A BINARY ADDITIVE EQUATION INVOLVING FRACTIONAL POWERS

ANGEL V. KUMCHEV

1. Introduction

It is well-known that the number of integers $n \le x$ that can be expressed as sums of two squares is $O\big(x(\log x)^{-1/2}\big)$. On the other hand, Deshouillers [2] showed that when $1 < c < \frac{4}{3}$, every sufficiently large integer n can be represented in the form

$$[m_1^c] + [m_2^c] = n, \tag{1}$$

with integers m_1, m_2; henceforth, $[\theta]$ denotes the integral part of θ. Subsequently, the range for c in this result was extended by Gritsenko [3] and Konyagin [5]. In particular, the latter author showed that (1) has solutions in integers m_1, m_2 for $1 < c < \frac{3}{2}$ and n sufficiently large.

The analogous problem with prime variables is considerably more difficult, possibly at least as difficult as the binary Goldbach problem. The only progress in that direction is a result of Laporta [6], which states that if $1 < c < \frac{17}{16}$, then almost all n (in the sense usually used in analytic number theory) can be represented in the form (1) with primes m_1, m_2. Recently, Balanzario, Garaev and Zuazua [1] considered the equation

$$[m^c] + [p^c] = n, \tag{2}$$

where p is a prime number and m is an integer. They showed that when $1 < c < \frac{17}{11}$, this hybrid problem can be solved for almost all n. It should be noted that in regard to the range of c, this result goes even beyond Konyagin's. On the other hand, when c is close to 1, one may hope to solve (2) for all sufficiently large n, since the problem is trivial when $c = 1$. The main purpose of the present note is to address this issue. We establish the following theorem.

Theorem 1. *Suppose that $1 < c < \frac{16}{15}$. Then every sufficiently large integer n can be represented in the form* (2).

The main new idea in the proof of this theorem is to translate the additive equation (2) into a problem about Diophantine approximation. The same idea enables us to give also a simple proof of a slightly weaker version of the result of Balanzario, Garaev and Zuazua. For $x \ge 2$, let $E_c(x)$ denote the number of integers $n \le x$ that cannot be represented in the form (2). We prove the following theorem.

Theorem 2. *Suppose that $1 < c < \frac{3}{2}$ and $\varepsilon > 0$. Then*

$$E_c(x) \ll x^{3(1-1/c)+\varepsilon}.$$

Date: May 5, 2006.

We remark that Theorem 1 is hardly best possible. It is likely that more sophisticated exponential sum estimates and/or sieve techniques would have allowed us to extend the range of c. The resulting improvement, however, would have been minuscule; thus, we decided not to pursue such ideas.

Notation. Most of our notation is standard. We use Landau's O-notation, Vinogradov's $\ll$-symbol, and occasionally, we write $A \asymp B$ instead of $A \ll B \ll A$. We also write $\{\theta\}$ for the fractional part of θ and $\|\theta\|$ for the distance from θ to the nearest integer. Finally, we define $e(\theta) = \exp(2\pi i\theta)$.

2. Proof of Theorem 1: initial stage

In this section, we only assume that $1 < c < 2$. We write $\gamma = 1/c$ and set

$$X = \left(\tfrac{1}{2}n\right)^{\gamma}, \quad X_1 = \tfrac{5}{4}X, \quad \delta = \gamma X^{1-c}. \tag{3}$$

If n is sufficiently large, it has at most one representation of the form (2) with $X < p \le X_1$. Furthermore, such a representation exists if and only if there is an integer m satisfying the inequality

$$\big(n - [p^c]\big)^{\gamma} \le m < \big(n + 1 - [p^c]\big)^{\gamma}. \tag{4}$$

We now proceed to show that such an integer exists, if p satisfies the conditions

$$X < p \le X_1, \quad \{p^c\} < \tfrac{1}{2}, \quad 1 - \tfrac{5}{6}\delta < \big\{\big(n - p^c\big)^{\gamma}\big\} < 1 - \tfrac{2}{3}\delta. \tag{5}$$

Under these assumptions, one has

$$X^{1-c} = (n - X^c)^{\gamma-1} < \big(n - p^c\big)^{\gamma-1} \le (n - X_1^c)^{\gamma-1} < 1.1X^{1-c}.$$

Hence,

$$\begin{aligned}
\big(n - [p^c]\big)^{\gamma} &= \big(n - p^c\big)^{\gamma}\Big(1 + \gamma\{p^c\}\big(n - p^c\big)^{-1} + O\big(n^{-2}\big)\Big)\\
&< \big(n - p^c\big)^{\gamma} + \tfrac{1}{2}\gamma\big(n - p^c\big)^{\gamma-1} + O\big(n^{\gamma-2}\big)\\
&< \big(n - p^c\big)^{\gamma} + 0.55\delta + O\big(\delta n^{-1}\big)\\
&< \big[\big(n - p^c\big)^{\gamma}\big] + 1 - 0.1\delta,
\end{aligned}$$

and

$$\begin{aligned}
\big(n + 1 - [p^c]\big)^{\gamma} &= \big(n - p^c\big)^{\gamma}\Big(1 + \gamma(1 + \{p^c\})\big(n - p^c\big)^{-1} + O\big(n^{-2}\big)\Big)\\
&\ge \big(n - p^c\big)^{\gamma} + \gamma\big(n - p^c\big)^{\gamma-1} + O\big(n^{\gamma-2}\big)\\
&> \big(n - p^c\big)^{\gamma} + \delta + O\big(\delta n^{-1}\big)\\
&> \big[\big(n - p^c\big)^{\gamma}\big] + 1 + 0.1\delta.
\end{aligned}$$

Consequently, conditions (5) are indeed sufficient for the existence of an integer m satisfying (4). It remains to show that there exist primes satisfying the inequalities in (5). To this end, it suffices to show that

$$\sum_{X < p \le X_1} \Phi\big(p^c\big)\Psi\big((n - p^c)^{\gamma}\big) > 0 \tag{6}$$

for some smooth, non-negative, 1-periodic functions Φ and Ψ such that Φ is supported in $(0, 1/2)$ and Ψ is supported in $(1 - \frac{5}{6}\delta, 1 - \frac{2}{3}\delta)$.

Let ψ_0 be a non-negative C^∞-function that is supported in $[0,1]$ and is normalized in L^1: $\|\psi_0\|_1 = 1$. We choose Φ and Ψ to be the 1-periodic extensions of the functions

$$\Phi_0(t) = \psi_0(2t) \qquad \text{and} \qquad \Psi_0(t) = \psi_0\big(6\delta^{-1}(t-1)+5\big),$$

respectively. Writing $\hat{\Phi}(m)$ and $\hat{\Psi}(m)$ for the mth Fourier coefficients of Φ and Ψ, we can report that

$$\begin{aligned} \hat{\Phi}(0) &= \tfrac{1}{2}, \quad |\hat{\Phi}(m)| \ll_r (1+|m|)^{-r} \quad \text{for all } r \in \mathbb{Z}, \\ \hat{\Psi}(0) &= \tfrac{1}{6}\delta, \quad |\hat{\Psi}(m)| \ll_r \delta(1+\delta|m|)^{-r} \quad \text{for all } r \in \mathbb{Z}. \end{aligned} \tag{7}$$

Replacing $\Phi(p^c)$ and $\Psi((n-p^c)^\gamma)$ on the left side of (6) by their Fourier expansions, we obtain

$$\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) = \sum_{h\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{X<p\le X_1} \hat{\Phi}(h)\hat{\Psi}(j)\, e\big(hp^c + j(n-p^c)^\gamma\big). \tag{8}$$

Set $H = X^\varepsilon$ and $J = X^{c-1+\varepsilon}$, where $\varepsilon > 0$ is fixed. By (7) with $r = [\varepsilon^{-1}]+2$, the contribution to the the right side of (8) from the terms with $|h| > H$ or $|j| > J$ is bounded above by a constant depending on ε. Thus,

$$\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) = \tfrac{1}{12}\delta\big(\pi(X_1)-\pi(X)\big) + O\big(\delta\mathcal{R}+1\big),$$

where $\pi(X)$ is the number of primes $\le X$ and

$$\mathcal{R} = \sum_{\substack{|h|\le H \\ (h,j)\ne(0,0)}}\sum_{|j|\le J}\bigg|\sum_{X<p\le X_1} e\big(hp^c + j(n-p^c)^\gamma\big)\bigg|.$$

Thus, it suffices to show that

$$\sum_{X<p\le X_1} e\big(hp^c + j(n-p^c)^\gamma\big) \ll X^{2-c-3\varepsilon} \tag{9}$$

for all pairs of integers (h,j) such that $|h| \le H$, $|j| \le J$, and $(h,j) \ne (0,0)$.

3. Bounds on exponential sums

In this section, we establish estimates for bilinear exponential sums, which we shall need in the proof of (9). Our first lemma is a variant of van der Corput's third-derivative estimate (see [4, Corollary 8.19]).

Lemma 3. *Suppose that $2 \le F \le N^{3/2}$, $N < N_1 \le 2N$, and $0 < \delta < 1$. Let $f \in C^3[N,N_1]$ and suppose that we can partition $[N,N_1]$ into $O(1)$ subintervals so that on each subinterval one of the following sets of conditions holds:*

i) $\delta FN^{-2} \ll |f''(t)| \ll FN^{-2}$;
ii) $\delta FN^{-3} \ll |f'''(t)| \ll FN^{-3}$, $|f''(t)| \ll \delta FN^{-2}$.

Then

$$\sum_{N<n\le N_1} e(f(n)) \ll \delta^{-1/2}\big(F^{1/6}N^{1/2} + F^{-1/3}N\big).$$

Proof. Let η be a parameter to be chosen later so that $0 < \eta \le \delta$ and let $\mathbf{I}$ be one of the subintervals of $[N, N_1]$ mentioned in the hypotheses. If i) holds in $\mathbf{I}$, then by [4, Corollary 8.13],

$$\sum_{n \in \mathbf{I}} e(f(n)) \ll \delta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big). \tag{10}$$

Now suppose that ii) holds in $\mathbf{I}$. We subdivide $\mathbf{I}$ into two subsets:

$$\mathbf{I}_1 = \big\{t \in \mathbf{I} : \eta F N^{-2} \le |f''(t)| \ll \delta F N^{-2}\big\}, \quad \mathbf{I}_2 = \mathbf{I} \setminus \mathbf{I}_1.$$

Since f'' is monotone on $\mathbf{I}$, the set $\mathbf{I}_1$ consists of at most two intervals and $\mathbf{I}_2$ is a (possibly empty) subinterval of $\mathbf{I}$. If $\mathbf{I}_2 = [a, b]$, then there is a $\xi \in (a, b)$ such that

$$f''(b) - f''(a) = (b - a)f'''(\xi) \qquad \Longrightarrow \qquad b - a \ll \eta\delta^{-1}N.$$

Thus, by [4, Corollary 8.13] and [4, Corollary 8.19],

$$\sum_{n \in \mathbf{I}_1} e(f(n)) \ll \eta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big), \tag{11}$$

$$\sum_{n \in \mathbf{I}_2} e(f(n)) \ll \eta\delta^{-4/3}F^{1/6}N^{1/2} + \eta^{1/2}\delta^{-2/3}F^{-1/6}N. \tag{12}$$

Combining (10)–(12), we get

$$\sum_{N < n \le N_1} e(f(n)) \ll \eta^{-1/2}\big(F^{1/2} + NF^{-1/2}\big) + \eta\delta^{-4/3}N^{1/2}F^{1/6} + \eta^{1/2}\delta^{-2/3}NF^{-1/6}. \tag{13}$$

We now choose

$$\eta = \delta \max\big(F^{-1/3}, F^{2/3}N^{-1}\big).$$

With this choice, (13) yields

$$\sum_{N < n \le N_1} e(f(n)) \ll \delta^{-1/2}\big(F^{1/6}N^{1/2} + F^{-1/3}N\big) + \delta^{-1/3}\big(F^{5/6}N^{-1/2} + F^{-1/6}N^{1/2}\big),$$

and the lemma follows on noting that, when $F \ll N^{3/2}$,

$$F^{-1/6}N^{1/2} \ll F^{-1/3}N, \quad F^{5/6}N^{-1/2} \ll F^{1/6}N^{1/2}.$$

□

Next, we turn to the bilinear sums needed in the proof of (9). From now on, X, X_1, N, H, J have the same meaning as in §2 and ε is subject to $0 < \varepsilon < \frac{1}{2}\big(\frac{16}{15} - c\big)$.

Lemma 4. *Suppose that* $1 < c < \frac{6}{5} - 6\varepsilon$, $M < M_1 \le 2M$, $2 \le K < K_1 \le 2K$, *and*

$$M \ll X^{1-2c/3-\varepsilon}. \tag{14}$$

Further, suppose that h, j *are integers with* $|h| \le H$, $|j| \le J$, $(h, j) \ne (0, 0)$, *and that the coefficients* a_m *satisfy* $|a_m| \le 1$. *Then*

$$\sum_{M < m \le M_1} \sum_{\substack{K < k \le K_1 \\ X < mk \le X_1}} a_m e\big(hm^ck^c + j(n - m^ck^c)^\gamma\big) \ll X^{2-c-4\varepsilon}.$$

Proof. We shall focus on the case $j \neq 0$, the case $j = 0$ being similar and easier. We set

$$y = jn^\gamma, \quad x = y^{-1}hn, \quad T = T_m = n^\gamma m^{-1} \asymp K.$$

With this notation, we have

$$f(k) = f_m(k) = hm^c k^c + j(n - m^c k^c)^\gamma = y\alpha(kT_m^{-1}),$$

where

$$\alpha(t) = \alpha(t; x) = xt^c + (1 - t^c)^\gamma. \tag{15}$$

We have

$$f''(k) = yT^{-2}\alpha''(kT^{-1}), \quad f'''(k) = yT^{-3}\alpha'''(kT^{-1}), \tag{16}$$

and

$$\alpha''(t) = (c - 1)t^{c-2}\big(cx - (1 - t^c)^{\gamma-2}\big), \tag{17}$$

$$\alpha'''(t) = -(c - 1)(2c - 1)t^{2c-3}(1 - t^c)^{\gamma-3} + (c - 2)t^{-1}\alpha''(t). \tag{18}$$

Moreover, by virtue of (3),

$$\tfrac{1}{2} < (kT^{-1})^c \leq \tfrac{1}{2}(1.25)^c < \tfrac{4}{5} \tag{19}$$

whenever $X < mk \leq X_1$.

Let $\delta_0 = X^{-\varepsilon/10}$. If $|x| \geq \delta_0^{-1}$, then by (16), (17), and (19),

$$|f''(k)| \asymp |xy|K^{-2} \asymp |h|nK^{-2} \qquad \Longrightarrow \qquad JX^{1-\varepsilon}K^{-2} \ll |f''(k)| \ll JXK^{-2}.$$

Thus, by Lemma 3 with $\delta = X^{-\varepsilon}$, $F = JX$ and $N = K$,

$$\sum_{\substack{M<m\leq M_1 \\ X<mk\leq X_1}} \sum_{K<k\leq K_1} a_m e\big(f_m(k)\big) \ll MX^{\varepsilon/2}\big(X^{(c+\varepsilon)/6}K^{1/2} + KX^{-c/3}\big). \tag{20}$$

Note that we need also to verify that $JX \leq K^{3/2}$. This is a consequence of (14).

Suppose now that $|x| \leq \delta_0^{-1}$. The set where $|\alpha''(kT^{-1})| \geq \delta_0$ consists of at most two intervals. Consequently, we can partition $[K, K_1]$ into at most three subintervals such that on each of them we have one of the following sets of conditions:

i) $\delta_0|y|K^{-2} \ll |f''(k)| \ll \delta_0^{-1}|y|K^{-2}$;
ii) $|y|K^{-3} \ll |f'''(k)| \ll |y|K^{-3}$, $|f''(k)| \ll \delta_0|y|K^{-2}$.

Thus, by Lemma 3 with $\delta = \delta_0^2$, $F = \delta_0^{-1}|y| \asymp \delta_0^{-1}|j|X$, and $N = K$,

$$\sum_{\substack{M<m\leq M_1 \\ X<mk\leq X_1}} \sum_{K<k\leq K_1} a_m e\big(f_m(k)\big) \ll MX^{\varepsilon/10}\big(X^{(c+2\varepsilon)/6}K^{1/2} + KX^{-1/3}\big). \tag{21}$$

Again, we have $\delta_0^{-1}|j|X \leq JX^{1+\varepsilon/10} \leq K^{3/2}$, by virtue of (14).

Combining (20) and (21), we obtain the conclusion of the lemma, provided that $c < \frac{4}{3} - 5\varepsilon$ and

$$M \ll X^{3-7c/3-10\varepsilon}.$$

Once again, the latter inequality is a consequence of (14). □

Lemma 5. *Suppose that* $1 < c < \frac{16}{15} - 2\varepsilon$, $M < M_1 \le 2M$, $K < K_1 \le 2K$, *and*

$$X^{2c-2+9\varepsilon} \ll M \ll X^{3-2c-9\varepsilon}. \tag{22}$$

Further, suppose that h, j *are integers with* $|h| \le H$, $|j| \le J$, $(h,j) \ne (0,0)$, *and that the coefficients* a_m, b_k *satisfy* $|a_m| \le 1$, $|b_k| \le 1$. *Then*

$$\sum_{\substack{M<m\le M_1 \\ X<mk\le X_1}} \sum_{K<k\le K_1} a_m b_k e\big(hm^c k^c + j(n - m^c k^c)^\gamma\big) \ll X^{2-c-4\varepsilon}.$$

Proof. As in the proof of Lemma 4, we shall focus on the case $j \ne 0$. By symmetry, we may assume that $M \ge X^{1/2}$. We set

$$y = jn^\gamma, \quad x = y^{-1}hn, \quad T = n^\gamma.$$

With this notation, we have

$$f(k,m) = hm^c k^c + j(n - m^c k^c)^\gamma = y\alpha(mkT^{-1}),$$

where $\alpha(t)$ is the function defined in (15).

By Cauchy's inequality and [4, Lemma 8.17],

$$\Bigg| \sum_{\substack{M<m\le M_1 \\ X<mk\le X_1}} \sum_{K<k\le K_1} a_m b_k e\big(f(k,m)\big) \Bigg|^2 \ll \frac{X}{Q} \sum_{|q|\le Q} \sum_{K<k\le 2K} \Bigg| \sum_{m\in \mathbf{I}(k,q)} e\big(g(m;k,q)\big) \Bigg|$$

$$\ll \frac{X^2}{Q} + \frac{X}{Q} \sum_{0<|q|\le Q} \sum_{K<k\le 2K} \Bigg| \sum_{m\in \mathbf{I}(k,q)} e\big(g(m;k,q)\big) \Bigg|, \tag{23}$$

where $g(m;k,q) = f(k+q,m) - f(k,m)$, $Q = J^2 X^{6\varepsilon}$, and $\mathbf{I}(k,q)$ is a subinterval of $[M, M_1]$ such that

$$X < mk, m(k+q) \le X_1$$

for all $m \in \mathbf{I}(k,q)$. We remark that the right inequality in (22) ensures that $Q \ll KX^{-\varepsilon}$. When $q \ne 0$, we write

$$g(m;k,q) = yT^{-1} \int_{mk}^{m(k+q)} \alpha'(tT^{-1})\, dt = qy \int_0^1 \beta(m(k+\theta q)T^{-1}) \frac{d\theta}{k+\theta q},$$

where $\beta(t) = t\alpha'(t)$. Introducing the notation

$$z_\theta = z_\theta(k,q) = yq(k+\theta q)^{-1}, \quad U_\theta = U_\theta(k,q) = T(k+\theta q)^{-1} \asymp M,$$

we find that

$$g''(m) = \int_0^1 z_\theta U_\theta^{-2} \beta''(mU_\theta^{-1})\, d\theta, \quad g'''(m) = \int_0^1 z_\theta U_\theta^{-3} \beta'''(mU_\theta^{-1})\, d\theta,$$

and

$$\beta''(t) = (c-1)t^{c-2}\big(c^2 x + (1-t^c)^{\gamma-3}(c + (c-1)t^c)\big), \tag{24}$$

$$\beta'''(t) = (c-1)(2c-1)t^{2c-3}(1-t^c)^{\gamma-4}\big((c-1)t^c + 2c\big) + (c-2)t^{-1}\beta''(t). \tag{25}$$

Let $\delta_0 = X^{-\varepsilon/10}$. If $|x| \ge \delta_0^{-1}$, then by (24) and a variant of (19),

$$|g''(m)| \asymp |qxy|(XM)^{-1} \quad \Longrightarrow \quad |q|JX^{-\varepsilon}M^{-1} \ll |g''(m)| \ll |q|JM^{-1}.$$

Thus, by Lemma 3 with $\delta = X^{-\varepsilon}$, $F = |q|JM$ and $N = M$,

$$\sum_{m\in\mathbf{I}(k,q)} e\big(g(m;k,q)\big) \ll (|q|J)^{1/6}M^{2/3}X^{\varepsilon/2}. \tag{26}$$

Note that we need also to verify that $F \le M^{3/2}$, which holds if

$$M \gg X^{6(c-1)+12\varepsilon}. \tag{27}$$

Suppose now that $|x| \le \delta_0^{-1}$. We then deduce from (24) and (25) that

$$|\beta''(mU_\theta^{-1})| \ll \delta_0^{-1}, \quad |\beta'''(mU_\theta^{-1})| \ll \delta_0^{-1},$$

whence

$$|\beta''(mU_\theta^{-1})| = |\beta''(mU_0^{-1})| + O\big(|q|K^{-1}\delta_0^{-1}\big) = |\beta''(mU_0^{-1})| + O\big(\delta_0^2\big).$$

We now note that the subset of $[M, M_1]$ where $|\beta''(mU_0^{-1})| \ge \delta_0$ consists of at most two intervals. Consequently, we can partition $[M, M_1]$ into at most three subintervals such that on each of them we have one of the following sets of conditions:

i) $\delta_0|qy|(XM)^{-1} \ll |g''(m)| \ll \delta_0^{-1}|qy|(XM)^{-1}$;

ii) $|qy|X^{-1}M^{-2} \ll |g'''(m)| \ll |qy|X^{-1}M^{-2}$, $|g''(m)| \ll \delta_0|qy|(XM)^{-1}$.

Thus, Lemma 3 with $\delta = \delta_0^2$, $F = \delta_0^{-1}|qj|M$, and $N = M$ yields (26), provided that (27) holds.

Combining (23) and (26), we get

$$\Bigg|\sum_{\substack{M<m\le M_1\\ X<mk\le X_1}}\sum_{K<k\le K_1} a_m b_k e\big(f(k,m)\big)\Bigg|^2 \ll X^2Q^{-1} + X^{2+\varepsilon/2}(QJ)^{1/6}M^{-1/3}. \tag{28}$$

In view of our choice of Q, the conclusion of the lemma follows from (28), provided that

$$M \gg X^{7.5(c-1)+10\varepsilon}.$$

Both (27) and the last inequality follow from the assumption that $M \ge X^{1/2}$ and the hypothesis $c < \frac{16}{15} - 2\varepsilon$. ☐

We close this section with a lemma that will be needed in the proof of Theorem 2.

Lemma 6. *Suppose that $1 < c < 2$, $2 \le X < X_1 \le 2X$, and $0 < \delta < \frac{1}{4}$. Let $\mathcal{S}_\delta$ denote the number of integers n such that $X < n \le X_1$ and $\|n^c\| < \delta$. Then*

$$\mathcal{S}_\delta \ll \delta(X_1 - X) + \delta^{-1/2}X^{c/2}.$$

Proof. Let Φ be the 1-periodic extension of a smooth function that majorizes the characteristic function of the interval $[-\delta, \delta]$ and is majorized by the characteristic function of $[-2\delta, 2\delta]$. Then

$$\mathcal{S}_\delta \le \sum_{X<n\le X_1} \Phi\big(n^c\big) = \sum_{X<n\le X_1} \hat{\Phi}(0) + \sum_{h\ne 0} \hat{\Phi}(h) \sum_{X<n\le X_1} e\big(hn^c\big). \tag{29}$$

If $h \ne 0$, [4, Corollary 8.13] yields

$$\sum_{X<n\le X_1} e\big(hn^c\big) \ll |h|^{1/2}X^{c/2},$$

whence

$$\sum_{h\neq 0}\hat\Phi(h)\sum_{X<n\le X_1} e\big(hn^c\big) \ll X^{c/2}\sum_{h\neq 0}|\hat\Phi(h)||h|^{1/2}$$
$$\ll X^{c/2}\sum_{h\neq 0}\frac{\delta|h|^{1/2}}{(1+\delta|h|)^2} \ll \delta^{-1/2}X^{c/2}. \tag{30}$$

Since $\hat\Phi(0)\le 4\delta$, the lemma follows from (29) and (30). □

4. Proof of Theorem 1: conclusion

Suppose that $1<c<\frac{16}{15}$ and $0<\varepsilon<\frac12\big(\frac{16}{15}-c\big)$. To prove (9), we recall Vaughan's identity in the form of [4, Proposition 13.4]. We can use it to express the sum in (9) as a linear combination of $O(\log^2 X)$ sums of the form

$$\sum_{\substack{M<m\le M_1\\ X<mk\le X_1}}\sum_{K<k\le K_1} a_m b_k e\big(hm^ck^c + j(n-m^ck^c)^\gamma\big),$$

where either

i) $|a_m|\ll m^{\varepsilon/2}$, $b_k=1$, and $M\ll X^{2/3}$; or
ii) $|a_m|\ll m^{\varepsilon/2}$, $|b_k|\ll k^{\varepsilon/2}$, and $X^{1/3}\ll M\ll X^{2/3}$.

A sum subject to conditions ii) is $\ll X^{2-c-3.5\varepsilon}$ by Lemma 5. A sum subject to conditions i) can be bounded using Lemma 4 if (14) holds and using Lemma 5 if (14) fails. In either case, the resulting bound is $\ll X^{2-c-3.5\varepsilon}$. Therefore, each of the $O(\log^2 X)$ terms in the decomposition of (9) is $\ll X^{2-c-3.5\varepsilon}$. This establishes (9) and completes the proof of the theorem.

5. Proof of Theorem 2

We can cover the interval $(x^{1/2},x]$ by $O((\log x)^3)$ subintervals of the form $(N,N_1]$, with $N_1 = N\big(1+(\log N)^{-2}\big)$. Thus, it suffices to show that

$$Z_c(N)\ll N^{3-3/c+5\varepsilon/6}, \tag{31}$$

where $Z_c(N)$ is the number of integers n in the range

$$N<n\le N\big(1+(\log N)^{-2}\big)$$

that cannot be represented in the form (2).

As in the proof of Theorem 1, we derive solutions of (2) from solutions of (4). We set $\gamma=1/c$, $\eta=(\log N)^{-2}$, and write

$$N_1=(1+\eta)N,\quad X=\big(\tfrac12 N\big)^\gamma,\quad X_1=(1+\eta)X,\quad \delta=\gamma X^{1-c}.$$

Suppose that $N<n\le N_1$ and $X<p\le X_1$. Then

$$(1-\eta)\delta<\gamma\big(n-p^c\big)^{\gamma-1}<(1+2\eta)\delta.$$

Assuming that p satisfies the inequalities

$$4\eta<\{p^c\}<1-4\eta,\quad 1-\delta-\eta\delta<\big\{\big(n-p^c\big)^\gamma\big\}<1-\delta+\eta\delta, \tag{32}$$

we deduce that

$$\begin{aligned}\big(n-[p^c]\big)^\gamma &< \big(n-p^c\big)^\gamma + (1-4\eta)(1+2\eta)\delta + O\big(\delta n^{-1}\big)\\ &< \big[\big(n-p^c\big)^\gamma\big] + 1 - \eta\delta,\\ \big(n+1-[p^c]\big)^\gamma &> \big(n-p^c\big)^\gamma + (1+4\eta)(1-\eta)\delta + O\big(\delta n^{-1}\big)\\ &> \big[\big(n-p^c\big)^\gamma\big] + 1 + \eta\delta.\end{aligned}$$

In particular, a prime p, $X < p \le X_1$, that satisfies (32) yields a solution m of (4) and a representation of n in the form (2).

Let Φ be the 1-periodic extension of a smooth function Φ_0 that majorizes the characteristic function of $[6\eta, 1-6\eta]$ and is majorized by the characteristic function of $[4\eta, 1-4\eta]$. Further, let Ψ be the 1-periodic extension of

$$\Psi_0(t) = \psi_0\big((2\eta\delta)^{-1}(t-1+\delta) + \tfrac{1}{2}\big),$$

where ψ_0 is the function appearing in the proof of Theorem 1. Then Ψ_0 is supported inside $[1-\delta-\eta\delta, 1-\delta+\eta\delta]$ and the Fourier coefficients of Ψ satisfy

$$\hat{\Psi}(0) = 2\eta\delta, \quad |\hat{\Psi}(h)| \ll_r \eta\delta(1+\eta\delta|h|)^{-r} \quad \text{for all } r \in \mathbb{Z}. \tag{33}$$

Hence,

$$\begin{aligned}\sum_{X<p\le X_1} \Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) &= \sum_{h\in\mathbb{Z}}\sum_{X<p\le X_1} \Phi\big(p^c\big)\hat{\Psi}(h)\, e\big(h(n-p^c)^\gamma\big)\\ &= \hat{\Psi}(0)\sum_{X<p\le X_1}\Phi\big(p^c\big) + \mathcal{R}(n)\\ &= 2\eta\delta\big(\pi(X_1)-\pi(X)+O(\mathcal{S})\big) + \mathcal{R}(n).\end{aligned} \tag{34}$$

Here,

$$\mathcal{R}(n) = \sum_{h\ne 0}\hat{\Psi}(h)\sum_{X<p\le X_1}\Phi\big(p^c\big)\, e\big(h(n-p^c)^\gamma\big)$$

and $\mathcal{S}$ is the number of integers m such that $X < m \le X_1$ and $\|m^c\| < 6\eta$. By Lemma 6,

$$\mathcal{S} \ll \eta(X_1 - X) + \eta^{-1/2}X^{c/2} \ll \eta^2 X. \tag{35}$$

Combining (34), (35) and the Prime Number Theorem, we find that

$$\sum_{X<p\le X_1}\Phi\big(p^c\big)\Psi\big((n-p^c)^\gamma\big) \gg X^{2-c}(\log X)^{-5} \tag{36}$$

for any n, $N < n \le N_1$, for which we have

$$\mathcal{R}(n) \ll X^{2-c-\varepsilon/12}. \tag{37}$$

Since the sum on the right side of (36) is supported on the primes p satisfying (32), (31) will follow if we show that (37) holds for all but $O\big(N^{3-3\gamma+5\varepsilon/6}\big)$ integers $n \in (N, N_1]$.

Set $H = X^{c-1+\varepsilon/6}$. By (33) with $r = 2 + [2\varepsilon^{-1}]$, the contribution to $\mathcal{R}(n)$ from terms with $|h| > H$ is bounded. Consequently,

$$Z_c(N) \ll X^{-2+\varepsilon/6}\sum_{N<n\le N_1}\mathcal{R}_1(n)^2,$$

where

$$\mathcal{R}_1(n) = \sum_{0<|h|\le H} \Big| \sum_{X<p\le X_1} \Phi\big(p^c\big) e\big(h(n-p^c)^\gamma\big) \Big|.$$

Appealing to Cauchy's inequality and the Weyl–van der Corput lemma [4, Lemma 8.17], we obtain

$$\begin{aligned} Z_c(N) &\ll X^{c-3+\varepsilon/3} \sum_{0<|h|\le H} \sum_{N<n\le N_1} \Big| \sum_{X<p\le X_1} \Phi\big(p^c\big) e\big(h(n-p^c)^\gamma\big) \Big|^2 \\ &\ll X^{c-2+\varepsilon/3} Q^{-1} \sum_{0<|h|\le H} \sum_{|q|\le Q} \sum_{X<p\le X_1} \Big| \sum_{N<n\le N_1} e(f(n)) \Big|, \end{aligned}$$

where $Q \le \eta X$ is a parameter at our disposal and

$$f(n) = qh\big((n-p^c)^\gamma - (n-(p+q)^c)^\gamma\big).$$

We choose $Q = \eta X^{1-\varepsilon/6}$. Then

$$|qh|N^{-1} \ll |f'(n)| \ll |qh|N^{-1} \ll \eta < \tfrac{1}{2},$$

so [4, Corollary 8.11] and the trivial bound yield

$$\sum_{N<n\le N_1} e(f(n)) \ll N(1+|qh|)^{-1}.$$

We conclude that

$$Z_c(N) \ll NX^{c-2+2\varepsilon/3} \sum_{0<|h|\le H} \sum_{|q|\le Q} (1+|qh|)^{-1} \ll NX^{2c-3+5\varepsilon/6}.$$

This establishes (31) and completes the proof of the theorem.

References

[1] E. P. Balanzario, M. Z. Garaev, and R. Zuazua, *Exceptional set of a representation with fractional powers*, preprint.

[2] J.-M. Deshouillers, *Un problème binaire en théorie additive*, Acta Arith. **25** (1973/74), 393–403.

[3] S. A. Gritsenko, *Three additive problems*, Izv. Ross. Akad. Nauk **41** (1992), 447–464, in Russian.

[4] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society, 2004.

[5] S. V. Konyagin, *An additive problem with fractional powers*, Mat. Zametki **73** (2003), 633–636, in Russian.

[6] M. B. S. Laporta, *On a binary problem with prime numbers*, Math. Balkanica (N.S.) **13** (1999), 119–123.

Department of Mathematics, Towson University, Towson, MD 21252-0001, U.S.A.

E-mail address: `akumchev@towson.edu`