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NOTES

A BOREL SET NOT CONTAINING A GRAPH[1]

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Examples of Borel sets X, Y, B such that (a) $B \subset X \times Y$, (b) the projection of B on X is X, but (c) for no Borel measurable d mapping X into Y is the graph of d a subset of B have been given by Novikoff [4], Sierpiński [5], and Addison [1]. Such examples are of interest in dynamic programming (see for instance [2]), since if we interpret X as the set of states of some system, Y as the set of available acts, and $I_B(x, y)$, where I_B is the indicator of B, as your income if the system is in state x and you choose act y, you can earn 1 in every state, but there is no Borel measurable plan, i.e. function d from X into Y, with $d(x)$ specifying the act to be chosen when the system is in state x, that earns 1 in every state.

This note presents a new example X, Y, B, simpler than those previously given. The proof that it *is* an example uses ideas from Addison's construction, and a theorem of Gale and Stewart [3] on infinite games of perfect information, and is somewhat more complicated than Addison's.

To construct X, Y, B, denote by U the set of all finite sequences $u = (n_1, \cdots, n_k)$ of positive integers, $k = 1, 2, \cdots$, and by X the set of subsets of U. We associate with each $x \varepsilon X$ a game $G(x)$ between two players α and β, as follows. The players alternately choose positive integers, α choosing first, each choice made with complete information about all previous choices. For a play

$$\omega = (n_1, n_2, \cdots),$$

define $k(\omega)$ as the first integer i for which $(n_1, \cdots, n_i) \not\varepsilon x$, $k(\omega) = \infty$ if $(n_1, \cdots, n_i) \varepsilon x$ for all i. A play ω is a *win for* α (in $G(x)$) if $k(\omega)$ is even, a *win for* β if $k(\omega)$ is odd, and a *draw* if $k(\omega) = \infty$. Informally, whoever first leaves x loses; if neither ever leaves x, they draw. Denote by Y_1 the set of (pure) strategies for α, and by B_1 the set of pairs $(x, y) \varepsilon X \times Y_1$ such that y guarantees α at least a draw in $G(x)$. Similarly Y_2 is the set of strategies for β and B_2 is the set of pairs $(x, y) \varepsilon X \times Y_2$ such that y guarantees β a draw or better in $G(x)$. Put $Y = Y_1 \cup Y_2$, $B = B_1 \cup B_2$. Then X, Y, B are our example.

In decision language, you are presented with an x. You must then choose which player you want to be in $G(x)$, and specify a strategy y for that player. If the strategy specified guarantees a draw or better for the chosen player, you earn a dollar; if not you earn nothing. The claim is that, while you can earn a dollar for

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every x, there is no Borel measurable way of associating with each x a y that earns a dollar for you at that x.

We sketch the proof. First, B_1 is Borel, since $B_1^c \cap X \times Y_1$ is the set of $(x, y) \varepsilon X \times Y_1$ for which there is a $u = (n_1, \cdots, n_k)$ of odd length, consistent with y, such that $u \not\varepsilon x$ but all proper segments $(n_1, \cdots, n_i)$, $1 \leqq i < k$, of u are in x. Similarly B_2 is Borel, so B is Borel.

Next, the projection of B on X is X, for the set of plays ω that win for α in $G(x)$ is open, so that, according to the Gale-Stewart theorem [3], either α can force a win, in which case x is in the projection of B_1 on X, or β can prevent α from winning, in which case x is in the projection of B_2 on X.

Finally, let d be any Borel measurable function from X into Y. We show that the graph of d is not a subset of B. We may and shall assume that the graph of d intersects both B_1 and B_2. The main steps in the proof are:

(1) Associate with each $x \varepsilon X$ an $x' \varepsilon X$ and two analytic subsets A_1, A_2 of X so that

$$G(x') \text{ is a win for } \alpha \text{ if } x \varepsilon A_1 - A_2,$$

$$G(x') \text{ is a win for } \beta \text{ if } x \varepsilon A_2 - A_1,$$

$$x' \text{ is a Borel measurable function of } x.$$

Every pair A_1, A_2 of non-empty analytic subsets of X is associated with some x.

(2) Denote by D the set of $x \varepsilon X$ for which $d(x) \not\varepsilon B_1$, by H the set of x for which $x' \varepsilon D$, and choose x_0 for which $A_1 = H$, $A_2 = H^c$.

(3) Put $x^* = x_0'$. Then

$$x^* \not\varepsilon D \Rightarrow x_0 \not\varepsilon H \Rightarrow \beta \quad \text{wins} \quad G(x^*) \Rightarrow d(x^*) \not\varepsilon B_1 \Rightarrow x^* \varepsilon D,$$

so $x^* \varepsilon D$. Now $x^* \varepsilon D \Rightarrow d(x^*) \not\varepsilon B_1$. But also $x^* \varepsilon D \Rightarrow x_0 \varepsilon H \Rightarrow \alpha$ wins $G(x^*)$, so $d(x^*) \not\varepsilon B_2$. We have found a state x^* for which $G(x^*)$ is a win for α, but $d(x^*)$ does not even guarantee him a draw.

Informally, we have produced an x^* such that (a) draws cannot occur in $G(x^*)$ (b) α can win $G(x^*)$ iff $d(x^*)$ is not a winning strategy for him in $G(x^*)$. So if α can't win $G(x^*)$, he can win $G(x^*)$ with $d(x^*)$, so he can win $G(x^*)$, so he can't win $G(x^*)$ with $d(x^*)$.

It remains to carry out the construction in (1). To do this, let ϕ map X Borel measurably *onto* the space $F \times G$ of pairs of f, g of continuous functions from Ω into X, where Ω is the space of infinite sequences $\omega = (n_1, n_2, \cdots)$ of positive integers. Let ψ be the following (Borel measurable) map of $F \times G \times X$ into X: $\psi(f, g, s)$ consists of all $u = (n_1, \cdots, n_k)$ such that either (a) k is odd and $x \varepsilon$ closure of $f\Omega(n_1, n_3, \cdots, n_k)$, where $\Omega(v)$ denotes the set of ω that begin with v or (b) k is even and $x \varepsilon$ closure of $g\Omega(n_2, n_4, \cdots, n_k)$. Then the association with x of $x' = \psi(\phi(x), x)$, $A_1 = f\Omega$, $A_2 = g\Omega$, where $(f, g) = \phi(x)$ has the required properties. For if $x \varepsilon A_1 - A_2$, there is an $\omega = (n_1, n_3, n_5, \cdots)$ with $f(\omega) = x$. If α plays $n_1, n_3, n_5, \cdots$ in $G(x')$, he never leaves x'. But no matter what sequence $n_2, n_4, \cdots \beta$ plays, $g(n_2, n_4, \cdots) \neq x$, so β must leave x' eventually. Similarly $G(x')$ is a win for β if $x \in A_2 - A_1$.

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