PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2005) **461**, 3231–3243
doi:10.1098/rspa.2005.1513
Published online 25 August 2005

A Borg–Levinson theorem for trees

By B. M. Brown[1] and R. Weikard[2]

[1]*School of Computer Science, University of Wales, Cardiff, PO Box 916, Cardiff CF2 3XF, UK*
(malcolm.brown@cs.cf.ac.uk)
[2]*Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294-1170, USA*
(rudi@math.uab.edu)

We prove that the Dirichelet-to-Neumann map for a Schrödinger operator on a finite simply connected tree determines uniquely the potential on that tree.

Keywords: Dirichelet-to-Neumann map; inverse spectral problems; trees

1. Introduction

Recently there has been much interest in the inverse spectral problem for the one dimensional Sturm–Liouville problem

$$-y'' + qy = \lambda y, \quad x \in [0, l),$$

where l may be either finite or infinite. More precisely the problem is to recover q from spectral data. Borg (1946) and Levinson (1949) showed when q was real that it is uniquely determined by the associated Titchmarsh–Weyl function. Local versions of this theorem have recently been proved by Simon (1999), Gesztesy & Simon (2000) and Bennewitz (2001), the latter theorem being the result of an elegant and short proof. In the case of complex q, Brown *et al.* (2002) have shown, using a method modelled on the approach in Bennewitz (2001), that the local Borg theorem is still true. In the case of real q mention must also be made of the famous result of Gel'fand & Levitan (1951) who showed that q could be determined by the spectral function which in the case of a finite interval can itself be determined by two sets of eigenvalues.

The study of the Sturm–Liouville problem on an interval has been extended to its consideration on trees. This has been motivated by quantum models in both physics and chemistry see for instance (Exner *et al.* 1988; Gerasimenko 1988; Gerasimenko & Pavlov 1988; Bulla & Trenkler 1990). In the case of real coefficients the self-adjoint extensions of symmetric operators on trees have been studied by Carlson in Carlson (1998) and their self-adjoint extensions characterized. Treelike domains have been used to model the scattering problem for partial differential equations (Melnikov & Pavlov 2001), while in a series of papers (Evans & Saitō 2000; Evans *et al.* 2001) the authors study

Received 26 November 2003
Accepted 6 May 2005

PDE problems on domains with fractal boundaries and reduce the study of these problems to ones on trees. Further, a recent result of Carlson (1997) has shown that on an infinite homogeneous tree the spectrum of the free Laplacian consists of bands and gaps and possibly eigenvalues. Sobolev & Solomyak (2002) have studied the effect of introducing a real perturbation of the zero potential on such an infinite homogeneous tree when the coupling constant tends to infinity. A further impetus has been given to these studies by the requirements in micro electronics fabrication problems, in particular the construction of quantum switches and other nano-computational devices (see Mikhailova *et al.* 2002; Mikhaylova & Pavlov 2002; Pavlov 2002) where scattering problems on trees have been studied. In a recent paper, Pivovarchik (2000) considers the inverse spectral problem of the recovery of the coefficients of the Sturm–Liouville problem on a domain consisting of three intervals together with appropriate interface and boundary conditions. He shows that given the spectrum of the problem on the tree together with the Dirichlet problems on each edge, the coefficients of the Sturm–Liouville problem may be recovered. For further references the reader may consult the January 2004 issue of Waves in Random Media where a special section on quantum graphs appeared.

The approach in Borg (1946) and Levinson (1949) is to show that the Titchmarsh–Weyl m-function determines uniquely the function q in the case of a one-dimensional Schrödinger equation. Nachman *et al.* (1988) have shown an analogous result in the case of a multi-dimensional Schrödinger equation on a bounded domain using the Dirichlet-to-Neumann map instead of the m-function. Curtis & Morrow (1990, 1991) have a similar result for the (combinatorial) problem of a resistor network. We will show here that this is the right idea also for treelike domains following the approach in Bennewitz (2001) and Brown *et al.* (2002).

More precisely, in this paper we shall study the inverse spectral problem for the Sturm–Liouville problem

$$-y'' + q_j y = \lambda y, \quad x \in [0,1],$$

on each branch of a finite tree where the q_j are complex-valued and integrable and where continuity and Kirchhoff type conditions for the solutions and their derivatives are imposed at the internal branch points. We shall follow the approach in Brown *et al.* (2002) adapted to the tree domain with the m-function being replaced by the Dirichlet-to-Neumann map. The main result of this paper is the following theorem (cf. §§2 and 3 for precise definitions).

Theorem 1.1. *Let q be a complex-valued integrable potential supported on a simply connected finite tree. Then the associated (generalized) Dirichlet-to-Neumann map uniquely determines the potential almost everywhere on the tree.*

Section 2 contains basic definitions and a formulation of the Sturm–Liouville problem on a tree through interface conditions between the branches. Section 3 defines the Dirichlet-to-Neumann map while §4 discusses the structure of the Green's function for this problem. The Weyl solution is introduced in §5 and its asymptotics are obtained. Section 6 shows how the Dirichlet-to-Neumann map may be used to recover the coefficients of boundary edges while an induction argument in §7 shows how the remaining coefficients are also determined by the map. An appendix contains the asymptotics of the basic solutions on an interval.

2. Preliminaries

(a) Trees

For $j=1,\ldots,r$ let ϵ_j be homeomorphisms defined on [0,1]. The Hausdorff space

$$T=\bigcup_{j=1}^{r}\{\epsilon_j(t):t\in[0,1]\},$$

is called a directed finite tree if it is simply connected and if $\epsilon_j(t)=\epsilon_k(s)$ happens only if $j=k$ or if $t,s\in\{0,1\}$. The points $\epsilon_1(0)$, $\epsilon_1(1),\ldots,\epsilon_r(0)$, $\epsilon_r(1)$ are called vertices. The set of all vertices is denoted by V. It contains precisely $r+1$ elements. The images of the functions ϵ_j are called edges. The structure of the tree is described uniquely by specifying the vertices connected by each edge, i.e. by the set

$$((\epsilon_1(0),\epsilon_1(1)),\ldots,(\epsilon_r(0),\epsilon_r(1)))\in(V\times V)^r.$$

A vertex is called a boundary vertex if it belongs to only one edge. Such an edge will be called a boundary edge. A vertex which belongs to several edges is called an internal vertex. An edge both of whose endpoints are internal vertices is called an internal edge. Without loss of generality we assume that the boundary edges are labelled $1,\ldots,n_0$ when n_0 denotes the number of boundary vertices. We will also assume that the boundary vertices are given by $v_j=\epsilon_j(0)$, $j=1,\ldots,n_0$.

(b) The interface conditions

An integrable function y on T may be represented as $\boldsymbol{y}=(y_1,\ldots,y_r)^{\mathrm{T}}$, where $y_j(t)=y(\epsilon_j(t))$.

If q is an integrable function on T we are interested in problems associated with the differential expression L given by $(Ly)(\epsilon_j(t))=-y_j''(t)+q_j(t)y_j(t)$. We impose the following requirements on y.

(i) For each j the functions y_j and y_j' are absolutely continuous on [0,1].
(ii) For each j the function $-y_j''+q_jy_j$ is in $L^2([0,1])$.
(iii) y is continuous on T.
(iv) For each internal vertex v_k the Kirchhoff condition

$$\sum_{\epsilon_j(1)=v_k}y_j'(1)-\sum_{\epsilon_j(0)=v_k}y_j'(0)=0$$

holds.

Conditions (iii) and (iv) are called interface conditions.

If ν_k edges meet at vertex v_k then these are precisely ν_k conditions. Therefore, every internal edge gives rise to two conditions and every boundary edge gives rise to one condition. Altogether there are therefore $2r-n_0$ interface conditions.

To represent the interface conditions we introduce next the following operators which map $C^1([0,1])^n$ to $\mathbb{C}^n$ (any n):

$$E_0y=y(0),\quad E_1y=y(1),\quad D_0y=y'(0),\quad D_1y=y'(1).$$

The interface conditions are then given as $\mathcal{I}\boldsymbol{y}=0$, where $\mathcal{I}$ is a $(2r-n_0)\times r$ matrix whose entries are zeros or $\pm E_0, \dots, \pm D_1$. Note that the first n_0 columns of $\mathcal{I}$ correspond to boundary edges. Therefore, these columns will not involve E_0 or D_0.

For example, for the tree given by $((v_1, v_4), (v_2, v_4), (v_3, v_4))$ we have $r=3$, $n_0=3$ and

$$\mathcal{I} = \begin{pmatrix} E_1 & -E_1 & 0 \\ 0 & E_1 & -E_1 \\ D_1 & D_1 & D_1 \end{pmatrix}.$$

For the tree given by $((v_1, v_3), (v_2, v_4), (v_3, v_4))$ we have $r=3$, $n_0=2$ and

$$\mathcal{I} = \begin{pmatrix} E_1 & 0 & -E_0 \\ D_1 & 0 & -D_0 \\ 0 & E_1 & -E_1 \\ 0 & D_1 & D_1 \end{pmatrix}.$$

Denote by $c_j(\lambda, \cdot)$ and $s_j(\lambda, \cdot)$ the solutions of $-y_j''+q_jy_j=\lambda y_j$ satisfying initial conditions $c_j(\lambda,0)=s_j'(\lambda,0)=1$ and $c_j'(\lambda,0)=s_j(\lambda,0)=0$. Note that $c_j(\cdot, x)$ and $s_j(\cdot, x)$ are entire functions of growth order 1/2 at most. Now any solution of $-y_j''+q_jy_j=\lambda y_j$ can be written as

$$y_j(x) = a_jc_j(\lambda, x) + b_js_j(\lambda, x),$$

when a_j and b_j denote appropriate constants. The column $(a_1, \dots, a_r, b_1, \dots, b_r)^{\mathrm{T}}$ will be denoted by ξ.

We introduce the following matrices

$$C_{\mathrm{e}}(\lambda, x) = \mathrm{diag}(c_1(\lambda, x), \dots, c_{n_0}(\lambda, x)),$$

$$S_{\mathrm{e}}(\lambda, x) = \mathrm{diag}(s_1(\lambda, x), \dots, s_{n_0}(\lambda, x)),$$

$$C_{\mathrm{i}}(\lambda, x) = \mathrm{diag}(c_{n_0+1}(\lambda, x), \dots, c_r(\lambda, x)),$$

$$S_{\mathrm{i}}(\lambda, x) = \mathrm{diag}(s_{n_0+1}(\lambda, x), \dots, s_r(\lambda, x)),$$

and the block matrices

$$C(\lambda, x) = \mathrm{diag}(C_e(\lambda, x), C_i(\lambda, x))$$

and

$$S(\lambda, x) = \mathrm{diag}(S_e(\lambda, x), S_i(\lambda, x)).$$

Then we have

$$\boldsymbol{y}(x) = (C(\lambda, x), S(\lambda, x))\xi.$$

3. The generalized Dirichlet-to-Neumann map

We will show below that for all but countably many values of λ there will be a unique solution of $Ly=\lambda y$ satisfying the Dirichlet boundary conditions $y_j(0)=f_j$ for $j=1, \dots, n_0$. One may then compute the values $g_j=-y_j'(0)$. The relationship between the f_j and the g_j is linear and is called the Dirichlet-to-Neumann map. Instead of Dirichlet and Neumann data we will consider general linear boundary conditions described by n_0 pairs (α'_j, α_j) and n_0 pairs (β'_j, β_j) which satisfy the relationship

$$\alpha'_j\beta_j - \alpha_j\beta'_j = 1, \tag{3.1}$$

for $j=1, \dots, n_0$. More precisely, given the unique solution of $Ly=\lambda y$ satisfying the non-homogeneous boundary conditions $\alpha'_j y_j(0) - \alpha_j y'_j(0) = f_j$ (which exists for all but countably many values of λ) we compute the values $g_j = \beta'_j y_j(0) - \beta_j y'_j(0)$ and call the corresponding linear map the generalized Dirichlet-to-Neumann map which we will denote by Λ. The goal of this section is to compute it.

The boundary conditions are described by the equation $\mathcal{A}\boldsymbol{y}=f$, where $\mathcal{A}$ is the $n_0 \times r$ matrix (boundary operator)

$$\mathcal{A} = \begin{pmatrix} \alpha'_1 E_0 - \alpha_1 D_0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha'_{n_0} E_0 - \alpha_{n_0} D_0 & 0 & \dots & 0 \end{pmatrix},$$

and where $f = (f_1, \dots, f_{n_0})^{\mathrm{T}} \in \mathbb{C}^{n_0}$. We also define the matrices

$$\mathcal{B} = \begin{pmatrix} \beta'_1 E_0 - \beta_1 D_0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \beta'_{n_0} E_0 - \beta_{n_0} D_0 & 0 & \dots & 0 \end{pmatrix},$$

as well as

$$A = \mathcal{A}(C, S) = (\mathrm{diag}(\alpha'_1, \dots, \alpha'_{n_0}), 0_{n_0\times(r-n_0)}, \mathrm{diag}(-\alpha_1, \dots, -\alpha_{n_0}), 0_{n_0\times(r-n_0)}),$$

and

$$B = \mathcal{B}(C, S) = (\mathrm{diag}(\beta'_1, \dots, \beta'_{n_0}), 0_{n_0\times(r-n_0)}, \mathrm{diag}(-\beta_1, \dots, -\beta_{n_0}), 0_{n_0\times(r-n_0)}).$$

Finally, let

$$\mathcal{M} = \begin{pmatrix} \mathcal{A} \\ \mathcal{I} \end{pmatrix},$$

and

$$M(\lambda) = \mathcal{M}(C(\lambda, x), S(\lambda, x)).$$

The $2r \times 2r$ matrix M is of central importance. Its determinant $\det M$ is an entire function of λ whose zeros are the eigenvalues of the boundary value problem determined by the α_j and α'_j. Hence there are at most countably many eigenvalues which cannot have a finite accumulation point. Note that $M\xi = Pf$, where

$$P = \begin{pmatrix} I_{n_0\times n_0} \\ 0_{(2r-n_0)\times n_0} \end{pmatrix},$$

when ξ is the vector of coefficient of the basis functions c_j and s_j as described in §2*b*.

Now we have

$$g = \mathcal{B}\boldsymbol{y} = \mathcal{B}(C, S)\xi = B\xi = BM^{-1}Pf.$$

Hence the generalized Dirichlet-to-Neumann map is

$$\Lambda(\lambda) = BM(\lambda)^{-1}P.$$

4. Green's function

We want to obtain a solution of the non-homogeneous system of equations $Ly=h$, where $h\in L^2(T)$ and where y is subject to the homogeneous boundary conditions $\mathcal{A}\boldsymbol{y}=0$ as well as the interface conditions $\mathcal{I}\boldsymbol{y}=0$.

Let $\boldsymbol{h}(t)$ denote the column $(h(\epsilon_1(t)),\dots,h(\epsilon_r(t)))^{\mathrm{T}}$. Define

$$\hat{K}(\lambda,x,t)=\operatorname{diag}(k_1(\lambda,x,t),\dots,k_r(\lambda,x,t))=(C(\lambda,x),S(\lambda,x))\begin{pmatrix}S(\lambda,t)\\-C(\lambda,t)\end{pmatrix},$$

where

$$k_j(\lambda,x,t)=c_j(\lambda,x)s_j(\lambda,t)-s_j(\lambda,x)c_j(\lambda,t)$$

and

$$K(\lambda,x)=\int_0^x\hat{K}(\lambda,x,t)\boldsymbol{h}(t)\mathrm{d}t.$$

Then the general solution of $Ly=h$ may then be represented as

$$\boldsymbol{y}(x)=(C(\lambda,x),S(\lambda,x))\xi+K(\lambda,x).$$

Since $\mathcal{M}\boldsymbol{y}=0$ is equivalent to $M\xi+\mathcal{M}K=0$ we obtain

$$\boldsymbol{y}=-(C,S)M^{-1}\mathcal{M}K+K.$$

Before we proceed we need to determine the vector $\mathcal{M}K$. To this end we write $\mathcal{M}=\mathcal{M}_0+\mathcal{M}_1$, where $\mathcal{M}_0$ involves the operators E_0 and D_0 only while $\mathcal{M}_1$ involves the operators E_1 and D_1 only. Then we have $\mathcal{M}_0K=0$. Suppressing the λ-dependence for a while we see that

$$E_1K=\int_0^1\hat{K}(1,t)\boldsymbol{h}(t)\mathrm{d}t=\int_0^1(E_1\hat{K}(\cdot,t))\boldsymbol{h}(t)\mathrm{d}t,$$

and, since $\hat{K}(x,x)=0$,

$$D_1K=E_1\left[\hat{K}(x,x)\boldsymbol{h}(x)+\int_0^x\hat{K}'(x,t)\boldsymbol{h}(t)\mathrm{d}t\right]=\int_0^1(D_1\hat{K}(\cdot,t))\boldsymbol{h}(t)\mathrm{d}t.$$

Therefore,

$$\mathcal{M}K=\mathcal{M}_1K=\int_0^1(\mathcal{M}_1\hat{K})\boldsymbol{h}\,\mathrm{d}t=\int_0^1(M-M_0)\begin{pmatrix}S\\-C\end{pmatrix}(t)\boldsymbol{h}(t)\mathrm{d}t,$$

where $M_0=\mathcal{M}_0(C,\,S)$.

Let H be the Heaviside function, i.e. $H(t)$ equals zero or one depending on whether t is negative or positive. Then

$$\boldsymbol{y}(x)=\int_0^1\left[-(C(\lambda,x),S(\lambda,x))(I-M(\lambda)^{-1}M_0(\lambda))\begin{pmatrix}S(\lambda,t)\\-C(\lambda,t)\end{pmatrix}\right.$$
$$\left.+H(x-t)\hat{K}(\lambda,x,t)\right]\boldsymbol{h}(t)\mathrm{d}t.$$

The Green's function Γ is now the term in brackets in the integrand on the right-hand side of this equation. Since $\hat{K}(\lambda, t, t) = 0$ we have

$$\Gamma(\lambda, t, t) = -(C(\lambda, t), S(\lambda, t))(I - M(\lambda)^{-1} M_0(\lambda)) \begin{pmatrix} S(\lambda, t) \\ -C(\lambda, t) \end{pmatrix}$$

$$= (C(\lambda, t), S(\lambda, t)) M(\lambda)^{-1} M_0(\lambda) \begin{pmatrix} S(\lambda, t) \\ -C(\lambda, t) \end{pmatrix}.$$

Next let $1 \le k \le n_0$. Then

$$\Gamma_{k,k}(\lambda, t, t) = c_k(\lambda, t) s_k(\lambda, t)[M(\lambda)^{-1} M_0(\lambda)]_{k,k} + s_k(\lambda, t)^2 [M(\lambda)^{-1} M_0(\lambda)]_{k+r,k}$$
$$- c_k(\lambda, t)^2 [M(\lambda)^{-1} M_0(\lambda)]_{k,k+r} - c_k(\lambda, t) s_k(\lambda, t)[M(\lambda)^{-1} M_0(\lambda)]_{k+r,k+r}.$$

Since the only entry in column k of $\mathcal{M}_0$ is $\alpha'_k E_0 - \alpha_k D_0$ situated in row k we obtain that the kth and $(k+r)$th columns of $M_0(\lambda)$ equal $\alpha'_k e_k$ and $-\alpha_k e_k$, respectively.[1] Hence

$$\Gamma_{k,k}(\lambda, t, t) = ((M^{-1})_{k,k} c_k(\lambda, t) + (M^{-1})_{k+r,k} s_k(\lambda, t))(\alpha'_k s_k(\lambda, t) + \alpha_k c_k(\lambda, t)).$$

If we compute the determinant of M by expanding with respect to row k (which contains at most two non-zero entries) we obtain

$$\det M = \alpha'_k \det \mathrm{mr}_{k,k}(M) - (-1)^r \alpha_k \det \mathrm{mr}_{k,k+r}(M),$$

where $\mathrm{mr}_{k,j}(M)$ denotes the minor of M obtained by deleting row k and column j. The minors may be expressed by the corresponding entries of M^{-1}. Therefore,

$$1 = \alpha'_k (M^{-1})_{k,k} - \alpha_k (M^{-1})_{k+r,k}.$$

Since we also have

$$\Lambda_{k,k} = \beta'_k (M^{-1})_{k,k} - \beta_k (M^{-1})_{k+r,k},$$

we obtain

$$(M^{-1})_{k,k} = \beta_k - \alpha_k \Lambda_{k,k} \quad \text{and} \quad (M^{-1})_{k+r,k} = \beta'_k - \alpha'_k \Lambda_{k,k}.$$

Hence

$$\Gamma_{k,k}(\lambda, t, t) = (\theta_k(\lambda, t) - \Lambda_{k,k} \varphi_k(\lambda, t)) \varphi_k(\lambda, t),$$

where

$$\theta_k(\lambda, t) = \beta_k c_k(\lambda, t) + \beta'_k s_k(\lambda, t)$$

and

$$\varphi_k(\lambda, t) = \alpha_k c_k(\lambda, t) + \alpha'_k s_k(\lambda, t).$$

Note $\theta_k(\cdot, t)$ and $\phi_k(\cdot, t)$ are entire functions of growth order 1/2 at most.

5. Weyl solutions

Fix $k \in \{1, \dots, n_0\}$. Let $\psi(k, \lambda, \cdot)$ be the solution of the problem $Ly = \lambda y$, $\mathcal{I}\boldsymbol{y} = 0$, $\mathcal{A}\boldsymbol{y} = e_k$. We call this solution the Weyl solution for the boundary vertex k. The Weyl solutions are uniquely determined for any λ which is not an eigenvalue of

[1] We denote the vector whose entries are zero save for a 1 at position k by e_k.

the boundary value problem determined by the α_j and α_j'. Recall that these are the roots of the determinant of M and hence the poles of Λ.

Lemma 5.1. *Fix k, $j \in \{1, \ldots, n_0\}$. The Weyl solution for the boundary vertex k satisfies*

$$\psi_j(k, \lambda, t) = \delta_{j,k}\theta_j(\lambda, t) - \Lambda_{j,k}(\lambda)\varphi_j(\lambda, t).$$

Proof. Note that $\psi_j(k, \lambda, \cdot) = a_j\theta_j(\lambda, \cdot) + b_j\varphi_j(\lambda, \cdot)$ for appropriate values of a_j and b_j. Hence, using equation (3.1)

$$\delta_{j,k} = \alpha_j'\psi_j(k, \lambda, 0) - \alpha_j\psi_j'(k, \lambda, 0) = a_j$$

and

$$\Lambda_{j,k} = (\Lambda e_k)_j = \beta_j'\psi_j(k, \lambda, 0) - \beta_j\psi_j'(k, \lambda, 0) = -b_j.$$

■

In the following lemma and its proof we will use different conventions on labelling and orienting boundary vertices and boundary edges of trees as before. We will designate one of the boundary vertices as the root of the tree and denote it by v_0. All other boundary vertices will then be called branch tips. The edge attached to the root, denoted by ϵ_0, will be called the stem. Nothing will be assumed about the orientation of boundary edges. Note that every vertex v is connected to another vertex v' by a unique sequence of edges. The number of these edges will be called the distance between v and v' and will be denoted by $d(v, v')$. The number

$$h = \max\{d(v, v_0) : v \text{ is a vertex of the tree}\}$$

is called the height of the tree with respect to the root v_0. (The height of a tree depends on which boundary vertex is designated as root.)

We denote the outward normal derivative of a differentiable function $y : (0,1) \to \mathbb{C}$ at one of the end points by $\dot{y}(0)$ and $\dot{y}(1)$, i.e. we define

$$\dot{y}(p) = \begin{cases} -\lim_{t\to 0} y'(t) & \text{if } p = 0, \\ \lim_{t\to 1} y'(t) & \text{if } p = 1. \end{cases}$$

Finally, we will call a ray in the complex plane admissible if it emanates from zero and lies otherwise in the open upper half plane.

Lemma 5.2. *Suppose T is a tree with root v_0 and $\psi(\lambda, \cdot)$ satisfies the differential equation $Ly = \lambda y$ and the interface conditions $\mathcal{I}\boldsymbol{y} = 0$. Also assume that $\psi(\lambda, \cdot)$ is zero at the branch tips but not identically zero on T. Then, as $\sqrt{\lambda}$ tends to infinity along an admissible ray,*

$$\frac{\dot{\psi}_0(p)}{\psi_0(p)} = -\mathrm{i}\sqrt{\lambda} + O(1), \tag{5.1}$$

where $\psi_0(\lambda, t) = \psi(\lambda, \epsilon_0(t))$ and $p \in \{0,1\}$ is such that $\epsilon_0(p) = v_0$.

Proof. The proof is by induction on the height of the tree. Assume that the height of T with respect to v_0 is one (i.e. T is an interval). If $v_0 = \epsilon_0(1)$ then

$\psi_0(\lambda,\ t)=b_0 s_0(\lambda,\ t)$ for some non-zero b_0 and, using lemma A 1,

$$\frac{\dot\psi_0(\lambda,1)}{\psi_0(\lambda,1)}=\frac{s'_0(\lambda,1)}{s_0(\lambda,1)}=-\mathrm{i}\sqrt{\lambda}+O(1).$$

If $v_0=\epsilon_0(0)$ then $\psi_0(\lambda,t)=a_0c_0(\lambda,t)+b_0s_0(\lambda,t)$, where $a_0c_0(\lambda,1)+b_0s_0(\lambda,1)=0$. Using again lemma A 1 we find

$$\frac{\dot\psi_0(\lambda,0)}{\psi_0(\lambda,0)}=\frac{-b_0}{a_0}=\frac{c_0(\lambda,1)}{s_0(\lambda,1)}=-\mathrm{i}\sqrt{\lambda}+O(1).$$

Next assume that equation (5.1) is true for every tree whose height is at most n and that T has height $n+1$ with respect to v_0. In addition to the stem itself there are k subtrees attached to the internal endpoint v_1 of the stem. We designate v_1 to be the root of each of these subtrees and we assign labels $1,\ldots,k$ to their stems. We also assume that $v_1=\epsilon_j(1)$ for $j=1,\ldots,\ell$ and $v_1=\epsilon_j(0)$ for $j=\ell+1,\ldots,k$.

First, assume again $v_0=\epsilon_0(1)$. Then $\psi_0(\lambda,t)=a_0c_0(\lambda,t)+b_0s_0(\lambda,t)$, where $a_0=\psi(\lambda,\ v_1)$ and where

$$b_0=\sum_{j=1}^{\ell}\psi'_j(\lambda,1)-\sum_{j=\ell+1}^{k}\psi'_j(\lambda,0).$$

Employing the induction hypothesis we obtain $b_0/a_0=-\mathrm{i}k\sqrt{\lambda}+O(1)$ and thus, using lemma A 1,

$$\frac{\dot\psi_0(\lambda,1)}{\psi_0(\lambda,1)}=\frac{a_0c'_0(\lambda,1)+b_0s'_0(\lambda,1)}{a_0c_0(\lambda,1)+b_0s_0(\lambda,1)}=-\mathrm{i}\sqrt{\lambda}+O(1).$$

If, however, $v_0=\epsilon_0(0)$ then $\psi_0(\lambda,t)=a_0c_0(\lambda,t)+b_0s_0(\lambda,t)$, where $a_0c_0(\lambda,1)+b_0s_0(\lambda,1)=\psi(\lambda,v_1)$ and where, using the induction hypothesis,

$$a_0c'_0(\lambda,1)+b_0s'_0(\lambda,1)=\sum_{j=\ell+1}^{k}\psi'_j(\lambda,0)-\sum_{j=1}^{\ell}\psi'_j(\lambda,1)=(\mathrm{i}k\sqrt{\lambda}+O(1))\psi(\lambda,v_1).$$

Solving for a_0 and b_0 and a final application of lemma A 1 gives

$$\frac{\dot\psi_0(\lambda,0)}{\psi_0(\lambda,0)}=\frac{-b_0}{a_0}=-\mathrm{i}\sqrt{\lambda}+O(1).$$

■

The preceding lemma has the following immediate corollary for a Weyl solution if $t=0$. For $t\in(0,1)$ one simply has to apply the lemma to the tree whose stem is $\epsilon_k([t,1])$ rather than $\epsilon_k([0,1])$.

Corollary 5.3. *If* $k\in\{1,\ldots,n_0\}$ *and* $t\in[0,1)$ *then*

$$\frac{\psi'_k(k,\lambda,t)}{\psi_k(k,\lambda,t)}=\mathrm{i}\sqrt{\lambda}+O(1),$$

as $\sqrt{\lambda}$ *tends to infinity along an admissible ray.*

Theorem 5.4. *If* $k\in\{1,\ldots,n_0\}$ *and* $t\in[0,1)$ *then the diagonal Green's function* $\Gamma_{k,k}(\lambda,\ t,\ t)$ *tends to zero as* $\sqrt{\lambda}$ *tends to infinity along an admissible ray.*

Proof. By lemma 5.1 we have $\Gamma_{k,k}(\lambda,t,t)=\psi_k(k,\lambda,t)\varphi_k(\lambda,t)$ and that the Wronskian of $\psi_k(k,\ \lambda,\ \cdot)$ and $\phi_k(\lambda,\ \cdot)$ equals the Wronskian of $\theta_k(k,\ \lambda,\ \cdot)$

and $\phi_k(\lambda,\ \cdot)$ and hence 1. Therefore, as $\sqrt{\lambda}$ tends to infinity,

$$\frac{1}{\Gamma_{k,k}(\lambda,t,t)} = \frac{1}{\psi_k(k,\lambda,t)\varphi_k(\lambda,t)} = \frac{\varphi_k'(\lambda,t)}{\varphi_k(\lambda,t)} - \frac{\psi'_k(k,\lambda,t)}{\psi_k(k,\lambda,t)} = -2\mathrm{i}\sqrt{\lambda} + O(1),$$

using lemma A 1 for the first term and corollary 5.3 for the second. ■

6. The potential on the boundary edges

Theorem 6.1. *The (generalized) Dirichlet-to-Neumann map determines uniquely the potential almost everywhere on the boundary edges.*

Proof. Fix $k\in\{1,\dots,n_0\}$ and $t\in[0,1)$. Suppose q and $\tilde{q}$ are two potentials on T giving rise to the same (generalized) Dirichlet-to-Neumann map. Associated with $\tilde{q}$ are the functions $\tilde{\theta}$, $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{\Lambda}$ just like θ, ϕ, ψ and Λ are associated with q. From lemma A 1 we know that $\varphi_k(\lambda,t)/\tilde{\varphi}_k(\lambda,t)$ tends to one as λ tends to infinity. This fact and theorem 5.4 yield

$$g(\lambda) = \tilde{\varphi}_k(\lambda,t)\psi_k(k,\lambda,t) - \varphi_k(\lambda,t)\tilde{\psi}_k(k,\lambda,t) \to 0,$$

as $\sqrt{\lambda}$ tends to infinity along an admissible ray. Recall that $\psi_k(k,\cdot,\cdot) = \theta_k - \Lambda_{k,k}\varphi_k$ and, by assumption, $\Lambda_{k,k} = \tilde{\Lambda}_{k,k}$. Therefore we find that

$$g(\lambda) = \tilde{\varphi}_k(\lambda,t)\theta_k(\lambda,t) - \varphi_k(\lambda,t)\tilde{\theta}_k(\lambda,t).$$

As noted earlier all of the four terms appearing here on the right-hand side are entire functions of growth order 1/2 when viewed as functions of λ. Thus g is an entire function of growth order 1/2, which tends to zero along the positive and the negative imaginary axis (for instance). The Phragmén–Lindelöf theorem implies that g is bounded in $\mathbb{C}$, so it is constant by Liouville's theorem and in fact identically equal to zero, i.e.

$$\frac{\theta_k(\lambda,t)}{\varphi_k(\lambda,t)} = \frac{\tilde{\theta}_k(\lambda,t)}{\tilde{\varphi}_k(\lambda,t)}.$$

Since $t\in[0,1)$ was arbitrary this equation holds for all $t\in[0,1)$ and for all $\lambda\in\mathbb{C}$. Differentiating both sides with respect to t gives $\varphi_k(\lambda,t)^2 = \tilde{\varphi}_k(\lambda,t)^2$. Differentiating once more gives $\varphi_k'(\lambda,t)/\varphi_k(\lambda,t) = \tilde{\varphi}'_k(\lambda,t)/\tilde{\varphi}_k(\lambda,t)$. Differentiating a third time gives finally

$$q_k(t) - \lambda = \frac{\varphi_k''(\lambda,t)}{\varphi_k(\lambda,t)} = \frac{\tilde{\varphi}_k''(\lambda,t)}{\tilde{\varphi}_k(\lambda,t)} = \tilde{q}_k(t) - \lambda,$$

almost everywhere on [0,1]. ■

7. Pruning the tree

Theorem 7.1. *Let T be a tree with n_0 boundary edges, q a potential on T and Λ the associated Dirichlet-to-Neumann map. Let v^* be a vertex such that all but one of the edges attached to v^* are boundary edges. Assume the number of these boundary edges is r^* and that the labels of their boundary vertices are $n_0 - r^* + 1, \dots, n_0$. Let T^* be the tree with the boundary edges just mentioned removed so that its boundary vertices are $v_1, v_2, \dots, v_{n_0-r^*}$, and v^*. Then the Dirichlet-to-Neumann map Λ^* for T^* is uniquely determined by Λ and the restriction of the potential q to the boundary edges attached to v^*.*

Proof. We have to compute the values $\Lambda^* f^*$ for an arbitrary vector $f^* \in \mathbb{C}^{n_0 - r^* + 1}$. We will do this for any value of λ which is not an eigenvalue for the Dirichlet problem for T nor an eigenvalue for the Dirichlet problem for T^*. It is clearly sufficient to consider only these λ as we are missing at most countably many. Note that we may assume that the functions $c_\ell(\lambda, \cdot)$ and $s_\ell(\lambda, \cdot)$, and hence the function $\psi_\ell(j, \lambda, \cdot) = \delta_{\ell,j} c_\ell(\lambda, \cdot) - \Lambda_{\ell,j}(\lambda) s_\ell(\lambda, \cdot)$ are known for all $\lambda \in \mathbb{C}$ and all $\ell \in \{n_0 - r^* + 1, \ldots, n_0\}$ since we know the potentials on the corresponding edges.

Next we want to show that there is a $k \in \{n_0 - r^* + 1, \ldots, n_0\}$ such that $\psi(k, \lambda, v^*) = \psi_k(k, \lambda, 1) \neq 0$. Assume that $\psi(n_0, \lambda, v^*) = 0$. Since λ is not an eigenvalue for T^* we have that $\psi(n_0, \lambda, \cdot)|_{T^*}$ is the zero function. Hence there must be a $k \in \{n_0 - r^* + 1, \ldots, n_0 - 1\}$ such that $\psi(n_0, \lambda, \cdot)|_{\epsilon_k([0,1])}$ is not identically equal to zero. For this k we have, using lemma 5.1,

$$0 = \psi(n_0, \lambda, v^*) = \psi_k(n_0, \lambda, 1) = -\Lambda_{k,n_0}(\lambda) s_k(\lambda, 1),$$

i.e. $s_k(\lambda, 1) = 0$ and thus $c_k(\lambda, 1) \neq 0$. Hence

$$\psi_k(k, \lambda, 1) = c_k(\lambda, 1) - \Lambda_{k,k}(\lambda) s_k(\lambda, 1) = c_k(\lambda, 1) \neq 0.$$

Now define

$$\chi(\lambda, \cdot) = \gamma \psi(k, \lambda, \cdot) + \sum_{j=1}^{n_0 - r^*} f_j^* \psi(j, \lambda, \cdot),$$

for some number γ yet to be determined. Then $\chi(\lambda, \cdot)$ has Dirichlet data given by the vector

$$f = (f_1^*, \ldots, f_{n_0 - r^*}^*, 0, \ldots, 0) + \gamma e_k \in \mathbb{C}^{n_0},$$

and $\chi(\lambda, \cdot)|_{T^*}$ has Dirichlet data given by the vector f^* provided that $f_{n_0 - r^* + 1}^* = \chi(\lambda, v^*)$, i.e. if

$$f_{n_0 - r^* + 1}^* = \gamma \psi_k(k, \lambda, 1) - s_k(\lambda, 1) \sum_{j=1}^{n_0 - r^*} \Lambda_{k,j} f_j.$$

Since $\psi_k(k, \lambda, 1) \neq 0$ we may (and will) choose γ such that this condition is satisfied. Therefore

$$(\Lambda^* f^*)_j = (\Lambda f)_j \quad \text{for } j = 1, \ldots, n_0 - r^*.$$

Moreover, by the Kirchhoff conditions,

$$\begin{aligned}(\Lambda^* f^*)_{n_0 - r^* + 1} &= -\sum_{\ell = n_0 - r^* + 1}^{n_0} \chi'_\ell(\lambda, 1) \\ &= -\sum_{\ell = n_0 - r^* + 1}^{n_0} \left[\gamma \psi'_\ell(k, \lambda, 1) + \sum_{j=1}^{n_0 - r^*} f_j \psi'_\ell(j, \lambda, 1) \right],\end{aligned}$$

where the right-hand side involves only known quantities. ■

We are now in a position to prove theorem 1.1.

Proof of theorem 1.1. Let Λ be the generalized Dirichlet-to-Neumann map given. Then the Dirichlet-to-Neumann map $\Lambda_{\text{D–N}}$ itself is given by

$$\Lambda_{\text{DN}} = (\beta - \Lambda\alpha)^{-1} (\beta' - \Lambda\alpha').$$

Hence we may assume without loss of generality that the given map is the Dirichlet-to-Neumann map. Theorem 6.1 gives q on the boundary edges. If we can show the existence of an internal vertex with the properties of the vertex v^* in theorem 7.1 we may use that theorem to show that the Dirichlet-to-Neumann map is now given on the tree where certain edges are removed. Induction then completes the proof.

To show the existence of v^* recall the concepts introduced before lemma 5.2 and designate any boundary vertex as the root of the tree. Suppose that the tree has height h with respect to the root. Then any vertex whose distance from the root is $h-1$ (we may, of course, assume that $h>1$) has the properties of v^*, i.e. all but one of the edges attached to it are boundary edges. ■

We are indebted to the referees for many useful comments which led to a greatly improved presentation and the abolition of a number of mistakes.

Appendix A. Asymptotics of basic solutions

Lemma A 1. *Let $q\in L^1([0,1])$. Suppose that $u(\lambda, \cdot)$ solves the equation $-y''+qy=\lambda y$. Define*

$$u_0(\lambda, x) = u(\lambda, 0)\cos(\sqrt{\lambda}x) + u'(\lambda, 0)\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$$

and $c(\lambda)=|u(\lambda,0)|+|u'(\lambda,0)|/|\sqrt{\lambda}|$. Then, for all $x\in[0,1]$ and all complex $\lambda\neq 0$,

$$|u(\lambda, x)-u_0(\lambda, x)|, \frac{|u'(\lambda, x)-u'_0(\lambda, x)|}{|\sqrt{\lambda}|}\leq c(\lambda)\mathrm{e}^{x|\mathrm{Im}(\sqrt{\lambda})|}\left(\exp\left\{\int_0^x\left|\frac{q(t)}{\sqrt{\lambda}}\right|\mathrm{d}t\right\}-1\right).$$

Proof. Without loss of generality we choose the root of λ in the open upper half plane or the positive real axis. Let $z=\sqrt{\lambda}$ and define

$$g(z, t) = \mathrm{e}^{-t\,\mathrm{Im}(z)}|u(\lambda, t)-u_0(\lambda, t)|.$$

Replacing u by $u_0+(u-u_0)$ in the variation of constants formula

$$u(\lambda, t) = u_0(\lambda, t) + \int_0^t \frac{\sin(z(x-t))}{z}q(t)u(\lambda, t)\mathrm{d}t,$$

one finds

$$g(z, t)\leq \frac{1}{|z|}\int_0^t|q(s)|g(z, s)\mathrm{d}s + \frac{c(\lambda)}{|z|}\int_0^t|q(s)|\mathrm{d}s. \tag{A 1}$$

Let $\phi(t)=\int_0^t|q(s)/z|\mathrm{d}s$, move the first term on the right of (A 1) to the left, and multiply with $|q(t)|\exp(-\phi(t))$. This will produce total derivatives on either side so that integration from 0 to x yields

$$\mathrm{e}^{-\phi(x)}\int_0^x|q(t)|g(z, t)\mathrm{d}t\leq c(\lambda)|z|\left(1-\mathrm{e}^{-\phi(x)}-\mathrm{e}^{-\phi(x)}\frac{1}{|z|}\int_0^x|q(t)|\mathrm{d}t\right).$$

Using this estimate in (A1) gives the desired estimate on $u-u_0$. The statement on $u'-u'_0$ follows from this and the derivative of the variation of constants formula. ■

We remark that this proof is fairly standard and that it is included here for the convenience of the reader.

References

Bennewitz, C. 2001 A proof of the local Borg–Marchenko theorem. *Commun. Math. Phys.* **218**, 131–132. (doi:10.1007/s002200100384.)

Borg, G. 1946 Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.* **78**, 1–96.

Brown, B. M., Peacock, R. A. & Weikard, R. 2002 A local Borg–Marchenko theorem for complex potentials. *J. Comput. Appl. Math.* **148**, 115–131. (doi:10.1016/S0377-0427(02)00577-0.).

Bulla, W. & Trenkler, T. 1990 The free Dirac operator on compact and noncompact graphs. *J. Math. Phys.* **31**, 1157–1163. (doi:10.1063/1.529025.)

Carlson, R. 1997 Hill's equation for a homogeneous tree. *Electron. J. Diff. Eqns*, no. 23.

Carlson, R. 1998 Adjoint and self-adjoint differential operators on graphs. *Electron. J. Diff. Eqns*, 6. 10 pp. (electronic.)

Curtis, E. B. & Morrow, J. A. 1990 Determining the resistors in a network. *SIAM J. Appl. Math.* **50**, 918–930. (doi:10.1137/0150055.)

Curtis, E. B. & Morrow, J. A. 1991 The Dirichlet to Neumann map for a resistor network. *SIAM J. Appl. Math.* **51**, 1011–1029. (doi:10.1137/0151051.)

Evans, W. D. & Saitō, Y. 2000 Neumann Laplacians on domains and operators on associated trees. *Q. J. Math.* **51**, 313–342. (doi:10.1093/qjmath/51.3.313.)

Evans, W. D., Harris, D. J. & Lang, J. 2001 The approximation numbers of Hardy-type operators on trees. *Proc. Lond. Math. Soc.* **83**, 390–418.

Exner, P., Šeba, P. & Štovíček, P. 1988 Quantum interference on graphs controlled by an external electric field. *J. Phys. A* **21**, 4009–4019.

Gel'fand, I. M. & Levitan, B. M. 1951 On the determination of a differential equation from its spectral function. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **15**, 309–360.

Gerasimenko, N. I. 1988 The inverse scattering problem on a noncompact graph. *Teor. Mat. Fiz.* **75**, 187–200.

Gerasimenko, N. I. & Pavlov, B. S. 1988 A scattering problem on noncompact graphs. *Teor. Mat. Fiz.* **74**, 345–359.

Gesztesy, F. & Simon, B. 2000 On local Borg–Marchenko uniqueness results. *Commun. Math. Phys.* **211**, 273–287. (doi:10.1007/s002200050812.)

Levinson, N. 1949 The inverse Sturm–Liouville problem. *Mat. Tidsskr. B* **1949**, 25–30.

Melnikov, Yu. & Pavlov, B. 2001 Scattering on graphs and one-dimensional approximations to *N*-dimensional Schrödinger operators. *J. Math. Phys.* **42**, 1202–1228. (doi:10.1063/1.1347395.)

Mikhailova, A., Pavlov, B., Popov, I., Rudakova, T. & Yafyasov, A. 2002 Scattering on a compact domain with few semi-infinite wires attached: resonance case. *Math. Nachr.* **235**, 101–128. (doi:10.1002/1522-2616(200202)235:1<101::AID-MANA101>3.3.CO;2-M.)

Mikhaylova, A. B. & Pavlov, B. S. 2002 Resonance triadic quantum switch. *Oper. Theory Adv. Appl.* **132**, 287–322.

Nachman, A., Sylvester, J. & Uhlmann, G. 1988 An *n*-dimensional Borg–Levinson theorem. *Commun. Math. Phys.* **115**, 595–605. (doi:10.1007/BF01224129.)

Pavlov, B. 2002 Resonance quantum switch: matching domains. *Proc. Centre Math. Appl.* **40**, 127–156.

Pivovarchik, V. 2000 Inverse problem for the Sturm–Liouville equation on a simple graph. *SIAM J. Math. Anal.* **32**, 801–819. (doi:10.1137/S0036141000368247.)

Simon, B. 1999 A new approach to inverse spectral theory. I. Fundamental formalism. *Ann. Math.* **150**, 1029–1057.

Sobolev, A. V. & Solomyak, M. 2002 Schrödinger operators on homogeneous metric trees: spectrum in gaps. *Rev. Math. Phys.* **14**, 421–467. (doi:10.1142/S0129055X02001235.)