

A Boosting Algorithm for Regression*

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Abstract. A new boosting algorithm ADABOOST-R Δ for regression problems is presented and upper bound on the error is obtained. Experimental results to compare ADABOOST-R Δ and other learning algorithms are given.

1 Introduction

Boosting refers to the general problem of producing a very accurate prediction algorithm by appropriately combining rough and moderately inaccurate ones. It works by calling repeatedly a given “weak” learning algorithm on various distributions on the training set, and combining the hypotheses obtained with a linearly separable boolean function.

The boosting algorithm ADABOOST proposed by Freund and Schapire [1] and presented in Section 2 has been successfully applied to improve the performance of different learning algorithms used for classification problems both binary and multiclass [2]. The first extension for regression problems $f : X \rightarrow [0, 1]$ is the algorithm ADABOOST-R [1]. In this paper we present ADABOOST-R Δ , a different and more general extension for problems $f : X \rightarrow [0, 1]^m$. The main theoretical result is an upper bound on the error.

To analyse the performance of ADABOOST-R Δ we have done experiments using backpropagation as “weak” learning algorithm. Preliminary results show good convergence properties of ADABOOST-R Δ ; notably it is able to lower both the mean and the maximum error on either the training and the test set. For regression problems $f : X \rightarrow [0, 1]$ it works better than ADABOOST-R.

2 Preliminary definitions and results

Given a set X and $Y = \{0, 1\}$, let \mathcal{P} be a probability distribution on $X \times Y$. An N -sample is a sequence $\langle (x_1, y_1), \dots, (x_N, y_N) \rangle$ with $(x_k, y_k) \in X \times Y$; we call $Samp_N$ the set of all the N -samples and $Samp = \cup_N Samp_N$.

Given a class of functions $H \subseteq \{f \mid f : X \rightarrow \{0, 1\}\}$, a learning algorithm \mathcal{A} on H is a function $\mathcal{A} : Samp \rightarrow H$.

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Let $S = \langle (x_1, y_1), \dots, (x_N, y_N) \rangle$ be a N -sample and let $h = \mathcal{A}_S$ be the hypothesis output by \mathcal{A} on input S , the empirical error $\hat{\epsilon}$ of \mathcal{A} on S is

$$\hat{\epsilon} = \frac{\#\{(x_k, y_k) \mid y_k \neq h(x_k)\}}{N}$$

while the generalization error is

$$\epsilon_g = \mathcal{P}\{y \neq h(x)\}.$$

Under weak conditions on H (that is the Vapnik-Chervonenkis dimension [3] of H is finite), choosing elements of $X \times Y$ randomly and independently according to \mathcal{P} , for sufficiently large samples, the empirical error is close to the generalization error with high probability [4]. For this reason a good learning algorithm should minimize the empirical error. Often this is a difficult task because of the large amount of computational resources required and computationally efficient algorithms are usually moderately accurate.

Boosting is a general method for improving the accuracy of a learning algorithm. In particular we refer to the algorithm ADABOOST presented in [1] and described below. It has in input a learning algorithm \mathcal{A} , a N -sample $S = \langle (x_1, y_1), \dots, (x_N, y_N) \rangle$, a distribution D on the elements of S , an integer T and gives in output a final hypothesis $h_f = HS(\sum_{k=1}^T w_k h_k - \lambda)$ with $h_k \in H$ ($k = 1, T$); HS denotes the function $HS(x) = \text{if } x \geq 0 \text{ then } 1 \text{ else } 0$. Even whether \mathcal{A} is moderately inaccurate, for a sufficiently large T the error made by the final hypothesis h_f on the sample S can be made close to 0. Besides, the generalization ability of h_f is good since the Vapnik-Chervonenkis dimension of the family of the final hypothesis does not grow too much [1].

Algorithm ADABOOST

Input: a N -sample $S = \langle (x_1, y_1), \dots, (x_N, y_N) \rangle$, a distribution D on S , a learning algorithm \mathcal{A} , an integer T .

Initialize the weight vector $w_i^1 = D(i)$ for $i = 1, \dots, N$

Do for $t = 1, 2, \dots, T$

1. Set $\mathbf{p}^{(t)} = \frac{\mathbf{w}^{(t)}}{\sum_{i=1}^N w_i^{(t)}}$

2. Choose randomly with distribution $\mathbf{p}^{(t)}$ the sample $S^{(t)}$ from S ; call the learning algorithm \mathcal{A} and get the hypothesis $h_t = \mathcal{A}_{S^{(t)}}$

3. Calculate the error $\epsilon_t = \sum_{i=1}^N p_i^{(t)} |h_t(x_i) - y_i|$

4. Calculate $\beta_t = \frac{\epsilon_t}{(1-\epsilon_t)}$

5. Set the new weights vector to be: $w_i^{(t+1)} = w_i^{(t)} \beta_t^{1-|h^{(t)}(x_i) - y_i|}$

Output the hypothesis $h_f = HS\left(\sum_{k=1}^T (\log \frac{1}{\beta_t}) h_t(x) - 1/2 \sum_{k=1}^T (\log \frac{1}{\beta_t})\right)$

An upper bound to the error $\epsilon = \sum_{k=1}^N D(k) \cdot |h_f(x_k) - y_k|$ is given by the following

Theorem 1 (Freund-Schapire). *Suppose the learning algorithm \mathcal{A} , when called by ADABOOST, generates hypotheses with errors $\epsilon_1, \dots, \epsilon_T$. Then the error $\epsilon = \sum_{k=1}^N D(k) \cdot |h_f(x_k) - y_k|$ of the final hypothesis h_f output by ADABOOST is bounded by*

$$\epsilon \leq 2^T \prod_{t=1}^T \sqrt{\epsilon_t(1 - \epsilon_t)}.$$

ADABOOST can be applied to classification problems with 2 classes. It has been generalized to multiclass problems ($Y = \{1, \dots, K\}$) and to regression problems ($Y = [0, 1]$) [1]. Roughly speaking, the main idea of the algorithm ADABOOST.R designed by Freund and Shapire for regression problems, is that of transforming the regression problem into a classification one using the total order relation \leq on the real numbers. For example, every hypothesis $h : X \rightarrow [0, 1]$ is transformed into the boolean function $\hat{h} : X \times [0, 1] \rightarrow \{0, 1\}$ with

$$\hat{h}(x, y) = \begin{cases} 1 & y \geq h(x) \\ 0 & \text{otherwise} \end{cases}$$

In the next paragraph we present a different way of transforming a regression problem for functions with values in $[0, 1]^m$, into a classification problem. It develops an idea presented in [5] and it is based on the notion of norm in R^m .

3 The algorithm ADABOOST-R Δ

In this section we show a boosting algorithm ADABOOST-R Δ for regression problems. In this setting, Y is $[0, 1]^m$ instead of $\{0, 1\}$; as before, a sample S , chosen at random according to a probability distribution \mathcal{P} on $X^m \times Y$, is given to a learning algorithm \mathcal{A} , that outputs a hypothesis $h : X \rightarrow [0, 1]^m$.

Given a norm $\| \cdot \|$ on R^m , fixed $\Delta > 0$, we say that x and \tilde{x} “ Δ -agree” if $\|x - \tilde{x}\| \leq \Delta$. We consider as generalization error ϵ_g^Δ of a hypothesis h the probability that y and $h(x)$ does not “ Δ -agree”, that is $\epsilon_g^\Delta = \int HS(\|h(x) - y\| - \Delta) dP$. Analogously, the empirical error ϵ^Δ on a sample $S = \langle (x_1, y_1), \dots, (x_N, y_N) \rangle$ is $\epsilon^\Delta = \sum (HS(\|h(x_k) - y_k\| - \Delta))$.

Algorithm ADABOOST-R Δ

Input: a N -sample $S = \langle (x_1, y_1), \dots, (x_N, y_N) \rangle$, a distribution D on S , a learning algorithm \mathcal{A} , an integer T , a real number Δ .

Initialize the weight vector $w_i^1 = D(i)$ for $i = 1, \dots, N$

Do for $t = 1, 2, \dots, T$

1. Set $\mathbf{p}^{(t)} = \frac{\mathbf{w}^{(t)}}{\sum_{i=1}^N w_i^{(t)}}$
2. Choose randomly with distribution $\mathbf{p}^{(t)}$ the sample $S^{(t)}$ from S ; call the learning algorithm \mathcal{A} and get the hypothesis $h_t = \mathcal{A}_{S^{(t)}}$
3. Calculate the error $\epsilon_t = \sum_{i=1}^N p_i^{(t)} HS(\|h_t(x_i) - y_i\| - \Delta)$
if $\epsilon_t > 1/2$ then $T = t - 1$ and abort loop

4. Calculate $\beta_t = \frac{\epsilon_t}{(1-\epsilon_t)}$

5. Set the new weights vector to be $w_i^{(t+1)} = w_i^{(t)} \beta_t^{1-HS(\|h_t(x_i)-y_i\|-\Delta)}$

Output the hypothesis $h_f = \operatorname{argmax}_{y \in [0,1]^m} \sum \alpha_t HS(\Delta - \|h_t(x_i) - y_i\|)$ where $\alpha_t = \log \frac{1}{\beta_t}$

An upper bound to the error $\epsilon^{2\Delta} = \sum (HS(\|h_f(x_k) - y_k\| - 2\Delta))$ is given by following

Theorem 2. *Suppose the learning algorithm \mathcal{A} , when called by ADABOOST-R Δ , generates hypotheses h_t with errors $\epsilon_1, \dots, \epsilon_T < 1/2$ and let h_f be the final hypothesis output by ADABOOST-R Δ . Then*

$$\sum_{\|h_f(x_k) - y_k\| > 2\Delta} D(k) \leq 2^T \prod_{t=1}^T \sqrt{\epsilon_t(1-\epsilon_t)}$$

Proof. (Outline) We transform the regression problem $X \rightarrow [0, 1]^m$ into a classification problem $X \times [0, 1]^m \rightarrow \{0, 1\}$

- the sample $S = \langle (x_i, y_i) \mid i = 1, N \rangle$ is transformed into the sample $\widehat{S} = \langle (x_i, y_i), 0 \mid i = 1, N \rangle$
- the distribution $D(i)$ is transformed into $\widehat{D}(i) = D(i)$ on the sample \widehat{S}
- the algorithm \mathcal{A} with input S is transformed into the algorithm $\widehat{\mathcal{A}}$ with input \widehat{S} with the rule: $\widehat{\mathcal{A}}_{\widehat{S}}(x_i, y_i) = HS(\|\mathcal{A}_S(x_i) - y_i\| - \Delta)$

Let $w^{(t)}$ and $\epsilon^{(t)}$ be respectively the weights vector and the error at step t of the algorithm ADABOOST-R Δ and let $\widehat{w}^{(t)}$ and $\widehat{\epsilon}^{(t)}$ be respectively the weight vector and the error at step t of the algorithm ADABOOST applied to the associated classification problem. By induction it can be proved that

$$\widehat{w}^{(t)} = w^{(t)}, \quad \widehat{\epsilon}^{(t)} = \epsilon^{(t)} \quad (1 \leq t \leq T)$$

Let $\widehat{h}_f(x, y)$ be the final hypothesis given by ADABOOST and $h_f(x)$ be the final hypothesis given by ADABOOST-R Δ . If $\|h_f(x_i) - y_i\| \geq 2\Delta$ then $\widehat{h}_f(x_i, y_i) = 1$. Let us suppose, on the contrary, that $\widehat{h}_f(x_i, y_i) = 0$, then

$$\sum_t \alpha_t HS(\|h_t(x_i) - y_i\| - \Delta) < 1/2 \sum_t \alpha_t \quad (1)$$

Let $I = \{t \mid \|h_t(x_i) - y_i\| < \Delta\}$, the inequality (1) becomes $\sum_{t \notin I} \alpha_t < 1/2 \sum_t \alpha_t$ and the following relation hold

$$\sum_{t \in I} \alpha_t > \sum_{t \notin I} \alpha_t \quad (2)$$

Since $h_f = \operatorname{argmax}_{y \in [0,1]^m} \sum \alpha_t HS(\Delta - \|h_t(x_i) - y_i\|)$ then

$$\sum_t \alpha_t HS(\Delta - \|h_t(x_i) - h_f(x_i)\|) \geq \sum_t \alpha_t HS(\Delta - \|h_t(x_i) - y_i\|) \quad (3)$$

From (2) and (3) follows

$$\sum_{t \in I} \alpha_t HS(\Delta - \|h_t(x_i) - h_f(x_i)\|) > \sum_{t \notin I} \alpha_t (1 - HS(\Delta - \|h_t(x_i) - h_f(x_i)\|)) \geq 0 \quad (4)$$

From the inequality (4) one can assert that there exists $\tilde{t} \in I$ such that $HS(\Delta - \|h_{\tilde{t}}(x_i) - h_f(x_i)\|) = 1$; for such \tilde{t} it holds that $\|h_f(x_i) - h_{\tilde{t}}(x_i)\| < \Delta$ and $\|h_{\tilde{t}}(x_i) - y_i\| < \Delta$. Hence by the triangular inequality we obtain: $\|h_f(x_i) - y_i\| < 2\Delta$ but this is against the hypothesis.

Since $\|\widehat{h}_f(x_i) - y_i\| > 2\Delta$ implies $\widehat{h}_f(x_i, y_i) = 1$, using Theorem 1 we conclude

$$\sum_{\|\widehat{h}_f(x_i) - y_i\| > 2\Delta} D(i) \leq \sum_{\widehat{h}_f(x_i, y_i) = 1} D(i) \leq 2^T \prod_{t=1}^T \sqrt{\epsilon_t (1 - \epsilon_t)}$$

□

4 Experimental results

In this section we present some preliminary results to evaluate the performance of ADABOOST- $R\Delta$ in terms of learning accuracy and computational efficiency. The experiments have been done as follows:

- the functions to be learned are functions $f : R \rightarrow R$ or $g : R^2 \rightarrow R^2$
- the “weak algorithm” \mathcal{A} , given in input to ADABOOST- $R\Delta$, is backpropagation on neural networks of fixed architecture
- the parameter Δ has been set at 1.5δ , where δ is the error on the training set made by backpropagation and preliminarily computed.

The experiments show that ADABOOST- $R\Delta$ exhibits better convergence properties than backpropagation: after the same number of epochs the accuracy on either the training and the test sets is higher. A qualitative example of this behaviour is shown in Figure 1. Figure 1a and 1b show respectively the interpolation of the function $f(x) = (\sin(10 \cdot x) + 2)/4 + \sin(50 \cdot (x + 0.5)^2)/15 + 0.1N(0, 0.1)$ made by backpropagation and by ADABOOST- $R\Delta$.

Quantitative results for a function $g : R^2 \rightarrow R^2$, described by the the expression $g(x, y) = \sin(\frac{6}{5 \cdot (x + 0.5)}) \cdot \sin(\frac{6}{5 \cdot (y + 0.5)})/3 + 0.5$ with gaussian noise $0.2 \cdot N(0, 0.2)$ added, are given in Table 1 (left). Two different network architectures (a denotes the architecture 2-10-10-2, b denotes the architecture 2-15-15-2) have been trained for different numbers of epochs.

For functions $f : R \rightarrow R$ we have also compared the performance of ADABOOST- $R\Delta$ with the algorithm ADABOOST-R. Table 1 (right) shows the results relative to the function $f(x) = \sin(\frac{1}{(0.03 \cdot x)})/5 + 0.2 \cdot N(0, 0.2)$

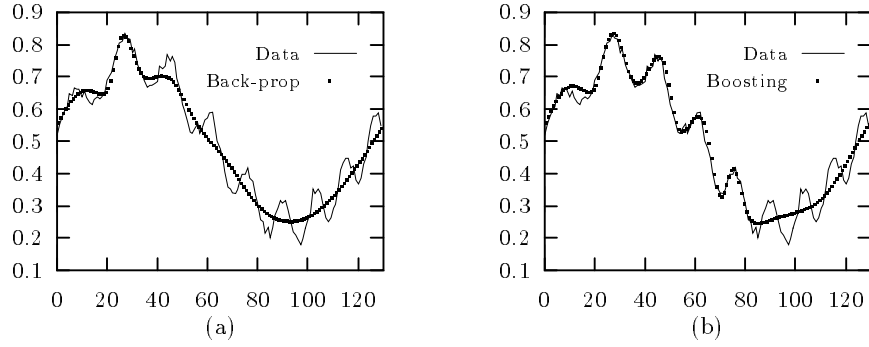


Fig. 1. Comparison between back-propagation and ADABOOST-R Δ .

str	data	epochs	T	mean err.	max err.
a	tr	1000	5	0.0428	0.1607
a	test	1000	5	0.0433	0.1545
a	tr	5000	1	0.0471	0.1739
a	test	5000	1	0.0475	0.1664
b	tr	1500	4	0.0417	0.1687
b	test	1500	4	0.0425	0.2101
b	tr	6000	1	0.0434	0.2054
b	test	6000	1	0.044 0	0.2404

alg.	data	epochs	T	mean err.	max err.
F.S.	tr	1000	6	0.021	0.111
F.S.	test	1000	6	0.020	0.128
Δ	tr	1000	5	0.017	0.058
Δ	test	1000	5	0.018	0.073
b.pr.	tr	6000	1	0.031	0.204
b.pr.	test	6000	1	0.036	0.228

Table 1.

References

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