A Benchmark Approach to Finance

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Abstract. This paper derives a unified framework for portfolio optimization, derivative pricing, financial modeling and risk measurement. It is based on the natural assumption that investors prefer more rather than less, in the sense that given two portfolios with the same diffusion coefficient value, the one with the higher drift is preferred. Each such investor is shown to hold an efficient portfolio in the sense of Markowitz with units in the market portfolio and the savings account. The market portfolio of investable wealth is shown to equal a combination of the growth optimal portfolio (GOP) and the savings account. In this setup the capital asset pricing model follows without the use of expected utility functions, Markovianity or equilibrium assumptions. The expected increase of the discounted value of the GOP is shown to coincide with the expected increase of its discounted underlying value. The discounted GOP has the dynamics of a time transformed squared Bessel process of dimension four. The time transformation is given by the discounted underlying value of the GOP. The squared volatility of the GOP equals the discounted GOP drift, when expressed in units of the discounted GOP. Risk neutral derivative pricing and actuarial pricing are generalized by the fair pricing concept, which uses the GOP as numeraire and the real world probability measure as pricing measure. An equivalent risk neutral martingale measure does not exist under the derived minimal market model.

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1 Introduction

The aim of this paper is to derive under the benchmark approach several relationships that are fundamental for the understanding of financial markets. This approach uses the *growth optimal portfolio* (GOP) as central building block.

In Markowitz (1959) a mean-variance theory with its well-known *efficient frontier* was introduced, thus opening the avenue to modern portfolio theory. This led to the *capital asset pricing model* (CAPM), see Sharpe (1964), Lintner (1965) and Merton (1973), which is based on the *market portfolio* as reference unit and represents an equilibrium model of exchange. For the continuous time setting Merton (1973) developed the *intertemporal* CAPM from the portfolio selection behavior of investors who maximize equilibrium expected utility. It is apparent that the choice of utility functions with particular time horizons introduces a subjective element and substantial computational challenges, which will be avoided in this paper. A practical problem for applications of the CAPM arises from the fact that the dynamics of the market portfolio with its stochastic volatility are difficult to specify from market data and consequently not easily modeled. The identification of the market portfolio and the dynamics of the GOP is of critical importance and a focus of this paper.

Of particular significance to derivative pricing has been the *arbitrage pricing theory* (APT) proposed in Ross (1976) and further developed in an extensive literature that includes Harrison & Kreps (1979), Harrison & Pliska (1981), Delbaen & Schachermayer (1994, 1998), Karatzas & Shreve (1998) and references therein. Under the APT, several authors refer to the *state price density*, *pricing kernel*, *deflator* or *stochastic discount factor* for the modeling of asset price dynamics, see, for instance, Constatinides (1992), Duffie (2001), Rogers (1997) or Cochrane (2001). The state price density is known to be the inverse of the discounted *numeraire portfolio*, introduced in Long (1990). The numeraire portfolio equals in a standard risk neutral setting the *growth optimal portfolio* (GOP), see Bajeux-Besnainou & Portait (1997), Becherer (2001), Platen (2004a) and Christensen & Larsen (2004). By using the numeraire portfolio as reference unit or *benchmark*, it makes sense to define benchmarked contingent claim prices as expectations of benchmarked contingent claims under the real world probability measure. The current paper emphasizes that this *fair pricing concept*, see Platen (2002), does not require the existence of an equivalent risk neutral martingale measure. It avoids any measure transformations, but is consistent with the APT when changes of numeraire with corresponding equivalent martingale measure changes can be performed, see Geman, El Karoui & Rochet (1995). To apply fair pricing effectively, it is useful if the GOP can be observed and modeled. This leads outside the standard pricing methodologies and is a challenge that will be addressed in the current paper.

The GOP was discovered in Kelly (1956) and is defined as the portfolio that maximizes expected logarithmic utility from terminal wealth. It has a myopic strategy and appears in a stream of literature including, for instance, Latané (1959), Breiman (1961), Long (1990), Artzner (1997), Bajeux-Besnainou & Portait (1997), Karatzas & Shreve (1998), Kramkov & Schachermayer (1999), Becherer (2001), Platen (2002) and Goll & Kallsen (2003). Collectively, this literature demonstrates that the GOP plays a natural unifying role in derivative pricing, risk management and portfolio optimization. The aim of this paper is to establish a relationship between the GOP and the market portfolio, under the natural assumptions that every investor prefers *more rather than less* and that the savings account is in *net zero supply*. Some of the resulting consequences are well known, but are derived here under weaker assumptions than usual, while others, as the interpretation of the discounted GOP drift as rate of increase of *underlying value*, are likely to stimulate further research.

The paper is structured as follows. Section 2 introduces a continuous benchmark model. Section 3 discusses the market portfolio in relation to the GOP and studies some applications. In Section 4 the expected value of the market portfolio is characterized. Finally, in Section 5 the dynamics of the market portfolio are modeled.

2 Continuous Benchmark Model

2.1 Primary Security Accounts

For the modeling of a financial market we rely on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0, \infty)$. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ is assumed to satisfy the usual conditions, see Karatzas & Shreve (1991). We consider that $\underline{\mathcal{A}}$ describes the structure of information entering the market, in the sense that the sigma-algebra $\mathcal{A}_t$ expresses the information available in the market at time t. For simplicity, we restrict ourselves to markets with continuous security prices. The trading uncertainty is expressed by the independent standard Wiener processes $W^k = \{W^k_t,\, t \in [0,T]\}$ for $k \in \{1, 2, \ldots, d\}$ and $d \in \{1, 2, \ldots\}$. The increments $W^k_{t+\varepsilon} - W^k_t$ are assumed to be independent of $\mathcal{A}_t$ for all $t \in [0,T]$, $\varepsilon > 0$ and $k \in \{1, 2, \ldots, d\}$.

We consider a continuous financial market model that comprises $d + 1$ primary security accounts. These include a savings account $S^{(0)} = \{S^{(0)}(t),\, t \in [0,T]\}$, which is a locally riskless primary security account whose value at time t is given by

$$S^{(0)}(t) = \exp\left\{\int_0^t r(s)\, ds\right\} \tag{2.1}$$

for $t \in [0,T]$, where $r = \{r(t),\, t \in [0,T]\}$ denotes the adapted short rate process. They also include d nonnegative, risky primary security account processes $S^{(j)} = \{S^{(j)}(t),\, t \in [0,T]\}$, $j \in \{1, 2, \ldots, d\}$, each of which contains units of one type of security with all proceeds reinvested. Typically, these securities are stocks. However, derivatives, including options, as well as foreign savings accounts and bonds, may also form primary security accounts.

To specify the dynamics of primary securities in the given financial market we assume, without loss of generality, that the jth primary security account value $S^{(j)}(t)$, $j \in \{1, 2, \dots, d\}$, satisfies the stochastic differential equation (SDE)

$$dS^{(j)}(t) = S^{(j)}(t) \left(a^j(t)\, dt + \sum_{k=1}^{d} b^{j,k}(t)\, dW_t^k \right) \tag{2.2}$$

for $t \in [0, T]$ with $S^{(j)}(0) > 0$. Here the process $b^{j,k} = \{b^{j,k}(t),\, t \in [0, T]\}$ can be interpreted as the (j, k)th *volatility* of the jth security account with respect to the kth Wiener process W^k. Suppose that the (j, k)th volatility $b^{j,k}$ is a given predictable process that satisfies the integrability condition

$$\int_0^T \sum_{j=1}^{d} \sum_{k=1}^{d} \left(b^{j,k}(t)\right)^2 \, dt < \infty \tag{2.3}$$

almost surely, for all $j, k \in \{1, 2, \dots, d\}$. Furthermore, we assume that the jth *appreciation rate* $a^j = \{a^j(t),\, t \in [0, T]\}$, $j \in \{1, 2, \dots, d\}$, is a predictable process such that

$$\int_0^T \sum_{j=0}^{d} |a^j(s)|\, ds < \infty \tag{2.4}$$

almost surely.

It is reasonable to use the same number d of Wiener processes for the modeling of trading uncertainty as there are risky primary security accounts. If the number of securities is greater than the number of Wiener processes, then we have redundant securities that can be removed from the set of primary security accounts. Alternatively, if there are fewer risky securities than Wiener processes, then the market is incomplete concerning trading uncertainty. The core analysis of this paper is then still valid. However, some additional considerations arise which are not the focus of the current paper. The following assumption avoids redundant primary security accounts.

Assumption 2.1 *Assume that the volatility matrix $b(t) = [b^{j,k}(t)]_{j,k=1}^d$ is invertible for Lebesgue-almost every $t \in [0, T]$ with inverse matrix $b^{-1}(t) = [b^{-1\,j,k}(t)]_{j,k=1}^d$.*

This allows us to introduce the kth *market price for risk* $\theta^k(t)$ with respect to the kth Wiener process W^k, according to the relation

$$\theta^k(t) = \sum_{j=1}^{d} b^{-1\,j,k}(t) \left(a^j(t) - r(t)\right) \tag{2.5}$$

for $t \in [0,T]$ and $k \in \{1,2,\ldots,d\}$. Now, we can rewrite the SDE (2.2) for the jth primary security account in the form

$$dS^{(j)}(t) = S^{(j)}(t)\left(r(t)\,dt + \sum_{k=1}^{d} b^{j,k}(t)\left(\theta^k(t)\,dt + dW_t^k\right)\right) \tag{2.6}$$

for $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$.

2.2 Portfolios

We call a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0,T]\}$ a *strategy* if for each $j \in \{0,1,\ldots,d\}$ the Itô stochastic integral

$$\int_0^t \delta^{(j)}(s)\,dS^{(j)}(s) \tag{2.7}$$

exists, see Karatzas & Shreve (1991). Here $\delta^{(j)}(t)$, $j \in \{0,1,\ldots,d\}$, is the number of units of the jth primary security account that are held at time $t \in [0,T]$ in the corresponding portfolio. We denote by

$$S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\,S^{(j)}(t) \tag{2.8}$$

the time t value of the portfolio process $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,T]\}$. A strategy δ and the corresponding portfolio $S^{(\delta)}$ are said to be *self-financing* if

$$dS^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\,dS^{(j)}(t) \tag{2.9}$$

for $t \in [0,T]$. This means that all changes in the portfolio value are due to gains or losses from trade in primary security accounts. In what follows we consider only self-financing strategies and portfolios and will therefore omit the phrase "self-financing".

Let $S^{(\delta)}$ be a portfolio process whose value $S^{(\delta)}(t)$ at time $t \in [0,T]$ is nonzero. In this case it is convenient to introduce the jth *fraction* $\pi_\delta^{(j)}(t)$ of $S^{(\delta)}(t)$ that is invested in the jth primary security account $S^{(j)}(t)$, $j \in \{0,1,\ldots,d\}$, at time t. This fraction is given by the expression

$$\pi_\delta^{(j)}(t) = \delta^{(j)}(t)\,\frac{S^{(j)}(t)}{S^{(\delta)}(t)} \tag{2.10}$$

for $j \in \{0,1,\ldots,d\}$. Note that fractions can be negative and always sum to one, that is

$$\sum_{j=0}^{d} \pi_\delta^{(j)}(t) = 1 \tag{2.11}$$

for $t \in [0,T]$. By (2.9), (2.6) and (2.10) we get for a nonzero portfolio value $S^{(\delta)}(t)$ the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} b_\delta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \right) \tag{2.12}$$

for a strictly nonzero portfolio. The kth *portfolio volatility* in (2.12) is given by

$$b_\delta^k(t) = \sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t) \tag{2.13}$$

and its appreciation rate is of the form

$$a_\delta(t) = r(t) + \sum_{k=1}^{d} b_\delta^k(t)\, \theta^k(t) \tag{2.14}$$

for $t \in [0,T]$ and $k \in \{1,2,\dots,d\}$. Given a strictly positive portfolio $S^{(\delta)}$, its discounted value

$$\bar{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(0)}(t)} \tag{2.15}$$

satisfies the SDE

$$d\bar{S}^{(\delta)}(t) = \sum_{k=1}^{d} \psi_\delta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \tag{2.16}$$

by (2.1), (2.12) and an application of the Itô formula with *kth portfolio diffusion coefficient*

$$\psi_\delta^k(t) = \bar{S}^{(\delta)}(t)\, b_\delta^k(t) = \sum_{j=1}^{d} \delta^{(j)}(t)\, S^{(j)}(t)\, b^{j,k}(t) \tag{2.17}$$

for $k \in \{1,2,\dots,d\}$ and $t \in [0,T]$. Note that $\psi_\delta^k(t)$ makes sense also in the case if $\bar{S}^{(\delta)}(t)$ equals zero. Obviously, by (2.16) and (2.17) the discounted portfolio process $\bar{S}^{(\delta)}$ has *discounted drift*

$$\alpha_\delta(t) = \sum_{k=1}^{d} \psi_\delta^k(t)\, \theta^k(t) \tag{2.18}$$

at time $t \in [0,T]$, which measures its trend at that time. The trading uncertainty of a discounted portfolio $\bar{S}^{(\delta)}$ at time $t \in [0,T]$ can be measured by its *aggregate diffusion coefficient*

$$\gamma_\delta(t) = \sqrt{\sum_{k=1}^{d} \left(\psi_\delta^k(t) \right)^2} \tag{2.19}$$

or its *aggregate volatility*

$$b_\delta(t) = \frac{\gamma_\delta(t)}{\bar{S}^{(\delta)}(t)} \tag{2.20}$$

for $\bar{S}^{(\delta)}(t) > 0$.

2.3 Growth Optimal Portfolio

It is well known, see Kelly (1956) or Long (1990), that the *growth optimal portfolio* (GOP), which maximizes expected logarithmic utility from terminal wealth, plays a central role in finance theory. To identify this important portfolio we apply for a strictly positive portfolio $S^{(\delta)}$ the Itô formula to obtain the SDE for $\ln(S^{(\delta)}(t))$ in the form

$$d\ln(S^{(\delta)}(t)) = g_\delta(t)\,dt + \sum_{k=1}^{d} b_\delta^k(t)\,dW_t^k \tag{2.21}$$

with *portfolio growth rate*

$$g_\delta(t) = r(t) + \sum_{k=1}^{d}\left(\sum_{j=1}^{d}\pi_\delta^{(j)}(t)\,b^{j,k}(t)\,\theta^k(t) - \frac{1}{2}\left(\sum_{j=1}^{d}\pi_\delta^{(j)}(t)\,b^{j,k}(t)\right)^2\right) \tag{2.22}$$

for $t \in [0,T]$.

Definition 2.2 *A strictly positive portfolio process* $S^{(\delta_*)} = \{S^{(\delta_*)}(t),\, t \in [0,T]\}$ *is called a* GOP *if, for all* $t \in [0,T]$ *and all strictly positive portfolios* $S^{(\delta)}$*, the inequality*

$$g_{\delta_*}(t) \geq g_\delta(t) \tag{2.23}$$

holds almost surely.

By using the first order conditions one can determine the *optimal fractions*

$$\pi_{\delta_*}^{(j)}(t) = \sum_{k=1}^{d}\theta^k(t)\,b^{-1\,j,k}(t) \tag{2.24}$$

for all $t \in [0,T]$ and $j \in \{1,2,\dots,d\}$, which maximize the portfolio growth rate (2.22). It is straightforward to show in the given continuous financial market, see Long (1990), Karatzas & Shreve (1998) or Platen (2002), that the GOP value $S^{(\delta_*)}(t)$ satisfies the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t)\left(r(t)\,dt + \sum_{k=1}^{d}\theta^k(t)\left(\theta^k(t)\,dt + dW_t^k\right)\right) \tag{2.25}$$

for $t \in [0,T]$ with some appropriate initial value $S^{(\delta_*)}(0) > 0$. Obviously, up to its initial value the GOP is uniquely determined. From now on we use the GOP as *benchmark* and refer to the above financial market model as a *benchmark model.*

We call any security expressed in units of the GOP a *benchmarked security.* For a portfolio $S^{(\delta)}$ the corresponding benchmarked portfolio value

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} \tag{2.26}$$

satisfies the SDE

$$d\hat{S}^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\,\hat{S}^{(j)}(t) \sum_{k=1}^{d} \left(b^{j,k}(t) - \theta^k(t)\right) dW_t^k \tag{2.27}$$

for $t \in [0,T]$. Note by (2.9) and (2.6) that this SDE holds also for portfolios that can become zero or negative. It follows from the driftless SDE (2.27) that any benchmarked portfolio is an $(\underline{\mathcal{A}}, P)$-local martingale. On the other hand, since any nonnegative local martingale is an $(\underline{\mathcal{A}}, P)$-supermartingale, see Karatzas & Shreve (1991), a nonnegative benchmarked portfolio is a supermartingale. The nonnegativity of the total portfolios of investors is natural, since it reflects their *limited liability* in the sense that their portfolio value remains at zero as soon as their total portfolio value becomes zero.

For any nonnegative benchmarked portfolio $\hat{S}^{(\delta)}$ with $\hat{S}^{(\delta)}(\tau) = 0$ almost surely at any stopping time $\tau \in [0,T]$ we have by the supermartingale property of $\hat{S}^{(\delta)}$ the relations

$$0 = \hat{S}^{(\delta)}(\tau) \geq E\left(\hat{S}^{(\delta)}(T) \,\middle|\, \mathcal{A}_\tau\right) \geq 0$$

and therefore

$$P(S^{(\delta)}(T) > 0) = P(\hat{S}^{(\delta)}(T) > 0) = 0. \tag{2.28}$$

This shows that in a benchmark model any nonnegative portfolio process $S^{(\delta)}$, which reaches at any stopping time $\tau \in [0,T]$ the value zero, is modeled in such a way that its trajectory remains at any later time $s \in [\tau, T]$ almost surely at the level zero. Therefore, the above described benchmark model does not allow any nonnegative portfolio process $S^{(\delta)}$ with strictly positive value $S^{(\delta)}(T)$ at the terminal time T if this portfolio has before time T reached the bankruptcy level zero. This means, the above benchmark framework leads via the supermartingale property of benchmarked securities to a natural no-arbitrage concept, which is discussed in more detail in Platen (2004a). Due to the inclusion of strict supermartingales as benchmarked nonnegative portfolios, the benchmark approach provides a richer modeling framework than is given, for instance, under the fundamental theorem of asset pricing derived in Delbaen & Schachermayer (1994, 1998). Later we will need to use this modeling freedom when we derive a parsimonious, realistic financial market model.

3 More rather than Less

3.1 Optimal Portfolios

It is now our aim to identify the typical structure of the SDE of a, so called, *optimal portfolio*. To describe an optimal portfolio we introduce the following definition, similar as in Platen (2002).

Definition 3.1 *We call a strictly positive portfolio* $S^{(\tilde{\delta})}$ optimal*, if for all* $t \in [0,T]$ *and all strictly positive portfolios* $S^{(\delta)}$ *when*

$$\gamma_{\tilde{\delta}}(t) = \gamma_{\delta}(t) \tag{3.1}$$

we have

$$\alpha_{\tilde{\delta}}(t) \geq \alpha_{\delta}(t). \tag{3.2}$$

Definition 3.1 specifies that a strictly positive portfolio is optimal if at all times its discounted drift is greater than or equal to that of every other discounted, strictly positive portfolio with the same value of the aggregate diffusion coefficient. Essentially, we are simplifying our analysis by discounting since this will not have an impact on our optimization procedure. Furthermore, our analysis is strongly simplified by comparing only locally in time discounted drift and aggregate diffusion coefficients. Note that we do not optimize risk premia or aggregate volatilities. During the optimization we leave the actual value of the portfolio open. This will lead us to a family of optimal portfolios.

Definition 3.1 encapsulates a precise and simple characterization of what means *more rather than less*. It is natural that investors, who keep the freedom to adjust at any time their strategies according to new information, would prefer to hold an optimal portfolio. Therefore, we make the following assumption.

Assumption 3.2 *Each investor holds with his or her investable wealth an optimal portfolio.*

Let us introduce the *total market price for risk*

$$|\theta(t)| = \sqrt{\sum_{k=1}^{d} \left(\theta^k(t)\right)^2} \tag{3.3}$$

for $t \in [0,T]$. If the total market price for risk is zero, then all discounted drifts are zero and all strictly positive portfolios are by Definition 3.1 optimal. We exclude this unrealistic case with the following assumption.

Assumption 3.3 *The total market price for risk is strictly greater than zero and finite with*

$$0 < |\theta(t)| < \infty \tag{3.4}$$

almost surely for all $t \in [0,T]$.

An important investment characteristic is the *Sharpe ratio* $s_\delta(t)$, see Sharpe (1964). It is defined for a portfolio $S^{(\delta)}$ with positive aggregate volatility $b_\delta(t) > 0$ at time t as the ratio of the *risk premium* of the discounted portfolio

$$p_\delta(t) = \frac{\alpha_\delta(t)}{\bar{S}^{(\delta)}(t)} \tag{3.5}$$

over its aggregate volatility $b_\delta(t) = \frac{\gamma_\delta(t)}{\bar{S}^{(\delta)}(t)}$, that is,

$$s_\delta(t) = \frac{p_\delta(t)}{b_\delta(t)} = \frac{\alpha_\delta(t)}{\gamma_\delta(t)} \tag{3.6}$$

for $t \in [0, T]$, see (2.18)–(2.20).

We can now formulate for our continuous benchmark model a *portfolio selection theorem* in the sense of Markowitz (1959) that identifies the structure of the drift and diffusion coefficients in the SDE of an optimal portfolio.

Theorem 3.4 *For any strictly positive portfolio $S^{(\delta)}$ the* Sharpe ratio $s_\delta(t)$ *satisfies the inequality*

$$s_\delta(t) \leq |\theta(t)| \tag{3.7}$$

for all $t \in [0, T]$, where equality arises when $S^{(\delta)}$ is an optimal portfolio. The value $\bar{S}^{(\delta)}(t)$ at time t of a discounted, optimal portfolio satisfies the SDE

$$d\bar{S}^{(\delta)}(t) = \bar{S}^{(\delta)}(t) \frac{b_\delta(t)}{|\theta(t)|} \sum_{k=1}^{d} \theta^k(t) \left(\theta^k(t)\, dt + dW_t^k\right), \tag{3.8}$$

with optimal fractions

$$\pi_\delta^{(j)}(t) = \frac{b_\delta(t)}{|\theta(t)|}\, \pi_{\delta_*}^{(j)}(t) \tag{3.9}$$

for all $j \in \{1, 2, \ldots, d\}$ and $t \in [0, T]$.

This means that the family of discounted, optimal portfolios $\bar{S}^{(\delta)}$ is at any time t parameterized by the aggregate volatility $b_\delta(t) \geq 0$. The proof of this theorem is given in the Appendix. Obviously, for $b_\delta(t) = 0$ one obtains the savings account as optimal portfolio, whereas in the case $b_\delta(t) = |\theta(t)|$ it is the GOP that arises.

By Theorem 3.4 and (2.25), any optimal portfolio value $S^{(\delta)}(t)$ can be decomposed into a fraction of wealth that is invested in the GOP and a remaining fraction that is held in the savings account. Therefore, Theorem 3.4 can also be interpreted as a *mutual fund theorem* or *separation theorem*, see Merton (1973). The maximum achievable Sharpe ratio is by (3.7) that of an optimal portfolio and equals the total market price for risk.

3.2 Markowitz Efficient Frontier

It follows from the SDE (3.8) that at time t the *risk premium* $p_\delta(t)$ of an optimal portfolio $S^{(\delta)}$ equals

$$p_\delta(t) = b_\delta(t)\, |\theta(t)| \tag{3.10}$$

for $t \in [0, T]$. Note that the risk premium of $S^{(\delta)}$ is the appreciation rate of $\bar{S}^{(\delta)}$. By analogy to the mean-variance portfolio theory in Markowitz (1959), one can introduce an *efficient portfolio*.

Definition 3.5 *An* efficient portfolio $S^{(\delta)}$ *is one whose appreciation rate* $a_\delta(t)$*, as a function of squared volatility* $(b_\delta(t))^2$*, lies on the* efficient frontier*, in the sense that*

$$a_\delta(t) = a_\delta(t, (b_\delta(t))^2) = r(t) + \sqrt{(b_\delta(t))^2}\ |\theta(t)| \tag{3.11}$$

for all times $t \in [0, T]$.

By relations (3.10) and (3.11) the following result is obtained.

Corollary 3.6 *An optimal portfolio is efficient.*

Corollary 3.6 can be interpreted as a "local in time" version of the Markowitz efficient frontier in a continuous time setting. Due to (3.7) and (3.6) it is not possible to form a strictly positive portfolio that produces an appreciation rate above the efficient frontier.

The Markowitz efficient frontier and the Sharpe ratio are important tools for investment management. They can probably be more efficiently exploited in practice if the stochastic nature of the volatility process of an efficient portfolio is properly understood. This is a problem that we address towards the end of the paper.

3.3 Capital Asset Pricing Model

We assume the existence of $n \in \{1, 2, \ldots\}$ investors who hold all investable wealth in the market. The portfolio of investable wealth of the ℓth investor is denoted by $S^{(\delta_\ell)}$, $\ell \in \{1, 2, \ldots, n\}$. The total portfolio $S^{(\underline{\delta})}(t)$ of the investable wealth of all investors is then the market portfolio of investable wealth, called *market portfolio*, and given by

$$S^{(\underline{\delta})}(t) = \sum_{\ell=1}^{n} S^{(\delta_\ell)}(t) \tag{3.12}$$

at time $t \in [0, T]$. To identify the SDE of the market portfolio we introduce the following minor technical condition.

Assumption 3.7 *The market portfolio* $S^{(\underline{\delta})}(t) > 0$ *is almost surely strictly positive and the GOP faction* $\pi^{(0)}_{\delta_*}(t) \neq 1$ *for the savings account is almost surely not equal to one for all* $t \in [0, T]$.

The discounted market portfolio $\bar{S}^{(\underline{\delta})}(t)$ at time t is by Assumptions 3.2 and 3.7, Theorem 3.4, (3.9), (2.11) and (3.12) determined by the SDE

$$\begin{aligned} d\bar{S}^{(\underline{\delta})}(t) &= \sum_{\ell=1}^{n} d\bar{S}^{(\delta_\ell)}(t) \\ &= \sum_{\ell=1}^{n} \frac{\left(\bar{S}^{(\delta_\ell)}(t) - \delta_\ell^{(0)}\right)}{\left(1 - \pi_{\delta_*}^{(0)}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\,(\theta^k(t)\,dt + dW_t^k) \\ &= \bar{S}^{(\underline{\delta})}(t)\, \frac{\left(1 - \pi_{\underline{\delta}}^{(0)}(t)\right)}{\left(1 - \pi_{\delta_*}^{(0)}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\,(\theta^k(t)\,dt + dW_t^k) \end{aligned} \tag{3.13}$$

for $t \in [0,T]$. This shows by (3.8) that the market portfolio $S^{(\underline{\delta})}(t)$ can be interpreted as an optimal portfolio.

The seminal *capital asset pricing model* (CAPM) was developed by Sharpe (1964), Lintner (1965) and Merton (1973) as an equilibrium model of exchange. By (3.5), (2.12), (2.21) and (3.13) the risk premium $p_\delta(t)$ of a portfolio $S^{(\delta)}$ can be expressed as

$$p_\delta(t) = \sum_{k=1}^{d}\sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t)\, \theta^k(t) = \frac{d\langle \ln(S^{(\delta)}), \ln(S^{(\underline{\delta})})\rangle_t}{dt}\, \frac{\left(1 - \pi_{\delta_*}^{(0)}(t)\right)}{\left(1 - \pi_{\underline{\delta}}^{(0)}(t)\right)} \tag{3.14}$$

at time t. Here $\langle \ln(S^{(\delta)}), \ln(S^{(\underline{\delta})})\rangle_t$ denotes the covariation at time t of the stochastic processes $\ln(S^{(\delta)})$ and $\ln(S^{(\underline{\delta})})$, see Karatzas & Shreve (1991). The time derivative of the covariation is the local in time analogue for continuous time processes of the covariance of log-returns. For a strictly positive portfolio $S^{(\delta)}$ the systematic risk parameter $\beta_\delta(t)$, the *portfolio beta*, is defined as the ratio of the covariations

$$\beta_\delta(t) = \frac{\frac{d\langle \ln(S^{(\delta)}), \ln(S^{(\underline{\delta})})\rangle_t}{dt}}{\frac{d\langle \ln(S^{(\underline{\delta})})\rangle_t}{dt}}, \tag{3.15}$$

for $t \in [0,T]$, where $S^{(\underline{\delta})}$ denotes again the market portfolio. Since by (3.13) the market portfolio can be interpreted as an optimal portfolio we get by (3.14) the following result.

Theorem 3.8 *For any strictly positive portfolio $S^{(\delta)}$ the portfolio beta has the form*

$$\beta_\delta(t) = \frac{p_\delta(t)}{p_{\underline{\delta}}(t)} \tag{3.16}$$

for $t \in [0,T]$.

The above expression of the portfolio beta is exactly what the intertemporal CAPM suggests. Thus, Theorem 3.8 provides a derivation of the intertemporal CAPM, see Merton (1973), without any equilibrium or utility assumptions. Note that, in general, if the market portfolio were not an optimal portfolio, then the intertemporal CAPM would not hold in a benchmark model.

3.4 GOP and Market Portfolio

Since any bond which has a buyer has also a seller in the market, we make the following assumption.

Assumption 3.9 *The savings account is in net zero supply, that is*

$$\pi^{(0)}_{\underline{\delta}}(t) = 0 \tag{3.17}$$

almost surely, for all $t \in [0, T]$.

Obviously, the market portfolio has by (3.13), (2.25) and (3.17) the fraction $\frac{1-\pi^{(0)}_{\underline{\delta}}(t)}{1-\pi^{(0)}_{\delta_*}(t)} = \frac{1}{1-\pi^{(0)}_{\delta_*}(t)}$ invested in the GOP. By (3.13) and (3.17) we get the following result.

Corollary 3.10 *The market portfolio* $S^{(\underline{\delta})}$ *is a combination of the GOP and the savings account, where*

$$d\bar{S}^{(\delta_*)}(t) = \left(1 - \pi^{(0)}_{\delta_*}(t)\right) d\bar{S}^{(\underline{\delta})}(t) \tag{3.18}$$

for all $t \in [0, T]$.

This statement has a number of important consequences. It says that the fractions of the GOP in the risky primary security accounts are, up to the factor $(1-\pi^{(0)}_{\delta_*}(t))$ equal to those of the market portfolio. In the case when the GOP is not allowed to invest in the savings account or has zero investment in the savings account, then the GOP equals the market portfolio. If, say, the monetary authorities optimize by their policy the growth rate of the market portfolio, then it equals the GOP. Further research will clarify when the market portfolio is a good proxy for the GOP. If one assumes that the market portfolio is observable, for instance, in the form of the MSCI world stock accumulation index (MSCI) and $\pi^{(0)}_{\delta_*}(\cdot)$ is also known, then the GOP can be observed, modeled and calibrated. It can then be used as benchmark in various ways, as described later.

Under Corollary 3.10 one can take directly the observed market capitalization of stocks to deduce the optimal fractions of the risky securities in the GOP. If one estimates additionally the volatilities from sufficient frequently observed stock prices, then the kth market price for risk can be obtained via the relation

$$\theta^k(t) = \left(1 - \pi^{(0)}_{\delta_*}(t)\right) \sum_{j=1}^{d} \pi^{(j)}_{\underline{\delta}}(t)\, b^{j,k}(t) \tag{3.19}$$

for $t \in [0,T]$ and $k \in \{1,2,\dots,d\}$, see (2.24). This then allows by (3.14) also to estimate the risk premium for a strictly positive portfolio $S^{(\delta)}$ in the form

$$p_\delta(t) = \left(1 - \pi^{(0)}_{\delta_*}(t)\right) \sum_{k=1}^{d} b^k_\delta(t) \sum_{j=1}^{d} \pi^{(j)}_{\underline{\delta}}(t)\, b^{j,k}(t) \tag{3.20}$$

for $t \in [0,T]$. Forthcoming work will demonstrate the feasibility of this new method to estimate risk premia and market prices for risk.

3.5 Fair Pricing

The direct use of the GOP allows us to generalize in a practical way the well-known *arbitrage pricing theory* (APT), introduced by Ross (1976) and further developed by Harrison & Kreps (1979), Harrison & Pliska (1981) and many others. Under the benchmark approach one can use the GOP $S^{(\delta_*)}$ as numeraire along the lines of Long (1990). By Corollary 3.10 the numeraire would become the market portfolio. Note that the Radon-Nikodym derivative process $\Lambda_Q = \{\Lambda_Q(t),\, t \in [0,T]\}$ for the candidate risk neutral measure Q can be expressed as inverse of the discounted GOP

$$\Lambda_Q(t) = \frac{dQ}{dP}\Big|_{\mathcal{A}_t} = \frac{\bar{S}^{(\delta_*)}(0)}{\bar{S}^{(\delta_*)}(t)} \tag{3.21}$$

for $t \in [0,T]$, see Karatzas & Shreve (1998). For $\Lambda_Q(t)$ we obtain the SDE

$$d\Lambda_Q(t) = -\Lambda_Q(t) \sum_{k=1}^{d} \theta^k(t)\, dW^k_t \tag{3.22}$$

for $t \in [0,T]$ with $\Lambda_Q(0) = 1$ by the Itô formula and (2.25). This demonstrates that Λ_Q is an $(\underline{\mathcal{A}}, P)$-local martingale. Furthermore, by (2.27) it follows that $\bar{S}^{(\delta)}(t)\Lambda_Q(t) = S^{(\delta_*)}(0)\frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} = S^{(\delta_*)}(0)\hat{S}^{(\delta)}(t)$ forms an $(\underline{\mathcal{A}}, P)$-local martingale for any portfolio $S^{(\delta)}$. We emphasize that in a benchmark model this does *not* mean that $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-martingale. One obtains the following result.

Corollary 3.11 *If an equivalent risk neutral martingale measure Q exists and a given benchmarked portfolio $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-martingale, then the* risk neutral pricing formula

$$\begin{aligned} S^{(\delta)}(t) &= S^{(\delta_*)}(t)\, E\left(\hat{S}^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{\Lambda_Q(s)}{\Lambda_Q(t)}\, \frac{S^{(0)}(t)}{S^{(0)}(s)}\, S^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) \\ &= E_Q\left(\frac{S^{(0)}(t)}{S^{(0)}(s)}\, S^{(\delta)}(s) \,\middle|\, \mathcal{A}_t\right) \end{aligned} \tag{3.23}$$

holds for all $t \in [0,T]$ *and* $s \in [t,T]$*. Here* E_Q *denotes expectation under the risk neutral measure* Q*.*

According to Corollary 3.11, $\hat{S}^{(\delta)}$ needs to be an $(\underline{\mathcal{A}}, P)$-martingale for $\bar{S}^{(\delta)}$ to be an $(\underline{\mathcal{A}}, Q)$-martingale, such that the risk neutral pricing formula holds. Note that even if an equivalent martingale measure exists, then *not all* benchmarked portfolios are automatically $(\underline{\mathcal{A}}, P)$-martingales and not all discounted portfolios are $(\underline{\mathcal{A}}, P)$-martingales. This point is sometimes overlooked in the literature. However, it is mentioned, for instance, in Delbaen & Schachermayer (1994). In the above sense one recovers the risk neutral pricing methodology of the APT when assuming the existence of a risk neutral equivalent martingale measure Q. Relations similar to (3.23) also appear in the literature in connection with pricing kernels, state price densities, deflators, stochastic discount factors and numeraire portfolios, see, for instance, Long (1990), Constatinides (1992), Duffie (2001) and Cochrane (2001).

Since a benchmark model does not require the existence of an equivalent risk neutral martingale measure it provides a more general modeling framework than the standard risk neutral setup. In particular, the benchmarked savings account does not need to be a true martingale and (3.23) may not hold. As we will see, this is important for realistic modeling.

An indication for the need to go beyond the APT is given by the fact that by (3.21) the candidate Radon-Nikodym derivative Λ_Q for a benchmark model equals the ratio of the savings account over the GOP. Thus, it can be interpreted as benchmarked savings account. In the long run the market portfolio, and even more the GOP, is expected by investors to outperform the savings account. This means that the trajectory of the discounted market portfolio should rise systematically over longer periods of time. However, this means that the presumed Radon-Nikodym derivative should decrease systematically over longer time periods. Empirical evidence supports such systematic long term decline for all major currency denominations, when using the MSCI world stock accumulation index as proxy for the GOP. For the last century this has been empirically documented in Dimson, Marsh & Staunton (2002).

Therefore, it is not likely that Λ_Q is in reality well modeled as a true $(\underline{\mathcal{A}}, P)$-martingale, thereby contradicting the standard APT assumptions. The reader may be surprised by this observation, however, it appears to be the reality and has to be taken into account for advanced financial market modeling. Note that a decreasing graph for Λ_Q is still consistent with it being a nonnegative, strict $(\underline{\mathcal{A}}, P)$-local martingale and hence a strict supermartingale, see Karatzas & Shreve (1991). We will come back to this point towards the end of the paper, when we derive a model for the market portfolio.

For derivative pricing in a benchmark model, where no equivalent risk neutral martingale measure exists, the *fair pricing concept* has been proposed in Platen (2002).

Definition 3.12 *A benchmarked portfolio process* (2.26) *is called* fair *if it forms an* $(\underline{\mathcal{A}}, P)$*-martingale.*

In practice, it appears that fair pricing is appropriate for determining the competitive price of a contingent claim.

Definition 3.13 *The fair price* $U_{H_\tau}(t)$ *at time* $t \in [0, \tau]$ *of an* $\mathcal{A}_\tau$*-measurable contingent claim* H_τ*, payable at a stopping time* τ*, is defined by the* fair pricing formula

$$U_{H_\tau}(t) = E\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(\tau)}\, H_\tau \,\Big|\, \mathcal{A}_t\right). \tag{3.24}$$

Note that fair prices are uniquely determined even in incomplete markets. Under the existence of a minimal equivalent martingale measure, see Föllmer & Schweizer (1991), fair prices have been shown to correspond to local risk minimizing prices, see Christensen & Larsen (2004) and Platen (2004c). Corollary 3.10 makes fair pricing via (3.24) practicable since one can observe, model and calibrate the GOP via the market portfolio. This enables us to calculate the real world expectations in (3.24). It is clear from (3.24), (3.23) and (3.21) that fair pricing generalizes risk neutral pricing. The previous observation that the presumed Radon-Nikodym derivative process may, in reality, be a strict supermartingale suggests that a benchmarked savings account is a strict supermartingale and therefore not a fair price process. This observation is not surprising in the light of the model that we will derive later. Note that the savings account is the limit of a rollover short term bond account. It is only the theoretical limit of this account that is not fair because limits of martingales may become strict local martingales if certain integrability conditions are not satisfied.

For the practically important case where a contingent claim is independent of the GOP, one obtains the following result by the fair pricing formula (3.24).

Corollary 3.14 *For a contingent claim* H_T *that is independent of the GOP value* $S^{(\delta_*)}(T)$*, the fair price* $U_{H_T}(t)$ *satisfies the* actuarial pricing formula

$$\begin{aligned} U_{H_T}(t) &= E\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(T)} \,\Big|\, \mathcal{A}_t\right) E\left(H_T \,\big|\, \mathcal{A}_t\right) \\ &= P(t,T)\, E\left(H_T \,\big|\, \mathcal{A}_t\right), \end{aligned} \tag{3.25}$$

where $P(t, T)$ *denotes the fair price at time* $t \in [0, T]$ *of a zero coupon bond with maturity date* T*.*

The formula (3.25) has been widely used in insurance and other areas of risk management, see, for instance, Bühlmann (1995) and Gerber (1990). One may regard (3.25) as a generalized actuarial pricing formula that is still valid when interest rates are stochastic.

4 Expectation of Discounted GOP

It is important to have an idea about the typical dynamics of the GOP. The SDE (2.25) for the GOP reveals a close link between its drift and diffusion coefficients. More precisely, the risk premium of the GOP equals the square of its volatility. To see this, let us rewrite (2.25) in the discounted form

$$d\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(t)\,|\theta(t)|\,(|\theta(t)|\,dt + dW_t), \tag{4.1}$$

where

$$dW_t = \frac{1}{|\theta(t)|}\sum_{k=1}^{d} \theta^k(t)\,dW_t^k \tag{4.2}$$

is the stochastic differential of a standard Wiener process W. This reveals a clear structural relationship between the drift and diffusion coefficients.

To emphasize this relationship let us reparameterize the GOP dynamics in a natural way. The *discounted GOP drift*

$$\alpha(t) = \bar{S}^{(\delta_*)}(t)\,|\theta(t)|^2 \tag{4.3}$$

is the average change per unit of time of the discounted GOP. Using the parametrization (4.3), we get the total market price for risk in the form

$$|\theta(t)| = \sqrt{\frac{\alpha(t)}{\bar{S}^{(\delta_*)}(t)}}. \tag{4.4}$$

By substituting (4.3) and (4.4) into (4.1) we obtain the following SDE for the discounted GOP

$$d\bar{S}^{(\delta_*)}(t) = \alpha(t)\,dt + \sqrt{\bar{S}^{(\delta_*)}(t)\,\alpha(t)}\,dW_t \tag{4.5}$$

for $t \in [0, T]$. This is a *time transformed squared Bessel process of dimension four*, see Revuz & Yor (1999). Its *transformed time* $\varphi(t)$ is at time t given by the expression

$$\varphi(t) = \varphi(0) + \int_0^t \alpha(s)\,ds \tag{4.6}$$

with $\varphi(0) \geq 0$ as a possibly hidden random initial value. We will call later $\varphi(t)$ also the underlying value of $\bar{S}^{(\delta_*)}$ because it represents the integral over the discounted drift $\alpha_{\delta_*}(t)$, see (2.18).

Since one obtains the SDE

$$d\sqrt{\bar{S}^{(\delta_*)}(t)} = \frac{3\,\alpha(t)}{8\,\sqrt{\bar{S}^{(\delta_*)}(t)}}\,dt + \frac{1}{2}\sqrt{\alpha(t)}\,dW_t \tag{4.7}$$

from (4.5) by the Itô formula, the increase of the transformed time $\varphi(t)$ can be *directly observed* as

$$\varphi(t) - \varphi(0) = 4\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t \tag{4.8}$$

for $t \in [0,T]$. This emphasizes the fact that we are able to observe the time integral over the drift of the discounted GOP. It is well known that even under the simplest stochastic dynamics the estimation of a drift parameter alone for the market portfolio needs several hundred years of data to guarantee any reasonable level of confidence. The benchmark approach resolves the problem of identifying the drift of the market portfolio by exploiting the link between its drift and diffusion coefficient. Of course, for the reliable evaluation of the quadratic variation in (4.8) one needs still a sufficiently large number of data but drawn only over a relatively short time period.

For the analysis that follows, let us decompose the discounted GOP value at time $t \in [0,T]$ as

$$\bar{S}^{(\delta_*)}(t) = \bar{S}^{(\delta_*)}(0) + \varphi(t) - \varphi(0) + M(t). \tag{4.9}$$

Here $M = \{M(t),\, t \in [0,T]\}$ is the $(\underline{\mathcal{A}}, P)$-local martingale

$$M(t) = \int_0^t \bar{S}^{(\delta_*)}(s)\, |\theta(s)|\, dW_s \tag{4.10}$$

for $t \in [0,T]$. The quantity $\bar{S}^{(\delta_*)}(t)$ in (4.9) consists of a part $M(t)$, which reflects the trading uncertainty of the discounted GOP and a part $\varphi(t) - \varphi(0)$ that expresses the increase of its *underlying value*. Since the trading uncertainty models the speculative component of the discounted GOP the underlying value can be interpreted by (4.6) as accumulated discounted economic value of the market portfolio. We have shown by (4.6) and (4.8) that it takes naturally the form of a transformed time.

The above relationships lead directly to the following result, which exploits equations (4.9) and (4.6) and a realistic martingale assumption for M.

Corollary 4.1 *If the local martingale M in* (4.10) *is a true $(\underline{\mathcal{A}}, P)$-martingale, then the expected change of the discounted GOP value over a given period equals the expected change of its underlying value. That is,*

$$E\left(\bar{S}^{(\delta_*)}(s) - \bar{S}^{(\delta_*)}(t) \,\middle|\, \mathcal{A}_t\right) = E\left(\varphi(s) - \varphi(t) \,\middle|\, \mathcal{A}_t\right) \tag{4.11}$$

for all $t \in [0,T]$ and $s \in [t,T]$.

By Corollaries 4.1 and 3.10 it follows that if the present value of the discounted GOP is interpreted as its underlying value, then the expected future underlying value equals the expected future value of the discounted GOP. In this sense the increase of the martingale part of the discounted GOP fluctuates around its increase in underlying value, which one would naturally expect. It is of practical importance that one can observe in a benchmark model the latter increase via the quadratic variation (4.8), see Platen (2004b). We emphasize that we have still not made any major assumptions about the particular dynamics of the GOP.

5 Dynamics of the GOP

In Platen (2004b) relation (4.11) has been illustrated for the US market by assuming that the market portfolio equals the GOP. It appears in reality that the underlying value of the discounted GOP, which can be observed via (4.8), evolves relatively smoothly as a monotone increasing function of time, see Platen (2004b). This matches the fact that only a rather predictable amount of discounted wealth is on average generated worldwide per unit of time, which increases smoothly the underlying value of the discounted GOP. Based on this observation we make the following assumption.

Assumption 5.1 *The underlying value $\alpha(\cdot)$ of the discounted GOP is twice differentiable with respect to time.*

Without loss of generality, the discounted GOP drift can then be expressed as

$$\alpha(t) = \alpha_0 \, \exp\left\{ \int_0^t \eta(s) \, ds \right\} \tag{5.1}$$

for $t \in [0,T]$. The two parameters in (5.1) are a nonnegative, potentially random initial value $\alpha_0 > 0$ and an adapted process $\eta = \{\eta(t),\, t \in [0,T]\}$, called the *net growth rate*. This expression takes the typical growth nature of the GOP into account. According to (4.4) the parametrization (5.1) allows us to study the dynamics of the *normalized GOP*

$$Y(t) = \frac{\bar{S}^{(\delta_*)}(t)}{\alpha(t)} = \frac{1}{|\theta(t)|^2} \tag{5.2}$$

for $t \in [0,T]$. By application of the Itô formula and using (4.4), (4.11) and (4.5), we obtain the SDE

$$dY(t) = (1 - \eta(t)\, Y(t))\, dt + \sqrt{Y(t)}\, dW_t \tag{5.3}$$

for $t \in [0,T]$ with $Y(0) = \frac{\bar{S}^{(\delta_*)}(0)}{\alpha_0}$. It follows that the normalized GOP is a *square root process* with the inverse of the net growth rate $\frac{1}{\eta(t)}$ as reference level for its linear mean-reverting drift. The net growth rate $\eta(t)$ is then the speed of adjustment parameter for the mean-reversion. Note that besides initial values, the net growth rate is the only parameter process needed to characterize the dynamics of the normalized GOP and its stochastic volatility. Therefore, one obtains a parsimonious model for the GOP dynamics, namely

$$S^{(\delta_*)}(t) = Y(t)\, \alpha(t)\, S^{(0)}(t) \tag{5.4}$$

for $t \in [0,T]$. It only remains to specify the initial values $S^{(\delta_*)}(0)$ and α_0 and the net growth rate process η as well as the short rate process r. The net growth rate for the world stock portfolio, when denominated in units of a US-Dollar savings account, has been estimated for the entire last century in Dimson, Marsh & Staunton (2002) to be on average close to $\eta \approx 4.9\%$.

The resulting model for the GOP when interpreted as market portfolio is called the *minimal market model* (MMM), see Platen (2001), which has been studied, for instance, in Platen (2002, 2004b) and Heath & Platen (2004). It is important to note that under the MMM the presumed Radon-Nikodym derivative for the candidate risk neutral measure equals the inverse of a squared Bessel process of dimension four, which is a nonnegative, strict local martingale and thus a strict supermartingale, see Karatzas & Shreve (1991). This potentially explains the observed systematic decline in the observed candidate Radon-Nikodym derivative, that is, the benchmarked savings account, when interpreting the MSCI as GOP. Obviously, under the MMM the APT is not applicable. However, the fair pricing formula (3.24) makes perfect sense for the competitive pricing of derivatives and can be directly applied by using the explicitly known transition density of the squared Bessel process of dimension four, see Revuz & Yor (1999).

The above analysis raises the question of whether the distribution of log-returns of the market portfolio, as above predicted, are actually observed. If the MMM is a reasonably accurate description of reality, then estimated log-returns of the market portfolio, based on long periods of observed data, should appear to be Student t distributed with four degrees of freedom. This follows because the squared volatility of the GOP $|\theta(t)|^2 = \frac{1}{Y(t)}$ has a stationary inverse gamma density with four degrees of freedom, when assuming a constant net growth rate η. For a sufficiently long observed time series of market portfolio log-returns, this inverse gamma density acts as mixing density for the resulting normal-mixture distribution, yielding the above mentioned Student t distribution.

This theoretical feature of the MMM is rather clear and testable. Importantly, it has already been documented in the literature as an empirical stylized fact for log-returns of large stock market indices. In an extensive Baysian estimation within a wide class of Pearson distributions, Markowitz & Usmen (1996) found that the Student t distribution with about 4.3 degrees of freedom matches well the daily S&P500 log-return data from 1962 until 1983. Independently, in Hurst & Platen (1997) it was found by maximum likelihood estimation within the rich class of symmetric generalized hyperbolic distributions that, not only for the S&P500 but also for most other regional stock market indices, daily log-returns for the period from 1982 until 1996 are likely to be Student t distributed with about four degrees of freedom. Another recent study by Breymann, Fergusson & Platen (2004) confirms, for the daily log-returns of the world stock market portfolio in 34 different currency denominations for the period from 1973 until 2003, that in all cases the Student t distribution provides the best fit in the rich class of symmetric generalized hyperbolic distributions. Furthermore, for the majority of the currency denominations the Student t hypothesis cannot be rejected at the 99% confidence level. The average estimated number of degrees of freedom in this study is with about 3.94 very close to the theoretical value of 4.00 predicted by the MMM.

Furthermore, empirical evidence from index and FX derivatives strongly support the MMM, see Heath & Platen (2004). Forthcoming work will extend the above benchmark approach to models with event driven jumps and general semimartingale dynamics.

Conclusion

This paper demonstrates that the growth optimal portfolio (GOP) plays a major role in various areas of finance, including portfolio optimization, derivative pricing and risk management. We assume that investors always prefer more rather than less. The paper then identifies optimal portfolios as combinations of the market portfolio and the savings account. The Markowitz efficient frontier and Sharpe ratio can then be derived naturally. Under the additional assumptions that the savings account is in net zero supply, it is shown that the GOP equals a combination of the market portfolio and the savings account.

The discounted GOP can be realistically modeled as a time transformed squared Bessel process of dimension four. The transformed time can be interpreted as its underlying value. The increase in expected discounted GOP value is shown to equal that of its expected discounted underlying value, which can be observed via some covariance process. A particular dynamics of the normalized GOP is derived. It is related to a square root process of dimension four, which appears to provide realistic log-returns for the market portfolio.

For the pricing of contingent claims the GOP is nominated as numeraire for fair pricing, with expectations to be taken under the real world probability measure. In fact, for the minimal market model described here, no equivalent risk neutral martingale measure exists.

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A Appendix

Proof of Theorem 3.4

To identify a discounted optimal portfolio we maximize the drift (2.18) subject to the constraint (3.1), locally in time, according to Definition 3.1. For this purpose we use the Lagrange multiplier λ and consider the function

$$G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\tilde{\delta}}, \dots, \psi^d_{\tilde{\delta}}) = \sum_{k=1}^{d} \psi^k_{\tilde{\delta}} \, \theta^k + \lambda \left((\gamma_{\tilde{\delta}})^2 - \sum_{k=1}^{d} \left(\psi^k_{\tilde{\delta}}\right)^2 \right) \tag{A.1}$$

by suppressing time dependence. For $\psi^1_{\tilde{\delta}}, \psi^2_{\tilde{\delta}}, \dots, \psi^d_{\tilde{\delta}}$ to provide a maximum for $G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\tilde{\delta}}, \dots, \psi^d_{\tilde{\delta}})$ it is necessary that the first-order conditions

$$\frac{\partial G(\theta^1, \dots, \theta^k, \gamma_{\tilde{\delta}}, \lambda, \psi^1_{\tilde{\delta}}, \dots, \psi^d_{\tilde{\delta}})}{\partial \psi^k_{\tilde{\delta}}} = \theta^k - 2\,\lambda\, \psi^k_{\tilde{\delta}} = 0 \tag{A.2}$$

are satisfied for all $k \in \{1, 2, \dots, d\}$. Consequently, an optimal portfolio $S^{(\tilde{\delta})}$, which maximizes the drift, must have

$$\psi^k_{\tilde{\delta}} = \frac{\theta^k}{2\,\lambda} \tag{A.3}$$

for all $k \in \{1, 2, \dots, d\}$. We can now use the constraint (3.1) together with (2.19) and (3.3) to obtain from (A.3) the relation

$$(\gamma_{\tilde{\delta}})^2 = \sum_{k=1}^{d} \left(\psi^k_{\tilde{\delta}}\right)^2 = \left(\frac{|\theta|}{2\,\lambda}\right)^2. \tag{A.4}$$

By (3.4) we have $|\theta(t)| > 0$ and obtain the equation

$$\psi^k_{\tilde{\delta}}(t) = \frac{\gamma_{\tilde{\delta}}(t)}{|\theta(t)|}\, \theta^k(t) \tag{A.5}$$

for $t \in [0, T]$ and $k \in \{1, 2, \dots, d\}$, from (A.3) and (A.4). This yields by (2.18) the discounted drift

$$\alpha_{\tilde{\delta}}(t) = \frac{|\gamma_{\tilde{\delta}}(t)|\, |\theta(t)|^2}{|\theta(t)|} = |\gamma_{\tilde{\delta}}(t)|\, |\theta(t)|, \tag{A.6}$$

which leads for the case of an optimal portfolio to the equality in (3.7). Due to our optimization it follows the inequality (3.7) for any strictly positive portfolio.

Equation (A.5) substituted into (2.16) provides by (2.19) the SDE (3.8). Furthermore, it follows from (A.5), (2.17), (2.13) and (2.19) that

$$\bar{S}^{(\tilde{\delta})}(t) \sum_{j=1}^{d} \pi^{(j)}_{\tilde{\delta}}(t)\, b^{j,k}(t) = \frac{\gamma_{\tilde{\delta}}(t)}{|\theta(t)|}\, \theta^k(t) = \bar{S}^{(\tilde{\delta})}(t)\, b_{\tilde{\delta}}(t)\, \frac{\theta^k(t)}{|\theta(t)|}. \tag{A.7}$$

This yields by Assumption 2.1 the optimal fraction

$$\pi_{\tilde{\delta}}^{(j)}(t) = \frac{b_{\tilde{\delta}}(t)}{|\theta(t)|} \sum_{k=1}^{d} \theta^k(t)\, b^{-1\, j,k}(t) \tag{A.8}$$

and, thus, by (2.24) the equation (3.9) for all $j \in \{1, 2, \ldots, d\}$ and $t \in [0, T]$. $\square$

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