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A BLOW-UP CRITERION FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

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Abstract. Let q, a, b, and T be real numbers with $q \geq 0$, $a > 0$, $0 < b < 1$, and $T > 0$. This article studies the following degenerate semilinear parabolic first initial-boundary value problem,

$$x^q u_t(x,t) - u_{xx}(x,t) = a\delta(x-b) f\left(u(x,t)\right) \text{ for } 0 < x < 1,\ 0 < t \leq T,$$
$$u(x,0) = \psi(x) \text{ for } 0 \leq x \leq 1,\ u(0,t) = u(1,t) = 0 \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function, and f and ψ are given functions. It is shown that for a sufficiently large, there exists a unique number $b^* \in (0, 1/2)$ such that u never blows up for $b \in (0, b^*] \cup [1-b^*, 1)$, and u always blows up in a finite time for $b \in (b^*, 1-b^*)$. To illustrate our main results, two examples are given.

1. Introduction. Let q, a, b, and T be real numbers with $q \geq 0$, $a > 0$, $0 < b < 1$, and $T > 0$, $Lu = x^q u_t - u_{xx}$, $D = (0,1)$, $\bar{D} = [0,1]$, and $\Omega = D \times (0,T]$. We consider the following degenerate semilinear parabolic first initial-boundary value problem,

$$\left.\begin{array}{c} Lu = a\delta(x-b) f\left(u(x,t)\right) \text{ in } \Omega, \\ u(x,0) = \psi(x) \text{ on } \bar{D},\ u(0,t) = u(1,t) = 0 \text{ for } 0 < t \leq T, \end{array}\right\} \tag{1.1}$$

where $\delta(x)$ is the Dirac delta function, and f and ψ are given functions. This model is motivated by applications in which the ignition of a combustible medium is accomplished through the use of either a heated wire or a pair of small electrodes to supply a large amount of energy to a very confined area. When $q = 1$, the model may also be used to describe the temperature u of the channel flow of a fluid with temperature-dependent

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viscosity in the boundary layer (cf. Chan and Kong [2]) with a concentrated nonlinear source at b; here, x and t denote the coordinates perpendicular and parallel to the channel wall respectively. When $q = 0$, it can be used to describe the temperature of a one-dimensional strip of a finite width that contains a concentrated nonlinear source at b. The case when $q = 0$ was studied by Olmstead and Roberts [4] by analyzing its corresponding nonlinear Volterra equation of the second kind at the site of the concentrated source. A problem due to a source with local and nonlocal features was also studied by Olmstead and Roberts [5] by analyzing a pair of coupled nonlinear Volterra equations with different kernels. We assume that $f(0) \geq 0$, $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $u > 0$, and $\psi(x)$ is nontrivial, nonnegative and continuous such that ψ attains its maximum at b, $\psi(0) = \psi(1) = 0$, and

$$\psi'' + a\delta(x-b)f(\psi) \geq 0 \text{ in } D.$$

Chan and Tian [3] proved that under certain conditions, the solution u of the problem (1.1) blows up in a finite time. For $q = 0$, they and Olmstead and Roberts [4] showed that if the position b of the concentrated source is sufficiently close to $x = 0$ or $x = 1$, then u never blows up. The main purpose here is to find the exact position b^* for the problem (1.1) with $q \geq 0$ such that u never blows up for $b \in (0, b^*] \cup [1-b^*, 1)$, and u always blows up in a finite time for $b \in (b^*, 1-b^*)$. This also implies that u does not blow up in infinite time.

2. Critical position b^*.

A proof similar to that used for Theorem 4 of Chan and Jiang [1] gives the following result.

THEOREM 2.1. *If $\lim_{t\to\infty} u(x,t) < \infty$, then $u(x,t)$ converges uniformly on $\bar{D}$ from below to a solution*

$$U(x) = ag(x;b)f(U(b)) \tag{2.1}$$

of the nonlinear two-point boundary value problem

$$-U'' = a\delta(x-b)f(U(x)) \text{ in } D,\ U(0) = U(1) = 0, \tag{2.2}$$

where

$$g(x;\xi) = \begin{cases} \xi(1-x) \text{ for } 0 \leq \xi \leq x, \\ x(1-\xi) \text{ for } x < \xi \leq 1, \end{cases}$$

is Green's function corresponding to the problem (2.2).

When ψ is sufficiently large and f is sufficiently nonlinear, it follows from Theorem 3.3 of Chan and Tian [3] that there is a position b to place the nonlinear concentrated source such that (1.1) blows up in a finite time. To find a position b so that the solution u exists for all $t > 0$, let us first consider the problem (1.1) with $q = 0$, namely,

$$\left.\begin{array}{c} \mu_t - \mu_{xx} = a\delta(x-b)f(\mu(x,t)) \text{ in } \Omega, \\ \mu(x,0) = \psi(x) \text{ on } \bar{D},\ \mu(0,t) = \mu(1,t) = 0 \text{ for } 0 < t \leq T. \end{array}\right\} \tag{2.3}$$

From Theorem 3.2 of Chan and Tian [3], the blow-up set is the single point $x = b$. From Chan and Tian [3],

$$\mu(b,t) = a\int_0^t G(b,t;b,\tau)f(\mu(b,\tau))d\tau + \int_D G(b,t,\xi,0)\psi(\xi)d\xi, \tag{2.4}$$

where

$$G(x,t;\xi,\tau) = 2\sum_{n=1}^{\infty}(\sin n\pi x)(\sin n\pi\xi)e^{-n^2\pi^2(t-\tau)} \text{ for } t > \tau.$$

From Olmstead and Roberts [4],

$$\int_0^t G(b,t;b,\tau)d\tau = b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi b}{n^2}e^{-n^2\pi^2 t}.$$

Since $\sum_{n=1}^{\infty}(\sin^2 n\pi b)e^{-n^2\pi^2 t}/n^2$ and $2\sum_{n=1}^{\infty}(\sin^2 n\pi b)e^{-n^2\pi^2 t}$ converge uniformly in $(0,t)$, we have

$$\frac{\partial}{\partial t}\left(\int_0^t G(b,t;b,\tau)d\tau\right) = 2\sum_{n=1}^{\infty}(\sin^2 n\pi b)e^{-n^2\pi^2 t} > 0,$$

$$\lim_{t\to\infty}\int_0^t G(b,t;b,\tau)d\tau = b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi b}{n^2}\lim_{t\to\infty}e^{-n^2\pi^2 t} = b(1-b).$$

The solution v of the linear problem,

$$v_t - v_{xx} = 0 \text{ in } \Omega,\ v(x,0) = \psi(x) \text{ on } \bar{D},\ v(0,t) = v(1,t) = 0 \text{ for } 0 < t \le T,$$

is given by

$$v(x,t) = \int_D G(x,t;\xi,0)\psi(\xi)d\xi. \tag{2.5}$$

Since ψ attains its maximum at b, it follows from the weak maximum principle that v attains its maximum, denoted by k_0, at $(b,0)$. From Theorem 2.4 of Chan and Tian [3], μ is a nondecreasing function of t. Hence, it follows from Theorem 2.6 of Chan and Tian [3] that for $0 \le t \le \theta$, $\mu(x,t)$ attains its maximum at (b,θ). Thus given any number $M > k_0$, it follows from (2.4) and (2.5) that

$$\mu(b,t) \le k_0 + af(M)\int_0^t G(b,t;b,\tau)d\tau \le k_0 + af(M)\,b(1-b).$$

In order that $k_0 + af(M)\,b(1-b) \le M$ so that μ exists for all $t > 0$, we choose b in such a way that

$$0 < b \le \frac{1}{2}\left[1 - \sqrt{1 - \frac{4(M-k_0)}{af(M)}}\right] \text{ or } \frac{1}{2}\left[1 + \sqrt{1 - \frac{4(M-k_0)}{af(M)}}\right] \le b < 1. \tag{2.6}$$

Since μ is a nondecreasing function of t, we have for $0 \le x \le 1$ and $q > 0$,

$$x^q\mu_t - \mu_{xx} \le \mu_t - \mu_{xx},$$

which implies that the solution of the problem (1.1) is a lower solution of the problem (2.3). Thus under the above condition (2.6) on b, the solution of (1.1) also exists globally.

Let us consider a positive function $S(b)$ satisfying

$$S(b) = ag(b;b)f(S(b)) = ab(1-b)f(S(b)).$$

Since $f(s)$ and its derivative $f'(s)$ are positive for $s > 0$, it follows that $f(S(b))$ and its derivative $f'(S(b))$ are positive. We would like to know for a sufficiently large, how $S(b)$ behaves as b varies.

THEOREM 2.2. If $a > 4\sup_{S(b)\in(0,\infty)}(1/f'(S(b)))$, then

$$S \text{ is a strictly increasing function of } b \text{ for } 0 < b < \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{af'(S(b))}}\right),$$

$$S \text{ is a strictly decreasing function of } b \text{ for } \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{af'(S(b))}}\right) < b < 1.$$

Proof. A direct calculation gives

$$S'(b) = \frac{a(1-2b)f(S(b))}{1 - ab(1-b)f'(S(b))}.$$

For $b \in \left(0, \left(1 - \sqrt{1 - 4/(af'(S(b)))}\right)/2\right)$, it follows from $a > 4\sup_{S(b)\in(0,\infty)}(1/f'(S(b)))$ that $b < 1/2$, and hence $1 - 2b > 0$. Also,

$$b < \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{af'(S(b))}}\right)$$

gives

$$b - \frac{1}{2} < -\frac{1}{2}\sqrt{1 - \frac{4}{af'(S(b))}} < 0.$$

Hence,

$$\left(b - \frac{1}{2}\right)^2 > \frac{1}{4} - \frac{1}{af'(S(b))},$$

which gives

$$1 - ab(1-b)f'(S(b)) > 0.$$

Thus, we have $S'(b) > 0$, and $S(b)$ is a strictly increasing function of b.

For $b \in \left(\left(1 + \sqrt{1 - 4/(af'(S(b)))}\right)/2, 1\right)$, we have

$$\frac{1}{2} < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{af'(S(b))}} < b,$$

and hence,

$$1 - 2b < 0,$$

$$b - \frac{1}{2} > \frac{1}{2}\sqrt{1 - \frac{4}{af'(S(b))}} > 0.$$

An argument as before gives

$$1 - ab(1-b)f'(S(b)) > 0.$$

Thus, $S'(b) < 0$, and $S(b)$ is a strictly decreasing function of b. □

If $a > 4\sup_{U(b)\in(0,\infty)}(1/f'(U(b)))$, then it follows from (2.1) and Theorem 2.2 that on the interval $(0,1/2)$, the position b for global existence of u is closer to 0 than the position b for the blow-up of u in a finite time. On the other hand, on the interval $(1/2,1)$, the position b for u to exist globally is closer to 1 than the position b for u to blow up in a finite time. Thus, there exists $b^* \in (0,1/2)$ such that the steady state $U(x)$ exists for $b \in (0,b^*)\cup(1-b^*,1)$, and does not exist for $b \in (b^*,1-b^*)$. Since $u(x,t) \le U(x) = \lim_{t\to\infty} u(x,t)$ in $D\times(0,\infty)$ when U exists, we have for $b \in (0,b^*)\cup(1-b^*,1)$, u exists for $0 \le t < \infty$, and for $b \in (b^*,1-b^*)$, u always blows up in a finite time.

To calculate b^*, let us consider the steady state $U(x)$ of the problem (1.1). Since $u(x,t)$ attains its maximum at (b,t), it follows that $U(x)$ attains its maximum at $x = b$. Also, $U(b)$ and $f(U(b))$ are positive. From (2.1),

$$U(b) = ag(b;b)f(U(b)) = ab(1-b)f(U(b)).$$

Since $af(U(b)) \neq 0$, we have

$$b^2 - b + \frac{U(b)}{af(U(b))} = 0,$$

which gives

$$b = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4U(b)}{af(U(b))}}\right).$$

Since b is real, we have

$$a > 4 \sup_{U(b)\in(0,\infty)} \frac{U(b)}{f(U(b))}.$$

For the case $b = \left(1 - \sqrt{1 - 4U(b)/\left(af(U(b))\right)}\right)/2$, b^* is the supremum of all b such that $U(x)$ exists. Thus,

$$b^* = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{a}\sup_{U(b)\in(0,\infty)} \frac{U(b)}{f(U(b))}}\right). \tag{2.7}$$

Let us consider the function $g(s) = s/f(s)$. Then, $g'(s) = 0$ if and only if $s = f(s)/f'(s)$. Since f is superlinear for u to blow up in a finite time (cf. Theorem 3.3 of Chan and Tian [3]), we have $\lim_{s\to\infty} s/f(s) = 0$. If we impose $f(0) > 0$, then $\lim_{s\to 0^+} s/f(s) = 0$. By Rolle's Theorem, $\sup_{U(b)\in(0,\infty)}(U(b)/f(U(b)))$ occurs when

$$f\left(U(b)\right) = U(b)f'(U(b)), \tag{2.8}$$

where $0 < U(b) < \infty$. This implies that $U(b)$ exists at $b = b^*$, and hence, u does not blow up in infinite time.

In the case $b = \left(1 + \sqrt{1 - 4U(b)/\left(af(U(b))\right)}\right)/2$, $1 - b^*$ is the infimum of all b for which $U(x)$ exists. Thus,

$$1 - b^* = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{a}\inf_{U(b)\in(0,\infty)} \frac{-U(b)}{f(U(b))}}\right).$$

An argument as above gives the conclusion that if $f(0) > 0$, then $\inf_{U(b)\in(0,\infty)}(-U(b)/f(U(b)))$ occurs when (2.8) holds with $0 < U(b) < \infty$. Again, this implies that $U(b)$ exists at $b = 1-b^*$, and hence, u does not blow up in infinite time.

We note that if (2.8) holds, then $U(b)$ can be determined. The above discussion gives the following result.

THEOREM 2.3. If $a > 4\max\left\{\sup_{U(b)\in(0,\infty)}\left(1/f'(U(b))\right),\ \sup_{U(b)\in(0,\infty)}\left(U(b)/f(U(b))\right)\right\}$, then there exists b^* given by (2.7) such that for $b \in (0, b^*) \cup (1-b^*, 1)$, u exists for $0 \le t < \infty$, and for $b \in (b^*, 1-b^*)$, u always blows up in a finite time. If in addition, $f(0) > 0$, then $\sup_{U(b)\in(0,\infty)}(U(b)/f(U(b)))$ occurs when (2.8) holds with $U(b) \in (0,\infty)$, and u does not blow up in infinite time.

3. Examples. For illustration, we give below two examples on calculating b^* for some given functions f.

EXAMPLE 3.1. Let $f(u) = (1+u)^p$, where p is a real number greater than 1. Since $f(0) > 0$, it follows from (2.8) that $U(b) = (1+U(b))/p$, which gives

$$U(b) = \frac{1}{p-1}.$$

From Theorem 2.3, if $a > 4(p-1)^{p-1}/p^p$, then

$$b^* = \frac{1}{2}\left[1 - \sqrt{1 - \frac{4(p-1)^{p-1}}{ap^p}}\right].$$

Thus for $b \in (0, b^*] \cup [1-b^*, 1)$, u exists for $0 \le t < \infty$, and for $b \in (b^*, 1-b^*)$, u always blows up in a finite time.

EXAMPLE 3.2. Let $f(u) = ke^u$, where k is a positive number. Since $f(0) > 0$, it follows from (2.8) that $U(b) = 1$. From Theorem 2.3, if $a > 4/(ke)$, then

$$b^* = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{ake}}\right).$$

Thus for $b \in (0, b^*] \cup [1-b^*, 1)$, u exists for $0 \le t < \infty$, and for $b \in (b^*, 1-b^*)$, u always blows up in a finite time.

A phenomenon related to blow-up is quenching. Chan and Jiang [1] studied the quenching phenomenon for the following semilinear problem with a concentrated nonlinear source at b,

$$Lw = a\delta(x-b)f(w(x,t)) \text{ in } \Omega,$$
$$w(x,0) = 0 \text{ on } \bar{D},\ w(0,t) = w(1,t) = 0 \text{ for } 0 < t \le T,$$

where $\lim_{w\to c^-} f(w) = \infty$ for some positive constant c, and $f(w)$ and $f'(w)$ are positive for $0 \le w < c$. They showed that there exists a critical length a^* such that for $a \le a^*$, w exists for $0 \le t < \infty$, and for $a > a^*$, $\max\{w(x,t) : 0 \le x \le 1\}$ reaches c^-, namely quenching occurs, in a finite time. Thus, existence of a critical domain of length a^* in quenching is equivalent to existence of b^* in blow-up in the sense that for $b \in (0, b^*] \cup [1-b^*, 1)$, the solution exists for $0 \le t < \infty$, and for $b \in (b^*, 1-b^*)$, the solution always blows up in a finite time.

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