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# A BIVARIATE MARSHALL AND OLKIN EXPONENTIAL MINIFICATION PROCESS

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**Abstract**

In this paper we present a stationary bivariate minification process with Marshall and Olkin exponential distribution. The process is given by

$$\begin{aligned} X_n &= K\min(X_{n-1}, Y_{n-1}, \eta_{n1}), \\ Y_n &= K\min(X_{n-1}, Y_{n-1}, \eta_{n2}), \end{aligned}$$

where $\{(\eta_{n1}, \eta_{n2}), n \geq 1\}$ is a sequence of independent and identically distributed random vectors, the random vectors $(X_m, Y_m)$ and $(\eta_{n1}, \eta_{n2})$ are independent for $m < n$ and $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $K > (\lambda_1 + \lambda_2 + \lambda_{12})/\lambda_{12}$. The innovation distribution, the joint distribution of random vectors $(X_n, Y_n)$ and $(X_{n-j}, Y_{n-j})$, $j > 0$, the autocovariance and the autocorrelation matrix are obtained. The unknown parameters are estimated and their asymptotic properties are obtained.

## 1. INTRODUCTION

A minification process of the first-order is given by

$$X_n = K\min(X_{n-1}, \varepsilon_n), n \geq 1,$$

where $K > 1$ and $\{\varepsilon_n,\ n \geq 1\}$ is an innovation process of independent and identically distributed (i.i.d.) random variables. Several authors have introduced minification processes with given marginals. Tavares [13] introduced the minification process with exponential marginal distribution. Sim [10] introduced the minification process with Weibull marginal distribution. Yeh, Arnold and Robertson [14] introduced a Pareto minification process. Arnold and Robertson [1] introduced a logistic minification process. Pillai [7] and Pillai, Jose and Jayakumar [8] introduced semi-Pareto minification processes. Balakrishna [2] considered some properties of

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the semi-Pareto minification process of Pillai [7] and estimated the unknown parameters of the model. Lewis and McKenzie [5] introduced the minification process with marginal distribution function $F_{X_0}(x)$. Some bivariate and multivariate minification processes are introduced by Balakrishna and Jayakumar [3], Thomas and Jose [11], [12] and Ristic [9].

In this paper we consider a stationary bivariate minification process of Ristić [9] with the bivariate Marshall and Olkin exponential distributions BVE$(\lambda_1, \lambda_2, \lambda_{12})$ and $K = L$. Motivated by situations that arise in reliability theory such as the failure of paired jet engines or the registration of an event by two adjacent geiger counters, Marshall and Olkin [6] introduced the bivariate exponential distribution with survival function

$$P\{X > x, Y > y\} = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)}, \quad x, y > 0,$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$. The random variables are constructed such that $X$ and $Y$ are dependent exponentially distributed random variables with parameters $\lambda_1 + \lambda_{12}$ and $\lambda_2 + \lambda_{12}$, respectively. The important property of this distribution is that it is not absolutely continuous distribution, since the probability $P\{X = Y\} = \lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_2)$ is non-negative. The density function $f(x, y)$ of the BVE$(\lambda_1, \lambda_2, \lambda_{12})$ distribution is given by

$$f(x,y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_{12})e^{-\lambda_1 x - (\lambda_2 + \lambda_{12})y}, & y > x > 0, \\ \lambda_2(\lambda_1 + \lambda_{12})e^{-(\lambda_1 + \lambda_{12})x - \lambda_2 y}, & x > y > 0, \\ \lambda_{12} e^{-(\lambda_1 + \lambda_2 + \lambda_{12})x}, & x = y > 0. \end{cases}$$

This paper is organized as follows. The properties of the process are considered in Section 2. In Section 3 we give the estimates of the parameters of the process.

## 2. PROPERTIES OF THE PROCESS

In this section we consider a stationary bivariate minification process with bivariate Marshall and Olkin exponential distribution BVE$(\lambda_1, \lambda_2, \lambda_{12})$. The process is given by

$$\begin{aligned} X_n &= K \min(X_{n-1}, Y_{n-1}, \eta_{n1}), \\ Y_n &= K \min(X_{n-1}, Y_{n-1}, \eta_{n2}), \end{aligned} \tag{1}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $K > \lambda/\lambda_{12}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, $\{(\eta_{n1}, \eta_{n2}),\ n \geq 1\}$ is a sequence of i.i.d. random vectors and the random vectors $(X_m, Y_m)$ and $(\eta_{n1}, \eta_{n2})$ are independent for $m < n$.

Ristić [9] derived the innovation distribution of the random vector $(\eta_{n1}, \eta_{n2})$. The random vector $(\eta_{n1}, \eta_{n2})$ has the bivariate Marshall and Olkin exponential distribution BVE$(\lambda_1 K, \lambda_2 K, \lambda_{12} K - \lambda)$. The marginal distributions of the random variables $\eta_{n1}$ and $\eta_{n2}$ are $\varepsilon(c_1)$ and $\varepsilon(c_2)$, respectively, where $c_1 = (\lambda_1 + \lambda_{12})K - \lambda$ and $c_2 = (\lambda_2 + \lambda_{12})K - \lambda$. Following Ristić [9], we obtain the joint survival function of the random vectors $(X_n, Y_n)$ and $(X_{n-j}, Y_{n-j})$, $j > 0$. Denote the joint survival function of $(X_n, Y_n)$ and $(X_{n-j}, Y_{n-j})$ by

$$S_j(x_1, y_1, x_2, y_2; K) = P\{X_n > x_1, Y_n > y_1, X_{n-j} > x_2, Y_{n-j} > y_2\}. \quad (2)$$

The joint survival function $S_j(x_1, y_1, x_2, y_2; K)$, $j \geq 1$, can be obtained recursively as

$$\begin{aligned} S_j(x_1, y_1, x_2, y_2; K) &= \frac{\overline{F}\left(\max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}, x_2\right), \max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}, y_2\right)\right) \cdot \overline{F}(x_1, y_1)}{\overline{F}\left(\max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}\right), \max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}\right)\right)}, \\ &= S_1(x_1, y_1, x_2, y_2; K^j). \end{aligned}$$

It is obvious that the joint distribution and the properties of the random vector $(X_n, Y_n,\ X_{n-j}, Y_{n-j})$ can be derived from the joint distribution and the properties of the random vector $(X_n, Y_n, X_{n-1}, Y_{n-1})$ replacing $K$ by $K^j$.

Now we discuss the autocovariance structure of the bivariate Marshall and Olkin exponential minification process. We define the autocovariance matrix of a bivariate process $\{(X_n, Y_n),\ n \geq 0\}$ by

$$\Gamma(j) = \begin{bmatrix} Cov(X_n, X_{n-j}) & Cov(X_n, Y_{n-j}) \\ Cov(Y_n, X_{n-j}) & Cov(Y_n, Y_{n-j}) \end{bmatrix}.$$

To derive the autocovariance matrix $\Gamma(j)$ it suffices to derive the autocovariance matrix $\Gamma(1)$.

In order to compute the moment $E(X_n X_{n-1})$, we consider the conditional expectation $E(X_n|X_{n-1}, Y_{n-1})$. From the definition of the process $\{(X_n, Y_n),\ n \geq 0\}$, we have that conditional distribution for $X_n$, given $X_{n-1} = x$ and $Y_{n-1} = y$, is given by

$$P\{X_n \leq z | X_{n-1} = x, Y_{n-1} = y\} = \begin{cases} 1 - e^{-\frac{c_1 z}{K}} & , \quad z < K\min(x, y), \\ 1 & , \quad z \geq K\min(x, y). \end{cases}$$

Note that this is not an absolutely continuous distribution, since the probability

$$\begin{aligned} P\{X_n = K\min(X_{n-1}, Y_{n-1}) | X_{n-1} = x, Y_{n-1} = y\} &= P\{\eta_{n1} > \min(x, y)\} \\ &= e^{-c_1 \min(x,y)} \end{aligned}$$

is non-negative. Now, the conditional expectation is

$$\begin{aligned} E(X_n|X_{n-1} &= x, Y_{n-1} = y) = \frac{c_1}{K} \int_0^{K\min(x,y)} z e^{-\frac{c_1 z}{K}} dz \\ &+ K\min(x, y) e^{-c_1 \min(x,y)} = \frac{K}{c_1}\left(1 - e^{-c_1 \min(x,y)}\right). \end{aligned}$$

Using this it is easy to verify that

$$E(X_n X_{n-1}) = \frac{K}{c_1} \cdot E\left[X_{n-1}\left(1 - e^{-c_1 \min(X_{n-1}, Y_{n-1})}\right)\right]. \quad (3)$$

In order to compute the moment $E(X_nX_{n-1})$ we will need the following lemma.

**Lemma 1.** Let $(X,Y)$ be a random vector with bivariate Marshall and Olkin exponential distribution $BVE(\lambda_1,\lambda_2,\lambda_{12})$. Let $U=X$, $W=Y$ and $V=\min(X,Y)$. Then:

$(i)$ the random vector $(U,V)$ has the survival function

$$P\{U>x,V>y\}=e^{-(\lambda_1+\lambda_{12})\max(x,y)-\lambda_2 y},$$

and

$$P\{U=V\}=\frac{\lambda_1+\lambda_{12}}{\lambda},$$

$(ii)$ the random vector $(W,V)$ has the survival function

$$P\{W>x,V>y\}=e^{-\lambda_1 y-(\lambda_2+\lambda_{12})\max(x,y)},$$

and

$$P\{W=V\}=\frac{\lambda_2+\lambda_{12}}{\lambda}.$$

*Proof.* $(i)$ From the definition of the random variables $U$ and $V$, we have that

$$\begin{aligned} P\{U>x,V>y\} &= P\{X>\max(x,y),Y>y\}\\ &= e^{-\lambda_1\max(x,y)-\lambda_2 y-\lambda_{12}\max(\max(x,y),y)}\\ &= e^{-(\lambda_1+\lambda_{12})\max(x,y)-\lambda_2 y}, \end{aligned}$$

and

$$P\{U=V\}=P\{X\le Y\}=\frac{\lambda_1+\lambda_{12}}{\lambda}.$$

$(ii)$ The proof is very similar to the proof of $(i)$. □

Now, setting $U=X_{n-1}$ and $V=\min(X_{n-1},Y_{n-1})$ in (3) and using Lemma 1, we have that

$$\begin{aligned} E(X_nX_{n-1}) &= \frac{K}{c_1}\cdot E\left[U\left(1-e^{-c_1V}\right)\right]\\ &= \frac{K}{c_1}\lambda_2(\lambda_1+\lambda_{12})\int_0^\infty\int_0^u u(1-e^{-c_1v})e^{-(\lambda_1+\lambda_{12})u-\lambda_2 v}dvdu\\ &+ \frac{K}{c_1}(\lambda_1+\lambda_{12})\int_0^\infty u(1-e^{-c_1u})e^{-\lambda u}du\\ &= \frac{K+1}{K(\lambda_1+\lambda_{12})^2}. \end{aligned}$$

Using this result, we conclude that

$$Cov(X_n,X_{n-1})=\frac{1}{K(\lambda_1+\lambda_{12})^2}.$$

Similarly, we can verify that

$$E(X_n Y_{n-1}) = \frac{K}{c_1} \cdot E\left[W\left(1 - e^{-c_1 V}\right)\right] = \frac{K(\lambda_1 + \lambda_{12}) + \lambda_2 + \lambda_{12}}{K(\lambda_1 + \lambda_{12})^2(\lambda_2 + \lambda_{12})},$$

and

$$Cov(X_n, Y_{n-1}) = \frac{1}{K(\lambda_1 + \lambda_{12})^2}.$$

Let us consider now $Cov(Y_n, X_{n-1})$ and $Cov(Y_n, Y_{n-1})$. The conditional distribution for $Y_n$, given $X_{n-1} = x$ and $Y_{n-1} = y$, is given by

$$P\{Y_n \leq z | X_{n-1} = x, Y_{n-1} = y\} = \begin{cases} 1 - e^{-\frac{c_2 z}{K}} & , \quad z < K \min(x, y), \\ 1 & , \quad z \geq K \min(x, y). \end{cases}$$

Also, we have $P\{Y_n = K \min(X_{n-1}, Y_{n-1}) | X_{n-1} = x, Y_{n-1} = y\} = e^{-c_2 \min(x,y)}$. Finally, we obtain

$$Cov(Y_n, X_{n-1}) = Cov(Y_n, Y_{n-1}) = \frac{1}{K(\lambda_2 + \lambda_{12})^2}$$

in a similar way as we have obtained $Cov(X_n, X_{n-1})$ and $Cov(X_n, Y_{n-1})$.

Thus we obtain the autocovariance matrix $\Gamma(1)$ as

$$\Gamma(1) = \frac{1}{K} \begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} \\ \frac{1}{(\lambda_2+\lambda_{12})^2} & \frac{1}{(\lambda_2+\lambda_{12})^2} \end{bmatrix}.$$

If we replace $K$ by $K^j$ in $\Gamma(1)$, then we will obtain the autocovariance matrix $\Gamma(j)$ as

$$\Gamma(j) = \frac{1}{K^j} \begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} \\ \frac{1}{(\lambda_2+\lambda_{12})^2} & \frac{1}{(\lambda_2+\lambda_{12})^2} \end{bmatrix}.$$

We will now discuss the autocorrelation structure of the bivariate minification process with bivariate Marshall and Olkin exponential distribution. We define the autocorrelation matrix by

$$R(j) = \begin{bmatrix} Corr(X_n, X_{n-j}) & Corr(X_n, Y_{n-j}) \\ Corr(Y_n, X_{n-j}) & Corr(Y_n, Y_{n-j}) \end{bmatrix}.$$

After elementary calculation we get

$$R(j) = \frac{1}{K^j} \begin{bmatrix} 1 & \frac{\lambda_2+\lambda_{12}}{\lambda_1+\lambda_{12}} \\ \frac{\lambda_1+\lambda_{12}}{\lambda_2+\lambda_{12}} & 1 \end{bmatrix}.$$

Now, we will derive the range of the correlations $Corr(X_n, X_{n-1})$, $Corr(X_n, Y_{n-1})$, $Corr(Y_n, X_{n-1})$ and $Corr(Y_n, Y_{n-1})$. Since $K > \lambda/\lambda_{12}$, it follows that

$$0 < Corr(X_n, X_{n-1}) = Corr(Y_n, Y_{n-1}) < \frac{\lambda_{12}}{\lambda} < 1,$$

$$0 < Corr(X_n, Y_{n-1}) < \frac{\lambda_{12}(\lambda_2 + \lambda_{12})}{(\lambda_1 + \lambda_{12})\lambda} < 1,$$

$$0 < Corr(Y_n, X_{n-1}) < \frac{\lambda_{12}(\lambda_1 + \lambda_{12})}{(\lambda_2 + \lambda_{12})\lambda} < 1.$$

## 3. ESTIMATION OF THE PARAMETERS

In this section we will estimate the unknown parameters $K$, $\lambda_1$, $\lambda_2$ and $\lambda_{12}$. Ristić [9] showed that our bivariate minification process is ergodic and uniformly mixing. Let us consider the estimation of the unknown parameters. Let $\{(X_0,Y_0), (X_1,Y_1), \ldots, (X_{N-1},Y_{N-1})\}$ be a sample of size $N$. First, we estimate the parameter $K$. Ristić [9] used the estimate

$$\hat{K}_N = \max_{1\le n\le N-1}\left\{\frac{X_n}{\min(X_{n-1},Y_{n-1})}\right\}.$$

He showed that the estimate $\hat{K}_N$ is strongly consistent estimate and is not asymptotically normal. As an alternative strongly consistent estimator of $K$, we can consider

$$\widetilde{K}_N = \max_{1\le n\le N-1}\left\{\frac{Y_n}{\min(X_{n-1},Y_{n-1})}\right\}.$$

Both estimators $\widehat{K}_N$ and $\widetilde{K}_N$ can be used in practical situation, since the true values of the parameters can be obtained for small $N$. Now we consider the estimation of the parameters $\lambda_1$, $\lambda_2$ and $\lambda_{12}$. We will use the estimates

$$\begin{aligned}\overline{X}_N &= \frac{1}{N}\sum_{i=0}^{N-1} X_i,\\ \overline{Y}_N &= \frac{1}{N}\sum_{i=0}^{N-1} Y_i,\\ \overline{I}_{N-1} &= \frac{1}{N-1}\sum_{i=1}^{N-1} I(X_i>\min(X_{i-1},Y_{i-1})),\end{aligned}$$

where

$$I(X_i>\min(X_{i-1},Y_{i-1})) = \begin{cases} 1, & X_i>\min(X_{i-1},Y_{i-1}),\\ 0, & X_i\le\min(X_{i-1},Y_{i-1}).\end{cases}$$

Since the bivariate minification process with bivariate Marshall and Olkin exponential distribution is ergodic, it follows that the estimates $\overline{X}_N$, $\overline{Y}_N$ and $\overline{I}_{N-1}$ are strongly consistent estimates of the parameters $1/(\lambda_1+\lambda_{12})$, $1/(\lambda_2+\lambda_{12})$ and $K\lambda/(K\lambda+K(\lambda_1+\lambda_{12})-\lambda)$. Also, since the bivariate minification process is stationary and uniformly mixing and $\sum_{i=1}^{\infty}\phi^{1/2}(h)<\infty$, it follows from Theorem 20.1 (Billingsley [4]) that

$$\sqrt{N}\begin{bmatrix}\overline{X}_N-\frac{1}{\lambda_1+\lambda_{12}}\\ \overline{Y}_N-\frac{1}{\lambda_2+\lambda_{12}}\end{bmatrix}$$

has asymptotically bivariate normal distribution $\mathcal{N}_2(\mathbf{0},\Sigma)$, as $N\to\infty$, where

$$\Sigma = \begin{bmatrix}\frac{K+1}{(K-1)(\lambda_1+\lambda_{12})^2} & \sigma_{xy}\\ \sigma_{xy} & \frac{K+1}{(K-1)(\lambda_2+\lambda_{12})^2}\end{bmatrix}$$

and

$$\sigma_{xy} = \frac{\lambda_{12}}{\lambda(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} + \frac{1}{K-1}\left[\frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{1}{(\lambda_2+\lambda_{12})^2}\right].$$

Following Balakrishna and Jacob (2003), we can show that

$$\sqrt{N-1}\left(\overline{I}_{N-1} - \frac{\lambda K}{c_1+\lambda K}\right)$$

has asymptotically normal distribution with zero mean and variance

$$\sigma^2 = \frac{c_1\lambda K}{(c_1+\lambda K)^2} + 2c_1^2\sum_{h=1}^{\infty}\left[\frac{K^h+1}{(c_1+\lambda)((c_1+\lambda)K^h+c_1)} - \frac{1}{(c_1+\lambda K)^2}\right] > 0\,.$$

So, we can take the estimates of the parameters $\lambda_1$, $\lambda_2$ and $\lambda_{12}$ as the solutions of the system of the equations

$$\begin{aligned}\overline{X}_N &= \frac{1}{\lambda_1+\lambda_{12}}\,,\\ \overline{Y}_N &= \frac{1}{\lambda_2+\lambda_{12}}\,,\\ \overline{I}_{N-1} &= \frac{K\lambda}{K\lambda+K(\lambda_1+\lambda_{12})-\lambda}\,.\end{aligned}$$

Now we present some numerical results. We simulated 10000 realizations of our process for the true values: a) $K=2$, $\lambda_1=0.2$, $\lambda_2=0.5$, $\lambda_{12}=0.8$, b) $K=3$, $\lambda_1=0.5$, $\lambda_2=1.5$, $\lambda_{12}=2.5$. The simulation was replicated 100 times and for each data set we computed sample means of the estimates $\hat{K}_N$, $\widetilde{K}_N$, $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_{12}$ and the standard errors (SE). The results are summarized in Table 1.

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| $N$ | $\widehat{K}_N$ | $\widetilde{K}_N$ | $\lambda_1$ | SE($\lambda_1$) | $\lambda_2$ | SE($\lambda_2$) | $\lambda_{12}$ | SE($\lambda_{12}$) |
|---|---|---|---|---|---|---|---|---|
| 100 | 2 | 2 | 0.2164 | 0.1152 | 0.5214 | 0.1314 | 0.8316 | 0.1592 |
| 500 | 2 | 2 | 0.2309 | 0.0742 | 0.5145 | 0.0708 | 0.8073 | 0.1163 |
| 1000 | 2 | 2 | 0.2319 | 0.0745 | 0.5179 | 0.0713 | 0.8114 | 0.1139 |
| 5000 | 2 | 2 | 0.2308 | 0.0659 | 0.5163 | 0.0649 | 0.8083 | 0.1049 |
| 10000 | 2 | 2 | 0.2268 | 0.0577 | 0.5139 | 0.0575 | 0.8069 | 0.0958 |
| $N$ | $\widehat{K}_N$ | $\widetilde{K}_N$ | $\lambda_1$ | SE($\lambda_1$) | $\lambda_2$ | SE($\lambda_2$) | $\lambda_{12}$ | SE($\lambda_{12}$) |
| 100 | 3 | 3 | 0.5627 | 0.3330 | 1.6093 | 0.4489 | 2.4995 | 0.4645 |
| 500 | 3 | 3 | 0.5734 | 0.2314 | 1.5891 | 0.2457 | 2.4878 | 0.3293 |
| 1000 | 3 | 3 | 0.5675 | 0.2278 | 1.5844 | 0.2425 | 2.4938 | 0.3285 |
| 5000 | 3 | 3 | 0.5739 | 0.2033 | 1.5891 | 0.2087 | 2.4737 | 0.3117 |
| 10000 | 3 | 3 | 0.5812 | 0.1580 | 1.6014 | 0.1748 | 2.4782 | 0.2989 |

Table 1: Some numerical results of the estimations (a) $K = 2$, $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, $\lambda_{12} = 0.8$, b) $K = 3$, $\lambda_1 = 0.5$, $\lambda_2 = 1.5$, $\lambda_{12} = 2.5$).


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