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A best proximity point theorem for Geraghty-contractions

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Abstract

The purpose of this paper is to provide sufficient conditions for the existence of a unique best proximity point for Geraghty-contractions.

Our paper provides an extension of a result due to Geraghty (Proc. Am. Math. Soc. 40:604-608, 1973).

Keywords: fixed point; Geraghty-contraction; *P*-property; best proximity point

1 Introduction

Let A and B be nonempty subsets of a metric space (X, d).

An operator $T: A \to B$ is said to be a k-contraction if there exists $k \in [0,1)$ such that $d(Tx, Ty) \leq kd(x,y)$ for any $x, y \in A$. Banach's contraction principle states that when A is a complete subset of X and T is a k-contraction which maps A into itself, then T has a unique fixed point in A.

A huge number of generalizations of this principle appear in the literature. Particularly, the following generalization of Banach's contraction principle is due to Geraghty [1].

First, we introduce the class $\mathcal{F}$ of those functions $\beta: [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0.$$

Theorem 1.1 ([1]) *Let (X,d) be a complete metric space and $T: X \to X$ be an operator. Suppose that there exists $\beta \in \mathcal{F}$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta\big(d(x,y)\big) \cdot d(x,y). \tag{1}$$

Then T has a unique fixed point.

Since the constant functions $f(t) = k$, where $k \in [0,1)$, belong to $\mathcal{F}$, Theorem 1.1 extends Banach's contraction principle.

Remark 1.1 Since the functions belonging to $\mathcal{F}$ are strictly smaller than one, condition (1) implies that

$$d(Tx, Ty) < d(x,y) \quad \text{for any } x, y \in X \text{ with } x \neq y.$$

Therefore, any operator $T\colon X \to X$ satisfying (1) is a continuous operator.

The aim of this paper is to give a generalization of Theorem 1.1 by considering a non-self map T.

First, we present a brief discussion about a best proximity point.

Let A be a nonempty subset of a metric space (X,d) and $T\colon A \to X$ be a mapping. The solutions of the equation $Tx = x$ are fixed points of T. Consequently, $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point for the operator T. If this necessary condition does not hold, then $d(x,Tx) > 0$ for any $x \in A$ and the mapping $T\colon A \to X$ does not have any fixed point. In this setting, our aim is to find an element $x \in A$ such that $d(x,Tx)$ is minimum in some sense. The best approximation theory and best proximity point analysis have been developed in this direction.

In our context, we consider two nonempty subsets A and B of a complete metric space and a mapping $T\colon A \to B$.

A natural question is whether one can find an element $x_0 \in A$ such that $d(x_0,Tx_0) = \min\{d(x,Tx)\colon x \in A\}$. Since $d(x,Tx) \geq d(A,B)$ for any $x \in A$, the optimal solution to this problem will be the one for which the value $d(A,B)$ is attained by the real valued function $\varphi\colon A \to \mathbb{R}$ given by $\varphi(x) = d(x,Tx)$.

Some results about best proximity points can be found in [2–9].

2 Notations and basic facts

Let A and B be two nonempty subsets of a metric space (X,d).

We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A\colon d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B\colon d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where $d(A,B) = \inf\{d(x,y)\colon x \in A \text{ and } y \in B\}$.

In [8], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Now, we present the following definition.

Definition 2.1 Let A,B be two nonempty subsets of a metric space (X,d). A mapping $T\colon A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \mathcal{F}$ such that

$$d(Tx,Ty) \leq \beta\bigl(d(x,y)\bigr) \cdot d(x,y) \quad \text{for any } x,y \in A.$$

Notice that since $\beta\colon [0,\infty) \to [0,1)$, we have

$$d(Tx,Ty) \leq \beta\bigl(d(x,y)\bigr) \cdot d(x,y) < d(x,y) \quad \text{for any } x,y \in A \text{ with } x \neq y.$$

Therefore, every Geraghty-contraction is a contractive mapping.

In [10], the author introduces the following definition.

Definition 2.2 ([10]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1,x_2 \in A_0$

and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) &= d(A,B) \\ d(x_2, y_2) &= d(A,B) \end{aligned}\right\} \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

It is easily seen that for any nonempty subset A of (X,d), the pair (A,A) has the P-property.

In [10], the author proves that any pair (A,B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

3 Main results

We start this section presenting our main result.

Theorem 3.1 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T\colon A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.*

Proof Since A_0 is nonempty, we take $x_0 \in A$.

As $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$. Similarly, since $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Repeating this process, we can get a sequence (x_n) in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \text{for any } n \in \mathbb{N}.$$

Since (A,B) has the P-property, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \quad \text{for any } n \in \mathbb{N}.$$

Taking into account that T is a Geraghty-contraction, for any $n \in \mathbb{N}$, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \beta\bigl(d(x_{n-1}, x_n)\bigr) \cdot d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$.

In this case,

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}),$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$.

Therefore,

$$d(A,B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0})$$

and this is the desired result.

In the contrary case, suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$.

By (2), $(d(x_n, x_{n+1}))$ is a decreasing sequence of nonnegative real numbers, and hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

In the sequel, we prove that $r = 0$.

Assume $r > 0$, then from (2) we have

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta\bigl(d(x_{n-1}, x_n)\bigr) < 1 \quad \text{for any } n \in \mathbb{N}.$$

The last inequality implies that $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$ and since $\beta \in \mathcal{F}$, we obtain $r = 0$ and this contradicts our assumption.

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Notice that since $d(x_{n+1}, Tx_n) = d(A, B)$ for any $n \in \mathbb{N}$, for $p, q \in \mathbb{N}$ fixed, we have $d(x_p, Tx_{p-1}) = d(x_q, Tx_{q-1}) = d(A, B)$, and since (A, B) satisfies the P-property, $d(x_p, x_q) = d(Tx_{p-1}, Tx_{q-1})$.

In what follows, we prove that (x_n) is a Cauchy sequence.

In the contrary case, we have that

$$\limsup_{m,n\to\infty} d(x_n, x_m) > 0.$$

By using the triangular inequality,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m).$$

By (2) and since $d(x_{n+1}, x_{m+1}) = d(Tx_n, Tx_m)$, by the above mentioned comment, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \beta\bigl(d(x_n, x_m)\bigr) \cdot d(x_n, x_m) + d(x_{m+1}, x_m), \end{aligned}$$

which gives us

$$d(x_n, x_m) \leq \bigl(1 - \beta\bigl(d(x_n, x_m)\bigr)\bigr)^{-1}\bigl[d(x_n, x_{n+1}) + d(x_{m+1}, x_m)\bigr].$$

Since $\limsup_{m,n\to\infty} d(x_n, x_m) > 0$ and by (3), $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$, from the last inequality it follows that

$$\limsup_{m,n\to\infty} \bigl(1 - \beta\bigl(d(x_n, x_m)\bigr)\bigr)^{-1} = \infty.$$

Therefore, $\limsup_{m,n\to\infty} \beta(d(x_n, x_m)) = 1$.

Taking into account that $\beta \in \mathcal{F}$, we get $\limsup_{m,n\to\infty} d(x_n, x_m) = 0$ and this contradicts our assumption.

Therefore, (x_n) is a Cauchy sequence.

Since $(x_n) \subset A$ and A is a closed subset of the complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$.

Since any Geraghty-contraction is a contractive mapping and hence continuous, we have $Tx_n \to Tx^*$.

This implies that $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$.

Taking into account that the sequence $(d(x_{n+1}, Tx_n))$ is a constant sequence with value $d(A, B)$, we deduce

$$d(x^*, Tx^*) = d(A, B).$$

This means that x^* is a best proximity point of T.

This proves the part of existence of our theorem.

For the uniqueness, suppose that x_1 and x_2 are two best proximity points of T with $x_1 \neq x_2$.

This means that

$$d(x_i, Tx_i) = d(A, B) \quad \text{for } i = 1, 2.$$

Using the P-property, we have

$$d(x_1, x_2) = d(Tx_1, Tx_2).$$

Using the fact that T is a Geraghty-contraction, we have

$$d(x_1, x_2) = d(Tx_1, Tx_2) \leq \beta\big(d(x_1, x_2)\big) \cdot d(x_1, x_2) < d(x_1, x_2),$$

which is a contradiction.

Therefore, $x_1 = x_2$.

This finishes the proof. □

4 Examples

In order to illustrate our results, we present some examples.

Example 4.1 Consider $X = \mathbb{R}^2$ with the usual metric.

Let A and B be the subsets of X defined by

$$A = \{0\} \times [0, \infty) \quad \text{and} \quad B = \{1\} \times [0, \infty).$$

Obviously, $d(A, B) = 1$ and A, B are nonempty closed subsets of X.

Moreover, it is easily seen that $A_0 = A$ and $B_0 = B$.

Let $T : A \to B$ be the mapping defined as

$$T(0, x) = \big(1, \ln(1 + x)\big) \quad \text{for any } (0, x) \in A.$$

In the sequel, we check that T is a Geraghty-contraction.

In fact, for $(0,x),(0,y) \in A$ with $x \neq y$, we have

$$\begin{aligned} d\big(T(0,x),T(0,y)\big) &= d\big(\big(1,\ln(1+x)\big),\big(1,\ln(1+y)\big)\big) \\ &= \big|\ln(1+x)-\ln(1+y)\big| \\ &= \left|\ln\left(\frac{1+x}{1+y}\right)\right|. \end{aligned} \tag{4}$$

Now, we prove that

$$\left|\ln\left(\frac{1+x}{1+y}\right)\right| \leq \ln\big(1+|x-y|\big). \tag{5}$$

Suppose that $x > y$ (the same reasoning works for $y > x$).

Then, since $\phi(t) = \ln(1+t)$ is strictly increasing in $[0,\infty)$, we have

$$\ln\left(\frac{1+x}{1+y}\right) = \ln\left(\frac{1+y+x-y}{1+y}\right) = \ln\left(1+\frac{x-y}{1+y}\right) \leq \ln(1+x-y) = \ln\big(1+|x-y|\big).$$

This proves (5).

Taking into account (4) and (5), we have

$$\begin{aligned} d\big(T(0,x),T(0,y)\big) &= \left|\ln\left(\frac{1+x}{1+y}\right)\right| \\ &\leq \ln\big(1+|x-y|\big) \\ &= \frac{\ln(1+|x-y|)}{|x-y|}\cdot|x-y| \\ &= \frac{\phi(d((0,x),(0,y)))}{d((0,x),(0,y))}\cdot d\big((0,x),(0,y)\big) \\ &= \beta\big(d\big((0,x),(0,y)\big)\big)\cdot d\big((0,x),(0,y)\big), \end{aligned} \tag{6}$$

where $\phi(t) = \ln(1+t)$ for $t \geq 0$, and $\beta(t) = \frac{\phi(t)}{t}$ for $t > 0$ and $\beta(0) = 0$.

Obviously, when $x = y$, the inequality (6) is satisfied.

It is easily seen that $\beta(t) = \frac{\ln(1+t)}{t} \in \mathcal{F}$ by using elemental calculus.

Therefore, T is a Geraghty-contraction.

Notice that the pair (A,B) satisfies the P-property.

Indeed, if

$$\begin{aligned} d\big((0,x_1),(1,y_1)\big) &= \sqrt{1+(x_1-y_1)^2} = d(A,B) = 1, \\ d\big((0,x_2),(1,y_2)\big) &= \sqrt{1+(x_2-y_2)^2} = d(A,B) = 1, \end{aligned}$$

then $x_1 = y_1$ and $x_2 = y_2$ and consequently,

$$d\big((0,x_1),(0,x_2)\big) = |x_1-x_2| = |y_1-y_2| = d\big((1,y_1),(1,y_2)\big).$$

By Theorem 3.1, T has a unique best proximity point.

Obviously, this point is $(0,0) \in A$.

The condition A and B are nonempty closed subsets of the metric space (X,d) is not a necessary condition for the existence of a unique best proximity point for a Geraghty-contraction $T: A \to B$ as it is proved with the following example.

Example 4.2 Consider $X = \mathbb{R}^2$ with the usual metric and the subsets of X given by

$$A = \{0\} \times [0, \infty) \quad \text{and} \quad B = \{1\} \times \left[0, \frac{\pi}{2}\right).$$

Obviously, $d(A,B) = 1$ and B is not a closed subset of X.

Note that $A_0 = 0 \times [0, \frac{\pi}{2})$ and $B_0 = B$.

We consider the mapping $T: A \to B$ defined as

$$T(0,x) = (1, \arctan x) \quad \text{for any } (0,x) \in A.$$

Now, we check that T is a Geraghty-contraction.

In fact, for $(0,x), (0,y) \in A$ with $x \neq y$, we have

$$d\big(T(0,x), T(0,y)\big) = d\big((1, \arctan x), (1, \arctan y)\big) = |\arctan x - \arctan y|. \tag{7}$$

In what follows, we need to prove that

$$|\arctan x - \arctan y| \leq \arctan|x - y|. \tag{8}$$

In fact, suppose that $x > y$ (the same argument works for $y > x$).

Put $\arctan x = \alpha$ and $\arctan y = \beta$ (notice that $\alpha > \beta$ since the function $\phi(t) = \arctan t$ for $t \geq 0$ is strictly increasing).

Taking into account that

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

and since $\alpha, \beta \in [0, \frac{\pi}{2})$, we have that $\tan\alpha, \tan\beta \in [0, \infty)$, and consequently, from the last inequality it follows that

$$\tan(\alpha - \beta) \leq \tan\alpha - \tan\beta.$$

Applying ϕ (notice that $\phi(t) = \arctan t$) to the last inequality and taking into account the increasing character of ϕ, we have

$$\alpha - \beta \leq \arctan(\tan\alpha - \tan\beta),$$

or equivalently,

$$\arctan x - \arctan y = \alpha - \beta \leq \arctan(x - y),$$

and this proves (8).

By (7) and (8), we get

$$\begin{aligned} d\big(T(0,x),T(0,y)\big) &= |\arctan x - \arctan y| \\ &\le \arctan|x-y| \\ &= \frac{\arctan|x-y|}{|x-y|}\cdot|x-y| \\ &= \beta\big(d(0,x),d(0,y)\big)\cdot d\big((0,x),(0,y)\big), \end{aligned} \tag{9}$$

where $\beta(t) = \frac{\arctan t}{t}$ for $t > 0$ and $\beta(0) = 0$. Obviously, the inequality (9) is satisfied for $(0,x),(0,y) \in A$ with $x = y$.

Now, we prove that $\beta \in \mathcal{F}$.

In fact, if $\beta(t_n) = \frac{\arctan t_n}{t_n} \to 1$, then the sequence (t_n) is a bounded sequence since in the contrary case, $t_n \to \infty$ and thus $\beta(t_n) \to 0$. Suppose that $t_n \nrightarrow 0$. This means that there exists $\epsilon > 0$ such that, for each $n \in \mathbb{N}$, there exists $p_n \ge n$ with $t_{p_n} \ge \epsilon$. The bounded character of (t_n) gives us the existence of a subsequence (t_{k_n}) of (t_{p_n}) with (t_{k_n}) convergent. Suppose that $t_{k_n} \to a$. From $\beta(t_n) \to 1$, we obtain $\frac{\arctan t_{k_n}}{t_{k_n}} \to \frac{\arctan a}{a} = 1$ and, as the unique solution of $\arctan x = x$ is $x_0 = 0$, we obtain $a = 0$.

Thus, $t_{k_n} \to 0$ and this contradicts the fact that $t_{k_n} \ge \epsilon$ for any $n \in \mathbb{N}$.

Therefore, $t_n \to 0$ and this proves that $\beta \in \mathcal{F}$.

A similar argument to the one used in Example 4.1 proves that the pair (A,B) has the P-property.

On the other hand, the point $(0,0) \in A$ is a best proximity point for T since

$$d\big((0,0),T(0,0)\big) = d\big((0,0),(1,\arctan 0)\big) = d\big((0,0),(1,0)\big) = 1 = d(A,B).$$

Moreover, $(0,0)$ is the unique best proximity point for T.

Indeed, if $(0,x) \in A$ is a best proximity point for T, then

$$1 = d(A,B) = d\big((0,x),T(0,x)\big) = d\big((0,x),(1,\arctan x)\big) = \sqrt{1+(x-\arctan x)^2},$$

and this gives us

$$x = \arctan x.$$

Taking into account that the unique solution of this equation is $x = 0$, we have proved that T has a unique best proximity point which is $(0,0)$.

Notice that in this case B is not closed.

Since for any nonempty subset A of X, the pair (A,A) satisfies the P-property, we have the following corollary.

Corollary 4.1 *Let (X,d) be a complete metric space and A be a nonempty closed subset of X. Let $T: A \to A$ be a Geraghty-contraction. Then T has a unique fixed point.*

Proof Using Theorem 3.1 when $A = B$, the desired result follows. □

Notice that when $A = X$, Corollary 4.1 is Theorem 1.1 due to Gerahty [1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors have contributed equally in this paper. They read and approval the final manuscript.

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