A Block Rational Arnoldi Algorithm for Multipoint Passive Model-Order Reduction of Multiport RLC Networks

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Abstract

Recent work in the area of model-order reduction for RLC interconnect networks has been focused on building reduced-order models that preserve the circuit-theoretic properties of the network, such as stability, passivity, and synthesizability [1, 2, 3, 4, 5]. Passivity is the one circuit-theoretic property that is vital for the successful simulation of a large circuit netlist containing reduced-order models of its interconnect networks. Non-passive reduced-order models may lead to instabilities even if they are themselves stable. In this paper, we address the problem of guaranteeing the accuracy and passivity of reduced-order models of multiport RLC networks at any finite number of expansion points. The novel passivity-preserving model-order reduction scheme is a block version of the rational Arnoldi algorithm [6, 7]. The scheme reduces to that of [5] when applied to a single expansion point at zero frequency. Although the treatment of this paper is restricted to expansion points that are on the negative real axis, it is shown that the resulting passive reduced-order model is superior in accuracy to the one that would result from expanding the original model around a single point. Nyquist plots are used to illustrate both the passivity and the accuracy of the reduced-order models.

1   Introduction

It is well known [8] that multiport RLC networks are passive, in the sense that they are energy dissipators. Passive networks are necessarily stable, but the converse is not true. Passivity and stability differ in the following fundamental way: while the connection of two stable networks is not necessarily stable, any multiport connection of passive networks is guaranteed to be passive.

This closure property is of paramount importance from a practical point of view for the following reason. The reduced-order model is intended to replace the original interconnect in the global netlist. It will have the same drivers and loads as the original model. The output impedances of the drivers and the input impedances of the loads are represented by passive elements. If the reduced-order model is only stable but not passive, there is no guarantee that the network composed of output impedances, reduced-order model, and input impedances is stable.[1] In the absence of such a guarantee the simulation and analysis of the circuit may become problematic. A *passive* reduced-order model eliminates this concern.

One of the first attempts at designing passivity-preserving model-order reduction algorithms for multiport RC interconnect networks is the PACT algorithm of [11] where matrix congruence transformations are used to preserve the positivity of the energy-storage and DC matrices, a fundamental requirement for passivity. As yet unpublished is the work described in [3, 10] which addresses the issues of passivity for both RC and RLC networks and proposes, for multiport RC networks, an algorithm based on the Choleski decomposition that makes the PVL algorithm [12] into a passivity-preserving one. Recently [4], the PACT algorithm has been extended to the RLC case using *split congruence transformations*. The splitting operation is designed to decouple the capacitive behavior of the RLC network from its inductive one, thus preserving the algebraic properties of the MNA matrices of the original network model. More recently [5], a reduction algorithm based on the Arnoldi iteration [13, 1] has been shown to guarantee the passivity of the reduced-order model without requiring a "splitting operation" for the congruence transformation matrix. It is worth noting, in this passivity context, that a symmetrized version of the PVL algorithm [2] was shown to preserve the passivity of one-port RC networks using the stronger requirement of RC Cauer-synthesizability.

One engineering concern in the use of model-order reduction algorithms is that the model be accurate not just at a single point in the frequency spectrum

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[1] An example of such a situation is given in [9]. See also [3, 10] for interesting theoretical insights.

but over a whole range of frequencies. This situation typically arises when dealing with microwave circuits. Reduction algorithms that address this concern are the complex frequency hopping algorithm [14] and the multipoint rational Krylov algorithm [7].

The main contribution of this paper is to extend the recent passivity preserving Arnoldi algorithm to the multipoint expansion case. The vehicle of the extension is the use of the block *rational* Arnoldi algorithm [6], the classical Arnoldi iteration being a *polynomial algorithm*. It is rigorously shown that for any multiport RLC network, the block rational Arnoldi reduced-order model is passive and satisfies the required accuracy at all the expansion points. Nyquist plots are used to illustrate both the passivity and the accuracy of the resulting models. Recall that a Nyquist plot contains both the magnitude and phase information of the network driving-point impedance (admittance). Moreover, the Nyquist plot of a passive network is entirely contained in the right-half complex plane.

The next section of this paper introduces the basic circuit notation and the multipoint model-order reduction formulation. In Section 3, the block rational Arnoldi model-order reduction algorithm is described, its accuracy properties at the expansion points are proved, and the passivity of the rational-Arnoldi reduced-order model is rigorously shown. Numerical results along with their Nyquist plot illustrations are presented in Section 4.

2 Background

The modified nodal analysis (MNA) equations of a multiport, linear, time-invariant RLC network are given by

$$
\begin{aligned}
\mathcal{L}\, \dot{x} &= -\mathcal{G}\, x + P\, u \\
y &= P^T x + D u
\end{aligned}
\tag{1}
$$

where, $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^p$ are, respectively, the state-space, input and output vectors of the p-port network, and $\mathcal{L}, \mathcal{G} \in \mathbb{R}^{n \times n}$ are, respectively, its energy-storage and DC matrices. [2] The $p \times p$ matrix D accounts for the direct gain from the input to the output. In terms of the circuit elements and variables, the above quantities are expressed as follows:

$$
x \equiv \begin{bmatrix} v \\ i \end{bmatrix} \quad \mathcal{L} \equiv \begin{bmatrix} C & O \\ O & L \end{bmatrix} \quad \mathcal{G} \equiv \begin{bmatrix} G & B \\ -B^T & O \end{bmatrix}
\tag{2}
$$

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[2] These matrices will be assumed nonsingular, which would exclude networks with resistive meshes, capacitive loops, or inductive cutsets.

with v and i being the capacitor node voltages and the inductor branch currents, respectively. The square matrices C, L, and G, are all symmetric, positive definite and denote, respectively, the nodal capacitance, inductance, and conductance matrices of the RLC circuit. It follows that the matrix $\mathcal{L}$ is symmetric, positive definite, and that the symmetric part of $\mathcal{G}$ is positive semidefinite. The matrix B is rectangular and denotes the incidence matrix of the inductor branches.

Assuming that p currents are injected at the input ports, the network driving-point impedance is the $p \times p$ transfer matrix given by

$$
Z(s) = P^T \left(\mathcal{G} + s\mathcal{L} \right)^{-1} P
\tag{3}
$$

In multipoint model-order reduction, we are given m distinct points $s_1, s_2, \ldots, s_m$ and m integers $n_1, n_2, \ldots, n_m$, and we are asked to find a reduced-order driving-point impedance $\tilde{Z}(s)$ such that

$$
\frac{d^{k-1}\tilde{Z}}{ds^{k-1}}(s_i) = \frac{d^{k-1}Z}{ds^{k-1}}(s_i), \quad 1 \leq k \leq n_i, \quad 1 \leq i \leq m.
\tag{4}
$$

In other words, at each point s_i the original and the reduced-order model must have equal moments up to order n_i. Or, equivalently, both models must have the same Taylor series expansion up to order $(n_i - 1)$ at each point s_i. Matching at multiple points should intuitively result in a reduced-order model that has better accuracy over a wider region of the complex plane than what would be obtained with a single-point matching at, say, the DC point.

Assume that the quadruplet $[\tilde{\mathcal{G}}, \tilde{\mathcal{L}}, \tilde{P}, D]$ is a state-space realization of $\tilde{Z}$, then the matching conditions can be expressed by the matrix equalities

$$
\tilde{P}^T \left\{ \left(\tilde{\mathcal{G}} + s_i\tilde{\mathcal{L}} \right)^{-1} \tilde{\mathcal{L}} \right\}^{k-1} \left(\tilde{\mathcal{G}} + s_i\tilde{\mathcal{L}} \right)^{-1} \tilde{P}
\tag{5}
$$

$$
= P^T \left\{ \left(\mathcal{G} + s_i\mathcal{L} \right)^{-1} \mathcal{L} \right\}^{k-1} \left(\mathcal{G} + s_i\mathcal{L} \right)^{-1} P
$$

Note that the matrix D is the same for both models and therefore it does not appear in these equalities.

For each point s_i, there is a Krylov subspace $\mathcal{K}_{n_i}$ spanned by the columns of the matrices

$$
span\left(N_i, M_i N_i, M_i^2 N_i, \ldots, M_i^{n_i-1} N_i \right)
$$

where $M_i \equiv \left(\mathcal{G} + s_i\mathcal{L} \right)^{-1} \mathcal{L}$ and $N_i \equiv \left(\mathcal{G} + s_i\mathcal{L} \right)^{-1} P$. Note that these subspaces arise naturally from the matrix moment matching formulas (5).

In the context of direct Padé approximation reduction methods like Asymptotic Waveform Evaluation (AWE) [15], the complex frequency hopping (CFH)

algorithm [14] has been proposed to deal with multipoint expansions. In CFH, the expansion points lie on the imaginary axis of the complex plane, and a binary search strategy for choosing these points based on the required accuracy of the reduced-order model is used. The main objective in CFH is to improve the accuracy of the reduced-model *poles* compared to those obtained using AWE.

In the context of indirect, iterative Krylov subspace methods, an algorithm for multipoint model-order reduction using a rational Lanczos process was developed in [7]. A family of algorithms employing unions of Krylov subspaces to produce multipoint moment matching is proposed in [16].

In both the direct and indirect approaches to multipoint expansion, the problem of preserving the passivity of interconnect network models has not been addressed. The algorithm of the next section can be viewed as a synthesis of the rational Lanczos algorithm of [7] that preserves multipoint moments with the Arnoldi algorithm of [5] that preserves passivity. This synthesis results in a model-order reduction algorithm that preserves the passivity and the moments up to a given order for multiport RLC interconnect networks.

3 Block Rational Arnoldi

The block rational Arnoldi algorithm proposed in this paper is an adaptation of the rational Krylov algorithm introduced in [6] in the context of the non-symmetric eigenvalue problem. In order to simplify the presentation, only real-valued expansion points are used in Algorithm 1.

To obtain the points $\sigma_1, \sigma_2, \ldots, \sigma_N$, (refer to Algorithm 1) from the points $s_1, s_2, \ldots, s_m$, each point s_i is repeated a number of times equal to n_i, the expansion order at the i-th point. In other words, $N = n_1 + n_2 + \ldots + n_m$.

Note that at each iteration $i \in [1, N-1]$, Algorithm 1generates p columns of the matrix $\boldsymbol{W}$, where p is the number of ports. Therefore, the order of the reduced-order model, which is equal to the number of columns of $\boldsymbol{W}$, is pN. A basic requirement is that pN must be much smaller than n, the number of energy-storage elements in the network.

The operator *orth* refers to the Gramm-Schmidt orthonormalization procedure. This operation is used to orthonormalize the vectors of each $n \times p$ matrix block $\boldsymbol{U}_i$, $0 \leq i \leq N-1$, resulting from running the block Arnoldi algorithm. Another way of ensuring that the columns of $\boldsymbol{W}_i$ are orthonormal is by setting

Algorithm 1 (**Block Rational Arnoldi**)

```
arnoldi(input G, L, P, σ₁, σ₂,...,σN;
   output W, G̃, L̃, P̃)
{
   Initialize:
      Solve :  (G + σ₁L)U₀ = P
      W₀ = orth(U₀)
   for (i = 1;  i <= N − 1;  i++)
   {
      if σᵢ₊₁ == σᵢ,  Vᵢ = LWᵢ
      else Vᵢ = P
      Solve (G + σᵢ₊₁L)Uᵢ = Vᵢ
      for (j = 0;  j <= i − 1;  j++)
      {
         Hⱼᵢ = WⱼᵀUᵢ
         Uᵢ = Uᵢ − WⱼHⱼᵢ
      }
      Wᵢ = orth(Uᵢ)
   }
   W = [W₀, W₁, ⋯, Wₙ₋₁]
   G̃ = WᵀGW
   L̃ = WᵀLW
   P̃ = WᵀP
}
```

$\boldsymbol{W}_i = \boldsymbol{U}_i \boldsymbol{Z}_i^{-1}$ where $\boldsymbol{Z}_i$ is the Choleski factor of the symmetric, positive definite matrix $\boldsymbol{U}_i^T \boldsymbol{U}_i$.

The innermost iteration of Algorithm 1, which runs over $j \in [0, i-1]$, is nothing but a classical *block* orthogonalization process that orthogonalizes the new matrix block $\boldsymbol{U}_i$ with respect to *all* the previous matrices $\boldsymbol{W}_0, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_{i-1}$. The net result of the algorithm is an $n \times pN$ matrix $\boldsymbol{W}$ with orthonormal columns, i.e., $\boldsymbol{W}^T \boldsymbol{W} = \boldsymbol{I}_{pN}$, the identity matrix of order pN.

The (*if, else*) condition at the start of each major iteration controls when and how to pass from one expansion point to another. As long as $\sigma_{i+1} = \sigma_i$ (we are still at the same expansion point), the algorithm proceeds as in any Krylov subspace iteration, i.e., it generates the next block of Krylov vectors and orthogonalizes it with respect to *all* the previous blocks. When $\sigma_{i+1} \neq \sigma_i$, the algorithm restarts at the input matrix $\boldsymbol{P}$ so as to produce the Krylov iterates corresponding to the expansion point at σ_{i+1}. In other words, the occurrence of the condition $\sigma_{i+1} \neq \sigma_i$ signals that the Krylov subspace $\mathcal{K}_{\sigma_i}$ has been completely spanned and that the spanning of the Krylov subspace $\mathcal{K}_{\sigma_{i+1}}$ is about to start.

Note finally that the block Arnoldi algorithm of [5]

can be derived as a special case from Algorithm 1 by setting $\sigma_1 = \sigma_2 = \ldots = \sigma_N = 0$.

Based on this algorithm, the matrices of the reduced-order model are given by[3]

$$\tilde{\mathcal{G}} = \boldsymbol{W}^T \mathcal{G} \boldsymbol{W}, \quad \tilde{\mathcal{L}} = \boldsymbol{W}^T \mathcal{L} \boldsymbol{W}, \quad \tilde{\boldsymbol{P}} = \boldsymbol{W}^T \boldsymbol{P} \quad (6)$$

The driving-point matrix impedance is then given by

$$\tilde{\boldsymbol{Z}}(s) = \tilde{\boldsymbol{P}}^T \left(\tilde{\mathcal{G}} + s\tilde{\mathcal{L}} \right)^{-1} \tilde{\boldsymbol{P}} \quad (7)$$

In fact, due to the symmetry in the problem, the reduced-order model can be alternatively and more efficiently computed via a symmtric Lanczos method [16].

The moment-matching properties of Algorithm 1 hold due to the following two facts:

1. The full rank of $\boldsymbol{W}$ which is guaranteed by its column orthonormality.

2. The union of the Krylov subspaces generated at the different expansion points satisfies the inclusion formula:

$$\cup_{k=1}^m \mathcal{K}_{n_k} \subseteq span\{\boldsymbol{W}\}. \quad (8)$$

The sufficiency of these conditions for multipoint moment matching is rigorously explored and proven in [16]. The seminal connections between Krylov subspaces and moment matching are presented in [17].

The extra computational cost to obtain the additional moment matching accuracy is a new LU factorization for each expansion point. Note that the number of expansion points can be traded off with the matching order at each point, thus resulting in (perhaps significantly) fewer Krylov iterations at each point. The total computational cost of the new algorithm to obtain a given level of accuracy may be less than that incurred by a single-point algorithm.

Passive networks are networks whose net electrical energy balance is nonpositive, i.e., the energy that is dissipated by the network is at least equal to the energy supplied by the sources. The fundamental theorem relating passivity to the network's linear response is the following [8]:

Theorem 1 *A one-port network is passive if and only if its driving-point impedance (admittance), denoted by $F(\sigma + j\omega)$, is positive real, i.e,*

$$(pr1) \qquad \forall \sigma \in \mathbb{R}, F(\sigma) \in \mathbb{R}$$
$$(pr2) \qquad \forall \sigma \geq 0, Re\{F(\sigma + j\omega)\} \geq 0$$

---

[3]We have dropped the matrix $\boldsymbol{D}$ as it is the same for both the original and the reduced-order models.

For a multiport network with driving-point matrix impedance, $\boldsymbol{G}(s)$, a sufficient condition for passivity [8] is that for any real vector $\boldsymbol{u} \in \mathbb{R}^p$, the transfer *function* $F(s) = \boldsymbol{u}^T \boldsymbol{G}(s)\boldsymbol{u}$ is positive real.

The main theoretical result of this paper is an extension to multipoint model reduction of the one obtained in [5] for the single-point case. It is stated as follows:

Theorem 2 *The multiport reduced-order model of driving-point impedance $\tilde{\boldsymbol{Z}}(s)$ is passive.*

Proof. Let $\boldsymbol{u} \in \mathbb{R}^p$, and let $z(s) \equiv \boldsymbol{u}^T \tilde{\boldsymbol{Z}}(s)\boldsymbol{u}$. First it is clear that $z(s)$ is real whenever its argument s is real. Next let $s \equiv \sigma + j\omega$ with $\sigma > 0$, $\boldsymbol{x} \equiv \tilde{\boldsymbol{P}}\boldsymbol{u}$, and

$$\boldsymbol{y} \equiv \left(\tilde{\mathcal{G}} + \bar{s}\tilde{\mathcal{L}} \right)^{-T} \boldsymbol{x},$$

where $\bar{s}$ is the complex conjugate of s. Then

$$
\begin{aligned}
2Re\{z(s)\} &= z(s) + z(\bar{s}) \\
&= \boldsymbol{x}^T \left(\tilde{\mathcal{G}} + s\tilde{\mathcal{L}} \right)^{-1} \boldsymbol{x} + \boldsymbol{x}^T \left(\tilde{\mathcal{G}} + \bar{s}\tilde{\mathcal{L}} \right)^{-T} \boldsymbol{x} \\
&= \boldsymbol{x}^T \left(\tilde{\mathcal{G}} + s\tilde{\mathcal{L}} \right)^{-1} \left[\left(\tilde{\mathcal{G}} + s\tilde{\mathcal{L}} \right) \right. \\
&\quad + \left. \left(\tilde{\mathcal{G}} + \bar{s}\tilde{\mathcal{L}} \right)^T \right] \left(\tilde{\mathcal{G}} + \bar{s}\tilde{\mathcal{L}} \right)^{-T} \boldsymbol{x} \\
&= \boldsymbol{y}^* \left(\mathcal{G} + s\mathcal{L} + \mathcal{G}^T + \bar{s}\mathcal{L}^T \right) \boldsymbol{y} \\
&= \boldsymbol{y}^* \left(\mathcal{G} + \mathcal{G}^T + \sigma(\mathcal{L} + \mathcal{L}^T) \right) \boldsymbol{y}.
\end{aligned}
$$

The conclusion that $Re\{F(s)\} \geq 0$, whenever $Re\{s\} \geq 0$ results from the fact that the symmetric matrices $\mathcal{G} + \mathcal{G}^T$ and $\mathcal{L} + \mathcal{L}^T$ are both positive semidefinite. $\quad\square$

4 Results

In this section, the Nyquist plot is used to illustrate, for some numerical examples, both the accuracy and the passivity of the reduced-order model obtained with a MATLAB implementation of the passive block rational Arnoldi Algorithm 1.

The circuit used in these examples is a 4-port RLC network with 2500 capacitors and 25 inductors. Two 8-pole reduced-order models are computed for this network. The first model is obtained using the PRIMA algorithm [5] by matching two moments at DC, while the second model is computed using the block rational Arnoldi algorithm of this paper by matching one moment at DC and another moment at the real shift $s = 10$. A set of magnitude Bode plots, step-response plots, and Nyquist plots are given.
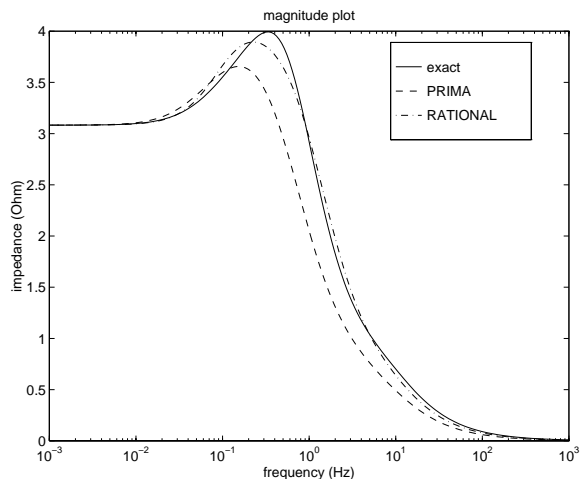
Figure 1: Driving-point impedance at port 1: Magnitude Bode plot

Figure 2: Driving-point impedance at port 1: Step response

Figure 1 is the magnitude plot of the driving-point impedance of port 1. It clearly shows that the resonant peak of the network is more closely approximated with the multipoint algorithm than with the single-point algorithm. This increased accuracy is also reflected in the step response plot. Note that both the transient and the overshoot are approximated rather well with the block rational Arnoldi. This better approximation is essential when using the model in the context of timing analysis (transient) or noise analysis (overshoot).

It has been the tradition, in model-order reduction publications, to display only these two types of plots. In Figure 3, the Nyquist plot[4] of the driving-point impedance at port 1 is shown. This plot has the following two advantages. First, it contains both the magnitude and phase information about the network impedance. Second, it provides a graphical test of port passivity. Indeed, it is well known [8] that the Nyquist plots of positive real transfer functions lie entirely in the right half of the complex plane. The accuracy of the block rational Arnoldi algorithm vs. PRIMA is again illustrated in Figure 3. Moreover, both algorithms lead to driving-point impedances whose Nyquist plots lie entirely in the right-half plane.

The reader is invited to contrast the behavior of the Nyquist plot in the driving-point case (Figure 3) vs. the transfer function case (Figure 4). In the latter, the reduced-order models may cross into the left-half plane. Note that this is not a violation of passivity or of Theorem 2, as passivity is a driving-point con-

Figure 3: Driving-point impedance at port 1: Nyquist plot

cept, i.e., electrical quantities and their duals must be measured at the *same* ports.

5 Conclusions

In this paper, the passivity-preserving Arnoldi algorithm of [5] was extended to the case where the reduced-order model is required to satisfy accuracy requirements at more than one expansion point. The vehicle of this extension was a block version of the *rational* Arnoldi algorithm [6] adapted to the modeling context so as to satisfy predefined moment matching requirements. Those requirements are met via the Krylov subspace inclusion formula (8) [16]. Nyquist plots were used to illustrate the accuracy and the pas-

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[4] The Nyquist plot displays the imaginary part of the transfer function $Z(j\omega)$ computed on the $j\omega$ axis vs. its real part.

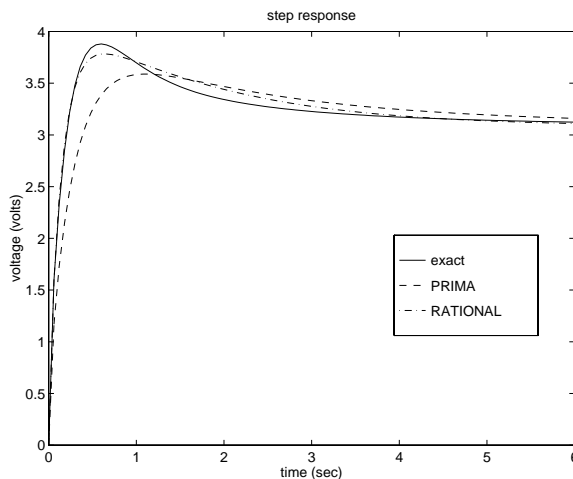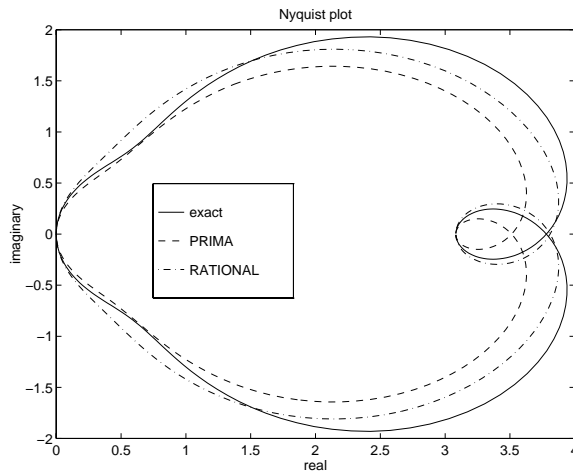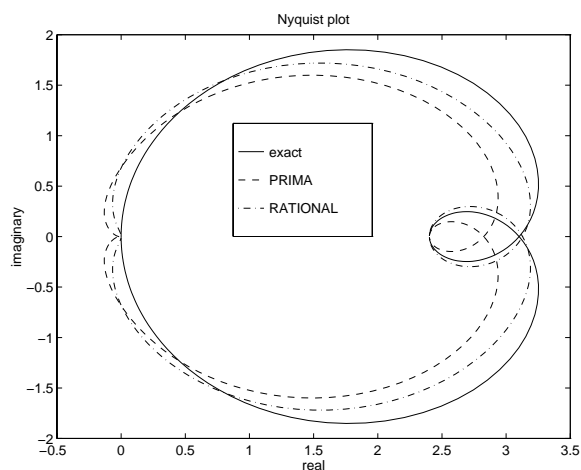Figure 4: Transfer function from Port 1 to Port 2: Nyquist plot

sivity of the driving-point impedances of the reduced-order models.

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