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A bidirectional traveling plane wave representation of exact solutions of the scalar wave equation

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A new decomposition of exact solutions to the scalar wave equation into bidirectional, forward and backward, traveling plane wave solutions is described. The resulting representation is a natural basis for synthesizing pulse solutions that can be tailored to give directed energy transfer in space. The development of known free-space solutions, such as the focus wave modes, the electromagnetic directed energy pulse trains, the spinor splash pulses, and the Bessel beams, in terms of this decomposition will be given. The efficacy of this representation in geometries with boundaries, such as a propagation in a circular waveguide, will also be demonstrated.

I. INTRODUCTION

The possibility of solutions of the wave equation that describe localized, slowly decaying transmission of energy in space-time has been suggested by several groups in recent years. These include efforts on "focus wave modes",[1–5] "EDEPT's,"[6–9] "splash modes",[10,11] "EM missiles",[12–16] "Bessel beams",[17–22] "EM bullets"[23,24] and "transient beams."[8,25–29] Much of this work was actually motivated by the pioneering work of Brittingham.[1] It has been recently discovered that these original focus wave modes represent Gaussian beams that translate through space with only local deformations and are the fundamental modes of a class of solutions that describe fields that originate from moving complex sources.[2] In particular, the scalar wave equation in real space, viz.,

$$[\partial_t^2 - \nabla^2]\Psi(\mathbf{r},t) = 0\,, \tag{1.1}$$

with a wave speed normalized to unity, has as an exact solution, the moving, modified Gaussian pulse

$$\Psi_\beta(\mathbf{r},t) = e^{i\beta(z+t)}(e^{-\beta\rho^2/V}/4\pi V)\,. \tag{1.2}$$

The complex variance $1/V = 1/A - i/R$ yields the beam spread $A = a_1 + \zeta^2/a_1$, the phase front curvature $R = \zeta + a_1^2/\zeta$, and beam waist $w = (A/\beta)^{1/2}$. Here, $\zeta = z - t$ and ρ denotes the radial cylindrical coordinate. The fundamental pulse (1.2) describes a Gaussian beam that translates through space-time with only local variations. It represents a generalization of earlier work by Deschamps[30] and Felsen[31] describing Gaussian beams as fields radiated from stationary complex source points.

As discussed in Ref. 9, the fundamental Gaussian pulse has either a plane wave or a particlelike character depending on whether β is small or large. Moreover, for all β it shares with the plane wave the property of having infinite energy. However, as with the plane waves, this is not to be considered as a drawback *per se*. The above solution procedure has introduced an added degree of freedom into the solution through the variable β that can be exploited. As shown in Refs. 2 and 5–9, fundamental Gaussian pulse fields, corresponding to different values of β, can be used as basis functions to represent new transient solutions of Eq. (1.1). In particular, the general electromagnetic directed energy pulse train (EDEPT) solution

$$\Psi(\mathbf{r},t) = \int_0^\infty d\beta\,\Psi_\beta(\mathbf{r},t)F(\beta) = \frac{1}{4\pi[a_1 + i(z-t)]}\int_0^\infty d\beta\,F(\beta)e^{-\beta s(\rho,z,t)}\,, \tag{1.3}$$

where

$$s(\rho,z,t) = \rho^2/[a_1 + i(z-t)] - i(z+t)\,, \tag{1.4}$$

is an exact source-free solution of the wave equation. This representation, in contrast to a plane wave decomposition, utilizes basis functions that are more localized in space and hence, by their very nature, are better suited to describe the directed transfer of electromagnetic energy in space. The resulting pulses have finite energy if the function $F(\beta)\beta^{-1/2}$ is square integrable.[9]

As reported in Ref. 2, the superposition (1.3), with the "complex traveling center wave" basis functions, has an inverse. The functions

$$\Phi_\beta(\rho,\zeta,\eta) = 8\sqrt{\pi}\,e^{-(\zeta/4\beta a_1)^2}\Psi_\beta(\rho,\zeta,\eta)\,, \tag{1.5}$$

with $\eta = z + t$, are orthogonal to the Ψ_β. This means these basis functions satisfy the completeness relation

$$\int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\zeta \int_0^\infty d\rho\,\rho\,\Phi_\beta^*(\rho,\zeta,\eta)\Psi_{\beta'}(\rho,\zeta,\eta) = \delta(\beta - \beta')\,, \tag{1.6}$$

where Φ_β^* is the complex conjugate of Φ_β. Hence an inversion of the superposition (1.3) exists.

Clearly, different spectra $F(\beta)$ in Eq. (1.3) lead to different wave equation solutions, and hence, to different solutions of Maxwell's equations. Many interesting solutions of the wave equation can be created by simply referring to a

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Laplace transform table. One particular interesting spectrum selection, recognized by Ziolkowski,[6] is the "modified power spectrum" (MPS)

$$F(\beta) = [p/\Gamma(q)](p\beta - b)^{q-1}e^{-[(p\beta - b)a_2]}, \quad \beta > b/p,$$
$$= 0, \quad b/p > \beta \geqslant 0. \tag{1.7}$$

It is so named because it is derived from the power spectrum $F(\beta) = \beta^{q-1}\exp(-\beta a_2)$ by a scaling and a truncation. This choice of spectrum leads to the MPS pulse

$$\Psi(\mathbf{r},t) = \frac{1}{4\pi(a_1 + i\zeta)} \frac{e^{-bs/p}}{[a_2 + s/p]^q}. \tag{1.8}$$

Solutions to Maxwell's equations follow naturally from these scalar wave equation solutions using a Hertz potential formulation.

The MPS pulse, for example, can be optimized so that it is localized near the direction of propagation and its original amplitude can be recovered out to extremely large distances from its initial location. This is demonstrated in Fig. 1, which shows surface plots and the corresponding contours plots of the electromagnetic energy density U of a TE electromagnetic MPS pulse relative to the pulse center locations at $z = 0.0$ km and $z = 9.42 \times 10^9$ km. The MPS parameters are $a_2 = 1.0$ m, $q = 1.0$, $b = 1.0 \times 10^{14}\ \mathrm{m}^{-1}$, $p = 6.0 \times 10^{15}$, and $a_1 = 1.0 \times 10^{-2}$ m. The energy density U is normalized to its maximum value at $t = 0$. The transverse space coordinate ρ is measured in meters; the longitudinal space coordinate $\zeta = z - t$ is the distance in meters along the direction of propagation away from the pulse center $z = ct$. These results definitively show the localization of the field near the direction of propagation over very large distances.

The MPS pulses are being characterized further and potential launching mechanisms are under investigation. However, it was recognized by Besieris and Shaarawi[32] that the representation (1.3) and its inverse has a generalization that can be exploited to explain these and other localized, slowly decaying solutions in a single framework. This new representation is the main purpose of this paper. It is based on a decomposition of exact solutions of the scalar wave equation into bidirectional, forward and backward, traveling plane wave solutions. The resulting representation is a natural basis for synthesizing pulse solutions. The derivation of this representation from a general operator embedding scheme will be described in Sec. II. The connections between this decomposition and various localized, slowly decaying solutions will be made explicit in Sec. III. In Sec. IV, the bidirectional representation will be extended to other classes of equations, e.g., the Klein–Gordon and the dissipative scalar wave equations that model wave propagation in dispersive and dissipative media, respectively. A specific demonstration of the efficacy of the new representation will be given in connection with an initial-boundary value modeling an infi-

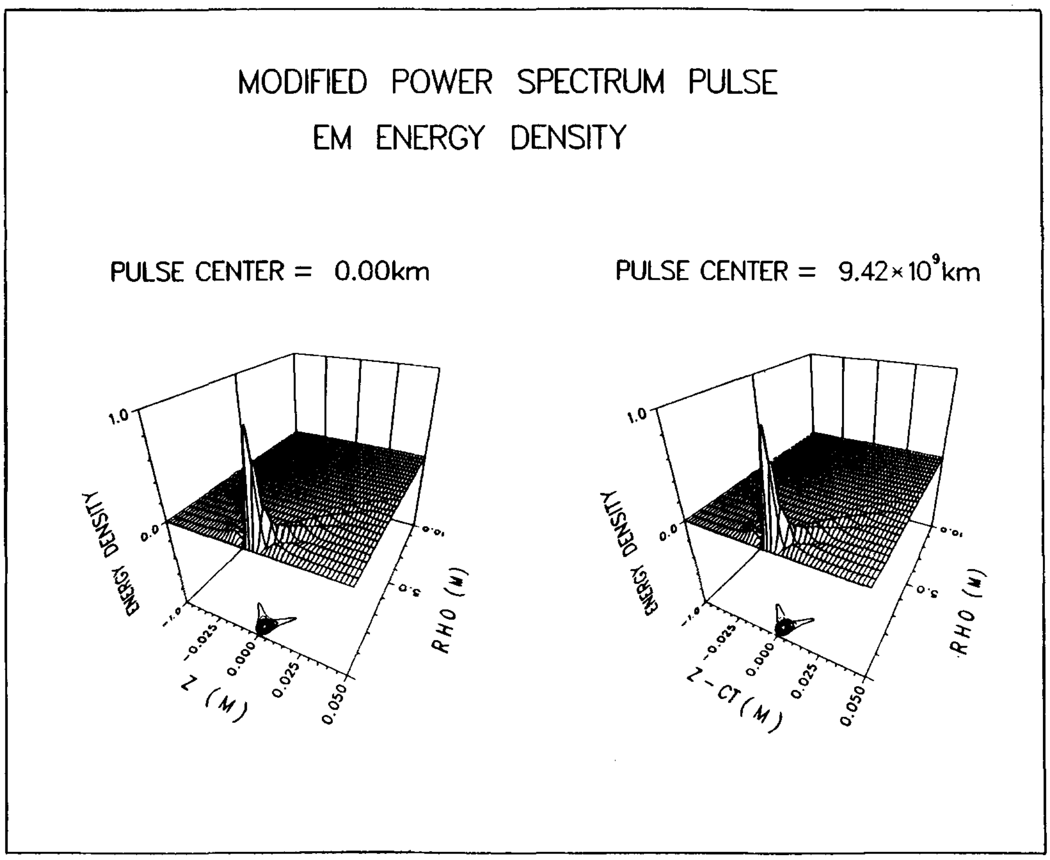

FIG. 1. The field energy of the electromagnetic MPS pulse is shown for the parameters: $a_1 = 1.0$ m, $q = 1.0$, $b = 1.0 \times 10^{14}\ \mathrm{m}^{-1}$, $p = 6.0 \times 10^{15}$, and $a_2 = 0.01$ m.

nite waveguide excited by a localized initial pulse. A summary of the results in this paper will be provided in Sec. V.

II. BIDIRECTIONAL PLANE WAVE DECOMPOSITION

The Cauchy problem

$$[\partial_t^2 + \hat{\Omega}(-i\nabla)]u(\mathbf{r},t) = 0, \quad \mathbf{r}\in R^3, \quad t>0, \tag{2.1a}$$

$$u(\mathbf{r},0) = u_0(\mathbf{r}), \quad u_t(\mathbf{r},0) = u_1(\mathbf{r}), \tag{2.1b}$$

where u is a real scalar-valued function and $\hat{\Omega}$ is a positive, self-adjoint, possibly pseudodifferential operator, can be used as a mathematical model for a large number of physical situations.

A Fourier synthesis of the solution to the Cauchy problem (2.1) can be effected as follows[33]:

$$u(\mathbf{r},t) = 2\,\mathrm{Re}\{\Psi(\mathbf{r},t)\}; \tag{2.2a}$$

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int_{R^3} d\mathbf{k}\, F(\mathbf{k}) e^{-i(\mathbf{k}\cdot\mathbf{r} - \Omega^{1/2}(\mathbf{k})t)}; \tag{2.2b}$$

$$F(\mathbf{k}) = \frac{1}{2}\left[\tilde{u}_0(\mathbf{k}) - i\frac{\tilde{u}_1(\mathbf{k})}{\Omega^{1/2}(\mathbf{k})}\right]. \tag{2.2c}$$

The complex value signal Ψ is generated via a linear superposition of plane waves propagating in the $\mathbf{k}$ direction with phase speeds $\Omega^{1/2}(\mathbf{k})/|\mathbf{k}|$. These plane waves are characterized by wave vectors $\mathbf{k}$ and they are weighted by the Fourier spectrum $F(\mathbf{k})$.

Equation (2.2) constitutes a mathematical solution to the Cauchy problem (2.1). However, for purposes of later comparison, the superposition (2.2b) can be recast in a more general form as follows:

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^4}\int_{R^3} d\mathbf{k}\int_{R^1} d\omega\, \hat{F}(\mathbf{k},\omega) \times e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\delta[-\omega^2 + \Omega(\mathbf{k})]. \tag{2.3}$$

The spectra entering into Eqs. (2.2b) and (2.3) are linked through the relationship

$$F(\mathbf{k}) = \frac{\hat{F}(\mathbf{k},\Omega(\mathbf{k}))}{4\pi|\Omega^{1/2}(\mathbf{k})|}. \tag{2.4}$$

Conditions can also be specified under which Ψ is square integrable, or, even further, under which the solution $u(\mathbf{r},t)$ of (2.1) is a finite energy signal. There is, however, a basic drawback associated with the Fourier method; namely, that in most cases the integral for Ψ can be computed only approximately by a variety of asymptotic approaches, such as the method of stationary phase/saddle point,[34,35] ray-theoretic techniques[36,37] and phase space methods,[38] or can be carried out numerically. Very few exact analytical solutions to (2.2b) are available, even for the simple, single mode dispersion relationship $\omega \equiv \Omega^{1/2}(\mathbf{k}) = (k^2+\mu^2)^{1/2}$ corresponding to the Klein–Gordon equation.

The Cauchy problem (2.1) will be used in the sequel as a vehicle for presenting a new principle of superposition that provides more freedom and flexibility when dealing with certain classes of solutions, e.g., the EDEPT solutions to the scalar wave equation.

Different types of superpositions are obtained by dividing the operator $L = [\partial_t^2 + \hat{\Omega}(-i\nabla)]$ into parts, each having its own eigenfunctions. A general solution can be constructed from the product of such eigenfunctions, together with a constraint relationship between their eigenvalues. The manner in which the operator L is partitioned determines the form of the final superposition. For example, the Fourier decomposition follows from partitioning L into two parts: $L_1 = \partial_t^2$ and $L_2 = \hat{\Omega}(-i\nabla)$. The superposition (2.3) contains the constraint $\omega = \Omega^{1/2}(\mathbf{k})$ relating the eigenvalues of L_1 and L_2 corresponding to the eigenfunctions $\exp(+i\omega t)$ and $\exp(-i\mathbf{k}\cdot\mathbf{r})$, respectively.

In general, the operator L can be partitioned in many different ways. Consider, for example, the preliminary splitting of the operator $\hat{\Omega}(-i\nabla)$ as follows:

$$\hat{\Omega}(-i\nabla) = \hat{A}(-i\partial_z) + [\hat{\Omega}(-i\nabla) - \hat{A}(-i\partial_z)] \equiv \hat{A}(-i\partial_z) + \hat{B}(-i\nabla_T, -i\partial_z). \tag{2.5}$$

The operator $\hat{A}(-i\partial_z)$, which may or may not be a natural part of $\hat{\Omega}(-i\nabla)$, is assumed to be positive, self-adjoint and the choice of the preferred variable z is arbitrary. By taking the Fourier transform with respect to the transverse components, the complex wave function $\Psi(\mathbf{r},t)$ can be expressed as

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_{R^2} d\boldsymbol{\kappa}\, \tilde{\psi}(\boldsymbol{\kappa},z,t)e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}, \tag{2.6}$$

with $\tilde{\psi}(\boldsymbol{\kappa},z,t)$ governed by the equation

$$[\partial_t^2 + \hat{A}(-i\partial_z) + \hat{B}(\boldsymbol{\kappa}, -i\partial_z)]\tilde{\psi}(\boldsymbol{\kappa},z,t) = 0. \tag{2.7}$$

The operator $L \equiv \partial_t^2 + \hat{\Omega}(-\boldsymbol{\kappa}, -i\partial_z)$ can now be partitioned as follows:

$$L_1 = \partial_t^2 + \hat{A}(-i\partial_z), \tag{2.8a}$$

$$L_2 = \hat{B}(-\boldsymbol{\kappa}, -i\partial_z). \tag{2.8b}$$

The most natural eigenfunctions of the operator L_1 are given by

$$\psi_e(z,t) = e^{-i\alpha\zeta}e^{i\beta\eta}, \tag{2.9}$$

where ζ and η are defined as follows:

$$\zeta = z - t\,\mathrm{sgn}(\alpha)\alpha^{-1}A^{1/2}(\alpha), \tag{2.10a}$$

$$\eta = z + t\,\mathrm{sgn}(\beta)\beta^{-1}A^{1/2}(\beta). \tag{2.10b}$$

The corresponding eigenvalues, denoted by $\lambda(\alpha,\beta)$, are given explicitly as follows:

$$\lambda(\alpha,\beta) = A(\beta-\alpha) - [A(\alpha) + A(\beta) + 2\,\mathrm{sgn}(\alpha)A^{1/2}(\alpha)\mathrm{sgn}(\beta)A^{1/2}(\beta)]. \tag{2.11}$$

The elementary functions (2.9) consist of products of two plane waves traveling in opposite directions, with wavenumber-dependent phase speeds equal to $\mathrm{sgn}(\alpha)\alpha^{-1}A^{1/2}(\alpha)$ and $\mathrm{sgn}(\beta)\beta^{-1}A^{1/2}(\beta)$, respectively.

The bilinear functions (2.9) are also eigenfunctions of L_2, with corresponding eigenvalues equal to $B(-\boldsymbol{\kappa},\beta-\alpha)$. As a consequence, a linear superposition of the bidirectional elementary solutions ψ_e results in a solution of Eq. (2.7), viz.,

$$\tilde{\psi}(\boldsymbol{\kappa},z,t) = \int_{R^1} d\alpha\int_{R^1} d\beta\, C(\boldsymbol{\kappa},\alpha,\beta) \times e^{-i\alpha\zeta}e^{i\beta\eta}\delta[\lambda(\alpha,\beta) + B(-\boldsymbol{\kappa},\beta-\alpha)], \tag{2.12}$$

where the constraint $\lambda(\alpha,\beta) + B(-\boldsymbol{\kappa},\beta-\alpha) = 0$ is included in the integration. A general solution to Eq. (2.1) can be obtained by resorting to a transverse Fourier inversion [cf. Eq. (2.6)]; specifically,

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_{R^2} d\boldsymbol{\kappa}\, e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\int_{R^1} d\alpha \int_{R^1} d\beta\, C(\boldsymbol{\kappa},\alpha,\beta) \times e^{-i\alpha\zeta}e^{i\beta\eta}\delta[\lambda(\alpha,\beta) + B(-\boldsymbol{\kappa},\beta-\alpha)]\,. \quad (2.13)$$

This representation constitutes a generalization of the three-dimensional Fourier synthesis [cf. Eq. (2.3)]; in the latter, the operator $\hat{A}(-i\partial_z)$ was chosen to be a constant given by the relations

$$\operatorname{sgn}(\alpha)A^{1/2}(\alpha) + \operatorname{sgn}(\beta)A^{1/2}(\beta) = \omega \quad (2.14a)$$

and

$$\alpha - \beta = k_z\,. \quad (2.14b)$$

The main advantage of this decomposition is the introduction of the embedded operator $\hat{A}(-i\partial_z)$. This provides a fresh approach for addressing different classes of problems. At the same time, the flexibility that one can enjoy through a clever choice of $\hat{A}(-i\partial_z)$ may open the way to approach some of the more impenetrable problems.

To clarify these ideas, consider specifically the case of the three-dimensional scalar wave equation for which $\hat{\Omega}(-i\nabla) = -\nabla^2$. The operator L, in this case, assumes the form $L = \partial_t^2 - \nabla^2$ and Eq. (2.1a) simplifies to

$$[\partial_t^2 - \nabla^2]u(\mathbf{r},t) = 0\,. \quad (2.15)$$

In cylindrical coordinates, the Laplacian ∇^2 can be written as follows:

$$\nabla^2 = \partial_z^2 + \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2\,.$$

In the usual Fourier decomposition, the operator L is divided into two parts:

$$L_1 = -[\partial_z^2 + \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2]\,, \quad (2.16a)$$

$$L_2 = \partial_t^2\,. \quad (2.16b)$$

The eigenfunctions of L_1 are $J_n(\kappa\rho)e^{\pm in\phi}e^{\pm ik_z z}$ and $N_n(\kappa\rho)e^{\pm in\phi}e^{\pm ik_z z}$, where $J_n(\kappa\rho)$ and $N_n(\kappa\rho)$ are Bessel functions of the first and second kind, respectively, and the eigenvalues equal $\kappa^2 + k_z^2$. The operator L_2 has eigenfunctions $e^{\pm i\omega t}$ with eigenvalues $-\omega^2$. An elementary solution to the scalar wave equation (2.15) can be written as

$$\Psi_e(\mathbf{r},t) = [A_n J_n(\kappa\rho) + B_n N_n(\kappa\rho)]e^{\pm in\phi}e^{-i(k_z z\pm\omega t)}\,, \quad (2.17a)$$

with the constraint

$$\kappa^2 + k_z^2 - \omega^2 = 0\,. \quad (2.17b)$$

Neglecting the terms $N_n(\kappa\rho)$ because of their infinite values at $\rho = 0$, one obtains a special case of the superposition (2.3) that gives the general Fourier synthesis solution to the scalar wave equation:

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} \times dk_z\, A_n(\omega,k_z,\kappa)\kappa J_n(\kappa\rho)e^{\pm in\phi} \times e^{-ik_z z}e^{+i\omega t}\delta(\omega^2 - \kappa^2 - k_z^2)\,. \quad (2.18)$$

Next consider the choice $\hat{A}(-i\partial_z) = -\partial_z^2$ which reduces Eqs. (2.10) to

$$\zeta = z - t \quad \text{and} \quad \eta = z + t\,. \quad (2.19)$$

The operator L can be written, in this case, as

$$L = -[4\partial_{\zeta\eta}^2 + \partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2]\,,$$

and it can be partitioned as follows:

$$L_1 = -[\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\partial_\phi^2]\,, \quad (2.20a)$$

$$L_2 = -4\partial_{\zeta\eta}^2\,. \quad (2.20b)$$

The eigenfunctions of L_1 are given now by $J_n(\kappa\rho)e^{\pm in\phi}$ and $N_n(\kappa\rho)e^{\pm in\phi}$, and its eigenvalues equal $+\kappa^2$. The operator L_2 has eigenfunctions $e^{-i\alpha\zeta}e^{i\beta\eta}$ with eigenvalues $-4\alpha\beta$. An elementary solution to the scalar wave equation (2.15) can be written as

$$\Psi_e(\mathbf{r},t) \equiv \Psi_e(\rho,\zeta,\eta) = [C_n J_n(\kappa\rho) + D_n N_n(\kappa\rho)]e^{\pm in\phi}e^{-i\alpha\zeta}e^{i\beta\eta}\,, \quad (2.21a)$$

with the constraint

$$\alpha\beta = \kappa^2/4\,. \quad (2.21b)$$

This constraint limits the value of α and β either to be both negative or both positive. A general solution to the scalar wave equation can be written in the nonconventional form

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\sum_{\substack{l=-1\\ l\neq 0}}^{+1}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_0^\infty d(l\alpha) \times \int_0^\infty d(l\beta)\, C_n(l\alpha,l\beta,\kappa)\kappa J_n(\kappa\rho) \times e^{\pm in\phi}e^{-il\alpha\zeta}e^{il\beta\eta}\delta(\alpha\beta - \kappa^2/4)\,. \quad (2.22)$$

The two representations [cf. Eqs. (2.18) and (2.22)] may appear to be very different. There exists, however, a one to one correspondence between these two superpositions through the change of variables

$$k_z = \alpha - \beta\,, \quad \omega = \alpha + \beta\,. \quad (2.23)$$

By using these relationships, the new representation (2.22) can be transformed into the Fourier synthesis given in Eq. (2.18), with the following connection between their spectra:

$$A_n(\omega,k_z,\kappa) = 2C_n[\tfrac{1}{2}(\omega + k_z),\tfrac{1}{2}(\omega - k_z),\kappa]\,. \quad (2.24)$$

It should be noted that this transformation requires a careful handling of the limits of integration. A complete discussion of this point will be given later when dealing with specific examples.

The representation (2.22) provides a fresh path through which exact solutions to the scalar wave equation can be obtained. Although such a representation is not a familiar one, solutions obtained using (2.22) can still be easily transformed into the more popular Fourier superposition, and one can link the bidirectional results to the more conventional Fourier interpretation. To emphasize these ideas, one can remove the constraint in (2.18) by integrating over ω, hence reducing (2.18) to a form similar to (2.2), with

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_{-\infty}^{+\infty} \times dk_z\Big[A_n\big[\sqrt{\kappa^2 + k_z^2},k_z,\kappa\big]e^{-i(k_z z - \sqrt{\kappa^2 + k_z^2}\,t)} + A_n\big[-\sqrt{\kappa^2 + k_z^2},k_z,\kappa\big]$$

$$\times e^{-i(k_z z+\sqrt{\kappa^2+k_z}\,t)}\Big]\frac{\kappa J_n(\kappa\rho)}{2\sqrt{\kappa^2+k_z}}e^{\pm in\phi}. \quad (2.25)$$

This is a special case of the superposition (2.2) and consists of the sum of two components, one traveling in the positive z direction and the other in the negative z direction. A common problem that arises when dealing with such integrals is associated with the branch-cut type singularities. These can pose significant difficulties even when the integrals are solved either asymptotically or computed numerically. On the other hand, when the constraint is integrated out of Eq. (2.22), one obtains

$$\Psi(\rho,\zeta,\eta)=\frac{1}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^{\infty}d\kappa\int_0^{\infty}$$
$$\times d\beta\left[C_n\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)e^{(-i\kappa^2/4\beta)\zeta}e^{i\beta\eta}\right.$$
$$\left.-C_n\left(-\frac{\kappa^2}{4\beta},-\beta,\kappa\right)e^{(i\kappa^2/4\beta)\zeta}e^{-i\beta\eta}\right]$$
$$\times\frac{\kappa}{\beta}J_n(\kappa\rho)e^{\pm in\phi}. \quad (2.26)$$

Similar to the Fourier synthesis where one can choose either the positive or negative ω branch, we can choose to work with either the positive or the negative branch of α and β. In what follows, for convenience only, we choose the positive branch. Notice that, unlike Eq. (2.25), the terms in the above integral consist of products of two plane waves traveling in opposite directions. An important characteristic of the representation (2.26) is that the branch-cut singularities in Eq. (2.25) have been converted into algebraic singularities. This provides a novel approach to finding solutions to the scalar wave equation. New exact solutions can be obtained by choosing appropriate spectra C_n, for which the corresponding Fourier spectra A_n might be very complicated and could not have been guessed. Moreover, because of the nature of the branch-cut singularities in the Fourier synthesis, problems arise because of their multivaluedness and because large oscillations accompany any attempt to evaluate them either numerically or asymptotically. We have found that one can circumvent such problems by dealing with the bidirectional synthesis and its tame algebraic singularities.

A number of important mathematical issues dealing with the new bidirectional synthesis will be considered at this point. These will include the completeness of the expansions (2.22) and (2.26), the inversion properties for $C_n(\alpha,\beta,\kappa)$, and conditions that must be imposed on the spectrum in order to ensure square integrability. The feasibility of solving Cauchy initial value problems on the basis of the new representation will be addressed in Sec. IV.

Completeness follows directly from the fact that the superimposed functions are either exponential or Bessel functions, which are both orthogonal functions and form their own complete sets. However, the inversion of $C_n(\alpha,\beta,\kappa)$ is not obvious. A generalization of Ziolkowski's formula (1.6) had to be used. Using the positive β branch in (2.26), $\Psi(\rho,\zeta,\eta)$ can be constructed as follows:

$$\Psi(\rho,\zeta,\eta)=\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\int_0^{\infty}d\kappa\int_0^{\infty}d\beta\frac{1}{\beta}$$
$$\times C_n\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)\kappa J_n(\kappa\rho)e^{in\phi}e^{-(i\kappa^2/4\beta)\zeta}e^{i\beta\eta}. \quad (2.27)$$

The inversion formula corresponding to this superposition is given by

$$C_m\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)=\frac{1}{4\sqrt{\pi}}\int_{-\pi}^{+\pi}d\phi\int_{-\infty}^{+\infty}d\zeta$$
$$\times e^{-\zeta^2/16\beta^2}\int_{-\infty}^{+\infty}d\eta\int_0^{\infty}d\rho\,\rho\,J_m(\kappa\rho)$$
$$\times\Psi(\rho,\zeta,\eta)e^{-im\phi}e^{(i\kappa^2/4\beta)\zeta}e^{-i\beta\eta}. \quad (2.28)$$

Note the appearance of the Gaussian measure over ζ. A similar measure occurred in Ziolkowski's inversion (1.6). However, in the more general inversion (2.28) the additional parameter a_1 has disappeared. The validity of the inversion will be demonstrated below in connection to specific examples.

To investigate the possible restrictions on the spectrum C_0 that would ensure square integrability of the solution, one can consider the integral over β in Eq. (2.27), namely,

$$\psi(z,t)=\int_0^{\infty}d\beta\frac{1}{\beta}C_0\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)e^{-i\kappa^2(z-t)/4\beta}e^{i\beta(z+t)}. \quad (2.29)$$

By rearranging the variables in the exponentials, one obtains

$$\psi(z,t)=\int_0^{\infty}d\beta\frac{1}{\beta}C_0\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)$$
$$\times\exp\left[-iz\left(\frac{\kappa^2}{4\beta}-\beta\right)\right]\exp\left[it\left(\frac{\kappa^2}{4\beta}+\beta\right)\right],$$

or

$$\psi(z,t)=\int_0^{\infty}d\beta\frac{1}{\beta}C_0\left(\frac{\kappa^2}{4\beta},\beta,\kappa\right)$$
$$\times\exp\left[\frac{i\kappa z}{2}\left(\frac{2\beta}{\kappa}-\frac{\kappa}{2\beta}\right)\right]$$
$$\times\exp\left[\frac{\kappa t}{2}\left(\frac{i2\beta}{\kappa}-\frac{\kappa}{i2\beta}\right)\right]. \quad (2.30)$$

By using the Laurent expansion of the Bessel generating function, viz.,

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty}J_n(x)t^n, \quad (2.31)$$

the exponentials in (2.30) can be rewritten as

$$\exp\left[\frac{i\kappa z}{2}\left(\frac{2\beta}{\kappa}-\frac{\kappa}{2\beta}\right)\right]=\sum_{n=-\infty}^{+\infty}(i)^nI_n(\kappa z)\left[\frac{2\beta}{\kappa}\right]^n,$$
$$\exp\left[\frac{\kappa t}{2}\left(\frac{i2\beta}{\kappa}-\frac{\kappa}{i2\beta}\right)\right]=\sum_{m=-\infty}^{+\infty}(i)^mJ_m(\kappa t)\left[\frac{2\beta}{\kappa}\right]^m.$$

By using these expansions, Eq. (2.30) can be rewritten as

$$\psi(z,t)=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}C_{mn}I_n(\kappa z)J_m(\kappa t), \quad (2.32a)$$

where

$$C_{mn} = \int_0^\infty d\beta \left[\frac{2\beta}{\kappa}\right]^{m+n} i^{m+n} C_0\left(\frac{\kappa^2}{2\beta}, \beta, \kappa\right). \quad (2.32b)$$

A necessary condition for the convergence of (2.32a) is that $C_{mn} < \infty$ for all values of m and n ranging from $-\infty$ to $+\infty$. By considering the integral (2.32b), it is then obvious that $C_0(\kappa^2/\beta,\beta,\kappa)$ should obey the conditions

$$\lim_{\beta\to 0} \frac{1}{\beta^r} C_0\left(\frac{\kappa^2}{4\beta}, \beta, \kappa\right) < \infty , \quad (2.33a)$$

$$\lim_{\beta\to\infty} \beta^r C_0\left(\frac{\kappa^2}{4\beta}, \beta, \kappa\right) < \infty , \quad (2.33b)$$

for arbitrary β, and $r = m + n$, for any integer values of m and n. A good candidate is a spectrum of the form

$$C_0\left(\frac{\kappa^2}{4\beta}, \beta, \kappa\right) = \beta^r \exp\left[-\beta a_1 + \frac{\kappa^2}{4\beta} a_2\right]. \quad (2.34)$$

This is similar to the one used for the splash pulses and EDEPT solutions, as will be demonstrated in Sec. III.

In summary, the procedure described in this section provides an alternate way of synthesizing solutions to different partial differential equations. Such representations are characterized by different types of singularities that may facilitate their asymptotic or numerical evaluation. This is a flexible procedure that changes with the types of equations considered. Moreover, solutions to the same equation may have different representations depending on how the operator L is partitioned. In Sec. III, the bidirectional representation (2.22) will be used as a natural superposition for the synthesis of Brittingham-like solutions, e.g., focus wave modes, splash pulses, Bessel beams, and EDEPT solutions. This will enable us to gain a better understanding of these unusual solutions, and by using the transformation (2.23), to obtain more information about their Fourier spectral content. Other types of equations, dealing with dispersive and dissipative problems, will be discussed in Sec. IV, where it will be demonstrated that the bidirectional representation can reduce the complexity level of such equations to that of the three-dimensional scalar wave equation.

III. BIDIRECTIONAL PLANE WAVE DECOMPOSITION OF KNOWN SOLUTIONS

It was demonstrated in the previous section that the main achievement of the embedding technique is to introduce a time-symmetric bidirectional representation. For the scalar wave equation, such a representation is given in Eq. (2.22). It turns out that such a superposition provides the most natural approach for synthesizing Brittingham-like solutions. This section is devoted, mainly, to substantiating this claim. Starting with the scalar analog of Brittingham's FWM's, it will be shown that by choosing very simple spectra $C_n(\alpha,\beta,\kappa)$, mostly of the type given in Eq. (2.34), all known Brittingham-like solutions can be synthesized. Because of the simple transformation (2.23), it will be easy to transform such solutions to their Fourier picture, from which a basic understanding of their spectral content can be achieved. These examples can also provide a vehicle through which the inversion formula can be checked. The following discussion will be restricted to the zeroth order mode ($n = 0$). This is a matter of convenience and does not affect the generality of the procedure.

A. Focus wave modes

The focus wave modes (FWM's) were originally stimulated by the work of Brittingham,[1] who derived their vector form in connection with Maxwell's equations. Their scalar form was derived by Belanger,[3] Sezinger,[4] and Ziolkowski.[2] These modes, the zeroth order of which is given in Eq. (1.2), are characterized by an infinite energy content. Motivated by the bidirectional character of the solution (1.2), it will be shown below that the representation (2.22) can be used to synthesize the FWM's associated with the scalar wave equation.

Consider the spectrum

$$C_0(\alpha,\beta,\kappa) = (\sqrt{\pi}/2)\sigma e^{-\sigma^2(\beta-\beta')^2} e^{-\alpha a_1}. \quad (3.1)$$

Substituting it into Eq. (2.22) results in the expression

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\int_0^\infty d\kappa \int_0^\infty d\beta \int_0^\infty d\alpha \times \frac{\sqrt{\pi}\sigma}{2} e^{-\sigma^2(\beta-\beta')^2} e^{-\alpha a_1} \kappa J_0(\kappa\rho) \times e^{-i\alpha\zeta} e^{i\beta\eta} \delta[\alpha\beta - \kappa^2/4]. \quad (3.2)$$

An integration over α reduces Eq. (3.2) to

$$\Psi(\rho,\zeta,\eta) = \frac{\sigma}{8\pi^{3/2}}\int_0^\infty d\kappa \int_0^\infty d\beta \frac{\kappa}{\beta} \times J_0(\kappa\rho) e^{-\sigma^2(\beta-\beta')^2} e^{-\kappa^2(a_1+i\zeta)/4\beta} e^{i\beta\eta}.$$

By using equation (6.631.4) in Gradshteyn and Ryzhik,[39] viz.,

$$\int_0^\infty dx\, x^{\nu+1} e^{-ax^2} J_\nu(\alpha x) = \frac{\alpha^\nu}{(2a)^{\nu+1}} e^{-\alpha^2/4a}, \quad (3.3)$$

the integration over κ can be carried out explicitly, yielding

$$\Psi(\rho,\zeta,\eta) = \frac{\sigma}{4\pi^{3/2}}\int_0^\infty d\beta \frac{1}{(a_1+i\zeta)} \times e^{-\sigma^2(\beta-\beta')^2} e^{-\beta\rho^2/(a_1+i\zeta)} e^{i\beta\eta}.$$

To carry out the final integration over β, Eq. (3.4621.1) in Gradshteyn and Ryzhik[39] is used, viz.,

$$\int_0^\infty dx\, x^{\nu-1} e^{-\gamma x} e^{-\beta x^2} = \frac{\Gamma(\nu)}{(2\beta)^{\nu/2}} e^{\gamma^2/8\beta} D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right),$$

to give the solution

$$\Psi(\rho,\zeta,\eta) = \frac{1}{4\pi(a_1+i\zeta)} \frac{e^{-\sigma^2\beta'^2}}{\sqrt{2\pi}} \times e^{(\Lambda - i\eta - 2\sigma^2\beta')^2/8\sigma^2} D_{-1}\left(\frac{\Lambda - i\eta - 2\sigma^2\beta'}{\sqrt{2}\sigma}\right), \quad (3.4a)$$

where D_{-1} is the parabolic cylinder function of order -1 and

$$\Lambda = \rho^2/(a_1 + i\zeta).$$

The solution (3.4a) is a generalization of the scalar FWM's; the latter can be recovered by taking the limit

$\sigma \to \infty$, for which the spectrum in (3.1) reduces to

$$C_0(\alpha,\beta,\kappa) = (\pi/2)\delta(\beta-\beta')e^{-\alpha a_1}.$$

In what follows, we shall use $\hat{\delta}$ to denote the part of the spectrum in (3.1) that reduces to the Dirac δ function as $\sigma \to \infty$. This will yield less cumbersome expressions and will make our discussion more transparent. As for the solution in (3.4a), it is more convenient to compute the limit $\sigma \to \infty$ after rewriting the parabolic cylinder function in an alternate form using the identity (9.254.1) in Gradshteyn and Ryzhik,[39] namely,

$$D_{-1}(z) = e^{z^2/4}\sqrt{\pi/2}\,[1-\Phi(z/\sqrt{2})],$$

where $\Phi(z)$ is the probability integral defined as follows:

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt.$$

Rewriting the wave function (3.4a) in terms of the probability integral, viz.,

$$\Psi(\rho,\zeta,\eta) = \frac{e^{-\beta'(\Lambda - i\eta)}}{8\pi(a_1+i\zeta)}\,e^{(\Lambda - i\eta)^2/4\sigma^2} \times\left[1-\Phi\left(\frac{\Lambda - i\eta}{2\sigma}-\sigma\beta'\right)\right],$$

facilitates taking the limit $\sigma\to\infty$ since $\Phi(-\infty) = -1$. Hence as σ goes to ∞, the above expression reduces to the scalar FWM solution

$$\Psi(\rho,\zeta,\eta) = [1/4\pi(a_1+i\zeta)]e^{-\beta'(\Lambda - i\eta)}. \tag{3.4b}$$

Felsen and Heyman[25,29] have established the acausal nature of the FWM's, in the limit where $\beta' a_1 \gg 1$, using their approximate STT theory. The causality issue can be handled in a more direct way by transforming (3.2) into the Fourier picture using the relationships

$$\beta = \tfrac{1}{2}(\omega - k_z), \quad \alpha = \tfrac{1}{2}(\omega + k_z). \tag{3.5}$$

If $\Psi(\rho,\zeta,\eta)$ in Eq. (3.2) is rewritten as

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\int_0^\infty d\kappa\,\kappa J_0(\kappa\rho)\tilde{\psi}(\kappa,\zeta,\eta), \tag{3.6a}$$

with

$$\tilde{\psi}(\kappa,\zeta,\eta) = \int_0^\infty d\beta \int_0^\infty d\alpha\,\frac{\pi}{2}\,\hat{\delta}(\beta-\beta') \times e^{-\alpha a_1}e^{-i\alpha\zeta}e^{i\beta\eta}\,\delta\left[\alpha\beta - \frac{\kappa^2}{4}\right], \tag{3.6b}$$

one can use Eq. (3.5) to express $\tilde{\psi}(\kappa,z,t)$ as follows:

$$\tilde{\psi}(\kappa,z,t) = \int_0^\infty dk_z \int_{k_z}^\infty d\omega\,\frac{\pi}{2}\,\hat{\delta}\left[\frac{(\omega-k_z)}{2}-\beta'\right]e^{-(a_1/2)(\omega+k_z)}\,\delta\left[\frac{\omega^2}{4}-\frac{k_z^2}{4}-\frac{\kappa^2}{4}\right]e^{-i(k_z z-\omega t)}$$
$$+\int_{-\infty}^0 dk_z \int_{-k_z}^\infty d\omega\,\frac{\pi}{2}\,\hat{\delta}\left[\frac{(\omega-k_z)}{2}-\beta'\right]e^{-(a_1/2)(\omega+k_z)}\,\delta\left[\frac{\omega^2}{4}-\frac{k_z^2}{4}-\frac{\kappa^2}{4}\right]e^{-i(k_z z-\omega t)}. \tag{3.7}$$

An integration over ω simplifies (3.7) to

$$\tilde{\psi}(\kappa,z,t) = \int_0^\infty dk_z\,\frac{2\pi}{\sqrt{k_z^2+\kappa^2}}\,\hat{\delta}\left[\sqrt{k_z^2+\kappa^2}-k_z-2\beta'\right]e^{-(a_1/2)[\sqrt{k_z^2+\kappa^2}+k_z]}e^{-i(k_z z-\overline{\omega}t)}$$
$$+\int_0^\infty dk_z\,\frac{2\pi}{\sqrt{k_z^2+\kappa^2}}\,\hat{\delta}\left[\sqrt{k_z^2+\kappa^2}+k_z-2\beta'\right]e^{-(a_1/2)[\sqrt{k_z^2+\kappa^2}-k_z]}e^{i(k_z z+\overline{\omega}t)}, \tag{3.8}$$

where $\overline{\omega} = \sqrt{k_z^2+\kappa^2}$. Referring to Fig. 2, it is clear that the first integral in (3.8) vanishes for $\beta' < \kappa/2$ while the second one vanishes for $\beta' > \kappa/2$. As a result, $\Psi(\mathbf{r},t)$ can be divided into two parts, one traveling in the positive z direction and the other in the negative z direction, viz.,

$$\Psi(\mathbf{r},t) = \Psi^+(\mathbf{r},t) + \Psi^-(\mathbf{r},t),$$

where

$$\Psi^+(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_{2\beta'}^\infty d\kappa\,\kappa J_0(\kappa\rho) \times\int_0^\infty dk_z\,F(k_z,\kappa)e^{-i(k_z z-\overline{\omega}t)}, \tag{3.9a}$$

$$\Psi^-(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^{2\beta'} d\kappa\,\kappa J_0(\kappa\rho) \times\int_0^{\beta'} dk_z\,F(-k_z,\kappa)e^{i(k_z z+\overline{\omega}t)}, \tag{3.9b}$$

and

$$F(k_z,\kappa) = (2\pi/\sqrt{k_z^2+\kappa^2})\hat{\delta}\left[\sqrt{k_z^2+\kappa^2}-k_z-2\beta'\right] \times e^{-[\sqrt{k_z^2+\kappa^2}+k_z]a_1/2}. \tag{3.10}$$

If the parameter a_1 is large, the spectrum $F(k_z,\kappa)$ in (3.10) has a very narrow bandwidth, while $F(-k_z,\kappa)$ can maintain a balance between $\sqrt{k_z^2+\kappa^2}$ and k_z in the exponential and, consequently, can have a much larger bandwidth bounded by the upper limits of integration over κ and k_z in (3.9b). In this case, the predominant contribution to $\Psi(\mathbf{r},t)$ comes from $\Psi^-(\mathbf{r},t)$. This contribution is primarily a nonlocalized plane wave moving in the negative z direction. If, on the other hand, a_1 is very small, both $F(k_z,\kappa)$ and $F(-k_z,\kappa)$ have large bandwidths and because of the limited range of integration in the expression for $\Psi^-(\mathbf{r},t)$ compared to the infinite range for $\Psi^+(\mathbf{r},t)$, one expects that $\Psi^+(\mathbf{r},t)$ becomes much larger than $\Psi^-(\mathbf{r},t)$. In this case, the solution $\Psi(\mathbf{r},t)$ behaves like a localized pulse moving in the positive z direction.

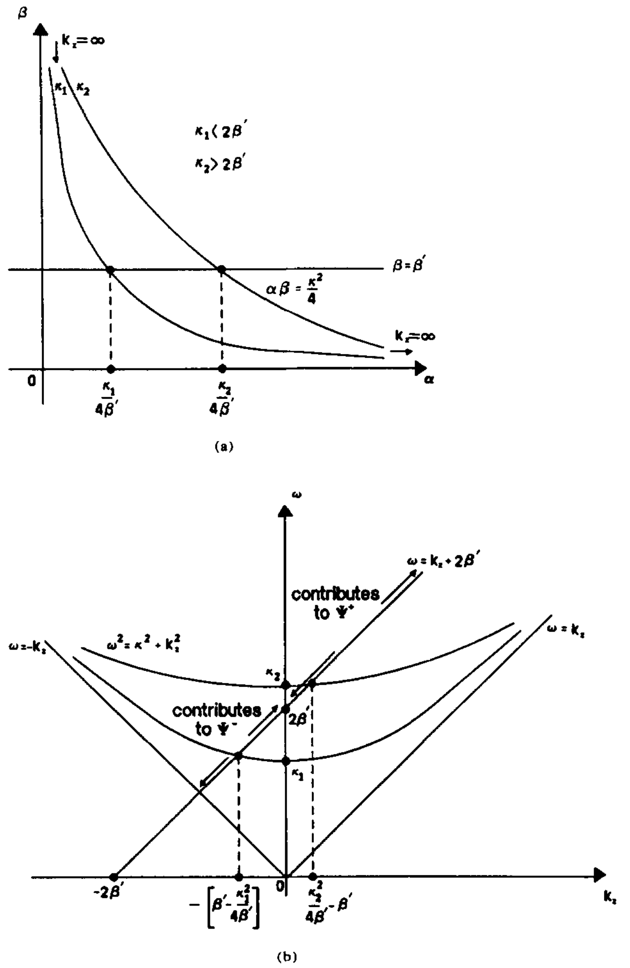

FIG. 2. The constraints $\alpha\beta = \kappa^2/4$ and $\beta = \beta'$ are shown in: (a) the $\alpha\beta$ plane and (b) the $k_z\omega$ plane. The contributions to Ψ^+ and Ψ^- are indicated by arrows.

In closing this subsection, we shall check the validity of the inversion given in Eq. (2.28). A substitution of (3.4b) into Eq. (2.28), leads to the expression

$$C_0\left(\frac{\kappa^2}{4\beta}, \beta, \kappa\right) = \frac{\sqrt{\pi}}{2} \int_{-\infty}^{+\infty} d\zeta \, e^{-\zeta^2/16\beta^2} \int_{-\infty}^{+\infty} d\eta \times \int_0^{+\infty} d\rho \, \rho J_0(\kappa\rho) \frac{e^{i\beta'\eta}}{4\pi(a_1 + i\zeta)} \times e^{-\beta'\rho^2/(a_1+i\zeta)} e^{(i\kappa^2/4\beta)\zeta} e^{-i\beta\eta}. \quad (3.11)$$

Integrating over η and making use of Eq. (3.3) in order to carry out the integration over ρ, it follows that

$$C_0\left(\frac{\kappa^2}{4\beta}, \beta, \kappa\right) = \frac{\sqrt{\pi}}{8} \int_{-\infty}^{+\infty} d\zeta \, e^{-\zeta^2/16\beta^2} \delta(\beta - \beta') \frac{1}{\beta'} e^{-\kappa^2 a_1/4\beta'}.$$

The relation

$$\int_{-\infty}^{+\infty} d\zeta \, e^{-\zeta^2/16\beta^2} = \sqrt{16\pi}\,\beta$$

yields, finally, the result

$$C_0(\kappa^2/4\beta, \beta, \kappa) = (\pi/2) e^{-\kappa^2 a_1/4\beta} \delta(\beta - \beta'), \quad (3.12)$$

which is identical to the spectrum given in Eq. (3.1) provided that $\alpha = \kappa^2/4\beta$. The latter follows from the constraint embodied in Eq. (2.21b).

B. Splash modes

The original "splash mode" was introduced by Ziolkowski[2] as the first example of the class of finite energy solutions constructed from superpositions of the original FWM's. Hillion[10,11] has extended the FWM and the splash mode concepts to the realm of spinors. Ziolkowski's splash pulse can be derived within the framework of the bidirectional representation by choosing the spectrum $C_0(\alpha,\beta,\kappa)$ as follows:

$$C_0(\alpha,\beta,\kappa) = (\pi/2)\beta^{q-1} e^{-(\alpha a_1 + \beta a_2)}. \quad (3.13)$$

It should be noted that this choice is a specific example of the general class of spectra given in Eq. (2.34). Substituting (3.13) into Eq. (2.22) yields

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2} \int_0^\infty d\kappa \int_0^\infty d\beta \int_0^\infty d\alpha \frac{\pi}{2} \beta^{q-1} \times e^{-(\alpha a_1 + \beta a_2)} \kappa J_0(\kappa\rho) e^{-i\alpha\zeta} e^{i\beta\eta} \delta\left[\alpha\beta - \frac{\kappa^2}{4}\right]. \quad (3.14)$$

The integration over α can be carried out explicitly, viz.,

$$\Psi(\rho,\zeta,\eta) = \frac{1}{8\pi} \int_0^\infty d\kappa \int_0^\infty d\beta \times \kappa J_0(\kappa\rho) \beta^{q-2} e^{-(\kappa^2/4\beta)(a_1 + i\zeta)} e^{-\beta(a_2 - i\eta)}.$$

Equation (3.471.9) in Gradshteyn and Ryzhik,[39] viz.,

$$\int_0^\infty d\beta \, \beta^{q-1} \exp\left[-\frac{a}{\beta} - b\beta\right] = 2\left[\frac{a}{b}\right]^{q/2} K_q \left[2\sqrt{ab}\right], \quad (3.15)$$

facilitates the integration over β; specifically,

$$\Psi(\rho,\zeta,\eta) = \frac{1}{8\pi} \int_0^\infty d\kappa \, \kappa^q J_0(\kappa\rho) 2\left(\frac{(a_1 + i\zeta)}{4(a_2 - i\eta)}\right)^{(q-1)/2} \times K_{q-1}\left[\kappa\sqrt{(a_1 + i\zeta)(a_2 - i\eta)}\right], \quad (3.16)$$

where K_q is the modified Bessel function of the second kind. To carry out the final integration over κ, formula (6.576.3) in Gradshteyn and Ryzhik[39] is used, viz.,

$$\int_0^\infty dx \, x^{-\lambda} K_\mu(ax) J_\nu(bx) = \frac{b^\nu \Gamma((\nu - \lambda + \mu + 1)/2) \Gamma((\nu - \lambda - \mu + 1)/2)}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(\nu + 1)} \times F\left(\frac{\nu - \lambda + \mu + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2}, 1, -\frac{b^2}{a^2}\right). \quad (3.17)$$

This leads to the result

$$\Psi(\rho,\zeta,\eta)=\frac{\Gamma(q)}{4\pi}\left(\frac{(a_1+i\zeta)}{(a_2-i\eta)}\right)^{(q-1)/2}\times\frac{F(q,1,1,-\rho^2/[(a_1+i\zeta)(a_2-i\eta)])}{[(a_1+i\zeta)(a_2-i\eta)]^{(q+1)/2}},\tag{3.18}$$

where $F(q,1,1,-p)$ is the hypergeometric function. The latter has the property that

$$F(q,1,1,-p)=1/(1+p)^q .$$

Hence (3.18) takes the form

$$\Psi(\rho,\zeta,\eta)=\frac{\Gamma(q)}{4\pi(a_1+i\zeta)}\left[(a_2-i\eta)+\frac{\rho^2}{(a_1+i\zeta)}\right]^{-q},\tag{3.19}$$

which is identical to that for the splash pulse introduced in Ref. 2.

It is interesting to note in connection with Eq. (3.16) that the transverse components are separated from ζ and η. The portion of $\Psi(\rho,\zeta,\eta)$ depending on ζ and η only, viz.,

$$\tilde{\psi}(\kappa,\zeta,\eta)\equiv\pi\left(\frac{(a_1+i\zeta)}{4(a_2-i\eta)}\right)^{(q-1)/2}\times\kappa^{q-1}K_{q-1}\left[\kappa\sqrt{(a_1+i\zeta)(a_2-i\eta)}\right]\tag{3.20}$$

is a solution to the one-dimensional Klein–Gordon equation.

The scalar wave equation analog to Hillion's splash modes can easily be derived by choosing the spectrum

$$C_0(\alpha,\beta,\kappa)=(\pi/2)J_\nu(\beta b)e^{-\alpha a_1}.\tag{3.21}$$

In this case,

$$\Psi(\rho,\zeta,\eta)=\frac{1}{(2\pi)^2}\int_0^\infty d\kappa\int_0^\infty d\beta\int_0^\infty d\alpha\,\frac{\pi}{2}\times J_\nu(\beta b)e^{-\alpha a_1}\kappa J_0(\kappa\rho)e^{-i\alpha\zeta}e^{i\beta\eta}\delta\left[\alpha\beta-\frac{\kappa^2}{4}\right],\tag{3.22}$$

or

$$\Psi(\rho,\zeta,\eta)=\frac{1}{8\pi}\int_0^\infty d\kappa\int_0^\infty d\beta\,\frac{\kappa}{\beta}\times J_0(\kappa\rho)J_\nu(\beta b)e^{-\kappa^2(a_1+i\zeta)/4\beta}e^{i\beta\eta}$$

upon integrating over α. The integration over κ can be carried out using Eq. (3.3). One finds

$$\Psi(\rho,\zeta,\eta)=\frac{1}{4\pi}\int_0^\infty d\beta\,\frac{J_\nu(\beta b)}{(a_1+i\zeta)}e^{-\beta s},$$

where

$$s=\rho^2/(a_1+i\zeta)-i\eta .\tag{3.23}$$

Using relation (6.611.1) in Gradshteyn and Ryzhik,[39] viz.,

$$\int_0^\infty dx\,e^{-\alpha x}J_\nu(\beta x)=\frac{\beta^{-\nu}[\sqrt{\alpha^2+\beta^2}-\alpha]^\nu}{\sqrt{\alpha^2+\beta^2}},\tag{3.24}$$

$\Psi(\rho,\zeta,\eta)$ assumes, finally, the form

$$\Psi(\rho,\zeta,\eta)=\frac{1}{4\pi(a_1+i\zeta)}\,\frac{b^{-\nu}[\sqrt{s^2+b^2}-s]^\nu}{\sqrt{s^2+b^2}},\tag{3.25}$$

which is a solution to the three-dimensional scalar wave equation analogous to Hillion's spinors. The Bessel function $J_\nu(\beta b)$ entering into the (3.21) forms a complete orthogonal set. This means that any spectrum expressed as

$$C_0(\alpha,\beta,\kappa)=(\pi/2)F(\beta)e^{-\alpha a_1},$$

with

$$F(\beta)=\int_0^\infty db\,B(b)J_\nu(\beta b),$$

can result in the solution

$$\Psi(\rho,\zeta,\eta)=\frac{1}{4\pi(a_1+i\zeta)}\int_0^\infty db B(b)\,\frac{b^{-\nu}[\sqrt{s^2+b^2}-s]^\nu}{\sqrt{s^2+b^2}},\tag{3.26}$$

which is a generalization of Hillion's result.

The Fourier spectral content of Ziolkowski's splash pulse will be discussed in the next section in conjunction with the "modified power spectrum" (MPS) pulse. The Fourier picture corresponding to Hillion's solution can be obtained using the same procedure as in Sec. III A. Starting with the function

$$\tilde{\psi}(\kappa,\zeta,\eta)=\int_0^\infty d\beta\int_0^\infty d\alpha\,\frac{\pi}{2}J_\nu(\beta b)\times e^{-\alpha a_1}e^{-i\alpha\zeta}e^{i\beta\eta}\delta\left[\alpha\beta-\frac{\kappa^2}{4}\right],\tag{3.27}$$

the relationships given in Eq. (3.5) can be used to find the corresponding Fourier representation; specifically,

$$\tilde{\psi}(\kappa,z,t)=\int_0^\infty dk_z\int_{k_z}^\infty d\omega\,\frac{\pi}{2}J_\nu\left[\frac{b}{2}(\omega-k_z)\right]e^{-(a_1/2)(\omega+k_z)}\delta\left[\frac{\omega^2}{4}-\frac{k_z^2}{4}-\frac{\kappa^2}{4}\right]e^{-i(k_z z-\omega t)}$$
$$+\int_{-\infty}^0 dk_z\int_{-k_z}^\infty d\omega\,\frac{\pi}{2}J_\nu\left[\frac{b}{2}(\omega-k_z)\right]e^{-(a_1/2)(\omega+k_z)}\delta\left[\frac{\omega^2}{4}-\frac{k_z^2}{4}-\frac{\kappa^2}{4}\right]e^{-i(k_z z-\omega t)}.\tag{3.28}$$

By integrating over ω, it follows that

$$\tilde{\psi}(\kappa,z,t)=\int_0^\infty dk_z\,\frac{\pi}{\sqrt{k_z^2+\kappa^2}}J_\nu\left[\frac{b}{2}\left\{\sqrt{k_z^2+\kappa^2}-k_z\right\}\right]e^{-(a_1/2)\left[\sqrt{k_z^2+\kappa^2}+k_z\right]}e^{-i(k_z z-\overline{\omega}t)}$$
$$+\int_0^\infty dk_z\,\frac{\pi}{\sqrt{k_z^2+\kappa^2}}J_\nu\left[\frac{b}{2}\left\{\sqrt{k_z^2+\kappa^2}+k_z\right\}\right]e^{-(a_1/2)\left[\sqrt{k_z^2+\kappa^2}-k_z\right]}e^{+i(k_z z-\overline{\omega}t)}.\tag{3.29}$$

It is seen, then, that $\Psi(\mathbf{r},t)$ can be divided into two portions, $\Psi^+(\mathbf{r},t)$ and $\Psi^-(\mathbf{r},t)$, given by

$$\Psi^+(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^\infty d\kappa\,\kappa J_0(\kappa\rho)\int_0^\infty dk_z \times F(k_z,\kappa)e^{-i(k_z z-\overline{\omega}t)} \tag{3.30a}$$

and

$$\Psi^-(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^\infty d\kappa\,\kappa J_0(\kappa\rho)\int_0^\infty dk_z \times F(-k_z,\kappa)e^{i(k_z z+\overline{\omega}t)}, \tag{3.30b}$$

where

$$F(k_z,\kappa) = \frac{\pi}{\sqrt{k_z^2+\kappa^2}}J_\nu\left[\frac{b}{2}\{\sqrt{k_z^2+\kappa^2} + k_z\}\right]e^{-(a_1/2)[\sqrt{k_z^2+\kappa^2}+k_z]}. \tag{3.31}$$

Unlike the FWM's, the spectrum in this case is not singular. As in the case of the FWM's, however, the $\Psi^-(\mathbf{r},t)$ part will predominate for large values of the parameter a_1. On the other hand, the contributions from both parts of the spectrum are almost equal for small values of a_1. This can be seen from the ratio

$$\frac{F(k_z,\kappa)}{F(-k_z,\kappa)} = \frac{J_\nu\left[\frac{b}{2}\{\sqrt{k_z^2+\kappa^2}-k_z\}\right]}{J_\nu\left[\frac{b}{2}\{\sqrt{k_z^2+\kappa^2}+k_z\}\right]}e^{-a_1k_z}.$$

As indicated earlier,

$$F(k_z,\kappa) \ll F(-k_z,\kappa)$$

for large values of a_1. This is true for most of the frequency range contributing to the integrations (3.30a) and (3.30b). On the other hand, for a_1 very small,

$$F(k_z,\kappa) \simeq F(-k_z,\kappa)$$

for the most significant components of this spectrum.

C. EDEPT's

These solutions, which were first introduced by Ziolkowski,[8,9] have finite energy, are extremely localized and they are highly directive. Another important feature of these solutions is that they contain certain parameters that can be "tweaked up" so that a pulse is predominantly propagating in one direction. An interesting example of the EDEPT solutions is the MPS pulse which can be synthesized in the context of the bidirectional representation using the shifted spectrum

$$\begin{aligned} C_0(\alpha,\beta,\kappa) &= [\pi p/2\Gamma(q)](p\beta-b)^{q-1} \\ &\quad\times e^{-[\alpha a_1+(p\beta-b)a_2]}, \quad \beta > b/p, \\ &= 0, \quad b/p > \beta \geqslant 0. \end{aligned} \tag{3.32}$$

A substitution of this spectrum into (2.22) leads to the following solution:

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\int_0^\infty d\kappa\int_{b/p}^\infty d\beta\int_0^\infty d\alpha \times\frac{\pi p}{2\Gamma(q)}(p\beta-b)^{q-1}e^{-[\alpha a_1+(p\beta-b)a_2]} \times\kappa J_0(\kappa\rho)e^{-i\alpha\zeta}e^{i\beta\eta}\delta\left[\alpha\beta-\frac{\kappa^2}{4}\right]. \tag{3.33}$$

By integrating over α, it follows that

$$\Psi(\rho,\zeta,\eta) = \frac{1}{8\pi}\int_0^\infty d\kappa\int_{b/p}^\infty d\beta\,\kappa J_0(\kappa\rho)\frac{p(p\beta-b)^{q-1}}{\beta\Gamma(q)} \times e^{-\kappa^2(a_1+i\zeta)/4\beta}e^{-(p\beta-b)a_2}e^{i\beta\eta}.$$

The integration over κ can be performed by resorting to the change of variables $\beta' = \beta - b/p$, and making use of Eq. (3.3):

$$\Psi(\rho,\zeta,\eta) = \frac{1}{4\pi}\int_0^\infty d\beta'\frac{p^q\beta'^{q-1}}{\Gamma(q)} \times e^{-\beta'(s+pa_2)}\frac{e^{-bs/p}}{(a_1+i\zeta)}.$$

The integration over β' can be carried out explicitly, resulting in the wave function

$$\Psi(\rho,\zeta,\eta) = \frac{1}{4\pi(a_1+i\zeta)}\frac{e^{-bs/p}}{[a_2+s/p]^q}, \tag{3.34}$$

which is identical to the MPS pulse introduced by Ziolkowski.[8,9]

A detailed analysis of the behavior of the MPS has been presented elsewhere.[8,9] Our main interest, at this point, is to transfer (3.33) into the corresponding Fourier representation in order to study the contributions from the positive- and negative-going components of the solution. A procedure identical to that introduced earlier yields, in this case,

$$\tilde{\psi}(\kappa,z,t) = \int_0^\infty d\omega\int_{-\infty}^{+\infty}dk_z\frac{\pi p}{2\Gamma(q)}\left[\frac{p}{2}(\omega-k_z) - b\right]^{q-1}e^{-a_1(\omega+k_z)/2}e^{-a_2(p(\omega-k_z)/2-b)} \times\delta\left[\frac{\omega^2}{4}-\frac{k_z^2}{4}-\frac{\kappa^2}{4}\right]e^{-i(k_z z-\omega t)} \tag{3.35a}$$

for $\frac{1}{2}(\omega-k_z) > b/p$, and

$$\tilde{\psi}(\kappa,z,t) = 0 \tag{3.35b}$$

for $\frac{1}{2}(\omega+k_z) < b/p$. The indicated ranges in the ω,k_z plane can be seen clearly by referring to Fig. (3b). Carrying out the integration over ω changes (3.35) to

$$\tilde{\psi}(\kappa,z,t) = \int_{-\infty}^{+\infty}dk_z\frac{\pi p}{\Gamma(q)}\left[\frac{p}{2}\{\sqrt{k_z^2+\kappa^2}-k_z\} - b\right]^{q-1}e^{ba_2}e^{-a_1\{\sqrt{k_z^2+\kappa^2}+k_z\}/2} \times e^{-a_2p\{\sqrt{k_z^2+\kappa^2}-k_z\}/2}e^{-i(k_z z-\overline{\omega}t)}$$

for $\sqrt{k_z^2+\kappa^2}-k_z > 2b/p$, and

$$\tilde{\psi}(\kappa,z,t) = 0$$

for $\sqrt{k_z^2+\kappa^2}-k_z < 2b/p$. Solving for k_z and splitting $\tilde{\psi}(\kappa,z,t)$ into positive- and negative-going parts, results, finally, in the components

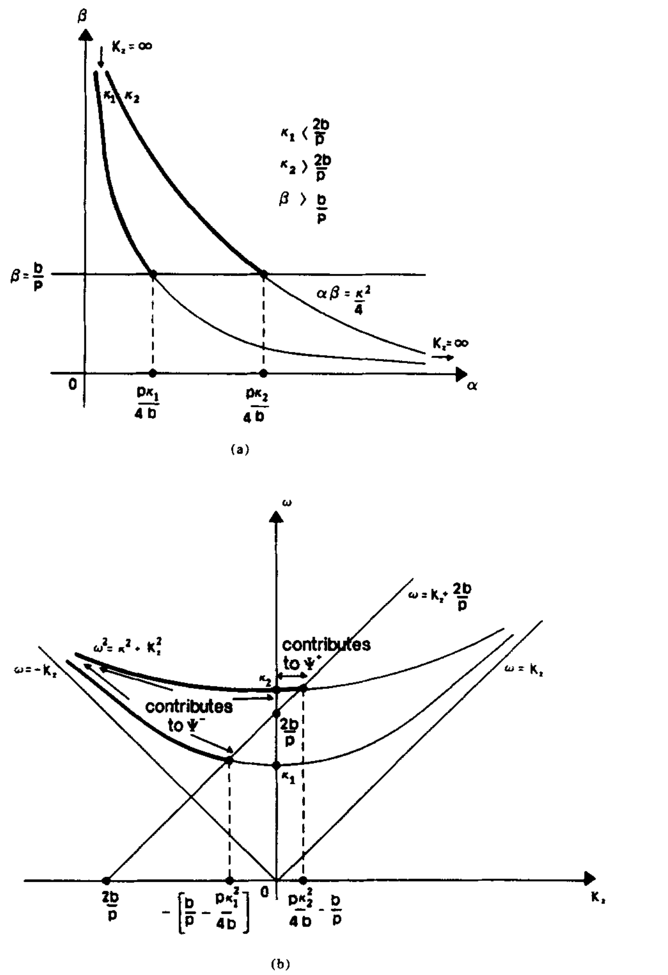

FIG. 3. The constraint $\alpha\beta = \kappa^2/4$ and the lower bound $\beta = b/p$ of the MPS pulse are shown in: (a) the $\alpha\beta$ plane and (b) the $k_z\omega$ plane. The contributions to Ψ^+ and Ψ^- are indicated by arrows.

$$\Psi^+(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_{2b/p}^{\infty} d\kappa\, \kappa J_0(\kappa\rho) \times \int_0^{\kappa^2 p/4b - b/p} dk_z\, F(k_z,\kappa) e^{-i(k_z z - \overline{\omega} t)}, \quad (3.36a)$$

$$\Psi^-(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^{2b/p} d\kappa\, \kappa J_0(\kappa\rho) \times \int_{b/p - \kappa^2 p/4b}^{\infty} dk_z\, F(-k_z,\kappa) e^{i(k_z z + \overline{\omega} t)} + \frac{1}{(2\pi)^2}\int_{2b/p}^{\infty} d\kappa\, \kappa J_0(\kappa\rho) \times \int_0^{\infty} dk_z\, F(-k_z,\kappa) e^{i(k_z z + \overline{\omega} t)}, \quad (3.36b)$$

where

$$F(k_z,\kappa) = \frac{\pi p}{\Gamma(q)}\left[\frac{p}{2}\left\{\sqrt{k_z^2+\kappa^2} - k_z\right\} - b\right]^{q-1} \times e^{ba_2} e^{-\sqrt{k_z^2+\kappa^2}(a_1 + a_2 p)/2} e^{-k_z(a_1 - a_2 p)/2}. \quad (3.37)$$

The strength of the MPS pulse arises from the introduced asymmetry in the positive- and negative-going components. This can be easily demonstrated by examining the ratio

$$F(k_z,\kappa)/F(-k_z,\kappa) = e^{(pa_2 - a_1)k_z}$$

for $q = 1$. By an appropriate choice of the parameters p, a_1, and a_2, the positive-going frequency components can be made much larger than the negative-going ones. This can be achieved by using large values of the product pa_2. It is also straightforward to demonstrate that in the limb $b \to 0$ and $p \to 1$ the MPS given in Eq. (3.34) is reduced to the splash pulse [cf. Eq. (3.19)]. Therefore, the Fourier spectral content of the splash pulse can be obtained directly from Eq. (3.36) by setting $b = 0$ and $p = 1$; specifically,

$$\Psi^+(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^{\infty} d\kappa\, \kappa J_0(\kappa\rho) \times \int_0^{\infty} dk_z\, F(k_z,\kappa) e^{-i(k_z z - \overline{\omega} t)}, \quad (3.38a)$$

$$\Psi^-(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^{\infty} d\kappa\, \kappa J_0(\kappa\rho) \times \int_0^{\infty} dk_z\, F(-k_z,\kappa) e^{i(k_z z + \overline{\omega} t)}, \quad (3.38b)$$

with

$$F(k_z,\kappa) = \pi\left[\tfrac{1}{2}\sqrt{k_z^2+\kappa^2} - k_z/2\right]^{q-1} \times e^{-\sqrt{k_z^2+\kappa^2}(a_1+a_2)/2} e^{-k_z(a_1 - a_2)/2}. \quad (3.39)$$

To compare $\Psi^+(\mathbf{r},t)$ to $\Psi^-(\mathbf{r},t)$, consider the following ratio for $q = 1$:

$$F(k_z,\kappa)/F(-k_z,\kappa) = e^{(a_2 - a_1)k_z}.$$

It is clear from this expression that one can have a predominantly positive component if a_2 is chosen to be large and a_1 very small. However, unlike the MPS pulse, the splash pulse is not localized in the transverse directions. This is due to the absence of the parameters b and p that provide some control over the transverse localization through the factor $\exp(-bs/p)$ in Eq. (3.34).

D. Bessel beams

The "Bessel beams" were introduced by Durnin[20] and, like Brittingham's FWM's, they are characterized by an infinite energy content. It is of interest that such beams have been realized experimentally,[21] primarily because of the manner in which the behavior of an infinite energy beam can be realized approximately. It is possible to show that these beams can be represented by the time-symmetric bidirectional superposition (2.22). One can choose, in this case,

$$C_0(\alpha,\beta,\kappa) = 4\pi\sigma\tau e^{-\sigma^2(\alpha+\beta-\omega_0)^2} e^{-\tau^2(\alpha-\beta-\lambda)^2}, \quad (3.40)$$

for which Eq. (2.22) specializes to

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\int_0^{\infty} d\kappa \int_0^{\infty} d\beta \int_0^{\infty} d\alpha \times 4\pi\sigma\tau e^{-\sigma^2(\alpha+\beta-\omega_0)^2} e^{-\tau^2(\alpha-\beta-\lambda)^2} \times \kappa J_0(\kappa\rho) e^{-i\alpha\zeta} e^{i\beta\eta} \delta\left[\alpha\beta - \frac{\kappa^2}{4}\right]. \quad (3.41)$$

Integrating first over κ, one has

$$\Psi(\rho,\zeta,\eta) = \frac{2\sigma\tau}{\pi}\int_0^\infty d\beta \int_0^\infty d\alpha\, e^{-\sigma^2(\alpha+\beta-\omega_0)^2} \times e^{-\tau^2(\alpha-\beta-\lambda)^2} J_0(2\sqrt{\alpha\beta}\,\rho)e^{-i\alpha\zeta}e^{i\beta\eta}. \quad (3.42)$$

This integration is very hard to evaluate exactly; nevertheless, an asymptotic solution can be obtained for large values of $\sigma\tau$. Without any loss of generality we can take $\sigma = \tau$ and Eq. (3.42) can be rewritten as follows:

$$\Psi(\rho,\zeta,\eta) = \frac{2\sigma^2}{\pi}\int_0^\infty d\beta \int_0^\infty d\alpha \times e^{-\sigma^2[\omega_0^2+\lambda^2-2(\lambda+\omega_0)\alpha-2(\omega_0-\lambda)\beta+2\alpha^2+2\beta^2]} \times J_0(2\sqrt{\alpha\beta}\,\rho)e^{-i\alpha\zeta}e^{i\beta\eta}. \quad (3.43)$$

This is a double integral of the Laplace type and can be evaluated asymptotically for large σ^2. Following Bleistein and Handelsman,[40] the function

$$\phi(\alpha,\beta) = -[\omega_0^2+\lambda^2-2(\lambda+\omega_0)\alpha - 2(\omega_0-\lambda)\beta+2\alpha^2+2\beta^2] \quad (3.44)$$

has critical points at $\phi_\alpha = \phi_\beta = 0$, or at

$$\alpha_0 = (\omega_0+\lambda)/2, \quad \beta_0 = (\omega_0-\lambda)/2. \quad (3.45)$$

Since $\phi_{\alpha\alpha} = \phi_{\beta\beta} = -4$ and $\phi_{\alpha\beta} = 0$, it follows that

$$\phi_{\alpha\alpha}(\alpha_0,\beta_0) < 0, \quad \phi_{\beta\beta}(\alpha_0,\beta_0) < 0,$$

$$\phi_{\alpha\alpha}(\alpha_0,\beta_0)\phi_{\beta\beta}(\alpha_0,\beta_0) - \phi_{\alpha\beta}^2(\alpha_0,\beta_0) = 16 > 0,$$

and the critical point given by (3.45) is a maximum. Hence the integration (3.43) can be approximated by

$$\Psi(\rho,\zeta,\eta) = \frac{2\pi\sigma^{-2}e^{\sigma^2\phi(\alpha_0,\beta_0)}}{\sqrt{\phi_{\alpha\alpha}(\alpha_0,\beta_0)\phi_{\beta\beta}(\alpha_0,\beta_0)-\phi_{\alpha\beta}^2(\alpha_0,\beta_0)}} \times \frac{2\sigma^2}{\pi}J_0(2\sqrt{\alpha_0\beta_0}\rho)e^{-i\alpha_0\zeta}e^{i\beta_0\eta} + O(\sigma^{-2}).$$

Rearranging the terms and using Eq. (3.45), one gets

$$\Psi(\rho,\zeta,\eta) = J_0\left[\sqrt{\omega_0^2-\lambda^2}\,\rho\right] \times e^{-i\lambda(\eta+\zeta)/2}e^{i\omega_0(\eta-\zeta)/2} + O(\sigma^{-2}), \quad (3.46)$$

which in the limit $\sigma\to\infty$ reduces to

$$\Psi(\rho,\zeta,\eta) = J_0\left[\sqrt{\omega_0^2-\lambda^2}\,\rho\right] e^{-i\lambda(\eta+\zeta)/2}e^{i\omega_0(\eta-\zeta)/2}. \quad (3.47a)$$

Although the wave function given in (3.47a) was obtained from the asymptotic evaluation of the double integration (3.43), it turns out to be an exact solution to the scalar wave equation. In fact, it is same as Durnin's Bessel beam, which can be obtained by substituting

$$\zeta = z - t, \quad \eta = z + t$$

into Eq. (3.47a) and rewriting it as follows:

$$\Psi(\rho,\zeta,\eta) = J_0\left[\sqrt{\omega_0^2-\lambda^2}\,\rho\right] e^{-i(\lambda z-\omega_0 t)}. \quad (3.47b)$$

As in the case of Brittingham's FWM's, the spectrum associated with a Bessel beam is singular; specifically, the spectrum given in (3.40) reduces to a product of two Dirac delta functions as σ goes to ∞. On the other hand, the conversion to a Fourier picture is trivial in this case since (3.47b) is totally traveling in the positive z direction.

IV. EXTENSIONS OF THE BIDIRECTIONAL SYNTHESIS

In this section, we shall extend the ideas discussed in Sec. II to other classes of problems. The most natural extension is an application involving the three-dimensional Klein–Gordon equation which describes the propagation of waves in a dispersive medium. Another one deals with the use of the bidirectional representation in connection with dissipative problems modeled, for example, by the three-dimensional dissipative scalar wave equation and the telegraph equation; in these cases the operator $\hat{\Omega}(-i\nabla)$ is nonpositive. Using these two classes of problems, we shall show that solutions obtained via the bidirectional representation will be as easy to evaluate asymptotically or numerically as those for the three-dimensional wave equation. By virtue of this observation, new exact solutions can be obtained trivially using spectra similar to those in Sec. III.

The spectral analysis in Sec. II was carried out over the β-dependent part of the integral in Eq. (2.27). We were led to this procedure because the three-dimensional wave equation has the same structure as the one-dimensional Klein–Gordon equation. In particular, a Fourier transformation with respect to the transverse coordinates x and y reduces the three-dimensional wave equation to a one-dimensional Klein–Gordon equation of the following form:

$$[\partial_t^2 - \partial_z^2 + \kappa^2]\tilde{u}(\boldsymbol{\kappa},z,t) = 0. \quad (4.1)$$

It should be observed that the functions $[I_n(\kappa z)J_m(\kappa t)]$ in the expression (2.32a) are not solutions to the one-dimensional Klein–Gordon equation. Only their sums over integer values of m and n constitute a solution to (4.1) and a delicate balance between the coefficients of $[I_n(\kappa z)J_m(\kappa t)]$ must be maintained in order to give finite solutions.

A natural extension is the three-dimensional Klein–Gordon equation describing the evolution of a signal propagating in dispersive media. For this case, the operator $\hat{\Omega}(-i\nabla)$ equals $-\nabla^2+\mu^2$ and Eq. (2.1a) takes the form

$$[\partial_t^2 - \nabla^2 + \mu^2]u(\mathbf{r},t) = 0. \quad (4.2)$$

A general solution to this equation is analogous to that given by (2.18), namely,

$$u(\mathbf{r},t) = 2\,\mathrm{Re}\{\Psi(\mathbf{r},t)\},$$

where $\Psi(\mathbf{r},t)$ can be represented by the following bidirectional superposition:

$$\Psi(\rho,\zeta,\eta) = \frac{1}{(2\pi)^2}\sum_{\substack{l=-1\\ l\neq 0}}^{+1}\sum_{n=0}^{\infty}\int_0^\infty d\kappa \int_0^\infty d(l\alpha)\int_0^\infty \times d(l\beta)C_n(l\alpha,l\beta,\kappa)\kappa J_n(\kappa\rho) \times e^{\pm in\phi}e^{-il\alpha\zeta}e^{il\beta\eta}\delta\left[\alpha\beta - \frac{1}{4}(\kappa^2+\mu^2)\right]. \quad (4.3)$$

In this case, a partitioning of $\hat{\Omega}(-i\nabla)$ was induced through the operators $\hat{A}(-i\partial_z) = -\partial_z^2$, $\hat{B}(-\boldsymbol{\kappa}-i\partial_z) = \kappa^2+\mu^2$, and the new constraint relation is given by

$$\alpha\beta=\tfrac{1}{4}(\kappa^2+\mu^2)\,. \tag{4.4}$$

Using the relationship (2.23), the representation (4.3) can be transformed into the conventional Fourier picture; specifically,

$$\Psi(\mathbf{r},t)=\frac{1}{(2\pi)^2}\sum_{n=0}^{\infty}\int_0^{\infty}d\kappa\int_{-\infty}^{+\infty}d\omega \times\int_{-\infty}^{+\infty}dk_z\,A_n(\omega,k_z,\kappa)\kappa J_n(\kappa\rho)e^{\pm in\phi} \times e^{-ik_z z}e^{+i\omega t}\delta(\omega^2-\kappa^2-k_z^2-\mu^2)\,. \tag{4.5}$$

The only difference between (4.5) and (2.18) is the more complicated constraint relationship. For the problem under consideration, the constraint requires that

$$\omega^2-\kappa^2-k_z^2=\mu^2\,, \tag{4.6}$$

which recovers the well known energy relation $E^2=p^2+\mu^2$. Recall that very few exact solutions to the three-dimensional Klein–Gordon equation are available. In this sense, the representation (4.3) is very valuable because it is characterized by the same algebraic singularities as (2.22). As a consequence, (4.3) allows the analytical computation of a rich class of novel exact and approximate solutions with as much facility as shown in Sec. III for the three-dimensional scalar wave equation. For example, all the spectra used in Sec. III can be used trivially to reproduce new solutions to the three-dimensional Klein–Gordon equation.

For physical situations requiring a nonpositive operator $\widehat{\Omega}(-i\nabla)$, e.g., those modeled by the dissipative scalar wave equation and the telegraph equation, one can still obtain novel, exact solutions using the bidirectional synthesis procedure. Along these lines, consider the three-dimensional dissipative scalar wave equation

$$[\partial_t^2-\nabla^2+(c_1+c_2)\partial_t+c_1c_2]\Psi(\mathbf{r},t)=0\,, \tag{4.7}$$

which describes a wave traveling in a dissipative medium. Although Eq. (4.7) has a different structure than Eq. (2.1), an exponential transformation of the form

$$\Psi(\mathbf{r},t)=\exp[-\tfrac{1}{2}(c_1+c_2)t]\widehat{\Psi}(\mathbf{r},t) \tag{4.8}$$

reduces it to

$$[\partial_t^2-\nabla^2-\tfrac{1}{2}(c_1-c_2)^2]\widehat{\Psi}(\mathbf{r},t)=0\,, \tag{4.9}$$

which is a special case of Eq. (2.1) with $\widehat{\Omega}(-i\nabla) = -\nabla^2-(c_1-c_2)^2/2$. Notice that the above equation is similar to the Klein–Gordon equation (4.2) with an imaginary μ [i.e., $\mu^2=-(c_1-c_2)^2/2$]. The bidirectional representation can be written directly as

$$\widehat{\Psi}(\rho,\zeta,\eta)=\frac{1}{(2\pi)}\sum_{\substack{l=-1\\ l\neq 0}}^{+1}\sum_{n=0}^{\infty}\int_0^{\infty}d\kappa\int_0^{\infty}d(l\alpha) \times\int_{-\infty}^{0}d(l\beta)\,C_n(l\alpha,l\beta,\kappa)\kappa J_n(\kappa\rho) \times e^{\pm in\phi}e^{-il\alpha\zeta}e^{il\beta\eta}\delta\Big[\alpha\beta +\frac{1}{8}(c_1-c_2)^2-\frac{\kappa^2}{4}\Big], \tag{4.10}$$

with the constraint

$$\alpha\beta=-\tfrac{1}{8}(c_1-c_2)^2+\kappa^2/4\,. \tag{4.11}$$

The same discussion concerning the nature of the singularities of this solution follows automatically, except for the fact that the hyperbolic constraint (4.11) can lie in the second or the fourth quadrants of the $\alpha\beta$ plane for $(c_1-c_2)^2/2>\kappa^2$, and in the first or third quadrants for $(c_1-c_2)^2/2<\kappa^2$. This also explains the difference in the α and β limits of the integration in Eq. (4.10).

The dissipative wave equation can be reduced to the telegraph equation by removing the dependence of Eq. (4.7) on the transverse coordinates x and y. The telegraph equation, which can be written as

$$[\partial_t^2-\partial_z^2+(c_1+c_2)\partial_t+c_1c_2]\psi(z,t)=0\,, \tag{4.12}$$

models the transmission of electromagnetic signals through wire cables. Using an exponential transformation of the form

$$\psi(z,t)=\exp[-\tfrac{1}{2}(c_1+c_2)t]\,\hat{\psi}(z,t)\,, \tag{4.13}$$

reduces the telegraph equation to

$$[\partial_t^2-\partial_z^2-\tfrac{1}{2}(c_1-c_2)^2]\,\hat{\psi}(z,t)=0\,. \tag{4.14}$$

A celebrated solution due to Lord Kelvin involves the choice $c_1=c_2$. This restriction reduces Eq. (4.14) to a one-dimensional scalar wave equation that has the distortion-free solutions $\hat{\psi}(z-t)$ and $\hat{\psi}(z+t)$. In an attempt to find solutions to Eq. (4.14) in the general case where $c_1\neq c_2$, one runs into the same complications as those discussed earlier in connection to the Fourier representation of the one-dimensional Klein–Gordon equation, or the three-dimensional scalar wave equation. An alternative is to use the bidirectional representation

$$\hat{\psi}(\zeta,\eta)=\frac{1}{(2\pi)}\sum_{\substack{l=-1\\ l\neq 0}}^{+1}\int_0^{\infty}d(l\alpha)\int_{-\infty}^{0}d(l\beta)\,c_0(\alpha,\beta) \times e^{-il\alpha\zeta}e^{il\beta\eta}\delta\Big[\alpha\beta+\frac{1}{8}(c_1-c_2)^2\Big], \tag{4.15}$$

with the constraint

$$\alpha\beta=-\tfrac{1}{8}(c_1-c_2)^2\,. \tag{4.16}$$

(Only the second or the fourth quadrants of the $\alpha\beta$ plane need be used in this case since α and β must have different signs.) The Fourier synthesis corresponding to (4.15) can be obtained by using the transformation (2.23). This leads to

$$\hat{\psi}(z,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}dk_z\,A_n(\omega,k_z) \times e^{-ik_z z}e^{i\omega t}\delta\Big(\omega^2-k_z^2+\frac{1}{2}(c_1-c_2)^2\Big), \tag{4.17}$$

with the constraint

$$\omega^2-k_z^2=-\tfrac{1}{2}(c_1-c_2)^2\,. \tag{4.18}$$

One should not get the wrong impression that the method introduced in this paper will replace the Fourier synthesis; on the contrary, the bidirectional synthesis complements it. As it was shown in Eq. (2.17), a Fourier decomposition is simply a special case of a general partitioning of the operator L. In many instances dealing with single frequency phenomena, the Fourier synthesis is the most intuitive one; however, this does not rule out all other representations, particularly when they can lead to new exact solutions. Consider, for instance, initial value problems. Even though the bidirectional representation is characterized by an implicit dependence on time through the variables ζ and η, an initial value problem can still be handled successfully. As an example, consider the[AV: specific problem of pulse propagation through an infinitely long cylindrical waveguide. This problem is modeled by the three-dimensional scalar wave equation

$$(\nabla^2 - \partial_t^2) u(\mathbf{r},t) = 0 ,$$

with the initial conditions

$$u(\mathbf{r},0) = F(\rho,z) , \tag{4.19a}$$

$$u_t(\mathbf{r},0) = G(\rho,z) , \tag{4.19b}$$

and the boundary condition

$$u(R,z,t) = 0 , \tag{4.19c}$$

where $\rho = R$ is the radius of the cross section of the waveguide. The functions $F(\rho,z)$ and $G(\rho,z)$ are assumed to be real. For this problem, it is advantageous to begin with the expression (2.27). A typical solution can then be written as

$$u(\mathbf{r},t) \equiv u(\rho,\zeta,\eta) = c_0(\kappa,\beta) J_0(\kappa\rho) e^{-(i\kappa^2/4\beta)\zeta} e^{i\beta\eta} . \tag{4.20}$$

Applying the boundary condition (4.19c), one obtains $J_0(\kappa R) = 0$. It immediately follows that $\kappa R = \kappa_{0m}$, where κ_{0m} are the zeros of the zeroth-order Bessel function. By summing over all modes and integrating over β, the general waveguide solution can be given as

$$u(\rho,\zeta,\eta) = \mathrm{Re} \sum_{m=1}^{\infty} \int_0^{\infty} d\beta\, c_0(\kappa_{0m},\beta) \times J_0\left(\frac{\kappa_{0m}\rho}{R}\right) e^{-i(\kappa_{0m}^2/4\beta R^2)\zeta} e^{i\beta\eta} . \tag{4.21}$$

The initial condition (4.19a) is satisfied if

$$F(\rho,z) = \mathrm{Re} \sum_{m=1}^{\infty} \int_0^{\infty} d\beta\, c_0(\kappa_{0m},\beta) \times J_0\left(\frac{\kappa_{0m}\rho}{R}\right) e^{-i(\kappa_{0m}^2/4\beta R^2 - \beta)z} . \tag{4.22}$$

The spectrum $c_0(\kappa_{0m},\beta)$, which is, in general, a complex function of κ_{0m} and β, can be determined by taking first the Fourier transform with respect to z and then the Hankel transform with respect to ρ in Eq. (4.22). This gives

$$f(\kappa_{0m},k_z) = \int_0^{\infty} d\beta \frac{R^2}{2} [J_1(\kappa_{0m})]^2 \int_{-\infty}^{+\infty} \frac{dz}{2} \times \left[c_0(\kappa_{0m},\beta) e^{-i(\kappa_{0m}^2/4\beta R^2 - \beta - k_z)z} + c_0^*(\kappa_{0m},\beta) e^{i(\kappa_{0m}^2/4\beta R^2 - \beta + k_z)z} \right] , \tag{4.23}$$

where $c_0^*(\kappa_{0m},\beta)$ is the Hermitian conjugate of $c_0(\kappa_{0m},\beta)$, and $f(\kappa_{0m},k_z)$ is defined as

$$f(\kappa_{0m},k_z) = \int_{-\infty}^{+\infty} dz \int_{\infty}^{R} d\rho\, \rho J_0\left(\frac{\kappa_{0m}\rho}{R}\right) F(\rho,z) e^{+ik_z z} . \tag{4.24}$$

By integrating the right-hand side of Eq. (4.23) over z, it follows that

$$f(\kappa_{0m},k_z) = \frac{\pi}{2} R^2 [J_1(\kappa_{0m})]^2 \int_0^{\infty} d\beta \times \left[c_0(\kappa_{0m},\beta) \delta\left(\frac{\kappa_{0m}^2}{4\beta R^2} - \beta - k_z\right) + c_0^*(\kappa_{0m},\beta) \delta\left(\frac{\kappa_{0m}^2}{4\beta R^2} - \beta + k_z\right) \right] . \tag{4.25}$$

By performing, finally, the integration over β, the following relation is obtained:

$$\frac{c_0(\kappa_{0m},\beta_1)}{\sqrt{k_z^2 + (\kappa_{0m}/R)^2}} \beta_1 + \frac{c_0^*(\kappa_{0m},\beta_2)}{\sqrt{k_z^2 + (\kappa_{0m}/R)^2}} \beta_2 = \frac{2f(\kappa_{0m},k_z)}{\pi R^2 [J_1(\kappa_{0m})]^2} . \tag{4.26}$$

Here,

$$\beta_1 = \tfrac{1}{2}\left[-k_z + \sqrt{(k_z^2 + (\kappa_{0m}/R)^2)} \right] ,$$

$$\beta_2 = \tfrac{1}{2}\left[+k_z + \sqrt{(k_z^2 + (\kappa_{0m}/R)^2)} \right] .$$

It turns out that the initial condition (4.19b) is satisfied if

$$c_0(\kappa_{0m},\beta_1)\beta_1 - c_0^*(\kappa_{0m},\beta_2)\beta_2 = 2g(\kappa_{0m},k_z)/\pi R^2 [J_1(\kappa_{0m})]^2 , \tag{4.27}$$

where

$$g(\kappa_{0m},k_z) = \int_{-\infty}^{+\infty} dz \int_{\infty}^{R} d\rho\, \rho J_0\left(\frac{\kappa_{0m}\rho}{R}\right) G(\rho,z) e^{+ik_z z} . \tag{4.28}$$

A combination of Eqs. (4.26) and (4.27) results in the spectrum

$$c_0(\kappa_{0m},\beta_1) = \frac{1}{\beta_1 \pi R^2 [J_1(\kappa_{0m})]^2} \left(f(\kappa_{0m},k_z) \times \sqrt{k_z^2 + \left(\frac{\kappa_{0m}}{R}\right)^2} + g(\kappa_{0m},k_z) \right) . \tag{4.29}$$

The relation $\beta_1 \equiv \beta(k_z)$ must be inverted in order to obtain $k_z \equiv k_z(\beta)$. Eq. (4.29) can be written, then, as

$$c_0(\kappa_{0m},\beta) = \frac{1}{\beta \pi R^2 [J_1(\kappa_{0m})]^2} \left(f(\kappa_{0m},k_z(\beta)) \times \sqrt{k_z^2(\beta) + \left(\frac{\kappa_{0m}}{R}\right)^2} + g(\kappa_{0m},k_z(\beta)) \right)$$

and the solution to the original problem can be expressed as

$$u(\rho,\zeta,\eta) = \mathrm{Re} \sum_{m=1}^{\infty} \frac{1}{\pi R^2 [J_1(\kappa_{0m})]^2} \times \int_0^{\infty} d\beta\, e^{-i(\kappa_{0m}^2/4\beta R^2)\zeta} e^{i\beta\eta} J_0\left(\frac{\kappa_{0m}\rho}{R}\right) \times \frac{1}{\beta} \left(f(\kappa_{0m},k_z(\beta)) \sqrt{k_z^2(\beta) + \left(\frac{\kappa_{0m}}{R}\right)^2} + g(\kappa_{0m},k_z(\beta)) \right) , \tag{4.30}$$

in terms of a superposition of the elementary blocks $e^{-i\alpha\zeta}e^{i\beta\eta}$. Obviously, the shape of the field $u(\rho,\zeta,\eta)$ depends on the choice of $F(\rho,z)$ and $G(\rho,z)$. If, for example, $F(\rho,z)$ is chosen in the separable form

$$F(\rho,z) = F_1(\rho)F_2(z) ,$$

and

$$G(\rho,z) = 0 ,$$

Eq. (4.30) can be rewritten as

$$u(\rho,\zeta,\eta) = \mathrm{Re} \sum_{m=1}^{\infty} \frac{1}{\pi R^2[J_1(\kappa_{0m})]^2} \times \int_0^\infty d\beta \frac{\hat{F}_2(k_z(\beta))}{\beta} \times \left[k_z^2(\beta) + \left(\frac{\kappa_{0m}}{R}\right)^2\right]^{1/2} e^{-i(\kappa_{0m}^2/4\beta R^2)\zeta} \times e^{i\beta\eta} J_0\left(\frac{\kappa_{0m}\rho}{R}\right) \int_0^R d\rho'\,\rho' F_1(\rho') J_0\left(\frac{\kappa_{0m}}{R}\rho'\right), \tag{4.31}$$

where $\hat{F}_2(k_z)$ is the Fourier transform of $F_2(z)$. To be more specific, let

$$F_1(\rho) = (1/4\pi)J_0(\kappa_{0m}\rho/R) ,$$

$$F_2(z) = K_0[(\kappa_{0m}/R)\sqrt{a^2+z^2}\,] ,$$

where K_0 is the zeroth-order modified Bessel function of the second kind. The initial conditions in this case have the form

$$u(\mathbf{r},0) = \frac{1}{4\pi} J_0\left(\frac{\kappa_{0m}\rho}{R}\right) K_0\left[\frac{\kappa_{0m}}{R}\sqrt{a^2+z^2}\right], \tag{4.32a}$$

$$u_t(\mathbf{r},0) = 0 . \tag{4.32b}$$

The Fourier transform of the function $F_2(z)$, required in Eq. (4.31), is given in this case by

$$\hat{F}_2(k_z) = \frac{\pi}{\sqrt{k_z^2 + \left(\frac{\kappa_{0m}}{R}\right)^2}} e^{-a\sqrt{k_z^2 + (\kappa_{0m}/R)^2}} . \tag{4.33}$$

The expression for the root β_1, viz.,

$$(2\beta - k_z)^2 = k_z^2 + \kappa_{0m}^2/R^2 ,$$

can be used to invert $k_z = k_z(\beta)$. This leads to the relations

$$k_z = +\kappa_{0m}^2/4\beta R^2 - \beta , \tag{4.34}$$

$$k_z^2 + \kappa_{0m}^2/R^2 = (\kappa_{0m}^2/4\beta R^2 + \beta)^2 . \tag{4.35}$$

Equations (4.33) and (4.35) can be used in conjunction with (4.31) to obtain

$$u(\rho,\zeta,\eta) = \mathrm{Re} \sum_{m=1}^{\infty} \frac{J_0(\kappa_{0m}\rho/R)}{\pi R^2[J_1(\kappa_{0m})]^2} \int_0^\infty d\beta \frac{\pi}{\beta} \times e^{-(\kappa_{0m}^2/4\beta R^2+\beta)a} e^{-i(\kappa_{0m}^2/4\beta R^2)\zeta} e^{i\beta\eta} \times \int_0^R d\rho' \frac{\rho'}{4\pi} J_0\left(\frac{\kappa_{0m}}{R}\rho'\right) J_0\left(\frac{\kappa_{0m}}{R}\rho'\right). \tag{4.36}$$

By integrating over ρ', Eq. (4.36) simplifies to

$$u(\rho,\zeta,\eta) = \mathrm{Re}\left\{\int_0^\infty \frac{d\beta}{8\pi\beta} \times e^{-(\kappa_{0m}^2/4\beta R^2)(a+i\zeta)} e^{-\beta(a-i\eta)} J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\right\}. \tag{4.37}$$

The remaining integration over β can be carried out using equation (3.478.4) in Gradshteyn and Ryzhik.[39] The solution to the initial boundary value problem under consideration then assumes the form

$$u(\mathbf{r},t) = \mathrm{Re}\left\{\frac{1}{4\pi} K_0\left[\frac{\kappa_{0m}}{R} \times \sqrt{(a+i\zeta)(a-i\eta)}\right] J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\right\}. \tag{4.38}$$

An interesting variation of the solution given in (4.38) can be obtained by simply replacing the single parameter a by two parameters a_1 and a_2, namely,

$$u(\mathbf{r},t) = \mathrm{Re}\left\{\frac{1}{4\pi} K_0\left[\frac{\kappa_{0m}}{R} \times \sqrt{(a_1+i\zeta)(a_2-i\eta)}\right] J_0\left(\frac{\kappa_{0m}\rho}{R}\right)\right\}. \tag{4.39}$$

Because of the asymmetric dependence on the values of a_1 and a_2, the pulse given in Eq. (4.39) can be made to travel mainly in one direction. On the other hand, the solution (4.38) represents a pulse that will split into two halves propagating in opposite directions. Such claims can be verified by referring to Eqs. (3.38) and (3.39) which give the positive-going and the negative-going components of the splash pulse. The only difference entails the replacement of the integration over κ by a summation over κ_{0m}. The backward and forward spectra have the following ratio:

$$F(k_z,\kappa_{0m})/F(-k_z,\kappa_{0m}) = e^{(a_2-a_1)k_z}.$$

It is seen that for $a_2 = a_1$, the positive and the negative components have the same strength. On the other hand, for $a_2 \gg 1$ and $a_1 \ll 1$, $F(k_z, \kappa_{0m}) \gg F(-k_z, \kappa_{0m})$ and the pulse moves predominantly in the positive z direction. Moreover, in contradistinction to the splash pulse (3.19), the solution (4.39) is localized by the walls of the waveguide and one does not have to worry about its localization in the transverse direction.

V. CONCLUDING REMARKS

A novel bidirectional decomposition of solutions to partial differential equations into backward and forward traveling plane waves was introduced in this paper. This technique is distinct from other factorization methods available in the literature (cf., e.g., Ref. 41). The main difference stems from the fact that it involves a product of plane waves propagating in opposite directions, while usual factorization techniques decompose the solutions into a sum of forward and backward traveling plane waves. The bidirectional decomposition, which was developed within the framework of a more general embedding procedure, allows the construction of general solutions by means of a superposition of elementary bidirectional blocks. Such a novel superposition differs significantly from the more conventional ones, e.g., the Fourier synthesis. In particular, it is characterized by algebraic singularities that can be much easier to handle than the branch-cut singularities arising usually in the Fourier synthesis. In spite of these differences, there is a one-to-one correspondence between the new synthesis and the Fourier method and these two methods complement each other.

Several mathematical aspects of the new synthesis were addressed. It was shown that the elementary blocks entering into this superposition are composed of exponential and Bessel functions which form complete sets of orthogonal functions. This led to an inversion formula, from which different spectra can be calculated from the knowledge of exact solutions. Necessary conditions for the choice of the spectra leading to convergent solutions were discussed.

The bidirectional decomposition was applied to the three-dimensional scalar wave equation, the three-dimensional Klein–Gordon equation, the three-dimensional dissipative wave equation, and the telegraph equation. For all these equations, it was demonstrated that new, exact solutions can be easily obtained. It was noted that the new synthesis provides the most natural basis for the construction of the unusual Brittingham-like solutions and that it can be used as a vehicle to find the Fourier spectral content of these solutions in order to gain a better understanding of their properties. Finally, it was shown that initial-boundary value problems can be solved using the bidirectional decomposition. A specific example was worked out in connection to an infinite waveguide and new solutions [cf. Eqs. (4.38) and (4.39)] were derived. These pulse solutions, especially (4.39), exhibit unusual decay patterns as they propagate down the waveguide. A detailed analysis of their properties has been published elsewhere.[42] A natural extension of this problem is the case of the open waveguide whose aperture is illuminated by the pulse given in Eq. (4.39). The solution to this problem is incorporated in Ref. 42.

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