interconnected system admits an autoregressive representation of the form

$$\begin{bmatrix} R_e(\sigma) & R_i(\sigma) \\ 0 & R_c(\sigma) \end{bmatrix} \begin{bmatrix} w_e \\ w_i \end{bmatrix} = 0. \tag{2.2}$$

Well-posedness of such an interconnection is easily checked.

Proposition 2.3 *The interconnection of* Σ_p *and* Σ_c *is well-posed if and only if the polynomial matrix*

$$\begin{pmatrix} R_i \\ R_c \end{pmatrix}$$

is injective as viewed as a matrix over the field of rational functions.

In a well-posed interconnection the controller has to add a minimal number of laws to guarantee that the interconnection variables are uniquely determined once the external variables are fixed. But we do not want to add too many control laws either. In a classical input-output framework a controller should guarantee that the interconnecting variables are uniquely determined once the external variables are fixed but it should not impose any constraints on the external inputs. To guarantee that also in the present setting we do not impose undue constraints on the external variables we introduce the concept of a *minimal controller.* For this, the input dimension of a dynamical system is the relevant integer invariant to consider.

Definition 2.4 *The* complexity *of a dynamical system* $\Sigma \in \mathcal{L}^q$ *is the pair of integers* $(m(\Sigma), n(\Sigma)) := (m, n)$ *which satisfy*

$$\dim(\mathcal{B}|_{[0,t-1]}) = mt + n \tag{2.3}$$

for all $t \geqslant n$.

The numbers $m(\Sigma)$ and $n(\Sigma)$ are well defined this way and correspond to the number of inputs and the minimal number of states in an input-state-output representation of Σ.

Definition 2.5 *The system $\Sigma_c \in \mathcal{L}^{q_i}$ is said to be a* minimal controller *for a plant $\Sigma_p \in \mathcal{L}^q$ if the interconnection $\Sigma_p \sqcap \Sigma_c$ is well-posed and if for any other controller Σ_c' which makes the interconnection well-posed we have*

$$m(\Sigma_p \sqcap \Sigma_c) \geqslant m(\Sigma_p \sqcap \Sigma_c')$$

Intuitively $m(\Sigma_p \sqcap \Sigma_c)$ is the number of free variables or inputs of the controlled systems and hence we require that we do not reduce the number of free variables any more than necessary to make the controlled system well-posed. A minimal controller therefore adds a minimal number of constraints to obtain a well-posed interconnection. Also minimality of an interconnection is easily checked.

Proposition 2.6 *Assume that the interconnection of $\Sigma_p \in \mathcal{L}^q$ and $\Sigma_c \in \mathcal{L}^{q_i}$ is well-posed. Then the interconnection is minimal if and only if*

$$q_i = \operatorname{normrank} R_i + \operatorname{normrank} R_c.$$

3 Control objectives

Let $\Sigma_p \in \mathcal{L}^{q_e+q_i}$, $\Sigma_c \in \mathcal{L}^{q_i}$ and consider the controlled plant $\Sigma_{cp} := (T, \mathbb{R}^{q_e+q_i}, \mathcal{B}_{cp}) = \Sigma_p \sqcap \Sigma_c$. Control objectives are usually specified as functionals defined on specific signals of the controlled plant behavior. Indeed, if initial conditions are known, if disturbances are known to be bounded in magnitude or if reference signals are specified, then only subsets of the controlled plant behavior are relevant for the specification of system performance. These restrictions will be formalized by considering trajectories $w \in \mathcal{B}_{cp}$ which satisfy

$$w \in \mathcal{R} \tag{3.1}$$

where $\mathcal{R}$ is a subset of $(\mathbb{R}^q)^T$. Thus, the intersection $\mathcal{B}_{cp} \cap \mathcal{R}$ is considered as the relevant set to verify control objectives. A *control objective* is another subset $\mathcal{S}$ of $(\mathbb{R}^q)^T$ which is assumed to be specified either in a qualitative or in a quantitative way. The controlled system Σ_{cp} achieves the control objective if its behavior $\mathcal{B}_{cp}$ satisfies the inclusion

$$\mathcal{B}_{cp} \cap \mathcal{R} \subseteq \mathcal{S}. \tag{3.2}$$

In that case, the controller Σ_c is said to achieve the control objective $\mathcal{S}$ for the plant Σ_p. We will outline how many different control objectives can be formulated in this framework.

3.1 Stability

External stability can be formulated quite easily. In general external stability depends on an input output structure imposed on the external variables. Suppose therefore that w_e is partitioned as

$$w_e = \begin{pmatrix} d \\ z \end{pmatrix} \tag{3.3}$$

where d denotes an input and z an output component of w_e. The classical definition of bounded-input, bounded-output stability then requires that $d \in \ell_\infty$ implies $z \in \ell_\infty$ in the controlled plant. This is imposed quite easily in our framework by choosing:

$$\mathcal{R} := \{ w = (d, z, w_i) \mid d \in \ell_\infty \}$$
$$\mathcal{S} := \{ w = (d, z, w_i) \mid z \in \ell_\infty \}$$

Similarly, *internal stability* can be imposed by considering well-posed interconnections and requiring (3.2) with

$$\mathcal{R} := \{ w = (d, z, w_i) \mid d = 0 \}$$
$$\mathcal{S} := \{ w = (d, z, w_i) \mid \lim_{t\to\infty} z(t) = 0, \text{ and } \lim_{t\to\infty} w_i(t) \}.$$

3.2 Linear Quadratic Control

The usual formulations of the linear quadratic control problem invariably start with state space descriptions of the system. Generally, an initial state is fixed and a quadratic functional defined on the state and input variables needs to be minimized subject to the state evolution equations. Here we depart from such a formulation. The idea of considering fixed initial states in LQ-type of problems can be imposed in our framework by fixing the past of the controlled plant. In other words we choose $T = \mathbb{Z}$ and assume that some function $\bar{w} : \mathbb{Z}_- \to \mathbb{R}^q$ is given. The linear quadratic control problem then amounts to finding a minimal controller Σ_c for Σ_p such that (3.2) is satisfied with

$$\mathcal{R} := \{ w \mid w(t) = \bar{w}(t) \text{ for } t \leqslant 0 \}$$
$$\mathcal{S} := \{ w \mid \sum_{t=0}^{\infty} w'(t) Q w(t) \leqslant 1 \}$$

where $Q \in \mathbb{R}^{q\times q}$ is some fixed positive semi-definite matrix.

This is obviously not really optimal control since a controller satisfying the above performance criterion does not minimimize a quadratic cost function but only guarantees that the cost function is smaller than some a priori fixed number. However, the essence of linear quadratic control can clearly be recovered via the above framework.

3.3 H_2 optimal control

In H_2 optimal control the H_2 norm of a closed-loop transfer function needs to be minimized over the class of internally stabilizing controllers.

We define the H_2 control problem in our framework by considering the control objective (3.2) with

$$\mathcal{R} := \{ w \mid \sum_{t=-\infty}^{-1} w'(t)Q_1w(t) \leqslant 1 \}$$

$$\mathcal{S} := \{ w \mid \sum_{t=0}^{\infty} w'(t)Q_2w(t) \leqslant 1 \}$$

for suitably chosen positive semi-definite matrices Q_1 and Q_2. The control objective therefore amounts to bounding a quadratic functional on those future trajectories which are compatible with pasts that satisfy a quadratic norm bound.

In a classical input-output setting this problem basically corresponds to a generalized H_2 control problem as studied in e.g. [7]. For non-fixed initial conditions, an H_2 problem can be formulated by introducing some measure on the size of the initial condition. With u denoting a control input signal and x the state of a linear, time-invariant system in input-state-output form, one can impose the following measure on initial states

$$m_1(x_0) = \min_u \{ \sum_{t=-\infty}^{-1} u'(t)u(t) \mid x(0) = x_0 \}.$$

For some given initial condition x_0 one could consider the following optimization criterion

$$m_2(x_0) = \min_u \{ \sum_{t=0}^{\infty} x'(t)Qx(t) \mid x(0) = x_0 \}.$$

where Q is a positive semi-definite matrix. The control objective amounts to finding a controller which achieves that for all x_0, $m_1(x_0) \leqslant 1$ implies $m_2(x_0) \leqslant 1$. For suitable choices of Q_1 and Q_2 this problem is equivalent to (3.2) with $\mathcal{R}$ and $\mathcal{S}$ as defined above.

3.4 H_∞ optimal control

In H_∞ control we impose conditions of the form

$$\|z\|_2 \leqslant \|d\|_2$$

on the external variables w_e which are assumed to be partitioned as in (3.3). This condition is obviously equivalent to

$$\sum_{t=0}^{\infty} \begin{pmatrix} d(t) \\ z(t) \end{pmatrix}' \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} d(t) \\ z(t) \end{pmatrix} \leqslant 0.$$

In other words, we can formulate the H_∞ control problem in our framework by choosing $\mathcal{R}$ and $\mathcal{S}$ as:

$$\mathcal{R} := \ell_2(\mathbb{Z}_+, \mathbb{R}^q)$$

$$\mathcal{S} := \{ w \mid \sum_{t=0}^{\infty} w'(t)Qw(t) \leqslant 0 \}$$

for some suitable chosen matrix Q which will in general be indefinite.

4 The ℓ_1 optimal control problem

L_1 optimal control is a more recent development (for extensive references see e.g. [2]). In its usual operator theoretic formulation, the ℓ_1 optimal control problem amounts to minimizing the ℓ_∞ induced norm of a closed-loop operator mapping disturbances to a to-be-controlled output variable.

We will address this problem in a behavioral framework. Recall that the classical ℓ_1 control problem amounts to finding a (stabilizing) controller for a plant so as to minimize

$$\sup_{d \in \ell_\infty} \frac{\|z\|_\infty}{\|d\|_\infty}$$

or so as to achieve that

$$\sup_{d \in \ell_\infty} \frac{\|z\|_\infty}{\|d\|_\infty} \leqslant 1.$$

Here, d and z denote a decomposition of the external variables in an input and an output component as in (3.3). The latter criterion will be formulated in our setting by imposing as an *a priori* condition that some components of the external variables, say d, have ℓ_∞ norm less than 1. This should imply that other components of the external variables, say z, have ℓ_∞ norm less than 1.

In addition, we would like to study the synthesis problem of constructing a controller which achieves this aim. For simplicity we consider in this paper a finite horizon version of this general problem.

Suppose that a polynomial $R \in \mathbb{R}^{p\times q}[z]$ of degree L is given and partitioned as $R = \begin{bmatrix} R_e & R_i \end{bmatrix}$. Let $N > L$ and let $T = [0, N]$ denote a finite time set. Then R defines a finite-time dynamical system $\Sigma_p = (T, \mathbb{R}^q, \mathcal{B}_p)$ with behavior

$$\mathcal{B}_p := \{ w \mid [R(\sigma)w](t) = 0,\ t = 0, \ldots, N - L \}.$$

Then $\mathcal{B}_p$ is a subspace of the linear space $\mathbb{R}^{q\times(N+1)}$. If we associate with $w : [0, N] \to \mathbb{R}^{q_e+q_i}$ the vector

$$(w_e'(0) \ \ \ldots \ \ w_e'(N) \ \ w_i'(0) \ \ \ldots \ \ w_i'(N))'$$

then the plant behavior can equivalently be described as

$$\mathcal{B}_p = \{ w \mid Cw = 0 \} = \{ (w_e, w_i) \mid C_1w_e + C_2w_i = 0 \} \tag{4.1}$$

where C is a real matrix acting on $\mathbb{R}^{q(N-L+1)}$ and is partitioned as $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ conformally with the partitioning of R.

In such a setting, controllers are finite time dynamical systems $\Sigma_c = (T, \mathbb{R}^{q_i}, \mathcal{B}_c)$ which can be represented by a real matrix C_3 acting on $\mathbb{R}^{q_i(N-L+1)}$, i.e,

$$\mathcal{B}_c = \{ w_i \mid C_3w_i = 0 \} \tag{4.2}$$

The interconnected system $\Sigma_p \sqcap \Sigma_c$ is well defined in this way and its behavior $\mathcal{B}_{cp}$ is represented as

$$\begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix} \begin{bmatrix} w_e \\ w_i \end{bmatrix} = 0.$$

The ℓ_1 control objective is specified by taking both $\mathcal{S}$ and $\mathcal{R}$ polyhedral sets in the finite dimensional space $\mathbb{R}^{q(N-L+1)}$ (which are not necessarily bounded). We will represent $\mathcal{S}$ and $\mathcal{R}$ by two real matrices $A_\mathcal{S}$ and $A_\mathcal{R}$, both having $q_e(N-L+1)$ columns, i.e.

$$\mathcal{R} = \{ w \mid A_\mathcal{R} w_e \leqslant \mathbf{1} \}$$
$$\mathcal{S} = \{ w \mid A_\mathcal{S} w_e \leqslant \mathbf{1} \}$$

where $\mathbf{1}$ denotes a vector of which each component is equal to 1 and of appropriate dimension. All inequalities should be interpreted element by element.

The ℓ_1 control objective is now formulated as in (3.2). The condition (3.2) can be checked via a simple linear programming problem by using the following lemma (see [3, 15]):

Lemma 4.1 (Farkas lemma) *Let a matrix A, column vectors z and a, and some real number c be given. Then $a'x \leqslant c$ for all x such that $Ax \leqslant z$, if and only if*

- *there exists a vector $t \geqslant 0$ such that $A't = a$ and $t'z \leqslant c$, or*
- *there exists a vector $t \geqslant 0$ such that $A't = 0$ and $t'z < 0$.*

Since our control objective translates into the implication

$$\begin{bmatrix} A_\mathcal{R} & 0 \\ C_1 & C_2 \\ 0 & C_3 \\ -C_1 & -C_2 \\ 0 & -C_3 \end{bmatrix} \begin{bmatrix} w_e \\ w_i \end{bmatrix} \leqslant \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \implies$$

$$\begin{bmatrix} A_\mathcal{S} & 0 \end{bmatrix} \begin{bmatrix} w_e \\ w_i \end{bmatrix} \leqslant \mathbf{1},$$

application of Farkas lemma yields the following result

Corollary 4.2 *Let Σ_p and Σ_c be finite time dynamical systems represented by* (4.1) *and* (4.2)*, respectively. Let $\mathcal{B}_{cp}$ denote the behavior of the interconnection $\Sigma_p \sqcap \Sigma_c$. Then the control objective*

$$\mathcal{B}_{cp} \cap \mathcal{R} \subseteq \mathcal{S}$$

is satisfied if and only if there exist matrices T_1, T_2 and T_3 such that:

$$T_1 A_\mathcal{R} + T_2 C_1 = A_\mathcal{S} \quad (4.3)$$
$$T_2 C_2 + T_3 C_3 = 0 \quad (4.4)$$
$$T_1 \mathbf{1} \leqslant \mathbf{1} \quad (4.5)$$
$$T_1 \geqslant 0 \quad (4.6)$$

To check for the existence of T_1, T_2 and T_3 is obviously a simple linear programming problem. This result therefore provides a linear programming type of test to check whether the interconnection of plant and control satisfy the control objective.

On the other hand, for the construction of a controller Σ_c, we have to make sure that there exists a suitable C_3 which guarantees the existence of appropriate T_1, T_2 and T_3 satisfying the conditions (4.3)-(4.6). We therefore need to search for T_1 and T_2 satisfying (4.3), (4.5) and (4.6) such that (4.4) is solvable for suitable T_3 and C_3. Noting that, except for dimensions, C_3 and T_3 are completely free, we obtain the following theorem:

Theorem 4.3 *There exists a controller Σ_c which yields a minimal, well-posed interconnection such that* (3.2) *is satisfied for the interconnected system if and only if there exists matrices T_1 and T_2 satisfying* (4.3), (4.5) *and* (4.6) *such that*

$$\operatorname{normrank}(T_2 C_2) \leqslant q_i - \operatorname{normrank} C_2$$

Proof. It is easy to check that to obtain a well-posed and minimal interconnection the number of rows of C_3 should be equal to $q_i - \operatorname{normrank} C_2$. It is then easy to check that (4.4) is solvable for suitable C_2 and C_3 if and only if

$$\operatorname{normrank}(T_2 C_2) \leqslant q_i - \operatorname{normrank} C_2.$$

□

The condition in the above lemma is not easy to check because of the rank condition. However, it is interesting to note that in a special case we do not need this rank condition:

Corollary 4.4 *Assume that*

$$q_i \geqslant 2 \operatorname{normrank} C_2. \quad (4.7)$$

In that case, there exists a controller which yields a minimal, well-posed interconnection such that (3.2) *is satisfied if and only if there exists matrices T_1 and T_2 satisfying* (4.3), (4.5) *and* (4.6).

Proof. If (4.7) is satisfied then

$$\operatorname{normrank}(T_2 C_2) \leqslant \operatorname{normrank} C_2 \leqslant q_i - \operatorname{normrank} C_2.$$

Hence the rank condition is always satisfied. The result is then an immediate consequence of theorem 4.3 □

The condition in the above corollary basically requires that the number of control inputs is larger than the number of measurements in any input-output decomposition of the interconnection variables w_i. Note that after we found T_1 and T_2, the construction of C_3 is a simple matrix factorization problem.

Remark 4.5 Throughout this paper, the interconnection variables w_i have not been partitioned in input and output components. After a minimal controller has been designed, such a decomposition can be made *a posteriori*. This would yield the usual causality structure of closed-loop configurations in which certain interconnection variables can be identified as inputs (outputs) for the controller and outputs (inputs) for the plant.

Remark 4.6 The results of this section are derived by applying Farkas lemma for polyhedral sets in finite dimensional spaces. This yields a characterization for the ℓ_1 control objective for finite time systems that has been translated in a controller synthesis procedure. We remark that a version of Farkas lemma for infinite dimensional vector spaces exists. See [1, 4]. A study of the infinite horizon ℓ_1 control problem is a topic of current research.

5 Conclusion

In this paper we have formulated a structure for control system design in a behavioral setting. Currently we are extending this work for the ℓ_1 optimal control problem. In particular, we are looking for techniques for the synthesis of minimal ℓ_1-optimal controllers if the rank condition is needed. Also extensions to the infinite horizon case is a topic of future research.

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