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# A Born-WKBJ inversion method for acoustic reflection data

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## ABSTRACT

Density and bulk modulus variations in an acoustic earth are separately recoverable from standard reflection surveys by utilizing the amplitude-versus-offset information present in the observed wave fields. Both earth structure and a variable background velocity can be accounted for by combining the Born and WKBJ approximations, in a "before stack" migration with two output sections, one for density variations and the other for bulk modulus variations.

For the inversion, the medium is considered to be composed of a known low-spatial frequency variation (the background) plus an unknown high-spatial frequency variation in bulk modulus and density (the reflectivity). The division between the background and the reflectivity depends upon the frequency content of the source.

For constant background parameters, computations are done in the Fourier domain, where the first part of the algorithm includes a frequency shift identical to that in an $F$–$K$ migration. The modulus and density variations are then determined by observing in a least-squares sense amplitude versus offset wavenumber.

For a spatially variable background, WKBJ Green's operators that model the direct wave in a medium with a smoothly varying background are used. A downward continuation with these operators removes the effects of variable velocity from the problem, and, consequently, the remainder of the inversion essentially proceeds as if the background were constant. If the background is strictly depth dependent, the inversion can be expressed in closed form.

The method neglects multiples and surface waves and it is restricted to precritical reflections. Density is distinguishable from bulk modulus only if a sufficient range of precritical incident angles is present in the data.


## INTRODUCTION

In seismic reflection data, there are basically two sources of information about the subsurface: traveltimes and amplitudes. Traveltimes of the various wavefronts in the wave field generally provide information about the low-spatial frequency components (the background) of the medium parameters. Amplitudes of the wavefronts, on the other hand, are most sensitive to the high-spatial frequency components (the reflectivity). The two types of information sample different aspects of the medium. The amplitude variations here are used to determine fine-scale variations in the density and modulus, and it will be assumed that the background can be determined by independent means. The field experiment necessary to provide data for the method is a "standard" (or perhaps slightly superstandard) reflection survey with multiple offset coverage.

Our basic approach is similar to that of Cohen and Bleistein (1977, 1979), Phinney and Frazer (1978), and Raz (1981). We use a Born approximation of the Lippmann-Schwinger equation to develop a forward equation relating the surface data to a scattering potential. The scattering potential is an operator which depends upon the medium parameters and essentially represents the reflectivity of the medium. The details of this approach are outlined in the second section of the paper.

The use of the Born approximation will entail several assumptions about the nature of the medium and the wave phenomena to be modeled. First, the Born approximation is limited to primary subcritical reflections only. Also, since it is based on a perturbation of the true medium about the background variations, it is necessary to be able to construct accurate solutions for the background variations. We use the WKBJ solutions for the background (as discussed in the third section).

The remaining sections of the paper deal with the inverse problem. In the fourth section, an inversion scheme is presented for the case when the background variations are assumed constant. In this case, the problem may be cast in the Fourier domain where the observed wave field can be algebraically related to variations in the medium parameters.

The inverse problem in a laterally varying medium is treated in the fifth section. It is shown that a "before stack" migration of the data essentially removes the effects of the variable background, and the remainder of the inversion proceeds as in the constant background case. A special case of this, where the background variation is strictly depth dependent, is given in the final section. This case is of interest because the WKBJ Green's operators are analytical.

We will assume the source used in the experiment is bandlimited. This usually causes problems with inversion methods because at some point in the inversion scheme, the source has to be deconvolved. This, of course, can only be successfully done within a limited passband, and attempts to invert data outside this passband will usually cause instabilities. We will bypass this prob-

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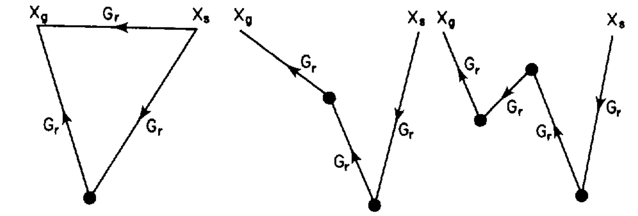


FIG. 1. A schematic interpretation of the Born series is shown. The left panel shows the first two terms in the Born series (the Born approximation). It contains a single scattering point, and hence models only the effects of the direct wave and primary reflections. The total response at the receiver $x_g$ due to the source at $x_s$ is the integration of the scattering point over all points in the subsurface. The addition of another term in the series adds another scattering point as shown in the center panel. This term accounts for first-order transmission effects. The right panel shows the next term, which includes the effects of first-order multiples.

lem by only reconstructing the parameter variations within a limited spatial frequency range.

We will also assume that the sources and receivers used in our experiment have no spatial extension (i.e., they are "points") and are of infinite aperture (that is, for a given source, receivers cover the whole of the earth's surface, and vice versa). This, of course, does not conform to current practice, and we acknowledge that some more analysis is required to establish the correspondence between our experiment and that actually performed.

Finally, we assume that the amplitude information in the data is retained. Since we are not attempting here to unite the rapid earth parameter variations with the slow ones, it is not necessary to know the absolute amplitude of the data. However, if we are to sort density from modulus variations, we must know accurately how amplitude varies with offset and, perhaps less accurately, how it varies with time.

## THE FORWARD SCATTERING EQUATION

In this section we derive the Lippmann-Schwinger equation for acoustic problems. The Born approximation of this equation will lead to a simple relationship between the observed data and the scattering potential.

The derivation starts with the linear isotropic acoustic wave equation

$$L\,P = \left(\frac{\omega^2}{K} + \nabla \cdot \frac{1}{\rho}\nabla\right) P = 0, \tag{1}$$

where $P$ is the pressure field, $K$ is the bulk modulus, and $\rho$ is the density. For development of the equivalent theory based on the elastic wave equation, see Clayton (1981). Associated with the wave operator $L$ is the Green's operator or resolvent, which we formally define as[1] (Taylor, 1972, p. 129)

$$G = -L^{-1}. \tag{2}$$

There are actually many Green's operators that satisfy equation (2). They are distinguished from one another by the manner in which the inverse of $L$ is evaluated. If we replace $-\omega^2$ in equation (1) with $(-i\omega + \varepsilon)^2$ and consider $L$ to be a function of the variable $\varepsilon$, then we can define two independent Green's operators

$$G^+ = \lim_{\varepsilon \downarrow 0} \frac{-I}{L(\varepsilon)}, \tag{3}$$

and

$$G^- = \lim_{\varepsilon \uparrow 0} \frac{-I}{L(\varepsilon)}. \tag{4}$$

The *exploding* Green's operator $G^+$ projects a wavefront a positive distance from the source point as time increases. The *imploding* Green's operator $G^-$ moves the wavefront a negative distance as time increases, or equivalently, if we keep distances positive, then $G^-$ projects backward in time.

We will employ free-space Green's operators. If the problem has external boundary conditions such as a free surface, then the Green's operators should satisfy them. For acoustic problems, this can usually be accomplished by a linear combination of the free-space Green's operators.

In general, we cannot analytically determine the Green's operator for arbitrary variations in $\rho$ and $K$. Instead, solutions are usually cast as a perturbation about a simpler problem for which analytic solutions are available, or at least can be easily computed. We will perturb about a reference problem for which the wave operator is

$$L_r = \left(\frac{\omega^2}{K_r} + \nabla \cdot \frac{1}{\rho_r}\nabla\right), \tag{5}$$

where $K_r$ and $\rho_r$ are the reference bulk modulus and density, respectively. The reference density and bulk modulus will be chosen to be the slow variations (the background) in the true density and bulk modulus. By slowly varying, we mean that the scale length of the variations is much greater than the wavelength of the waves under consideration.

To relate $G$ and $G_r$ (the Green's operator for $L_r$), we employ the simple identity

$$A = B + B(B^{-1} - A^{-1})A$$

and associate $G$ with $A$ and $G_r$ with $B$. Hence, if we define $V = L - L_r$, then

$$G = G_r + G_r V\,G. \tag{6}$$

Equation (6) is the Lippmann-Schwinger equation for $G$, and $V$ is termed the scattering potential. It is valid for any choice of $G_r$ that satisfies the same external boundary conditions as $G$.

As written, equation (6) is implicit in $G$, but it can be solved formally.

$$G = (I - G_r V)^{-1} G_r. \tag{7}$$

The Born series is an expansion of the right-hand side of equation (7) in powers of the operator $VG_r$.

$$G = G_r \sum_{i=0}^{\infty} (VG_r)^i. \tag{8}$$

The convergence properties of this series are discussed in Taylor (1972, p. 146) and Newton (1966, chapters 9–10). The Born approximation of the Lippmann-Schwinger equation is the first two terms of the series

$$G = G_r + G_r V G_r. \tag{9}$$

In this section we are constructing a model for the observed data,

[1] To express an operator in the abstract, we use a symbol without arguments (e.g., $G$) to represent the entire set of values the operator can assume. To perform calculations, we need to look at the individual elements of the set, which will be represented by the symbol with arguments [e.g., $G(x_g, z_g \,|\, x_s, z_s; \omega)$] where the left set of coordinates is the observation point $(x_g, z_g)$, the right set is the source point $(x_s, z_s)$, and $\omega$ is the frequency.

so it is appropriate to use the exploding Green's operators ($G^+$ and $G_r^+$).

In Figure 1 the Born series and the Born approximation are represented in terms of Feynman diagrams. According to this figure, if the source and receiver are above the scattering points, then the Born approximation models only the direct wave and primary reflections, while the next two terms include the effects of transmission and first-order multiples.

The suitability of the Born approximation depends upon how well the reference Green's operator models the direct wave between any two points in the medium. If it is a good approximation, then the higher order terms have the interpretation given in Figure 1. Thus it is clear what physical effects we are neglecting by omitting the higher order terms. If the reference Green's operator is a poor approximation to the direct wave, then the higher order terms contain corrections for the direct wave. In this case the series is very inefficient to sum up, and the suitability of the Born approximation is doubtful.

For acoustic problems, the scattering potential is simply the difference of the wave operators in equations (1) and (5)

$$V = \omega^2 \left(\frac{1}{K} - \frac{1}{K_r}\right) + \nabla \cdot \left(\frac{1}{\rho} - \frac{1}{\rho_r}\right) \nabla. \tag{10}$$

For convenience, we will introduce the dimensionless medium parameters

$$a_1 = \left(\frac{K_r}{K} - 1\right) \quad \text{and} \quad a_2 = \left(\frac{\rho_r}{\rho} - 1\right), \tag{11}$$

where $a_1$ represents the spatial variations in bulk modulus relative to the reference modulus, and $a_2$ represents the variations in density. For the remainder of the paper, we will consider $a_1$ and $a_2$ as the medium variations and not worry about reconstructing the actual modulus and density variations from them. With these definitions the scattering potential becomes

$$V(x, z) = \omega^2 \frac{a_1}{K_r} + \nabla \cdot \frac{a_2}{\rho_r} \nabla. \tag{12}$$

The presence of derivatives in equation (12) represents a departure from basic scattering theory, in which $V$ is a simple function of the spatial variables rather than a differential operator. As it turns out, however, the structure of $V$ will not greatly complicate the problem.

The observations of the wave field response are made on the horizontal surface ($z_s = z_g = 0$). In the following we will take the earth to be two-dimensional (2-D), making occasional note of the (straightforward) extensions to three-dimensions (3-D). In the 2-D problem, the response is a function of the receiver location $x_g$, the source location $x_s$, and frequency. It is convenient to define the data wave field $D$ as $D = (G - G_r)S(\omega)$, where $S(\omega)$ is the Fourier transform of the source time function. Thus, $D$ is the total recorded wave field minus the direct wave from the source to the receiver. Using the Born approximation, the relationship between the data field and the scattering potential is

$$\begin{aligned} D(x_g, x_s, \omega) &= \int dx' \int dz' \, G_r^+(x_g, 0 \,|\, x', z'; \omega) V(x', z'; \omega) \cdot \\ &\quad \cdot G_r^+(x', z' \,|\, x_s, 0; \omega) S(\omega). \end{aligned} \tag{13}$$

Equation (13) is a forward equation in the sense that given the parameter variations $a_1$ and $a_2$, the observed data wave field can be computed. Henceforth, we will be concerned with the inverse problem of finding $a_1$ and $a_2$ from measurements of $D$ on the surface.

## WKBJ SOLUTIONS FOR THE DIRECT WAVE

The suitability of the Born approximation depends upon how well the reference Green's operator models the direct wave in the medium. Since the effects of reflections, transmissions, and multipathing are best handled by the Born series itself (Stolt and Jacobs, 1980), we can ignore these effects when constructing the reference Green's operator. This makes the solution for the direct wave a candidate for the WKBJ approximation.

To find the 2-D Green's operators for the reference problem $L_r G_r^{\mp} = -\delta(x - x_s)\delta(z - z_s)$, they are cast as an asymptotic expansion of the form[2] (Yedlin, 1981)

$$\begin{aligned} G_r^{\mp}(x, z \,|\, x_s, z_s; \omega) &= \pm H_0^{(1)}_{(2)} [\omega \, \theta(x, z \,|\, x_s, z_s)] \sum_{n=0}^{\infty} \frac{A_n(x, z \,|\, x_s, z_s)}{(i\omega)^n}. \end{aligned} \tag{14}$$

Under the WKBJ approximation, we retain only the first term in the expansion. Hence,

$$\begin{aligned} G_r^{\mp}(x, z \,|\, x_s, z_s; \omega) &= \pm H_0^{(1)}_{(2)} [\omega \, \theta(x, z \,|\, x_s, z_s)] A_0(x, z \,|\, x_s, z_s). \end{aligned} \tag{15}$$

As $(x, z) \to (x_s, z_s)$, we require that $G_r^{\mp}$ approach the constant background form. Thus $\theta(x, z \,|\, x_s, z_s) \to \sqrt{x_s^2 + z_s^2}/v_r(x_s, z_s)$ and $A_0(x, z \,|\, x_s, z_s) \to \rho_r(x_s, z_s)/4i$. Applying the reference wave operator $L_r$ to equation (15), the following equations are generated for $\theta$ and $A_0$ by matching powers of $\omega$:

$$(\nabla \theta)^2 = \frac{\rho_r}{K_r} = \frac{1}{v_r^2} \tag{16}$$

and

$$\left(\nabla \cdot \frac{1}{\rho_r} \nabla \theta - \frac{1}{K_r \theta}\right) A_0 = \frac{-2}{\rho_r} \nabla \phi \cdot \nabla A_0. \tag{17}$$

The first equation is the Eikonal equation, and its solution for $\theta$ governs the traveltime of the wavefronts. The solution for $A_0$ from the second equation (the transport equation) determines the amplitudes of the wavefronts. The higher order terms in the expansion correct for the low-frequency behavior of the solution. The WKBJ solutions will be accurate if the wavelength of the waves is considerably shorter than the scale length of the variations in the medium. This is the motivation for choosing the background parameters to be slowly varying.

For a constant parameter medium, the Green's operators have a simple analytical form which is given in the next section. For the slightly more general case of a depth variable background, the Green's operators are

$$\begin{aligned} G_r^{\mp}(x, z \,|\, x_s, z_s; \omega) &= -\frac{\sqrt{\rho_r(z)\rho_r(z_s)}}{2\pi} \int dk_x e^{ik_x x} \frac{e^{\pm i \int_{z_s}^{z} dz' q(z')}}{\pm 2i\sqrt{q(z) q(z_s)}}, \end{aligned} \tag{18}$$

where

$$q(z) = \frac{\omega}{v_r(z)} \sqrt{1 - \frac{k_x^2 v_r^2(z)}{\omega^2}}. \tag{19}$$

[2] $H_0^{(1)}$ is the Hankel function of the first kind, and $H_0^{(2)}$ is the Hankel function of the second kind.

Equation (18) points out that the WKBJ solution is not valid near turning points $[q(z) = 0]$.

For a laterally variable background, the WKBJ solutions must be obtained numerically. The straightforward construction of $G_r^{\mp}$ using equations (15), (16), and (17) is certainly possible. However, finite-difference solutions of one-way wave equations (Claerbout, 1976; Clayton and Engquist, 1980) may provide a better approach if the tendency of current formulations to overlook amplitude effects is corrected or compensated for.

## CONSTANT BACKGROUND INVERSION

An inversion method is presented for the case when the reference parameters $K_r$ and $\rho_r$ are assumed to be constant. The solution in this case is simple because the WKBJ Green's operators have an exact analytical form. The resulting inversion will contain a frequency shift which is identical to $F$–$K$ migration on "unstacked" data (Stolt, 1978).

The first step is to Fourier transform[3] the data wave field [equation (13)] over $x_g$ and $x_s$.

$$D(k_g, k_s, \omega) = \frac{1}{2\pi} \int dx_g \int dx_s e^{-ik_g x_g} D(x_g, x_s, \omega) e^{ik_s x_s}$$
$$= \int dx' \int dz' G_r^{+}(k_g, 0 \mid x', z'; \omega) \cdot V(x', z'; \omega) G_r^{+}(x', z' \mid k_s, 0; \omega) S(\omega). \quad (20)$$

In the 2-D problem (line sources and receivers), $x_g$, $x_s$, $k_g$, and $k_s$ are scalars. If we consider them to be two-component vectors and adjust the occasional factor of $2\pi$, then the equations that follow will hold for the 3-D problem, too.

For constant background parameters, the Green's operators in equation (20) have the analytical expressions

$$G_r^{+}(k_g, 0 \mid x', z'; \omega) = \frac{i\rho_r}{\sqrt{2\pi}} \frac{e^{-i(k_g x' - q_g |z'|)}}{2q_g} \quad (21)$$

and

$$G_r^{+}(x', z' \mid k_s, 0; \omega) = \frac{i\rho_r}{\sqrt{2\pi}} \frac{e^{i(k_s x' + q_s |z'|)}}{2q_s}, \quad (22)$$

where

$$q_g = \frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2 k_g^2}{\omega^2}} \quad \text{and} \quad q_s = \frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2 k_s^2}{\omega^2}}. \quad (23)$$

In the expressions for $q_g$ and $q_s$, we have intentionally factored an $\omega$ outside the square roots to indicate that $q_g$, $q_s$, and $\omega$ have the same sign.

We will now use the fact that the Green's operators look very much like the kernel of a Fourier transform to obtain a simple equation relating the data field to the scattering potential. Substituting equations (21) and (22) into equation (20), we have

$$D(k_g, k_s, \omega) = \frac{-\rho_r^2}{2\pi} \int dx' \int dz' \frac{e^{-i(k_g x' - q_g |z'|)}}{2q_g} \cdot V(x', z'; \omega) \frac{e^{i(k_s x' + q_s |z'|)}}{2q_s} S(\omega). \quad (24)$$

We now assume that $a_1(x, z)$ and $a_2(x, z)$ are zero for $z < 0$. This will allow us to drop the absolute signs in equation (24) because $V(x', z'; \omega)$ will be zero for $z' < 0$. Actually, removing the absolute signs will mean that any scatterers located above the datum plane $z = 0$ will only contribute to $D$ in negative time. This point is discussed further in the next section. Using the definition (12) of $V$ and integrating equation (24) by parts yields

$$D(k_g, k_s, \omega) = \frac{-\rho_r}{2\pi} \int dx' \int dz' \frac{e^{-i[(k_g - k_s)x' - (q_g + q_s)z']}}{4q_g q_s} \cdot \left[\frac{\omega^2}{v_r^2} a_1(x', z') + (q_g q_s - k_g k_s) a_2(x', z')\right] S(\omega). \quad (25)$$

The two integrals in equation (25) are recognizable as Fourier transforms over $x'$ and $z'$. Thus,

$$D(k_g, k_s, \omega) = \frac{-\rho_r}{4q_g q_s} \left[\frac{\omega^2}{v_r^2} a_1(k_g - k_s, -q_g - q_s) + (q_g q_s - k_g k_s) a_2(k_g - k_s, -q_g - q_s)\right] S(\omega). \quad (26)$$

That is, the triple Fourier transform of $D$ is a linear combination of the double Fourier transforms of $a_1$ and $a_2$. Counting variables on both sides of equation (26) indicates the inverse problem is overdetermined. That is, there should be more than enough information in $D$ to solve for $a_1$ and $a_2$. If $V$ were a more general operator, things would have been different. $V$ would then be a function of two sets of coordinates $[V(x, z) \rightarrow V(x, z \mid x', z')]$, and equation (26) would have the form

$$D(k_g, k_s, \omega) = -\frac{2\pi\rho_r^2}{4q_g q_s} V(k_g, -q_g \mid k_s, q_s) S(\omega). \quad (27)$$

That is, the triple Fourier transform of $D$ would then be proportional to the quadruple Fourier transform of $V$. Counting variables again, we see the problem is underdetermined, and consequently there would be no way to calculate $V$ given $D$.

The first step to solving for $a_1$ and $a_2$ is to change to midpoint-offset coordinates. The midpoint wavenumber ($k_m$) and the half-offset wavenumber ($k_h$) are defined by[4]

$$k_m = k_g - k_s \quad \text{and} \quad k_h = k_g + k_s. \quad (28)$$

In the space domain, these substitutions correspond to a midpoint ($x_m$) and a half-offset ($x_h$) defined as

$$x_m = \frac{x_g + x_s}{2} \quad \text{and} \quad x_h = \frac{x_g - x_s}{2}. \quad (29)$$

Also, since $a_1$ and $a_2$ depend upon $-(q_g + q_s)$, a new independent variable ($k_z$) is defined

$$k_z = -q_g - q_s = -\frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2 k_g^2}{\omega^2}} - \frac{\omega}{v_r} \sqrt{1 - \frac{v_r^2 k_s^2}{\omega^2}}. \quad (30)$$

[3] In this paper Fourier transforms over source coordinates have the opposite sense to those over receiver coordinates. Also, rather than define a new symbol to express the Fourier transform of a quantity, we use the same symbol with a different argument. Thus,

$$G(k_g, 0 \mid x', z'; \omega) = \frac{1}{\sqrt{2\pi}} \int dx_g e^{-ik_g x_g} G(x_g, 0 \mid x', z'; \omega),$$

and

$$G(x', z' \mid k_s, 0; \omega) = \frac{1}{\sqrt{2\pi}} \int dx_s G(x', z' \mid x_s, 0; \omega) e^{ik_s x_s}.$$

These conventions may seen unnatural to some, but they are consistent with the treatment of $D$ and $G$ as linear operators.

[4] These definitions of midpoint and offset wavenumber differ from those of other authors (c.f., Yilmaz and Claerbout, 1980), because we have used a conjugate rather than a symmetric relationship between source and receiver. This arises directly from the operator notation used in this paper. In the physical domain [equation (29)], the relations for midpoint and offset are the same with both approaches.

After a little algebra, equations (28) and (30) may be combined to obtain expressions for $\omega$, $q_s$, and $q_g$ in terms of the new variables $k_m$, $k_h$, $k_z$.

$$\omega = -\frac{v_r k_z}{2}\sqrt{(1 + k_m^2/k_z^2)(1 + k_h^2/k_z^2)}$$

$$\equiv \omega(k_m, k_h, k_z), \tag{31}$$

$$q_g = -\frac{k_z}{2}(1 - k_m k_h/k_z^2), \tag{32}$$

and

$$q_s = -\frac{k_z}{2}(1 + k_m k_h/k_z^2). \tag{33}$$

Combining equations (26), (31), (32), and (33), we obtain

$$D(k_m, k_h, k_z) = -\rho_r\left[\sum_{i=1}^{2} A_i(k_m, k_h, k_z)a_i(k_m, k_z)\right]S(\omega), \tag{34}$$

where

$$A_1(k_m, k_h, k_z) = \frac{1}{4}\frac{(k_z^2 + k_h^2)(k_z^2 + k_m^2)}{k_z^4 - k_m^2 k_h^2} \tag{35}$$

and

$$A_2(k_m, k_h, k_z) = \frac{1}{4}\frac{(k_z^2 - k_h^2)(k_z^2 + k_m^2)}{k_z^4 - k_m^2 k_h^2}. \tag{36}$$

In equation (34), it is understood that $\omega$ obeys the functional relationship given in equation (31) which is identical to the frequency shift used in $F$–$K$ migration [Stolt, 1978, equation (60)].

To invert equation (34), we start by deconvolving the source $S(\omega)$. Thus we define

$$D'(k_m, k_h, k_z) = \frac{-1}{\rho_r}\frac{D(k_m, k_h, \omega)}{S(\omega)}. \tag{37}$$

Since in general $S(\omega)$ will be band-limited, this operation cannot be accomplished exactly without introducing instabilities. This is the point where Gel'fand-Levitan inverse methods (Ware and Aki, 1969; Jacobs and Stolt, 1980) have problems. To avoid the instabilities, we simply set $D'$ to zero outside the frequency bandwidth of $S(\omega)$, which means we will only be able to resolve the variations in $a_1$ and $a_2$ within the passband

$$\omega_1 \le \omega(k_m, k_h, k_z) \le \omega_2, \tag{38}$$

where $\omega_1$ and $\omega_2$ are the lower and upper limits of the passband of $S(\omega)$. In Figure 2 the region of resolution is illustrated for $k_h = 0$. It is interesting to note that by increasing the ratio $k_h/k_z$, the circles in this figure will shrink in radius. Hence, it appears possible to partially fill in the low-frequency variations in the parameters by increasing the offset in the experiment.

With the (partial) deconvolution of equation (37), the inverse problem reduces to

$$D'(k_m, k_h, k_z) = \sum_{i=1}^{2} A_i(k_m, k_h, k_z)a_i(k_m, k_z). \tag{39}$$

Since $a_i$ is independent of $k_h$, the measurement of $D'$ at any two distinct values of $k_h$ will suffice to determine $a_1$ and $a_2$. In a standard reflection survey, however, $D'$ is usually determined at many values of $k_h$, and therefore a more robust evaluation is possible. For example, a least-squares determination is given by the solution to the equation

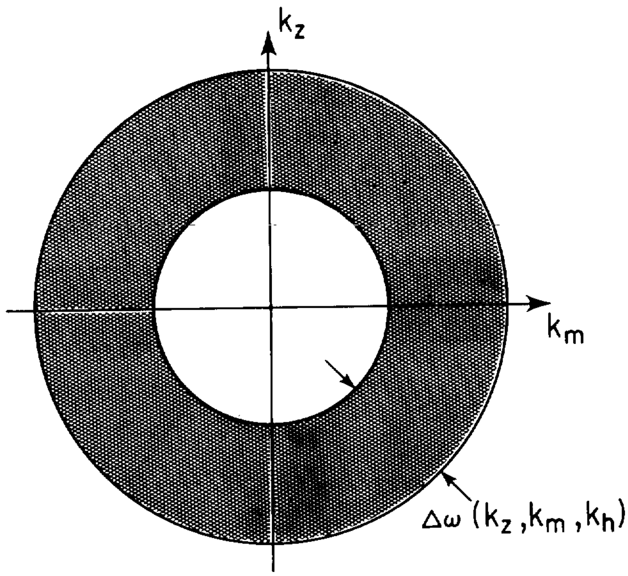


FIG. 2. The shaded ring shows the region of resolution of the bulk modulus and density variations. Here $k_z$, $k_m$, and $k_h$ are (respectively) the vertical, midpoint, and offset wavenumbers. The width and radius of the ring depend upon the passband $\Delta\omega$ of the source time function. As the ratio $k_h/k_z$ is increased, the radius of the ring shrinks. This corresponds in the physical domain to increasing the source-receiver offset relative to the depth to the reflector.

$$\begin{bmatrix} \sum A_1^2 & \sum A_1A_2 \\ \sum A_1A_2 & \sum A_2^2 \end{bmatrix}\begin{bmatrix} a_1(k_m, k_z) \\ a_2(k_m, k_z) \end{bmatrix} = \begin{bmatrix} \sum A_1D' \\ \sum A_2D' \end{bmatrix}. \tag{40}$$

In this equation, the summations are taken over $k_h$ with the restriction that

$$|k_h k_m| < |k_z|^2. \tag{41}$$

The necessity for the restriction lies in the fact that the Born approximation as used in this paper is not adequate in the evanescent zone. This restriction is sufficient to avoid both evanescent zones in equation (30), and to avoid turning points in both the up- and downgoing paths by keeping both $q_g$ and $q_s$ strictly negative in equations (32) and (33). In practice, the finite range of offsets attainable for a given experiment will likely impose a more severe restriction than equation (41). If the range of offsets is too small, the restrictions on usable $k_h$ may be so severe that $a_1$ cannot be distinguished from $a_2$. In this case the determinant of the matrix in equation (40) approaches zero.

Thus far we have been concerned with the 2-D problem which has line sources and receivers. The full 3-D problem with point sources and receivers is only slightly different. In the usual seismic experiment, the data are recorded with (assumed) point sources and receivers, but along a line on the free surface. In the Appendix, results presented in this section are modified for this case.

## INVERSION WITH A VARIABLE BACKGROUND

For a realistic earth model, we must assume that the background parameters will vary from one location to another. If we ignore this variation as we did in the previous section, then the inversion scheme will locate the parameter variations incorrectly. Fortunately, if the background variations are known, their effects may be removed from the inversion problem by a downward continuation. This step is actually a "before stack" migration of data prior to the inversion.

The migration is based on the representation integral over a closed surface $S$. If we assume that $P$ is a solution to the wave equation $L_r P = -F$, where $F$ is a volume source and $G_r^{\pm}$ are the Green's operators associated with $L_r$, then the representation integral is[5]

$$R^-(\mathbf{x}) = \int_S ds\, G_r^-(\mathbf{x} \mid s) T(s) P(s), \tag{42}$$

where

$$T(s) = \underset{\longleftarrow}{\frac{\partial}{\partial n}} \frac{1}{\rho_r} - \frac{1}{\rho_r} \underset{\longrightarrow}{\frac{\partial}{\partial n}}$$

and $n$ is the normal to the surface. The arrows in the definition of $T(s)$ have the following meaning

$$A \left( \underset{\longleftarrow}{\frac{\partial}{\partial n}} \frac{1}{\rho_r} - \frac{1}{\rho_r} \underset{\longrightarrow}{\frac{\partial}{\partial n}} \right) B = \left( \frac{\partial}{\partial n} A \right) \frac{B}{\rho_r} - \frac{A}{\rho_r} \left( \frac{\partial}{\partial n} B \right).$$

The imploding Green's operator is used in equation (42) because, since it projects backward in time, it is the proper operator to backtrack a wave to its point of origin. If we wanted to extrapolate waves away from their point of origin, then $G_r^+$ would replace $G_r^-$ in equation (42). Using the divergence theorem, we may convert $R^-$ to a volume integral

$$R^-(\mathbf{x}) = \int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') \left( \nabla \underset{\longleftarrow}{\frac{1}{\rho_r}} \nabla - \nabla \underset{\longrightarrow}{\frac{1}{\rho_r}} \nabla \right) P(\mathbf{x}')$$
$$= \int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') (\underset{\longleftarrow}{L_r} - \underset{\longrightarrow}{L_r}) P(\mathbf{x}'),$$

where $V$ is the volume bounded by $S$ and $\mathbf{x}' \in V$. Applying the fact that $L_r P = -F$ and $L_r G_r^- = -I$, $R^-$ is found to be

$$R^-(\mathbf{x}) = \begin{cases} P(\mathbf{x}) + \displaystyle\int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}') \text{ for } \mathbf{x} \in V \\ \displaystyle\int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}') \text{ for } \mathbf{x} \notin V. \end{cases} \tag{43}$$

If there are no sources inside the volume then $R^-(\mathbf{x})$ is a representation of $P(\mathbf{x})$ inside the volume and zero outside. When sources are present, they contribute to both the inner and outer solutions.

Equations (42) and (43) relate a volume integral to an integral over a closed surface containing the volume. To be useful for the seismic experiment, we will need an expression involving an integral over an open surface.

Consider applying the representation to a field point outside the volume. The geometry is shown in Figure 3. The closed surface integral can be broken up into two line integrals, if we assume that the edges are sufficiently far away that their contribution is zero. Hence we can write, by equation (43),

$$R^-(\mathbf{x}) = \int_{S_0} ds_0 G_r^-(\mathbf{x} | s_0) T(s_0) P(s_0)$$
$$- \int_{S_z} ds_z G_r^-(\mathbf{x} \mid s_z) T(s_z) P(s_z)$$
$$= R_0(\mathbf{x}) - R_z(\mathbf{x}) = \int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}'). \tag{44}$$

Suppose that $P$ has been generated by sources partly within $V$ and partly beneath it ($V^c$). That is,

$$P(\mathbf{x}) = \int_V d\mathbf{x}' G_r^+(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}') + \int_{V^c} d\mathbf{x}' G_r^+(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}')$$
$$= P_U(\mathbf{x}) + P_L(\mathbf{x}). \tag{45}$$

Note that the integral contains the exploding Green's operators because we are constructing a model of the wave field. If we now take $\mathbf{x}$ to lie infinitesimally below the surface $S_z$, then we can evaluate $R_z(\mathbf{x})$ to a very good approximation as

$$R_z(\mathbf{x}) = P_L(\mathbf{x}). \tag{46}$$

To obtain this result, assume that $P_L$ is *upgoing* at $S_z$ since it was created by sources beneath $S_z$, and similarly $P_U$ is *downgoing* at $S_z$. For $\mathbf{x}$ close enough to $S_z$, and for reasonable angles of propagation, the WKBJ Green's operator $G_r^-(\mathbf{x} \mid S_z)$ can be thought of as a constant velocity Green's operator. It is then easy to demonstrate that the surface integral $R_z(\mathbf{x})$ recreates the portion of $P(\mathbf{x})$ which was upgoing at $S_z$. The Green's operator $G_r^-$ loses the downgoing part because $P_U(\mathbf{x})$ reaches $\mathbf{x}$ after $S_z$.

With this result, equation (44) can be rewritten as

$$R_0(\mathbf{x}) = P_L(\mathbf{x}) + \int_V d\mathbf{x}' G_r^-(\mathbf{x} \mid \mathbf{x}') F(\mathbf{x}'). \tag{47}$$

To see what the volume integral in equation (47) really is, consider it in the time domain. Suppose that the source distribution $F$ is concentrated at zero time. Then, since $G_r^- = 0$ for $t > 0$, the integral itself is zero for $t > 0$; i.e., sources above $\mathbf{x}$ contribute to the downward continued field $R_0(\mathbf{x})$ only in negative time [and, in fact, are time reversals of their contribution $P_U(\mathbf{x})$ to the true wave field $P(\mathbf{x})$]. The substance of equation (47) is that it is a prescription for downward continuation of $P$ from the surface $S_0$ to the point $\mathbf{x}$.

Now we generalize the representation to reflection data. The appropriate surface integral in this case is

$$R^-(\mathbf{x}_g | \mathbf{x}_s) = - \int_S ds\, G_r^-(\mathbf{x}_g \mid s) T(s) \cdot$$
$$\cdot \int_S ds' D(s \mid s') T(s') G_r^-(s' \mid \mathbf{x}_s). \tag{48}$$

Applying the divergence theorem twice, the expression can be converted to the volume integral

$$R^-(\mathbf{x}_g \mid \mathbf{x}_s) = - \int_V d\mathbf{x} \int_V d\mathbf{x}' G_r^-(\mathbf{x}_g \mid \mathbf{x}) (\underset{\longleftarrow}{L_r} - \underset{\longrightarrow}{L_r}) \cdot$$
$$\cdot D(\mathbf{x} \mid \mathbf{x}') (\underset{\longleftarrow}{L_r} - \underset{\longrightarrow}{L_r}) G_r^-(\mathbf{x}' \mid \mathbf{x}_s), \tag{49}$$

where $\mathbf{x}$ and $\mathbf{x}' \in V$. For $\mathbf{x}_g$ and $\mathbf{x}_s$ below the volume, equation (49) reduces to

$$R^-(\mathbf{x}_g \mid \mathbf{x}_s) = \int_V d\mathbf{x} \int_V d\mathbf{x}' G_r(\mathbf{x}_g \mid \mathbf{x}) \cdot$$
$$\cdot L_r^{\rightarrow} D(\mathbf{x} \mid \mathbf{x}') L_r^{\leftarrow} G_r^-(\mathbf{x}' \mid \mathbf{x}_s).$$

Invoking the Born approximation for $D$ [equation (13) with $S(\omega) = 1$], this becomes

[5] The reader may interpret this and succeeding equations either as being in the frequency domain (in which case the ω-dependence of most quantities has been suppressed) or as vector equations in the time domain (in which case there will be an implied convolution over time in most of the following equations).

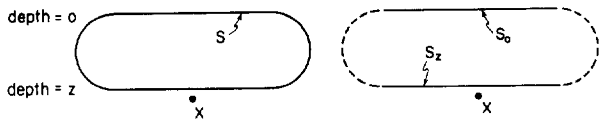


FIG. 3. The left panel shows the closed contour ($S$) to be used in the representation of the response at $\mathbf{x}$. The closed contour is then broken into two line integrals over $S_0$ and $S_z$ shown in the right panel. The contribution from the edges is assumed to be zero. In the text, it is shown that for positive time, response at $\mathbf{x}$ can be related to a line integral of recorded data along $S_0$.

$$R^-(\mathbf{x}_g \mid \mathbf{x}_s) = \int_V d\mathbf{x} G_r^-(\mathbf{x}_g \mid \mathbf{x}) V(\mathbf{x}) G_r^-(\mathbf{x} \mid \mathbf{x}_s). \quad (50)$$

We can repeat the previous analysis [equations (44) to (47)] to construct a downward continuation operator from the surface integral (48). The analogous results are (provided $\mathbf{x}_g$ and $\mathbf{x}_s$ are infinitesimally below the surface $S_z$)

$$\begin{aligned} R_z(\mathbf{x}_g \mid \mathbf{x}_s) &\equiv \int_{S_z} ds_z \int_{S_z} ds_z' G_r^-(\mathbf{x}_g \mid s_z) T(s_z) \cdot \\ &\quad \cdot D(s_z \mid s_z') T(s_z') G_r^-(s_z' \mid \mathbf{x}_s) \\ &= D_L(\mathbf{x}_g \mid \mathbf{x}_s), \end{aligned} \quad (51)$$

and

$$\begin{aligned} R_0(\mathbf{x}_g \mid \mathbf{x}_s) &\equiv \int_{S_0} ds_0 \int_{S_0} ds_0' G_r^-(\mathbf{x}_g \mid s_0) T(s_0) \cdot \\ &\quad \cdot D(s_0 \mid s_0') T(s_0') G_r^-(s_0' \mid \mathbf{x}_s) \\ &= D_L(\mathbf{x}_g \mid \mathbf{x}_s) + \int d\mathbf{x} G_r^-(\mathbf{x}_g \mid \mathbf{x}) \cdot \\ &\quad \cdot V(\mathbf{x}) G_r^-(\mathbf{x} \mid \mathbf{x}_s) \end{aligned} \quad (52)$$

where $D_L$ is the reflection data from points below $S_z$. The volume integral involving $G^- V G^-$ is zero in positive time. Thus for $t > 0$, $R_z$ and $R_0$ are identical, and $R_0$ is a formula for downward continuation of the data field. The $G^- V G^-$ term in $R_0$ simply represents the well known fact in downward continuation that the response from reflectors above the datum plane is pushed into negative time. This is because the downward continuation operator is unable to reflect the wave field. It can continue to a reflection point, but when continuing past a reflector, it extrapolates the waves instead of reflecting them, making them appear in negative time.

The result obtained in equation (52) can be easily generalized to move the data wave field between any two planes. Tc move $D$ from the depth $z - \varepsilon$ to the depth $z$, we have for $t > 0$

$$\begin{aligned} D_L(s_z \mid s_z') &= \int_{S_{z-\varepsilon}} ds \int_{S_{z-\varepsilon}} ds' G^-(s_z \mid s) T(s) \cdot \\ &\quad \cdot D(s \mid s') T(s') G^-(s' \mid s_z'). \end{aligned} \quad (53)$$

We are now in a position to invert the data for the scattering potential. First we define the migrated wave field $M$ at depth $z$ to be the zero time component of the downward continued field.

$$\begin{aligned} \rho_r(x_m, z) v_r(x_m, z) M(x_g, x_s, z) &= \lim_{t \downarrow 0} \int d\omega e^{-i\omega t} \cdot \\ &\quad \cdot D_L(x_g, z \mid x_s, z; \omega) \quad (54) \\ &= \int d\omega\, R_0(x_g, z \mid x_s, z; \omega). \end{aligned}$$

Except for the retention of data at $x_g \neq x_s$, this corresponds closely with the usual definition of migration of unstacked data. The presence of $\rho_r$ and $v_r$ in our definition will simplify things later on. For now we just note that with this definition, the triple Fourier transform of $M$ is dimensionless.

We now use equation (53) to relate the data field in equation (54) to the data field a small distance ($\varepsilon$) above. Writing this out for the 2-D case, we have

$$\begin{aligned} \rho_r v_r M(x_g, x_s, z) &= \int dx_g' \int dx_s' \lim_{t \downarrow 0} \int d\omega e^{-i\omega t} \cdot \\ &\quad \cdot G_r^-(x_g, z \mid x_g', z - \varepsilon; \omega) T(x_g', z - \varepsilon) \cdot \\ &\quad \cdot D_L(x_g', z - \varepsilon \mid x_s', z - \varepsilon) T(x_s', z - \varepsilon) \cdot \\ &\quad \cdot G_r^-(x_s', z - \varepsilon \mid x_s, z; \omega). \end{aligned} \quad (55)$$

As time goes to zero, causality requires that the region of support for the $x_g'$ and $x_s'$ integrals shrink to a small region centered around the midpoint between $x_g$ and $x_s$. Under the assumption of a smoothly varying background, $G_r^-$ and $D_L$ will assume in this region the constant parameter forms with the relevant parameters being $K_r(x_m, z)$ and $\rho_r(x_m, z)$. Substituting in the constant parameter Green's operators from the previous section [equations (21) and (22)] and performing the $T$ operator derivatives, we have

$$\begin{aligned} \rho_r v_r M(x_g, x_s, z) &= \int d\omega \int dk_g \int dk_s \cdot \\ &\quad \cdot D_L(k_g, z - \varepsilon \mid k_s, z - \varepsilon; \omega) \cdot \\ &\quad \cdot e^{i(k_g x_g - k_s x_s)} e^{-i\varepsilon(q_g + q_s)}. \end{aligned} \quad (56)$$

Substituting in the constant parameter form for $D_L$ [equation (34) multiplied by $e^{-i(z-\varepsilon)(q_g+q_s)}$ since the datum plane for $D_L$ is $z - \varepsilon$], we have

$$\begin{aligned} v_r M(x_g, x_s, z) &= -\int d\omega \int dk_g \int dk_s \cdot \\ &\quad \cdot e^{i(k_g x_g - k_s x_s)} e^{-iz(q_g + q_s)} \cdot \\ &\quad \cdot \sum_{i=1}^{2} A_i(k_g, k_s, q_g, q_s) \cdot \\ &\quad \cdot a_i(k_g - k_s, -q_g - q_s), \end{aligned} \quad (57)$$

where coefficients $A_i$ are defined in equations (35) and (36). Even though it is not explicitly mentioned in equation (57), coefficients $A_i$ depend upon $x_m$ and $z$ via the background modulus and density $K_r(x_m, z)$ and $\rho_r(x_m, z)$.

Equation (57) looks suspiciously like a Fourier transform, and indeed we can put it in that form. Changing integration variables in equation (57) from $(\omega, k_g, k_s)$ to $(k_z, k_m, k_h)$ as in equations (28) and (30) yields

$$\begin{aligned} M(x_m, x_h, z) &= -\int dk_m \int dk_h \frac{1}{2} \left| \frac{d\omega}{dk_z} \right| e^{i(k_m x_m + k_h x_h + k_z z)} \cdot \\ &\quad \cdot \sum_{i=1}^{2} \frac{A_i(k_g, k_s, q_g, q_s)}{v_r} a_i(k_m, k_z). \end{aligned} \quad (58)$$

With forms (35) and (36) for $A_1$ and $A_2$, plus the relation

$$\left| \frac{d\omega}{dk_z} \right| = \frac{v_r}{8} \sqrt{1 + \frac{k_m^2}{k_z^2}} \sqrt{1 + \frac{k_h^2}{k_z^2}} \frac{1}{A_1(k_g, k_s, q_g, q_s)} \quad (59)$$

obtained by differentiating equation (31), we have

$$M(x_m, x_h, z) = -\frac{1}{(2\pi)^{3/2}} \int dk_m \int dk_h \int dk_z \cdot \cdot e^{i(k_m x_m + k_h x_h + k_z z)} \cdot \cdot \sum_{i=1}^{2} a_i(k_m, k_z) B_i(k_m, k_h, k_z), \tag{60}$$

where

$$B_1(k_m, k_h, k_z) = \frac{(2\pi)^{3/2}}{16} \sqrt{1 + \frac{k_m^2}{k_z^2}} \sqrt{1 + \frac{k_h^2}{k_z^2}} \tag{61}$$

and

$$B_2(k_m, k_h, k_z) = B_1(k_m, k_h, k_z) \frac{k_z^2 - k_h^2}{k_z^2 + k_h^2}. \tag{62}$$

Note that $B_1$ and $B_2$ do not depend upon the spatial coordinates. Equation (60) is in fact a 3-D Fourier inverse transform over $k_m$, $k_h$, and $k_z$. Taking the Fourier transform of both sides, we arrive at the final result

$$M(k_m, k_h, k_z) = \sum_{i=1}^{2} a_i(k_m, k_z) B_i(k_m, k_h, k_z). \tag{63}$$

Thus, just as in the constant background case, the 3-D Fourier transform of the migrated field is a linear combination (with known coefficients) of the 2-D Fourier transforms of $a_1$ and $a_2$. Provided a sufficient range of $k_h$ (or $k_h/k_z = \tan\phi$, where $\phi$ is angle of incidence) exists in the data, equation (63) is solvable for $a_1$ and $a_2$.

## INVERSION WITH A DEPTH VARIABLE BACKGROUND

In this section we consider a special case of the previous section in which the background parameters are allowed to vary only in the depth direction. The WKBJ Green's operators in this case are analytic, and, consequently, explicit formulas can be derived for the inverse problem.

For a depth variable medium, the WKBJ Green's operators are given by equation (18). With these we can form the Born-WKBJ approximation of the data field

$$D(k_g, z_g = 0 \mid k_s, z_s = 0; \omega) = \frac{-\rho_r(0)}{8\pi\sqrt{q_g(0) q_s(0)}} \int_0^\infty dz \frac{e^{i\int_0^z dz'[q_g(z') + q_s(z')]}}{\sqrt{q_g(z) q_s(z)}} \cdot \left[\frac{\omega^2}{v_r^2(z)} a_1(k_g - k_s, z) + [q_g(z) q_s(z) - k_g k_s] a_2(k_g - k_s, z)\right], \tag{64}$$

where $q_g$ and $q_s$ are the same as in equation (23) except that now the velocity is a function of $z$.

Equation (52) for the downward continued field $R_0$ can be evaluated explicitly in this case. Fourier transforms over the lateral coordinates yield

$$R_0(k_g, z \mid k_s, z) = -\int_S ds G_r^-(k_g, z \mid s) \cdot T(s) \int_S ds' \cdot D(s \mid s') T(s') G_r^-(s' \mid k_s, z). \tag{65}$$

In this expression we have set the continuation depths for both the sources and receivers equal to $z$. To evaluate this expression, we need only substitute the explicit form (18) for each $G_r^-$ and do the derivatives in each $T$. The result is

$$R_0(k_g, z \mid k_s, z) = \frac{\rho_r(z)}{\rho_r(0)} \sqrt{\frac{q_g(0) q_s(0)}{q_g(z) q_s(z)}} e^{-i\int_0^z dz'(q_g + q_s)} \cdot D(k_g, 0 \mid k_s, 0). \tag{66}$$

In the derivation of this equation, the derivatives of $q_g$ and $q_s$ were neglected in comparison to the derivatives of the phase terms. Note that as $z \rightarrow 0$, the downward continued field $R_0$ approaches the data field $D$, as it must. According to equation (66), downward continuation is achieved in the vertically varying case by multiplying the data by the phase factor

$$\exp\left[-i \int_0^z dz'(q_g + q_g)\right],$$

and adjusting the amplitude of the data. The phase factor is that used in the Gazdag phase-shift migration method (Gazdag, 1978).

By equation (54), migration is achieved by integrating the downward continued field $R_0$ over all frequencies and dividing by $\rho_r v_r$. Formally,

$$M(k_g, k_s, z) = \frac{1}{v_r(z)\rho_r(0)} \int d\omega \sqrt{\frac{q_g(0) q_s(0)}{q_g(z) q_s(z)}} \cdot \cdot e^{-i\int_0^z dz'(q_g + q_s)} D(k_g, 0 \mid k_s, 0). \tag{67}$$

According to equation (63), the Fourier transform over $z$ of this quantity is a linear combination of the double Fourier transforms of the desired quantities $a_1$ and $a_2$.

In the Appendix, necessary modifications are given to incorporate point sources and receivers into the above solution.

## CONCLUSIONS

An inversion scheme has been presented to determine the rapid variations in bulk modulus and density from the amplitude versus offset information present in a seismic reflection survey. The procedure consists of two steps.

First, a before stack migration of the data is performed with WKBJ Green's operators for an assumed slowly varying background variation in the medium parameters. The migration essentially removes the effects of the background from the inversion by transforming the recorded wave field from the time domain to the depth domain. For a constant background, this step is similar to $F$–$K$ migration. For a depth variable background, a phase shift migration is used. For a laterally variable background, the WKBJ Green's operators have to be constructed numerically.

The second step is to determine the parameter variation from the migrated data. It is shown that the triple Fourier transform of the migrated data is a linear combination (with known coefficients) of the double Fourier transform of the bulk modulus and density variations. Thus, a simple least-squares solution can be used to invert the data.

## ACKNOWLEDGMENT

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## REFERENCES


Claerbout, J. F., 1976, Fundamentals of geophysical data processing: New York, McGraw-Hill Book Co., Inc.

Clayton, R. W., 1981, Wavefield inversion methods for refraction and reflection data: Ph.D. thesis, Stanford Univ.

Clayton, R. W., and Engquist, B., 1980, Absorbing boundary conditions for wave-equation migration: Geophysics, v. 45, p. 895–904.

Cohen, J. K., and Bleistein, N., 1977, An inverse method for determining small variations in propagation speed: SIAM J. of Appl. Math., v. 32, p. 784–799.

——— 1979, Velocity inversion procedure for acoustic waves: Geophysics, v. 44, p. 1077–1087.

Gazdag, J., 1978, Wave equation migration with the phase shift method: Geophysics, v. 43, p. 1342–1351.

Jacobs, A., and Stolt, R., 1980, Seismic inversion in a layered medium—The Gelfand-Levitan algorithm: Submitted to Geophysics.

Newton, R. G., 1966, Scattering theory of waves and particles: New York, McGraw-Hill Book Co., Inc.

Phinney, R. A., and Frazer, L. N., 1978, On the theory of seismic imaging by Fourier transform: Presented at the 48th Annual International SEG Meeting, October 31, in San Francisco.

Raz, S., 1981, Three-dimensional velocity profile inversion from finite-offset scattering data: Geophysics, v. 46, p. 837–842.

Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, v. 43, p. 23–48.

Stolt, R. H., and Jacobs, A., 1980, An approach to the inverse seismic problem: Submitted to Geophysics.

Taylor, J. R., 1972, Scattering theory: New York, John Wiley and Sons, Inc.

Ware, J. A., and Aki, K., 1969, Continuous and discrete inverse scattering problems in a stratified elastic medium, I: Plane waves at normal incidence: J. Acoust. Soc. Am., v. 45, p. 911–921.

Yedlin, M., 1981, Uniform asymptotic representation of the Green's function for the 2-D acoustic problem: to be submitted.

Yilmaz O., and Claerbout, J., 1980, Prestack partial migration: Geophysics, v. 45, p. 1753–1779.


## APPENDIX
## INCORPORATING POINT SOURCES AND RECEIVERS IN THE TWO-DIMENSIONAL SOLUTION

The solutions given in the text are for a 2-D medium. However, it is trivial to modify the solutions for the full 3-D case. For example, the 3-D equivalent of the constant background equation (26) is

$$\begin{aligned} &D(k_g, k_y \mid k_s, k_y'; \omega) \\ &= \frac{-1}{4\pi^2}\frac{\rho_r^2}{4qq'}\left[\frac{\omega^2}{v_r^2} a_1(k_g - k_s, k_y - k_y', -q - q') \right. \\ &\left. + (qq' - k_g k_s - k_y k_y') a_2(k_g - k_s, k_y - k_y', -q - q')\right] S(\omega), \end{aligned} \tag{A-1}$$

where

$$q = \frac{\omega}{v_r}\sqrt{1 - \frac{v_r^2}{\omega^2}(k_g^2 + k_y^2)} \quad \text{and}$$

$$q' = \frac{\omega}{v_r}\sqrt{1 - \frac{v_r^2}{\omega^2}(k_s^2 + k_y'^2)}.$$

In this equation primed variables refer to the source location, while unprimed variables refer to the receiver location.

The seismic experiment is usually conducted along a line (say, $y = y' = 0$), and the medium parameters are assumed to be invariant in the $y$-direction. In this case the $a_i$ have the form (in the wavenumber space)

$$\begin{aligned} &a_i(k_g - k_s, k_y - k_y', -q - q') \\ &\quad \rightarrow a_i(k_g - k_s, -q - q')\delta(k_y - k_y'). \end{aligned} \tag{A-2}$$

To restrict the 3-D problem to one that can be handled by the 2-D algorithm outlined in the text, we start by inverse transforming over $k_y$ and $k_y'$ and evaluating the data field along $y = y' = 0$.

$$D(k_g, 0 \mid k_s, 0; \omega) \equiv \int dk_y \int dk_y' D(k_g, k_y \mid k_s, k_y'; \omega). \tag{A-3}$$

The integral over $k_y'$ can be evaluated trivially because of the form of $a_i$ in equation (A–2).

$$D(k_g, 0 \mid k_s, 0; \omega) = \int dk_y D(k_g, k_y \mid k_s, k_y; \omega). \tag{A-4}$$

To remove the remaining integral over $k_y$, we express the $a_i$ as a Fourier transform over $z$. That is,

$$a_i(k_g - k_s, -q - q') = \int dz\, e^{-i(q+q')z} a_i(k_g - k_s, z). \tag{A-5}$$

Substituting equation (A–5) into equation (A–4) and interchanging the order of integration, we have

$$\begin{aligned} D(k_g, 0 \mid k_s, 0; \omega) = \int dz \sum_{i=1}^{2} \int dk_y A_i(k_g, k_s, k_y, q, q') \cdot \\ \cdot a_i(k_g - k_s, z) e^{-i(q+q')z}, \end{aligned} \tag{A-6}$$

where the $A_i$ are the 3-D analogs of the factors defined by equations (35) and (36). If we assume the $A_i$ are slowly varying compared to the exponential, then we can evaluate the $k_y$ integral by stationary phase. To do this, $q + q'$ is expanded about the point where its derivative with respect to $k_y$ is zero, which in this case is the point $k_y = 0$. Thus,

$$q + q' = k_z + k_y^2 k_z'',$$

where $k_z$ is given by equation (30), and $k_z'' = -k_z/q_g q_s$. In the last expression $q_g$ and $q_s$ are the 2-D vertical wavenumbers defined by equation (23).

Using the standard stationary phase formulas, equation (A–4) may be expressed as

$$D(k_g, 0 \mid k_s, 0; \omega) = \sum_{i=1}^{2} \tilde{A}_i \tilde{a}_i, \tag{A-7}$$

where the $\tilde{a}_i$ are scaled versions of the $a_i$ used in the text

$$\tilde{a}_i(x, z) = \frac{a_i(x, z)}{\sqrt{z}}, \tag{A-8}$$

and the factors $\tilde{A}_i$ are related to the $A_i$ of equations (35) and (36) by

$$\tilde{A}_i = \sqrt{\frac{q_g q_s}{ik_z}} A_i. \tag{A-9}$$

The result is, of course, subject to the approximations used in the stationary phase evaluation of the $k_y$ integral. However, since most seismic data are far-field, we expect the approximation to be reasonably accurate.

For the vertically varying medium, a similar argument leads to a modification of the multiplicative factor in the downward continuation algorithm. We obtain

$$\begin{aligned} &R_0(k_g, z \mid k_s, z) \\ &\rightarrow R_0(k_g, z \mid k_s, z)\left\{i\int_0^z dz' \left[\frac{1}{q_g(z')} + \frac{1}{q_s(z')}\right]\right\}^{1/2}. \end{aligned} \tag{A-10}$$

The rest of the inversion proceeds as before.

The modification required to adapt the laterally varying algorithm to point sources and receivers will be left as an exercise for the reader.