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A Bound for Global Solutions of Semilinear Heat Equations

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Abstract. We show that global positive solutions of the initial-boundary value problem for $u_t = \Delta u + u^p$ are bounded, provided that $p > 1$ is subcritical. Our bound depends only on sup norm of the initial data and is useful to classify initial data by the asymptotic behavior of the solutions as time tends to infinity.

1. Introduction

We consider the initial boundary value problem on Ω for a semilinear heat equation

$$u_t - \Delta u - u^p = 0, \tag{1}$$

with the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{on } \partial\Omega, \tag{2}$$

and initial data

$$u(x,0) = u_0(x) \geqq 0 \quad \text{in } \Omega. \tag{3}$$

Here Ω is a smoothly bounded domain in $\mathbb{R}^n$ and $p > 1$. By the maximum principle and (3) the solution $u(\cdot, t)$ is positive in Ω for $t > 0$ (unless $u_0(x) \equiv 0$); therefore the term u^p is well-defined for every p. The asymptotic behaviour of global solutions $u(\cdot, t)$ as $t \to \infty$ has been studied by many authors; see for example [1, 4, 5]. To classify initial data by the asymptotic behavior of the corresponding solutions, it is important to study whether global solutions satisfy a priori bounds for all time. In [5] Ni, Sacks and Tavantzis proved an a priori bound assuming that Ω is convex and $p < 1 + 2/n$. This was improved by Cazenave and Lions in [1]; they showed for a general domain Ω that a global solution is bounded in $\Omega \times (t_0, \infty)$ for every $t_0 > 0$, provided that $n/2 < (p+1)/(p-1)$ (equivalently, $p < (n+2)/(n-2)$ or $n \leqq 2$). For $p < 1 + 12/(3n-4)$ or $n = 1$, their bound depends explicitly on norm of the initial data. However, if $n > 1$ and $p \geqq 1 + 12/(3n-4)$, the dependence of their bound on the initial data is unclear.

Our main goal in this paper is to remove this technical restriction on p. We shall

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show that every global solution satisfies on a priori bound depending only on a norm of the initial data, provided only that $n/2 < (p+1)/(p-1)$. To avoid later confusion, by a "solution" of (1–3) in $\Omega \times [0, T]$ we shall always mean a continuous function in $\bar{\Omega} \times [0, T)$ which is smooth enough for each term appearing (1) to be continuous in $\Omega \times (0, T)$.

Main Theorem. *Let $n/2 < (p+1)/(p-1)$. Suppose that $u(x,t)$ is the solution of (1–3) in $\Omega \times [0, \infty)$ with $u_0 \in C(\bar{\Omega})$. Then there is a constant M depending only on* $\sup\{u_0(x); x \in \Omega\}$ *such that $u(x,t) \leqq M$ for $x \in \Omega$, $t \geqq 0$.*

As a simple application we obtain the closedness of the set of initial data which gives global existence. To be precise, we denote

$$C = \{u_0 \in C(\bar{\Omega});\ u_0 \geqq 0\},$$

$$K = \{u_0 \in C;\ \text{global solution of (1–3) exists with } u(x,0) = u_0(x)\}.$$

Corollary. *For $n/2 < (p+1)/(p-1)$, the set K is closed in C.*

Since $C \backslash K$ is not empty [1, 4, 5, 6], it is well-known that the Corollary provides "threshold" initial data $u_0 \in \partial K \cap K$ whose ω-limit set consists of nontrivial equilibrium solutions of (1–2). We note that this gives a dynamical way to construct a nontrivial equilibrium solution of (1–2). If Ω is star-shaped and $n/2 \geqq (p+1)/(p-1)$, it is well known that there is no solution of (1–2) other than zero. We conclude that the restriction on p in the Main Theorem is optimal.

To prove the Main Theorem we combine energy identities [1] with a scaling argument previously used by Gidas and Spruck [2] for elliptic equations. Our method works for all p, $n/2 < (p+1)/(p-1)$. Although we discuss only a special nonlinear term, the argument extends to a more general class of nonlinearity as will be briefly explained in Sect. 3.

In Sect. 2 we formulate and prove a lemma crucial to our method, and in Sect. 3 we prove the Main Theorem and its Corollary. As a by-product of the lemma we shall prove in Sect. 4 that the energy blows up if the solution blows up in finite time provided that $n/2 < (p+1)/(p-1)$.

2. A Priori Estimates

We shall derive a sup norm bound for solutions assuming boundedness of an integral norm.

Lemma. *Let u be a solution of (1–3) in $Q = \Omega \times [0, T)$. Suppose that u satisfies the estimate*

$$\int_0^T \int_\Omega |u_t|^2 \, dx dt < N < \infty, \tag{4}$$

and that for given $t_0 > 0$

$$\sup_\Omega u \text{ is attained in } \Omega \times (t_0, T). \tag{5}$$

Then there is a constant A independent of u, u_0 and T (depending on N and t_0) such that

$$u(x,t) \leqq A \quad \text{in } Q.$$

Our proof appeals to scaling and argues by contradiction, following the ideas introduced by Gidas and Spruck [2] for elliptic equations. Most of the argument is parallel to [2] except that we use parabolic theory and (4). However since the proof is by contradiction, we give not only the major part (the use of (4)) but also the outline of the whole proof.

Although we discuss only a special nonlinear term, the same proof works if one replaces u^p by any $f(u)$ with the property that

$$0 < \lim_{z \to +\infty} f(z)/z^p < \infty \tag{6}$$

for some p such that $n/2 < (p+1)/(p-1)$.

Proof. We first introduce a parabolic scaled function. Our proof is by contradiction.

Suppose the Lemma were false. Then there would exist a sequence of functions $u_k(x,t)$ satisfying (1–4) with $T = T_k > 0$ and a sequence of points (x_k, t_k), $t_k > t_0$ such that

$$M_k = \sup\{u_k(x,t);\ x \in \Omega,\ t \in (0, T_k)\} = u_k(x_k, t_k) \to \infty \quad \text{as} \quad k \to \infty.$$

Since $\bar{\Omega}$ is compact, we may assume that $x_k \to x_\infty \in \bar{\Omega}$ as $k \to \infty$, by choosing subsequences. Let λ_k be a sequence of positive numbers such that

$$\lambda_k^{2/(p-1)} M_k = 1.$$

Since $M_k \to \infty$, evidently $\lambda_k \to 0$. We now define the scaled function

$$v_k(y,s) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y,\ t_k + \lambda_k^2 s).$$

Clearly,

$$v_k(0,0) = 1, \tag{7}$$

and

$$v_k(y,s) \leqq 1, \text{ where } v_k \text{ is defined.} \tag{8}$$

Following [2] we divide the situation into two cases depending on whether $x_\infty \in \Omega$ or $x_\infty \in \partial\Omega$.

Case 1. $x_\infty \in \Omega$. (i) (*Domain of definition of* v^k). Let $Q(r)$ be a parabolic cylinder with radius r in $\mathbb{R}^{n+1}$,

$$Q(r) = \{(y,s) \in \mathbb{R}^{n+1},\ |y| < r,\ -r^2 < s \leqq 0\}.$$

Let $2d$ denote the minimum of $2\sqrt{t_0}$ and the distance of x_∞ and $\partial\Omega$. Since $x_k \to x_\infty$, we see $v_k(y,s)$ is defined in $Q(d/\lambda_k)$ for sufficiently large k.

(ii) (*The equation for* v_k). Because (1) is invariant under our scaling, v_k solves

$$v_{ks} - \Delta_y v_k - v_k^p = 0 \text{ in } Q(d/\lambda_k).$$

(iii) (*Convergence of* v_k). We apply parabolic L^p regularity theory [3] to get uniform bounds in some Sobolev space from (8). More precisely for any given R such that $Q(R) \subset Q(d/\lambda_k)$, $\{v_k\}$ is bounded in $W_p^{2,1}(Q(R))$ for any $q > n$. As is usual, we can find a subsequence (still denoted v_k) which converges uniformly to some function

$v \geqq 0$ in any $Q(R)$ and v is defined in $\mathbb{R}^n \times (-\infty, 0)$. We may assume, taking yet another subsequence if necessary that v_{ks} converges weakly to v_s in $L^2(Q(R))$.

(iv) (*The equation for* v). Combining (ii) and (iii) shows that v solves

$$v_s - \Delta_y v - v^p = 0 \text{ in } \mathbb{R}^n \times (-\infty, 0).$$

From (7) we see also

$$v(0,0) = 1.$$

(v) ($v_s = 0$). This is the only step we use (4). A simple manipulation shows that

$$\iint_{Q(d/\lambda_k)} |v_{ks}|^2 dyds = \lambda_k^\sigma \int_{t_k - d^2}^{t_k} \int_{|x - x_k| < d} |u_{kt}|^2 dxdt,$$

with $\sigma = -n + 2 + 4/(p-1)$. Applying (4), we get

$$\iint_{Q(R)} |v_{ks}|^2 dyds \leqq \lambda_k^\sigma N \to 0 \quad \text{as} \quad k \to \infty,$$

since $n/2 < (p+1)/(p-1)$ is equivalent to $\sigma > 0$. This yields $v_s = 0$ in $Q(R)$ because v_{ks} converges weakly in $L^2(Q(R))$ to v_s and the norm is lower semicontinuous under weak convergence. Since R is arbitrary, so v_s vanishes identically in $\mathbb{R}^n \times (-\infty, 0)$.

(vi) (*Application of a Liouville theorem*). Since $v_s = 0$, $v \geqq 0$ solves

$$\Delta v + v^p = 0 \text{ in } \mathbb{R}^n.$$

A Liouville theorem [2] asserts that there are no nontrivial solutions when $n/2 < (p+1)/(p-1)$, and so $v \equiv 0$. This contradicts the fact that $v(0,0) = 1$. Hence case 1 cannot occur.

Case 2. $x_\infty \in \partial\Omega$. In what follows we number the steps to correspond to those of Case 1. (O) (*Coordinate change*). We may assume the boundary $\partial\Omega$ is contained in the hyperplane $\{x^n = 0\}$ by taking a local coordinate $x = (x^1, \ldots, x^n)$ around x_∞. Unfortunately, because of the coordinate change the equation we must now discuss is a uniformly parabolic equation with smooth *variable* coefficients

$$u_t - \Sigma_{ij} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} u - \Sigma_i b^i(x) \frac{\partial u}{\partial x^i} - u^p = 0.$$

We may assume that $a^{ij}(0) = \delta^{ij}$.

(i) Let d_k be the distance from x_k to $\partial\Omega$. We see, at least for large k, that v_k is defined in $Q(\delta/\lambda_k) \cap \{y^n > -d_k/\lambda_k\}$ for some $\delta > 0$.

(ii) The equation for v_k is slightly more complicated because of (O),

$$v_{ks} - \Sigma_{ij} a^{ij}(\lambda_k y + x_k) \frac{\partial^2}{\partial y^i \partial y^j} v_k - \lambda_k \Sigma_i b^i(\lambda_k y + x_k) \frac{\partial}{\partial y^i} v_k - v_k^p = 0.$$

(iii) This time we should use parabolic L^p theory up to the boundary [3] to the equation in (ii). In particular, we get $|\nabla v_k|$ is uniformly bounded in $Q(R) \cap \{y^n > -d_k/\lambda_k\}$. Since $v_k(0,0) = 1$ by (7) and $v_k = 0$ on $\{y^n = 0\}$ we get $d_k/\lambda_k \geqq B > 0$. If $\limsup_k d_k/\lambda_k = \infty$, after a choice of subsequence the situation is reduced to case 1 suitably modified for variable coefficient Eq. (cf. [2]). We may therefore assume $d_k/\lambda_k \to c > 0$. Applying parabolic L^p theory up to the boundary yields that a subsequence (still denoted v_k) converges to v as in (iii) of case 1, except that we should replace $Q(R)$ by $Q(R) \cap \{y^n > -c + \varepsilon\}$ for arbitrary $\varepsilon > 0$ and $R > 0$. Note that $v = 0$ on $\{y^n = -c\}$ and $v(0,0) = 1$.

(iv) The equation for v is the same as (iv) of case 1, even though step (ii) is different (cf. [2]). Using $a^{ij}(0) = \delta^{ij}(0)$ and (iii), we get

$$v_t - \Delta v - v^p = 0 \quad \text{in } \{y \in \mathbb{R}^n; y^n > -c\} \times (-\infty, 0).$$

(v) This step is the same as case 1 with the necessary changes of the domain of integration.

(vi) Since $v_s = 0$, $v \geqq 0$ solves

$$\Delta v + v^p = 0 \quad \text{in } \{y^n > -c\},$$

with $v = 0$ on $\{y^n = -c\}$. Applying the Liouville theorem in a half space [2] yields $v \equiv 0$, which contradicts the fact that $v(0,0) = 1$. We thus have contradiction in both cases which completes the proof.

3. Proof of Main Results

Energy estimates, and the Lemma, lead easily to the main theorem. Although the argument is known [1], we outline it for the reader's convenience.

Proof of the Main Theorem. We may assume u_0 is not identically zero. We first note that there are constants B, $t' > 0$ depending only on $\sup_\Omega u_0$ such that

$$\sup_{0 \leqq \tau \leqq 2t'} \sup_\Omega u(x,\tau) \leqq B, \quad \int_\Omega |\nabla u|^2(x,t')dx \leqq B. \tag{9}$$

The first inequality is easy to prove. The second one follows from the regularizing property of parabolicity, which is well known (cf. [7]).

We next recall energy identities. Multiplying (1) with u and u_t and integrating over Ω yields

$$\frac{d}{dt}\int_\Omega |u|^2 dx = -4E[u](t) + \frac{2(p-1)}{p+1}\int_\Omega |u|^{p+1}dx \tag{10}$$

and

$$\int_\Omega |u_t|^2 dx = -\frac{d}{dt}E[u], \tag{11}$$

where E is the "energy," defined by

$$E[u] = \tfrac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega u^{p+1}dx. \tag{12}$$

Identity (11) says that the energy should decrease. This together with (10) shows that $E[u](t) \geqq 0$ for $t > 0$, since otherwise the solution must blow up in finite time.

Integrating (11) over (t', T) gives

$$\int_{t'}^{T}\int_{\Omega} |u_t|^2 \, dxdt \leqq E[u](t') \leqq B/2$$

by (9). Now by the Lemma, any solution in $\Omega \times [t', T)$ which attains its maximum outside $\Omega \times [t', 2t')$ is bounded from above by a constant depending only on B and t' (independent of T). By (9), a solution which takes its maximum inside $\Omega \times [t', 2t')$ is dominated by B. Hence in any case

$$u(x, t) \leqq M \text{ in } \bar{\Omega} \times [t', \infty),$$

where $M > B$ depends only on $\sup_{\Omega} u_0$. Of course $u \leqq B$ in $\bar{\Omega} \times [0, t')$ by (9), which completes the proof of the Main Theorem.

Remark. The nonlinear term u^p in (1) may be replaced the more general one $f(u)$ satisfying (6) and following conditions which are given in [1].

(a) $f(z) = -mz + g(z)$ and $g(z)$ is a nonnegative C^1 real-valued function on $z \geqq 0$.
(b) $m > -\lambda_1$ where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.
(c) There is a constant $\delta > 0$ such that

$$zg(z) \geqq (2 + \delta)\int_0^z g(s)ds.$$

Remark. In [1] Cazenave and Lions obtain global a priori bounds provided that $1 < p < 1 + 12/(3n-4)$ or $n = 1$. They do so by interpolating between estimates of $\int\int |u_t|^2 dxdt$ and $\int(\int|\nabla u|^2 dx)^2 dt$ to bound $\sup_t \int |u|^q dx$ for some $q > 1$. Unfortunately, this approach fails if $p \geqq 1 + 12/(3n-4)$ and $n \geqq 2$. Instead, for larger p they prove that a global solution is bounded by using that

$$\int_{T}^{\infty}\int_{\Omega} |u_t|^2 dxdt$$

is small if T is large. It seems that such an argument cannot give a bound depending only on a norm of the initial data.

Proof of Corollary. Let u_{m0} be a sequence in K such that $u_{m0} \to u_0$ in C. The Main Theorem implies that the solution u_m of (1–3) with $u_m(x, 0) = u_{m0}(x)$ satisfies the estimate $u_m \leqq M$ in Q independent of m. This gives a uniform bound for the solution with initial data u_0, which prevents the blow up in finite time. Therefore $u_0 \in K$, which completes the proof.

4. A Remark on Blow Up

It is well known that there are examples of initial data $u_0 \in K$, i.e., that solutions of (1–3) can blow up in finite time. For such u_0 there is a time $T_* < \infty$ such that u is the solution of (1–3) in $\Omega \times [0, T_*)$ and that $\sup_\Omega u \to \infty$ as $t \to T_*$. It is interesting to study what other quantities blow up at T_* (cf. [6]). As an application of the Lemma we claim the energy E defined by (12) blows up at T_*.

Corollary to Lemma. *If the solution u of (1–3) blows up at time $T_* < \infty$, then the energy $E[u](t) \to -\infty$ as $t \to T_*$ provided that $n/2 < (p+1)/(p-1)$.*

Proof. If not, (11) yields that for fixed t', $0 < t' < T < T_*$,

$$\int_{t'}^{T} \int_{\Omega} |u_t|^2 \, dx dt \leqq C < \infty,$$

where C is independent of T. If $\sup_{\Omega} u(x,t) \to \infty$ as $t \to T_*$, then the Lemma is contradicted. Therefore $E[u](t) \to -\infty$ as $t \to T_*$.

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