A Bound for the Least Prime Ideal in the Chebotarev Density Theorem

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1. Introduction

We first recall the Chebotarev Density Theorem. Let K be a finite algebraic extension of the rationals $\mathbf{Q}$, and L a Galois extension of K. To each prime ideal P of K unramified in L there corresponds a certain conjugacy class C of $\mathrm{Gal}(L/K)$ consisting of the set of Frobenius automorphisms σ attached to the prime ideals p of L which lie over P. Denote this conjugacy class by the Artin symbol $\left[\dfrac{L/K}{P}\right]$. For a given conjugacy class C of $\mathrm{Gal}(L/K)$ let $\pi_C(x)$ denote the number of prime ideals P of K unramified in L such that $\left[\dfrac{L/K}{P}\right] = C$ and $N_{K/\mathbf{Q}} P \leqq x$. The Chebotarev density theorem [6, 10] asserts that

$$\pi_C(x) \sim \frac{|C|}{|G|} Li(x) \tag{1.1}$$

as $x \to \infty$. In [7] two versions of the Chebotarev density theorem were proved, one unconditional and the other on the assumption of the Generalized Riemann Hypothesis (GRH), each of which expressed $\pi_C(x)$ as the sum of the main term $\dfrac{|C|}{|G|} Li(x)$ and an error term which is an effectively computable function of x, $\dfrac{|C|}{|G|}$, and the associated field constants $n_K = [K:\mathbf{Q}]$, $n_L = [L:\mathbf{Q}]$ and d_K, d_L (the absolute values of the discriminants of the two fields). Assuming the truth of the GRH for $\zeta_L(s)$, that paper also proved the existence of an effectively computable constant b (independent of K and L) such that for any conjugacy class C, there exists a prime ideal P in K with $\left[\dfrac{L/K}{P}\right] = C$ and

$$N_{K/\mathbf{Q}} P \leqq b(\log d_L)^2 (\log\log d_L)^4.$$

The method of proof showed that we might specify that $N_{K/\mathbf{Q}}P$ be a rational prime. Furthermore that paper sketched a proof that sharpened this result to:

$$N_{K/\mathbf{Q}}P \leqq b'(\log d_L)^2. \tag{1.2}$$

The main purpose of this paper is to supplement these results with an unconditional upper bound for the least prime ideal having a specified Artin symbol.

Theorem 1.1. *There is an absolute, effectively computable constant A_1 such that for every finite extension K of $\mathbf{Q}$, every finite Galois extension L of K and every conjugacy class C of $\mathrm{Gal}(L/K)$, there exists a prime ideal P of K which is unramified in L, for which $\left[\dfrac{L/K}{P}\right] = C$, for which $N_{K/\mathbf{Q}}P$ is a rational prime, and which satisfies the bound*

$$N_{K/\mathbf{Q}}P \leqq 2d_L^{A_1}. \tag{1.3}$$

(*Note.* The 2 in the bound (1.3) is there only to take care of the trivial case $L = K = \mathbf{Q}$.)

The primary technical difficulties in the proof of Theorem 1.1 center around the possible exceptional zero of $\zeta_L(s)$. As an auxiliary step we prove in Section 5 a version of the Deuring-Heilbronn phenomenon for $\zeta_L(s)$, which guarantees that, if the exceptional zero exists, then the other zeros of $\zeta_L(s)$ cannot lie very close to $s = 1$. The proof of this auxiliary result involves the use of a power-sum inequality which is relatively easy to prove and which may be used as a substitute for Turàn's second main theorem [11] in many L-function applications. A feature of the proofs which distinguishes them from those of [7] is the use of kernels which weight prime ideals of small norm very heavily. These kernels enable one to obtain good lower bounds for densities of prime ideals with specified properties at the cost of not being able to estimate their number accurately. The use of different kernels is related to the "explicit formulas" of Guinand [5] and Weil [12].

The proof of Theorem 1.1 uses classical analytic methods, and does not involve any deep zero-density estimates. In the general case the bound of Theorem 1.1 is probably the best that can be obtained by current methods, since for $K = \mathbf{Q}$, $L = \mathbf{Q}(\sqrt{D})$, ρ the non-identity in $\mathrm{Gal}(L/K)$, the bound (1.3) is equivalent to an upper bound for the least quadratic nonresidue modulo D, and even for this case no estimate better than (1.3) is known. However, in special cases, estimates sharper than (1.3) are possible. For example, Linnik's theorem [2] asserts the existence of an absolute effectively computable constant B such that for any cyclotomic field $L = \mathbf{Q}(\exp(2\pi i/N))$, $N \geqq 3$, and any $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$, there exists a rational prime p with $\left[\dfrac{L/\mathbf{Q}}{(p)}\right] = \sigma$ and $p \leqq (\log d_L)^B$. Zero-density results for zeros near the line $\mathrm{Re}(s) = 1$ are needed in all known proofs of Linnik's theorem. The methods of this paper do not suffice to prove Linnik's theorem, but they can be used to obtain improved bounds in some cases. For example, if $\mathrm{Gal}(L/K)$ contains a large abelian subgroup H, then $\zeta_L(s)$ can be factored into Hecke L-functions over the fixed field of H, which then leads to a better zero-free region for $\zeta_L(s)$, and thereby to improvements of the bound (1.3).

The proof of Theorem 1.1 uses the known zero-free region of Hecke L-functions. In addition, we prove a theorem which gives a bound for the least character nonresidue of a Hecke character χ that is an explicit function of the size of a hypothetical zero-free region near $s = 1$ for $\zeta_K(s) L(s, \chi, K)$.

Theorem 1.2. *There is an absolute, effectively computable positive constant A_2 such that for every finite extension K of $\mathbf{Q}$ and every Hecke character χ on K the following holds: Suppose that $\zeta_K(s) L(s, \chi, K)$ has no zeros $\rho = \beta + i\gamma$ in the region*

$$1 - \delta < \beta < 1,$$
$$0 < |\gamma| \leqq \delta^2 (\log A(\chi)) (1 + \delta (\log A(\chi))^{1/2}),$$

where

$$(c_2 \log A(\chi))^{-1} \leqq \delta \leqq \tfrac{1}{2},$$

and that $\zeta_K(s)$ has no zeros on the segment of the real axis $1 - \delta \leqq s \leqq 1$. (Here $A(\chi) = d_K N_{K/Q} f(\chi)$, $f(\chi)$ is the conductor of χ, and c_2 is a certain absolute, effectively computable constant given by Lemma 2.3.) Then there exists a prime ideal P of K of degree 1 over $\mathbf{Q}$ with $\chi(P) \neq 1$ such that

$$N_{K/Q} P < (A_2 \delta \log A(\chi))^{\delta^{-1}}.$$

Applying this theorem with $\delta = \frac{1}{2}$ we obtain the following corollary, which was first proved in the case $K = \mathbf{Q}$ by Ankeny [1].

Corollary 1.3. *There is an absolute, effectively computable positive constant A_3 such that for every finite extension K of $\mathbf{Q}$ and every Hecke character χ of K, if $\zeta_K(s) L(s, \chi, K)$ has no zeros $\rho = \beta + i\gamma$ in the region*

$$\tfrac{1}{2} < \beta < 1, \tag{1.4}$$

then there exists a prime ideal P of K of degree 1 over $\mathbf{Q}$ with $\chi(P) \neq 1$ such that

$$N_{K/Q} P < A_3 (\log A(\chi))^2. \tag{1.5}$$

It is an open question as to what kind of zero-free region suffices to prove (1.5). Using the kernel $k(s) = x^s \Gamma(s)$, we can show that (1.5) holds if $\zeta_K(s) L(s, \chi, K)$ has no zeros in $\frac{1}{2} < \beta < 1$, $|\gamma| \ll \log \log (A(\chi))$. We do not know whether (1.5) could be proved if we only knew there are no zeros in $\frac{1}{2} < \beta < 1$, $|\gamma| < \frac{1}{2}$.

Our proof of Theorem 1.1 does not suffice to show that

$$\pi_C(x) > \varepsilon \frac{|C|}{|G|} Li(x) \quad \text{for} \quad x > 2d_L^{A_1}, \tag{1.6}$$

for any fixed $\varepsilon > 0$. Such an estimate, or even a slightly weaker one, would be of great interest, but seems unattainable with our present techniques. However, we will sketch a proof of an unconditional upper bound estimate for $\pi_C(x)$ which is not difficult to prove and is of interest in some applications.

Theorem 1.4. *There exist absolute, effectively computable constants A_4 and A_5 such that if $L \neq \mathbf{Q}$ and*

$$x > \exp\{A_4 (\log d_L)(\log\log d_L)(\log\log\log d_L e^{20})\},$$

then

$$\pi_C(x) \leqq A_5 \frac{|C|}{|G|} Li(x).$$

The paper proves Theorem 1.2 first, since it exhibits some essential features of the proof of Theorem 1.1 with less complications from the possible exceptional zero.

Throughout this paper $c_1, c_2, \ldots$ as well as all constants implied by $\ll$ and O-notations will denote absolute and effectively computable positive constants. Background material on properties of Hecke and Artin L-functions is available in [6], [7].

2. Least Character Nonresidue

This section presents the proof of Theorem 1.2 on the least Hecke character nonresidue. We first give an outline. A weighted sum of the values of the Hecke character at prime ideals is evaluated as an inverse Mellin transform of $-\frac{L'}{L}(s, \chi, K)$ times a kernel function. The kernel weights only prime ideals of small norm. By contour integration this sum is equal to a sum of the kernel function over the zeros of $L(s, \chi, K)$, and this is estimated to be small. On the other hand, if $\chi(P) = 1$ for all small prime ideals P then the value of the Mellin transform is virtually the same as that obtained by replacing $\frac{L'}{L}(s, \chi, K)$ with $\frac{\zeta_K'}{\zeta_K}(s)$. This is evaluated by contour integration and is shown to be large, due to the contribution of the pole at $s = 1$, provided $\zeta_K(s)$ has no exceptional zero. This then yields a contradiction which proves the theorem.

Proof of Theorem 1.2. We first introduce our kernel function. If $k(s)$ is an entire function, say, such that $|k(\sigma + it)|$ is integrable as a function of t for $\sigma > -\frac{1}{2}$, let

$$\hat{k}(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+\infty} k(s) u^{-s} ds \tag{2.1}$$

be its inverse Mellin transform, where the integration is on a vertical line $\mathrm{Re}(s) = a > -\frac{1}{2}$. The kernel function we use here is

$$k(s) = k(s; x, y) = \left(\frac{y^{s-1} - x^{s-1}}{s-1} \right)^2. \tag{2.2}$$

A simple computation shows that for $y > x > 1$ and $u > 0$,

$$\hat{k}(u; x, y) = \begin{cases} 0 & \text{if } u > y^2, \\ u^{-1} \log \dfrac{y^2}{u} & \text{if } xy < u < y^2, \\ u^{-1} \log \dfrac{u}{x^2} & \text{if } x^2 < u < xy, \\ 0, & \text{if } u < x^2. \end{cases} \tag{2.3}$$

We next come to the Hecke L-function. Associated to a Hecke character χ over a field K is the constant

$$A(\chi) = d_K N_{K/\mathbf{Q}} f(\chi),$$

where $f(x)$ is the conductor of χ. We recall the Dirichlet series expansion,

$$-\frac{L'}{L}(s, \chi, K) = \sum_P \sum_{n=1}^{\infty} \chi(P^n) \Lambda(P^n)(NP)^{-ns} \tag{2.4}$$

where P runs over all prime ideals, and $\Lambda(I)$ is the generalized Von Mangoldt function

$$\Lambda(I) = \begin{cases} \log NP & \text{if } I = P^q, \\ 0 & \text{otherwise,} \end{cases}$$

and NP denotes $N_{K/\mathbf{Q}} P$. We now consider the inverse Mellin transform (with $y > x > 1$)

$$I_0 = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi, K) k(s; x, y)\, ds = \sum_{P, n} \chi(P^n) \Lambda(P^n)(NP)^{-n} \hat{k}(NP^n; x, y). \tag{2.5}$$

Note that the sum only counts prime-power ideals of norm $< y^2$. Furthermore the contribution of those ideals P^n for which NP^n is not a rational prime is

$$\ll n_K(\log y) \log\left(\frac{y}{x}\right) \sum_{\substack{n = p^a, a \geq 2 \\ x^2 \leq n \leq y^2}} n^{-1}$$

$$\ll n_K(\log y) \log\left(\frac{y}{x}\right)(x \log x)^{-1}. \tag{2.6}$$

Next we evaluate the integral (2.5) by contour integration. Consider the integrand of (2.5) integrated on the rectangular contour with vertices at $-N + iT$, $-N - iT$, $2 - iT$, $2 + iT$. Proceeding exactly as in [7, Sect. 6], the contribution of the horizontal sides of the box goes to zero as $T \to \infty$. The contribution of the side $-N + iT$, $-N - iT$ goes to zero as $N \to \infty$ through values $k + \frac{1}{4}$, k an integer. The contribution of the remaining side has as a limit the integral I_0, and by Cauchy's theorem we obtain

$$I_0 = -\sum_{\rho_\chi} k(\rho_\chi; x, y), \tag{2.7}$$

where ρ_χ runs over all zeros of $L(s, \chi, K)$.

We now use (2.7) to bound the real part of (2.5) from above. First $-k(\rho; x, y)$ is negative for real ρ, so we obtain a valid upper bound if we drop the terms corresponding to $\rho \in \mathbf{R}$. This takes care of the trivial zeros and any possible exceptional zero of $L(s, \chi, K)$. To proceed further we need two zero-density lemmas. The first is:

Lemma 2.1. ([7, Lemma 5.4]) *Let $n_\chi(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the Hecke L-function $L(s, \chi, K)$ in the rectangle $0 \leq \beta \leq 1$, $|\gamma - T| \leq 1$. Then*

$$n_\chi(T) \ll \log A(\chi) + n_K \log(|T| + 2), \tag{2.8}$$

where $A(X) = d_K N_{K/\mathbf{Q}} f(\chi)$.

Remark. Lemma 2.1 was proved in [7] under the tacit assumption that χ is primitive, since those were the only characters that played any role there. However, if χ' is the primitive character that induces χ, then

$$L(s, \chi, K) = L(s, \chi', K) \cdot \prod_{P | f(\chi)} (1 - \alpha(P)(NP)^{-s}), \tag{2.9}$$

where $\alpha(P) = 0$ or $\chi'(P)$. It follows that the zeros of $L(s, \chi, K)$ are the same as those of $L(s, \chi', K)$, except for $\ll \log A(\chi)$ arithmetic progressions of zeros on the line $\sigma = 0$, which together contain

$$\ll (1 + \Delta) \log A(\chi)$$

zeros in any interval of length Δ on the line $\sigma = 0$. Therefore the bound of the lemma above is true also for imprimitive characters. Furthermore, it is easily seen from (2.9) that

$$\frac{L'}{L}(s, \chi, K) = \frac{L'}{L}(s, \chi', K) + O(\log A(\chi)) \tag{2.10}$$

for $\sigma = \mathrm{Re}(s) \geq 1/2$, say.

The second lemma counts zeros in smaller regions. It generalizes known results for Dirichlet L-functions.

Lemma 2.2 *Let $n(r; s)$ denote the number of zeros ρ of $L(s, \chi, K)$ with $|s - \rho| \leq r$. Then for $\mathrm{Re}(s) \geq 1$, and all $r > 0$*

$$n(r; s) \ll 1 + r(\log A(\chi) + n_K \log(|s| + 2)). \tag{2.11}$$

Proof. The estimate (2.11) is true for $r \geq 1/2$ by Lemma 2.1. For the case $0 < r < 1/2$ it suffices to prove the result for $s = 1 + it$, because $n(r; 1 + it) \geq n(r; \sigma + it)$ for any $\sigma \geq 1$. We next note

$$n(r; 1 + it) \leq n(2r; (1 + r) + it). \tag{2.12}$$

We now apply the estimate [7, Lemma 5.6]

$$\left| \frac{L'}{L}(s, \chi, K) + \frac{\delta(\chi)}{s - 1} - \sum_{\substack{\rho_\chi \\ |\gamma - t| \leq 1}} \frac{1}{s - \rho} \right| \ll \log A(\chi) + n_K \log(|t| + 2), \tag{2.13}$$

valid for primitive χ, $-\frac{1}{2} \le \mathrm{Re}(s) \le 3$, and $|s| > \frac{1}{8}$, where $\delta(\chi) = 1$ if $\chi = \chi_0$ and 0 otherwise. By (2.10) this estimate (with a different constant implied by the $\ll$ notation) holds for any χ if $\mathrm{Re}(s) \ge 1$. Let $s = 1 + r + it$. Then

$$
\begin{aligned}
\left| \frac{L'}{L}(s, \chi, K) \right| &\le \left| \frac{L'}{L}(1 + r, \chi, K) \right| \\
&\le \left| \frac{\zeta_K'}{\zeta_K}(1 + r, \chi, K) \right| \le r^{-1} + c_1 \log d_K,
\end{aligned}
$$

the last inequality by the equation following (8.3) in [7]. Applying this to (2.13) we have

$$
\left| \sum_{\substack{\rho_\chi \\ |\gamma - t| \le 1}} \frac{1}{s - \rho} \right| \ll \log A(\chi) + n_K \log(|t| + 2) + r^{-1}. \tag{2.14}
$$

On the other hand, if $\rho = \beta + i\gamma$,

$$
\mathrm{Re} \frac{1}{s - \rho} = \frac{(1 + r) - \beta}{|s - \rho|^2} \ge \frac{r}{|s - \rho|^2}.
$$

Consequently

$$
\left| \sum_{\substack{\rho_\chi \\ |\gamma - t| \le 1}} \frac{1}{s - \rho} \right| \ge \sum_{\substack{\rho_\chi \\ |\gamma - t| \le 1}} \mathrm{Re} \frac{1}{s - \rho} \ge n(2r; 1 + r + it) \cdot \frac{1}{4r}.
$$

Substituting this result in (2.14) gives the required inequality for $n(2r; 1 + r + it)$, which with (2.12) completes the proof.

We also need a zero-free region for Hecke L-functions.

Lemma 2.3. *There is an effectively computable positive absolute constant c_2 such that for all finite extensions K of $\mathbf{Q}$ and Hecke characters χ on K, the Hecke L-function $L(s, \chi, K)$ has at most one zero $\rho = \beta + i\gamma$ with*

$$
|\beta| > 1 - (c_2 \log A(\chi))^{-1}
$$
$$
|\gamma| < (c_2 \log A(\chi))^{-1}. \tag{2.15}
$$

(If $L(s, \chi, K) = \zeta_{\mathbf{Q}}(s)$, we regard this statement as empty.) This zero can occur only if $\chi^2 = \chi_0$, and must be real and simple. It will be called the exceptional zero. Aside from this possible exceptional zero, $L(s, \chi, K)$ has no other zero $\rho = \beta + i\gamma$ with

$$
|\beta| > 1 - c_2^{-1} \{\log A(X) + n_K \log(|\gamma| + 2)\}^{-1}. \tag{2.16}
$$

Proof. Note first that

$$
f(\sigma) = 3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \ge 0.
$$

Let $\chi(P) = \exp(i\theta_p)$. Then for $\sigma > 1$,

$$
\begin{aligned}
&\mathrm{Re}\left\{ -3\frac{\zeta_K'}{\zeta_K}(\sigma) - 4\frac{L'}{L}(\sigma+it,\chi,K) - \frac{L'}{L}(\sigma+2it,\chi^2,K)\right\} \\
&= \sum_{P,m} f(m\theta_P - mt\log NP)(\log NP)(NP)^{-m\sigma} \geqq 0.
\end{aligned}
$$

By (2.13) (and the remark that follows (2.13)) and the fact that $A(\chi^2) \leqq A(\chi)$, we find that

$$
\begin{aligned}
&-3\frac{\zeta_K'}{\zeta_K}(\sigma) - 4\frac{L'}{L}(\sigma+it,\chi,K) - \frac{L'}{L}(\sigma+2it,\chi^2,K) \\
&= \frac{3}{\sigma-1} + \frac{4\delta(\chi)}{\sigma-1+it} + \frac{\delta(\chi^2)}{\sigma-1+2it} - 3\sum_{\substack{\rho_{\chi_0}\\|\gamma|\leqq 1}} \frac{1}{\sigma-\rho} \tag{2.17}\\
&\quad -4\sum_{\substack{\rho_\chi\\|\gamma-t|\leqq 1}} \frac{1}{\sigma+it-\rho} - \sum_{\substack{\rho_{\chi^2}\\|\gamma-2t|\leqq 1}} \frac{1}{\sigma+2it-\rho} + O(\log A(\chi) + n_K \log(|t|+2)).
\end{aligned}
$$

Now $\mathrm{Re}(s-\rho)^{-1} \geqq 0$ if $\mathrm{Re}(s) \geqq \mathrm{Re}(\rho)$. Hence if $\rho = \beta + i\gamma$ is any zero of $L(s,\chi,K)$, then the real part of the quantity in (2.17) (which is nonnegative) is

$$
\begin{aligned}
&\leqq \frac{3}{\sigma-1} + \mathrm{Re}\frac{4\delta(\chi)}{\sigma-1+it} + \mathrm{Re}\frac{\delta(\chi^2)}{\sigma-1+2it} - \mathrm{Re}\frac{4}{\sigma+it-\rho} \\
&\quad + c_3(\log A(\chi) + n_K \log(|t|+2)).
\end{aligned}
$$

By choosing $t = \gamma$, we obtain

$$
\begin{aligned}
0 &\leqq \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + \mathrm{Re}\frac{4\delta(\chi)}{\sigma-1+i\gamma} + \mathrm{Re}\frac{\delta(\chi^2)}{\sigma-1+2i\gamma} \tag{2.18}\\
&\quad + c_3(\log A(\chi) + n_K \log(|\gamma|+2)).
\end{aligned}
$$

Suppose first that $\chi^2 \neq \chi_0$. Then $\delta(\chi) = \delta(\chi^2) = 0$, and so

$$
0 \leqq \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + c_3(\log A(\chi) + n_K \log(|\gamma|+2)).
$$

Choosing

$$
\sigma = 1 + (100c_3)^{-1}(\log A(\chi) + n_K \log(|\gamma|+2))^{-1}, \tag{2.19}
$$

say, we obtain

$$
\beta \leqq 1 - (1000c_3)^{-1}(\log A(\chi) + n_K \log(|\gamma|+2))^{-1}. \tag{2.20}
$$

Suppose next that $\chi^2 = \chi_0$, but $|\gamma| \geqq c_3^{-1}(\log A(\chi) + n_K)^{-1}$. Then (2.18) gives us

$$
0 \leqq \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + 6c_3(\log A(\chi) + n_K \log(|\gamma|+2)),
$$

which gives (2.20) again if we choose σ according to (2.19).

It remains to consider the case $\chi^2 = \chi_0$ and $|\gamma| < c_3^{-1} (\log A(\chi))^{-1}$. Suppose first that $\chi \neq \chi_0$, but $L(s, \chi, K)$ has 2 zeros $\rho_i = \beta_i + i\gamma_i$, $1 \leq i \leq 2$, with

$$|\gamma_i| < c_3^{-1} (\log A(\chi) + n_K)^{-1}, \tag{2.21}$$

$$\beta_i > 1 - (1000 c_3)^{-1} (\log A(\chi) + n_K)^{-1}. \tag{2.22}$$

If we now go through the derivation of (2.18) from (2.17), but keep the contributions of both ρ_1 and ρ_2, we find that for $\sigma > 1$, $t = 0$, we have

$$0 \leq \frac{4}{\sigma - 1} - \mathrm{Re} \frac{4}{\sigma - \rho_1} - \mathrm{Re} \frac{4}{\sigma - \rho_2} + c_3 (\log A(\chi) + n_K).$$

If we now choose

$$\sigma = 1 + (100 c_3)^{-1} (\log A(\chi) + n_K)^{-1},$$

we obtain a contradiction. Thus there cannot be 2 zeros ρ_i as specified above. If there is a single zero satisfying (2.21) and (2.22), then it must be real, since $\chi^2 = \chi_0$ means that the complex zeros of $L(s, \chi, K)$ come in conjugate pairs.

The only remaining case is $\chi = \chi_0$, which is handled in an analogous way. To conclude the proof it therefore suffices to note that $n_K \leq c_4 \log A(\chi)$ if $A(\chi) \neq 1$ (i.e., if $L(s, \chi, K) \neq \zeta_\mathbf{Q}(s)$).

To continue bounding (2.7) we divide the zeros of $L(s, \chi, K)$ off the real axis into two classes, those with $\beta \leq 1 - \delta$ and the remainder, which must have $|\gamma| \geq \delta^2 (\log A(\chi)) (1 + \delta (\log A(\chi))^{1/2})$ by hypothesis.

Consider those zeros ρ with $\beta \leq 1 - \delta$. For those zeros

$$|k(\rho; x, y)| \leq \frac{x^{-2\delta}}{|\rho - 1|^2}.$$

Their contribution to the sum (2.7) is bounded by

$$\ll x^{-2\delta} \int_\delta^\infty \frac{1}{r^2} dn(r; 1) \ll x^{-2\delta} \left(\frac{-1}{r^2} n(r; 1) \right]_\delta^\infty + \int_\delta^\infty \frac{2}{r^3} n(r; 1) dr \right),$$

which by Lemma 2.2 is

$$\ll x^{-2\delta} (\delta^{-2} + \delta^{-1} \log A(\chi)) \tag{2.23}$$

since $n_K \ll \log |A(\chi)|$.

Next consider those zeros ρ with $\beta \geq 1 - \delta$, $|\gamma| \geq \delta^2 (\log A(\chi)) (1 + \delta (\log A(\chi))^{1/2})$. For each such zero

$$|k(\rho; x, y)| \leq (\rho - 1)^{-2}.$$

If $n^*(r; 1)$ denotes the number of such zeros with $|\rho - 1| < r$, then note $n^*(r; 1) = 0$ for

$$r < \delta^* = \mathrm{Max} \, (\delta^2 (\log A(x)) (1 + \delta (\log A(\chi))^{1/2}), \, (c_2 \log A(\chi))^{-1})$$

by Lemma 2.3 and by hypothesis. The contribution of these zeros is bounded by

$$\ll \int_{\delta*}^{\infty} \frac{1}{r^2} \, dn^*(r,1) \ll \frac{-1}{r^2} n(r;1) \Big]_{\delta*}^{\infty} + \int_{\delta*}^{\infty} \frac{2}{r^3} n(r;1) \, dt.$$

Using Lemma 2.2, this becomes

$$\ll \delta^{*-2} + \delta^{*-1} \log A(\chi) + \delta^{*-1} n_K (\delta^* + 2)$$
$$\ll \delta^{*-2} + \delta^{*-1} \log(\delta^* + 2) \log A(\chi).$$

In fact we will show this is

$$\ll \delta^{-2}. \tag{2.24}$$

To check this, note that if $\delta \geq (\log A(\chi))^{-1}$ then $\delta^* \geq \delta^2 \log A(\chi) \geq \delta$ while if $\delta < (\log A(\chi))^{-1}$ then $\delta^* \geq (c_2 \log A(\chi))^{-1} \geq c_2^{-1} \delta$ so that in either case $\delta^{*-2} = O(\delta^{-2})$. In order to deal with the second term, we note that $\delta^* \geq \delta^2 (\log A(\chi)) (1 + \delta(\log A(\chi))^{1/2})$, so it will suffice to show

$$1 + \delta(\log A(\chi))^{\frac{1}{2}} \gg \log(\delta^* + 2). \tag{2.25}$$

If $\delta^* = (c_2 \log A(\chi))^{-1} \leq c_2^{-1}$ this is immediate, so suppose $\delta^* = \delta^2 (\log A(\chi)) (1 + \delta(\log A(\chi))^{\frac{1}{2}})$. If $\delta \leq 2(\log A(\chi))^{-\frac{1}{2}}$ then $\delta^* \ll 1$ and (2.25) holds. If $\delta \geq 2(\log A(\chi))^{-\frac{1}{2}}$ then $\delta^* \geq 4$ and $\delta^* \leq 2\delta^3 (\log A(\chi))^{3/2}$, so

$$\log(\delta^* + 2) \ll \log \delta^* \ll (\delta^*)^{1/3} \ll \delta(\log A(\chi))^{1/2},$$

which again yields (2.25).

Next we introduce the integral

$$J = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(-\frac{\zeta_K'}{\zeta_K}(s) \right) k(s;x,y) \, ds = \sum_{P,n} \Lambda(NP^n)(NP)^{-n} \hat{k}(NP^n;x,y). \tag{2.26}$$

If we now suppose that $\chi(P) = 1$ for all first degree prime ideals with $NP = p < y^2$ for which $(P, f(\chi)) = 1$, then (2.26) agrees with (2.5) except possibly on those ideals P^n for which $NP^n \neq p$ and also for primes $P|f(\chi)$. The contribution of primes dividing the conductor is

$$\ll \sum_{P|f(\chi)} (\log NP) x^{-2} \log \left(\frac{y}{x} \right)$$

$$\ll (\log A(\chi)) x^{-2} \log \left(\frac{y}{x} \right), \tag{2.27}$$

and the contribution of the other primes is bounded by the quantity in (2.6), yielding

$$|J - I_0| \ll n_K x^{-1} \left(\log \frac{y}{x} \right) \frac{\log y}{\log x} + x^{-2} \log \left(\frac{y}{x} \right) \log A(\chi). \tag{2.28}$$

Finally, we compute a lower bound for the integral J. By an argument similar to that yielding (2.7) we obtain

$$J = \left(\log\frac{y}{x}\right)^2 - \sum_\rho k(\rho; x, y),$$

where ρ runs over all the zeros of $\zeta_K(s)$.

We now bound the contributions of the remaining zeros, by exactly the same argument as for $L(s, \chi, K)$, obtaining

$$\left| J - \left(\log\frac{y}{x}\right)^2 \right| \ll x^{-2\delta}\delta^{-1}\log A(X) + \delta^{-2}. \tag{2.29}$$

Note that the contribution of the zeros $\rho = \beta$ on the real axis with $\beta < 1 - \delta$ is actually included in the estimate (2.23), and there are no zeros of $\zeta_K(s)$ with $1 - \delta \le \beta < 1$ by hypothesis.

We now complete the proof of Theorem 1.2 by choosing

$$x = (B^2 c_2 \delta \log A(\chi))^{\frac{1}{2}\delta^{-1}}, \qquad y = B^{\frac{1}{2}\delta^{-1}}x,$$

and checking that for a sufficiently large value of B (fixed once and for all) the estimates (2.25), (2.28), and (2.29) are mutually inconsistent. Recall that

$$(c_2 \log A(\chi))^{-1} \le \delta \le \tfrac{1}{2}. \tag{2.30}$$

We must check that

$$\left(\log\frac{y}{x}\right)^2 > c_5 \left\{ x^{-2}\delta^{-1}\log A(\chi) + \delta^{-2} + n_K x^{-1}\left(\log\frac{y}{x}\right)(\log y)(\log x)^{-1} \right.$$

$$\left. + x^{-2}\log\frac{y}{x}\log A(\chi) \right\}, \tag{2.31}$$

where c_5 is the sum of the constants implied in (2.25), (2.28), (2.29). Now $(C\delta)^{\delta^{-1}}$ attains its maximum as a function of δ at eC^{-1}. Consequently x is a monotone decreasing function of δ on the interval (2.30). In particular

$$x \ge \tfrac{1}{2}B\log A(\chi) \tag{2.32}$$

and $y \le x^{3/2}$. From (2.32) we conclude

$$c_5 x^{-2}\delta^{-1}\log A(\chi) \le 2c_5\delta^{-1} \le c_5(\tfrac{1}{2}\log B)^{-1}\left(\log\frac{y}{x}\right)^2 \tag{2.33}$$

and also

$$c_5 n_K x^{-1}\left(\log\frac{y}{x}\right)\left(\frac{\log y}{\log x}\right) \ll B^{-1}\left(\log\frac{y}{x}\right)^2 \tag{2.34}$$

with the implied constant independent of B, since

$$\frac{\log y}{\log x} \le \log\frac{y}{x} \quad \text{when } e^2 \le x \text{ and } xe^2 \le y \le x^{3/2}.$$

Next

$$\delta^{-2} \leq (\tfrac{1}{4} \log B)^{-1} \left(\log \frac{y}{x}\right)^2, \tag{2.35}$$

and finally

$$x^{-2} \left(\log \frac{y}{x}\right) \log A(\chi) \leq 4B^{-2} (\log A(\chi))^{-1} \log \frac{y}{x} \leq B^{-1} \left(\log \frac{y}{x}\right)^2 \tag{2.36}$$

if $B \geq 100$, say. The inequalities (2.33)–(2.36) show that choosing B sufficiently large will guarantee that (2.31) holds. Hence there will be a prime ideal with the required properties satisfying

$$N_{K/\mathbf{Q}} P \leq y^2 \leq (B^3 c_2 \, \delta \log A(\chi))^{\delta-1}$$

We make some remarks on proving a result similar to Theorem 1.2 for Artin L-functions. By a result of Stark [9, Theorem 3] we can conclude that all Artin L-functions $L(s, \psi, L/K)$ are analytic at any exceptional zero of $\zeta_L(s)$, so this causes no new difficulties. The problem is in obtaining a good upper bound for the number of singularities of $\frac{L'}{L}(s, \psi, L/K)$ in the critical strip. Assuming Artin's conjecture on the analyticity of $L(s, \psi, L/K)$, we can get a good estimate from the functional equation, and immediately prove the analogue of Theorem 1.2, with $A(\psi)$ being the appropriate constant from the functional equation of the Artin L-function $L(s, \psi, L/K)$.

3. *L*-Functions and Mellin Transforms

The following sections give the proof of Theorem 1.1. As was the case with the previous proof, this proof also relies on estimates of inverse Mellin transforms

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k(s) \, ds \tag{3.1}$$

where $k(s)$ is a kernel function and $F_C(s)$ is a certain Dirichlet series which counts prime ideals whose Artin symbol is the conjugacy class C. We use two different kernel functions, one in the case that $\zeta_L(s)$ has no (exceptional) zero on the real line segment $1 - (\log d_L)^{-2} \leq s < 1$ and the other when it does. We first show that $F_C(s)$, defined as a sum involving Artin L-functions, can be expressed as a sum involving Hecke L-functions, thus eliminating Artin L-functions from the proof. In order to get better error estimates for (3.1) if an exceptional zero of $\zeta_L(s)$ occurs in the range $1 - (\log d_L)^{-1} \leq s < 1$, we show that its presence implies the existence of an enlarged zero-free region near $s = 1$ (for other zeros), a result of the kind usually called the *Deuring-Heilbronn phenomenon*. To do this we give a simple direct proof of a power-sum inequality which may be used instead of Turan's second main theorem in many L-function applications. Our second kernel has the property that it drops off very rapidly as we move to the left of $\sigma = 1$, thus attenuating the contribution of the

exceptional zero, at the cost of not dropping off so rapidly as we move vertically, thus necessitating a larger zero-free region to get good error bounds.

Start of Proof of Theorem 1.1. We define

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\psi} \bar{\psi}(g) \frac{L'}{L}(s, \psi, L/K), \tag{3.2}$$

where ψ runs over the irreducible characters of G and $L(s, \psi, L/K)$ is the Artin L-function attached to the character ψ. The orthogonality relations for the characters ψ imply (see [6, Sect. 3]) that for $\mathrm{Re}(s) > 1$,

$$F_C(s) = \sum_{P} \sum_{m=1}^{\infty} \theta(P^m)(\log NP)(NP)^{-s}, \tag{3.3}$$

where for prime ideals P of K unramified in L,

$$\theta(P^m) = \begin{cases} 1 & \text{if } \left[\frac{L/K}{P}\right]^m \in C, \\ 0 & \text{otherwise}, \end{cases}$$

and $0 \leq \theta(P^m) \leq 1$ if P ramifies in L.

Our first reduction uses a method due to Deuring [3] to obtain an expression for $F_C(s)$ which involves Hecke L-functions rather than Artin L-functions, and which eliminates Artin L-functions from the subsequent proof. We choose $g \in C$ and let $H = \langle g \rangle$ be the cyclic group generated by G, E the fixed field of H. Then [6, Lemma 4]

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/E), \tag{3.4}$$

where χ runs over the irreducible characters of H, which are 1-dimensional since H is cyclic. By the fundamental theorem of class field theory the Artin L-function $L(s, \chi, L/E)$ is a certain Hecke L-function $L(s, \chi, E)$ attached to the field E, and χ is a certain *primitive* Hecke character, which satisfies

$$\chi(P) = \chi\left(\left[\frac{L/E}{P}\right]\right)$$

for all prime ideals P of E unramified in L. Hence (3.4) becomes

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, E) \tag{3.5}$$

for certain Hecke L-functions $L(s, \chi, E)$.

We now discuss the kernel functions. The two kernels we shall use are

$$k_1(s) = k_1(s; x) = k(s; x, x^2) = \left(\frac{x^{2s} - x^s}{s-1}\right)^2,$$

$$k_2(s) = k_2(s; x) = x^{s^2 + s}.$$

In these kernels $x \geqq 2$ is an adjustable parameter whose value will be selected later. The dependence of the kernel on x will be suppressed in what follows for notational convenience. The inverse Mellin transforms $\hat{k}_i(u)$ for $u > 0$ of these kernels are:

$$
\hat{k}_1(u) = \begin{cases} u^{-1} \log \dfrac{x^4}{u} & x^3 \leqq u \leqq x^4, \\[2mm] u^{-1} \log \dfrac{u}{x^2} & x^2 \leqq u \leqq x^3, \\[2mm] 0 & \text{otherwise,} \end{cases}
\tag{3.6}
$$

$$
\hat{k}_2(u) = (4\pi \log x)^{-\frac{1}{2}} \exp \left[-\frac{\left(\log \dfrac{u}{x} \right)^2}{4 \log x} \right].
\tag{3.7}
$$

The sums we will estimate are

$$
I_i = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k_i(s) \, ds \qquad (i = 1, 2).
\tag{3.8}
$$

Using the uniform and absolute convergence of the Dirichlet series (2.4) for $\operatorname{Re}(s) > 1$, and the absolute integrability of the kernels on vertical lines, we find that

$$
I_i = \sum_P \sum_{m=1}^{\infty} \theta(P^m)(\log NP) \hat{k}_i(NP^m),
\tag{3.9}
$$

where the outer sum is over all prime ideals of K.

We next prove a series of three lemmas which bound the contributions to the sums in (3.9) coming from the ramified primes, the terms for which NP^m is not prime, and (in the case of the kernel k_2) prime ideals with large norms. These estimates could be easily improved, but this is unnecessary for our purpose, since other error terms will be larger.

Lemma 3.1. *Let $\sum^R$ denote summation over the prime ideals of K that ramify in L. Then*

$$
\sum_P^R \sum_{m=1}^{\infty} \theta(P^m) \log(NP) \hat{k}_1(NP^m) \ll \frac{\log x}{x^2} \log d_L,
\tag{3.10}
$$

$$
\sum_P^R \sum_{\substack{m=1 \\ NP^m \leqq x^{10}}}^{\infty} \theta(P^m) \log(NP) \hat{k}_2(NP^m) \ll (\log x)^{\frac{1}{2}} \log d_L.
\tag{3.11}
$$

Proof. We have

$$
\sum_P^R \sum_{m=1}^{\infty} \theta(P^m) \log(NP) \hat{k}_1(NP^m) \ll \sum_P^R \log(NP) \log x \sum_{\substack{m \geqq 1 \\ NP^m \geqq x^2}} (NP)^{-m}
$$

$$
\ll \frac{\log x}{x^2} \sum_P^R \log NP \ll \frac{\log x}{x^2} \log d_L,
$$

since all the ramified primes divide the different $\delta(L/K)$ so the product of their norms divides d_L. Similarly

$$\sum_P^R \sum_{\substack{m=1 \\ NP^m \leq x^{10}}}^{\infty} \theta(P^m) \log(NP) \hat{k}_2(NP^m) \ll \sum_P^R \log(NP) \sum_{\substack{m \\ NP^m \leq x^{10}}} (\log x)^{-\frac{1}{2}}$$

$$\ll (\log x)^{\frac{1}{2}} \sum_P^R \log(NP) \ll (\log x)^{\frac{1}{2}} \log d_L.$$

Lemma 3.2. *If $\sum^P$ denotes summation over those pairs (P, m) for which NP^m is not a rational prime, then*

$$\sum^P \theta(P^m) \log(NP) \hat{k}_1(NP^m) \ll n_K \frac{(\log x)^2}{x}, \tag{3.12}$$

$$\sum^P \theta(P^m) \log(NP) \hat{k}_2(NP^m) \ll n_K x^{7/4}. \tag{3.13}$$

Proof. If q is a positive integer, there are at most n_K distinct prime power ideals P^m with $NP^m = q$. Hence

$$\sum_{P,m}^P \theta(P^m) \log(NP) \hat{k}_1(NP^m) \ll n_K (\log x)^2 \sum_{\substack{x^2 \leq p^h \leq x^4 \\ p, h \geq 2}} p^{-h}$$

$$\ll n_K (\log x)^2 \sum_{\substack{m^h \geq x^2 \\ h \geq 2}} m^{-h} \ll n_K \frac{(\log x)^2}{x}.$$

For the other kernel,

$$\sum_{P,m}^P \theta(P^m) \log(NP) \hat{k}_2(NP^m) \ll n_K \sum_{p, h \geq 2} \log(p^h) \hat{k}_2(p^h)$$

$$\ll n_K \int_2^{\infty} \log(u) \hat{k}_2(u) \, dS(u)$$

where $S(u)$ counts the number of perfect h-th powers, $h \geq 2$, which are $\leq u$. To complete the proof we use the fact that $S(u) \ll u^{1/2}$ and integration by parts.

Lemma 3.3. *We have*

$$\sum_P \sum_{\substack{m=1 \\ NP^m \geq x^{10}}}^{\infty} \theta(P^m)(\log NP) \hat{k}_2(NP^m) \ll n_K x^{-10}. \tag{3.14}$$

Proof. We have

$$\sum_P \sum_{\substack{m=1 \\ NP^m \geq x^{10}}}^{\infty} \theta(P^m)(\log NP) \hat{k}_2(NP^m) \ll n_K \sum_{q \geq x^{10}} (\log q) \hat{k}_2(q)$$

$$\ll n_K \int_{x^{10}}^{\infty} (\log u) \hat{k}(u) \, du \ll n_K x^{-10}.$$

We define

$$P(C) = \left\{ P; P \text{ a prime ideal of } K, \text{ unramified in } L, \text{ of degree one over } \mathbf{Q}, \text{ and} \right.$$
$$\left[\frac{L/K}{P} \right] = C. \right\}$$

The results of the last three lemmas show that

$$\left| I_1 - \sum_{P \in P(C)} (\log NP) k_1(NP) \right| \ll n_K \frac{(\log x)^2}{x} + \frac{\log x}{x^2} \log d_L$$
$$\ll x^{-\frac{1}{2}} \log d_L \tag{3.15}$$

and

$$\left| I_2 - \sum_{\substack{P \in P(C) \\ NP < x^{10}}} (\log NP) \hat{k}_2(NP) \right| \ll n_K x^{-10} + (\log x)^{\frac{1}{2}} \log d_L + n_K x^{7/4}$$
$$\ll x^{7/4} \log d_L. \tag{3.16}$$

The next step of the proof consists of evaluating the integrals I_1, I_2 in (3.9) by contour integration. In view of the identity (3.5) expressing $F_C(s)$ in terms of Hecke L-functions it suffices to evaluate integrals of the form

$$J_j(\chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi, E) k_j(s) ds, \tag{3.17}$$

where χ is a primitive Hecke character.

In order to evaluate $J_j(x)$ we will consider

$$J_j(\chi, T) = \frac{1}{2\pi i} \int_{B(T)} -\frac{L'}{L}(s, \chi, E) k_j(s) ds, \tag{3.18}$$

where $B(T)$ is the positively oriented rectangle with vertices at $2-iT, 2+iT, -\frac{1}{2}+iT, -\frac{1}{2}-iT$, and where $T > 0$ does not equal the ordinate of any of the zeros of $L(s, \chi, E)$. By Cauchy's Theorem

$$J_j(\chi, T) = \delta(\chi) k_j(1) - a(\chi) k_j(0) - \sum_{\substack{\rho \\ |\gamma| < T}} k_j(\rho), \tag{3.19}$$

where the last sum is over the nontrivial zeros $\rho = \beta + i\gamma$ of $L(s, \chi, E)$, counted with multiplicity, and $a(\chi)$ is the order of the zero of $L(s, \chi, E)$ at $s = 0$. From the functional equation of $L(s, \chi, E)$ we deduce that $a(\chi) \le n_E$.

On the line segment from $-\frac{1}{2}+iT$ to $-\frac{1}{2}-iT$ we have [7; Lemma 6.2]

$$\left| \frac{L'}{L}(s, \chi, E) \right| \ll \log A(\chi) + n_E \log(|s| + 2),$$

so that for $j = 1$ or 2,

$$\left| \frac{1}{2\pi i} \int_{-\frac{1}{2}+iT}^{-\frac{1}{2}-iT} \frac{L'}{L}(s, \chi, E) k_j(s) ds \right| \ll k(-\tfrac{1}{2}) \{ \log A(\chi) + n_E \}. \tag{3.20}$$

On the two segments from $2 \pm iT$ to $-\frac{1}{2} \pm iT$ we use the method of Landau [8], [7; Sect. 6] to obtain the estimate

$$\ll |k_j(iT)| (\log A(\chi) + n_E \log T). \tag{3.21}$$

If we now combine (3.18)–(3.21) and let $T \to \infty$, we obtain

$$J_j(\chi) = \delta(\chi) k_j(1) - \sum_\rho k_j(\rho) + O(n_E k_j(0)) + O(k_j(-\tfrac{1}{2})(\log A(x) + n_E)). \tag{3.22}$$

The definition of $F_C(s)$ together with (3.22) and the conductor-discriminant formula:

$$\sum_\chi \log A(\chi) = \log d_L$$

now show that

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k_j(s)\, ds = \frac{|C|}{|G|} k_j(1) - \frac{|C|}{|G|} \sum_\chi \bar{\chi}(g) \sum_{\rho_\chi} k_j(\rho_\chi)$$

$$+ O\left(\frac{|C|}{|G|} n_L k_j(0) + \frac{|C|}{|G|} k_j(-\tfrac{1}{2}) \log d_L\right), \tag{3.23}$$

where ρ_χ runs over the zeros of $L(s, \chi, E)$ in the critical strip. From (3.23) we obtain a lower bound for I_j:

$$\frac{|G|}{|C|} I_j \geq k_j(1) - \sum_\rho |k_j(\rho)| - c_6 [n_L k_j(0) + k_j(-\tfrac{1}{2}) \log d_L], \tag{3.24}$$

where ρ runs over the zeros of $\zeta_L(s)$ lying in the critical strip.

To complete the proof we need good lower bounds for the sum

$$k_j(1) - \sum_\rho |k_j(\rho)| \quad (j = 1, 2)$$

where ρ runs over the nontrivial zeros of $\zeta_L(s)$. This requires additional information on the location of the zeros of $\zeta_L(s)$. The sum over the zeros could have a single large term coming from the exceptional zero (if it exists) and in that case we need an enlarged zero-free region to show the contribution of the other zeros is small. This enlarged region, and a lower bound on the distance of the exceptional zero from $s = 1$ are the subject of the next two sections.

4. A Power-Sum Inequality

In this section we prove a power-sum inequality (Theorem 4.2 below) which will be used in the next section to derive an enlarged zero-free region. This result may serve as a substitute for Turán's second main theorem [11] in many situations.

Lemma 4.1. *Let*

$$P(r, \theta) = \sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right) r^j \cos j\theta.$$

Then

(i) $P(r, \theta) \geq -\frac{1}{2}$ *for* $0 \leq r \leq 1$ *and all* θ,

(ii) $P(1, 0) = \dfrac{J}{2}$,

(iii) $|P(r, \theta)| \leq \frac{3}{2} r$ *for* $0 \leq r \leq \frac{1}{3}$.

Proof. $\frac{1}{2} + P(1, \theta)$ is Fejer's kernel, and hence ≥ 0 for all θ. But if $z = re^{i\theta}$, then $P(r, \theta)$ is a harmonic function of z, and hence $\geq -\frac{1}{2}$ for $|z| \leq 1$, proving (i). Assertion (ii) is trivial. For $0 \leq r \leq \frac{1}{3}$,

$$|P(r, \theta)| \leq \sum_{j=1}^{\infty} r^j \leq \frac{3}{2} r.$$

Theorem 4.2. *Let* $s_m = \sum\limits_{n=1}^{\infty} b_n z_n^m$ *and suppose that*

(i) $|z_n| \leq |z_1|$ *for all* $n \geq 1$,

(ii) *the* b_n *are real*,

(iii) $b_n \geq 0$ *for those* n *for which* $\frac{1}{3}|z_1| \leq |z_n| \leq |z_1|$.

Set

$$L = (b_1 |z_1|)^{-1} \sum_{n=1}^{\infty} |b_n z_n|.$$

Then there exists j_0 *with* $1 \leq j_0 \leq 24L$ *such that*

$$\operatorname{Re} s_{j_0} \geq \frac{b_1}{8} |z_1|^{j_0}.$$

Proof. By homogeneity we may suppose $|z_1| = 1$. Let $z_n = r_n \exp(i\theta_n)$. In the notation of Lemma 4.1,

$$\sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right) \operatorname{Re}(s_j)(1 + \cos j\theta_1) = \sum_{n=1}^{\infty} b_n \sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right) (\cos j\theta_n)(1 + \cos j\theta_1) r_n^j$$

$$= \sum_{n=1}^{\infty} b_n \{P(r_n, \theta_n) + \tfrac{1}{2} P(r_n, \theta_n - \theta_1) + \tfrac{1}{2} P(r_n, \theta_n + \theta_1)\}.$$

The term with $n = 1$ contributes $\geq \left(\dfrac{J}{4} - \dfrac{3}{4}\right) b_1 > \left(\dfrac{J+1}{4} - 3r_1\right) b_1$. Each term with $r_n > \frac{1}{3}$ contributes $\geq -b_n \geq -3r_n b_n$. Each term with $r_n \leq \frac{1}{3}$ contributes $\geq -3r_n |b_n|$. Thus

$$\sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right) (\operatorname{Re} s_j)(1 + \cos j\theta_1) \geq \frac{J+1}{4} b_1 - 3b_1 L. \qquad (4.1)$$

Let $J = [24L]$. Then the right side of (4.1) is $\geqq \dfrac{J}{8}b_1$. But $\left(1 - \dfrac{j}{J+1}\right)(1 + \cos j\theta_1) \geqq 0$ and

$$\sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right)(1 + \cos j\theta_1) \leqq 2P(1,0) = J,$$

which implies that there is at least one $j \leqq J$ such that $\operatorname{Re} s_j \geqq \dfrac{b_1}{8}$.

5. The Deuring-Heilbronn Phenomenon

The Deuring-Heilbronn phenomenon refers to the fact that if $\zeta_L(s)$ has a real zero β_0 close to $s = 1$, then $\zeta_L(s)$ can have no other zeros nearby. The precise result is:

Theorem 5.1. *There are positive, absolute, effectively computable constants c_7 and c_8 such that if $\zeta_L(s)$ has a real zero $\beta_0 > 0$ then $\zeta(\sigma + it) \neq 0$ for*

$$\sigma \geqq 1 - c_8 \frac{\log\left(\dfrac{c_7}{(1-\beta_0)\log d_L \tau^{n_L}}\right)}{\log d_L \tau^{n_L}}, \tag{5.1}$$

where $\tau = |t| + 2$, with the single exception $\sigma + it = \beta_0$.

Proof. Since $(s-1)\zeta_L(s)$ is an entire function of order one, we have the Hadamard product

$$(s-1)\zeta_L(s) = s^r e^{\alpha_1 + \alpha_2 s} \prod_\omega \left(1 - \frac{s}{\omega}\right) e^{s/\omega}, \tag{5.2}$$

where ω runs through all the zeros of $\zeta_L(s)$, $\omega \neq 0$, including the trivial ones. Differentiating (5.2) logarithmically yields

$$-\frac{\zeta_L'}{\zeta_L}(s) = \frac{1}{s-1} - \alpha_2 - \sum_\omega \left(\frac{1}{s-\omega} + \frac{1}{\omega}\right) - \frac{r}{s}.$$

The Euler product for $\zeta_L(s)$ gives

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_P \sum_{m=1}^{\infty} (\log NP)(NP)^{-ms}$$

for $\operatorname{Re} s > 1$. Equating these two expansions and differentiating $2j - 1$ times, we obtain

$$\frac{1}{(2j-1)!} \sum_P \sum_{m=1}^{\infty} (\log NP)(\log NP^m)^{2j-1}(NP)^{-ms} = \frac{1}{(s-1)^{2j}} - \sum_\omega \frac{1}{(s-\omega)^{2j}}, \tag{5.3}$$

where ω now runs also over the r trivial zeros $\omega = 0$, and this is valid for $\operatorname{Re} s > 1, j \geqq 1$. Therefore if $s = \sigma + it$, we have

$$\frac{1}{(2j-1)!} \sum_P \sum_{m=1}^{\infty} (\log NP)(\log NP^m)^{2j-1} NP^{-m\sigma}(1 + (NP^m)^{-it})$$

$$= \frac{1}{(\sigma-1)^{2j}} + \frac{1}{(s-1)^{2j}} - \frac{1}{(\sigma-\beta_0)^{2j}} - \frac{1}{(s-\beta_0)^{2j}} - \sum_{n=1}^{\infty} z_n^j \tag{5.4}$$

where the z_n are of the form $(\sigma - \omega)^{-2}$ or $(s - \omega)^{-2}$. The real part of the left side of (5.4) is nonnegative, so if we take $\sigma = 2$, then

$$\operatorname{Re} \sum_{n=1}^{\infty} z_n^j \leqq 1 - \frac{1}{(2-\beta_0)^{2j}} + \operatorname{Re} \left\{ \frac{1}{(1+it)^{2j}} - \frac{1}{(2-\beta_0+it)^{2j}} \right\} \leqq c_9 j(1-\beta_0). \tag{5.5}$$

Suppose that $\rho = \beta + i\gamma \neq \beta_0$ is a zero of $\zeta_L(s)$. Set $t = \gamma$ in (5.5). We now apply Theorem 4.2 to the left side of (5.5). First note

$$|z_1| \geqq (2-\beta)^{-2}$$

so that

$$L \ll (2-\beta)^2 \sum_{\omega} \left(\frac{1}{|2-\omega|^2} + \frac{1}{|2+i\gamma-\omega|^2} \right) \ll \int_0^{\infty} \frac{1}{u^2+1} \, dn(u) + \int_0^{\infty} \frac{1}{u^2+1} \, dn(u+\tau)$$

$$\ll \log d_L \tau^{n_L}, \tag{5.6}$$

using the zero-density estimates for the number of zeros $n(t)$ of $\zeta_L(s)$ from Lemma 2.1. By Theorem 4.2 there exists some j_0 with $1 \leqq j_0 \leqq 24L$ such that

$$\operatorname{Re} \sum_{n=1}^{\infty} z_n^{j_0} \geqq \tfrac{1}{8}(2-\beta)^{-2j_0} \geqq \tfrac{1}{8} \exp(-2j_0(1-\beta)).$$

Combining this with (5.5), we find that

$$\tfrac{1}{8} \exp(-2j_0(1-\beta)) \ll j_0(1-\beta_0).$$

Since $j_0 \leqq 24L$, which is bounded by (5.6), the desired bound follows.

From Theorem 5.1 we immediately obtain a bound on how close the exceptional zero β_0 can be to $s = 1$.

Corollary 5.2. *There is a positive, absolute, effectively computable constant c_{10} such that any real zero β_0 of $\zeta_L(s)$ satisfies*

$$1 - \beta_0 \geqq d_L^{-c_{10}}.$$

Proof. Choose c_{10} so large that whenever $n_L > 1$,

$$c_8 \log \left(\frac{c_7 d_L^{c_{10}}}{\log d_L \cdot 2^{n_L}} \right) > 3 \log(d_L 2^{n_L}).$$

This is possible since $n_L \ll \log d_L$ by discriminant bounds. If we had $1 - \beta_0 < d_L^{-c_{10}}$, then by Theorem 5.1 $\zeta_L(\sigma)$ would not vanish for $\sigma \geqq -2$, except at σ

$= \beta$. However $\zeta_L(s)$ always has a trivial zero at $s=0$ or $s=-1$, so this gives a contradiction.

Stark [9; Lemmas 8, 11] has given upper bounds for the exceptional zero that are better than the one of Corollary 5.2 in the special case that there is a sequence of fields $L_0 = \mathbf{Q} \subseteq L_1 \subseteq \ldots \subseteq L_m = L$ such that L_{j+1}/L_j is normal.

6. Final Estimates

In this section we complete the proof of Theorem 1.1 by obtaining good lower bounds for the sums

$$k_j(1) - \sum_\rho |k_j(\rho)| \quad (j=1,2)$$

where ρ runs over the nontrivial zeros of $\zeta_L(s)$.

The possible exceptional zero β_0 plays a special role in our estimation. To simplify the notation, we define β_0 to be the exceptional zero of $\zeta_L(s)$ if it exists, and $\beta_0 = 1 - (c_2 \log d_L)^{-1}$ otherwise. In either case

$$k_j(1) - \sum_\rho |k_j(\rho)| \geqq k_j(1) - k_j(\beta_0) - \sum_{\rho \neq \beta_0} |k_j(\rho)| \quad (j=1,2). \tag{6.1}$$

By using the Taylor series expansions for $k_j(s)$ about $s=1$, we find that for $0 < \beta_0 < 1$ and $x \geqq 2$,

$$k_1(1) - k_1(\beta_0) = (\log x)^2 - \left(\frac{x^{2\beta_0 - 2} - x^{\beta_0 - 1}}{\beta_0 - 1} \right)^2 \geqq \frac{(\log x)^2}{10} \operatorname{Min} \{1, (1-\beta_0) \log x\}$$
$$\tag{6.2}$$

and

$$k_2(1) - k_2(\beta_0) = x^2 - x^{\beta_0 + \beta_0^2} \geqq \frac{x^2}{10} \operatorname{Min} \{1, (1-\beta_0) \log x\}. \tag{6.3}$$

First suppose that

$$1 - \beta_0 \geqq c_7^2 (\log d_L \, 3^{n_L})^{-2}.$$

In this case we use the kernel $k_1(s)$. The contribution of the zeros ρ of $\zeta_L(s)$ with $|\rho - 1| \geqq 1$ is bounded by

$$\sum_{|\rho - 1| \geqq 1} |k_1(\rho)| \leqq \int_1^\infty \frac{2}{t^2} dn(t; 1) \ll \log d_L \tag{6.4}$$

by Lemma 2.2. Next suppose $|\rho - 1| < 1$ and $\rho = \beta + i\gamma \neq \beta_0$. If an exceptional zero β_0 exists with

$$1 - \beta_0 \leqq \frac{1}{18} c_2 c_6^2 (\log d_L)^{-1}$$

then, since $d_L \geqq 3^{n_L/2}$ for $n_L \geqq 2$, we have

$$\frac{c_6}{(1-\beta_0) \log d_L \, 3^{n_L}} \geqq \{(\tfrac{1}{2} c_2)(1-\beta_0) \log d_L\}^{-1/2},$$

and therefore by Theorem 5.1

$$\beta \leq 1 - c_8 \frac{\log\left\{\dfrac{c_7}{(1-\beta_0)\log d_L 3^{n_L}}\right\}}{\log d_L 3^{n_L}} \leq 1 - c_{11} \frac{\log\left\{(\tfrac{1}{2}c_2)(1-\beta_0)(\log d_L)\right\}^{-1}}{\log d_L}.$$

On the other hand, if

$$1 - \beta_0 \geq \frac{1}{18} c_2 c_6^2 (\log d_L)^{-1},$$

then the zero-free region of Lemma 2.3 gives

$$\beta \leq 1 - (c_2 \log d_L)^{-1}.$$

Hence

$$\beta \leq 1 - c_{12} \frac{\log\left\{(\tfrac{1}{2}c_2)(1-\beta_0)(\log d_L)\right\}^{-1}}{\log d_L} \tag{6.5}$$

for an effectively computable positive c_{12}, and we may require that $c_{12} < c_{11}$. Hence (6.5) holds in all cases. Let

$$B = c_{12} \frac{\log\left\{(\tfrac{1}{2}c_2)(1-\beta_0)(\log d_L)\right\}^{-1}}{\log d_L}. \tag{6.6}$$

From (6.5) we conclude

$$|k_1(\rho)| \ll x^{2(\beta-1)}|\rho-1|^{-2} \ll x^{-2B}|\rho-1|^{-2}.$$

By Lemma 2.2,

$$\sum_{\substack{|\rho-1| < 1 \\ \rho \neq \beta_0}} |k_1(\rho)| \leq x^{-2B} \int_B^1 \frac{1}{t^2} dn(t;1) \ll x^{-2B}\{B^{-2} + B^{-1}\log d_L\}$$
$$\ll x^{-2B} B^{-1}\log d_L,$$

since $B \gg (\log d_L)^{-1}$. Using (6.6) this becomes

$$\sum_{\substack{|\rho-1| < 1 \\ \rho \neq \beta_0}} |k_1(\rho)| \ll (\log d_L)^2 \left\{(\tfrac{1}{2}c_2)((1-\beta_0)\log d_L)\right\}^{2c_{12}\frac{\log x}{\log d_L}}. \tag{6.7}$$

We have thus shown that

$$k_1(1) - \sum_\rho |k_1(\rho)| \geq \tfrac{1}{10}\text{Min}\left\{(\log x)^2, (1-\beta_0)(\log x)^3\right\}$$
$$- c_{13}\log d_L - c_{14}(\log d_L)^2 \left[(\tfrac{1}{2}c_2)(1-\beta_0)\log d_L\right]^{2c_{12}\frac{\log x}{\log d_L}}. \tag{6.8}$$

We now complete the proof in the case $1 - \beta_0 \geqq c_7^2 (\log d_L 3^{n_L})^{-2}$. Combining (6.8) with (3.15) and (3.24) we have

$$\sum_{P \in P(C)} (\log NP) \hat{k}_1(NP) \geqq \frac{1}{10} \frac{|C|}{|G|} (\log x)^2 \operatorname{Min} \{1, (1 - \beta_0) \log x\}$$
$$- c_{13} \frac{|C|}{|G|} \log d_L - c_{14} \frac{|C|}{|G|} (\log d_L)^2 [(\tfrac{1}{2} c_2)(1 - \beta_0) \log d_L]^{2c_{12} \frac{\log x}{\log d_L}} - c_{15} x^{-\frac{1}{4}} \log d_L.$$

We choose $\log x = c_{16} \log d_L$ with c_{16} a sufficiently large absolute constant. Then the right side above is bounded below by

$$\geqq c_{17} \frac{|C|}{|G|} (\log d_L)^2 \operatorname{Min} \{1, (1 - \beta_0) c_{16} \log d_L\} \geqq c_{18} \frac{|C|}{|G|} (\log d_L),$$

since $1 - \beta_0 \gg (\log d_L)^{-2}$. In particular there is a $P \in P(C)$ with

$$NP \leqq x^4 \leqq d_L^{4c_{16}}.$$

Now consider the remaining case, when $1 - \beta_0 \leqq c_7^2 (\log d_L 3^{n_L})^{-2}$. In that case

$$\log \frac{c_7}{(1 - \beta_0) \log d_L 3^{n_L}} \geqq \tfrac{1}{2} \log (1 - \beta_0)^{-1}.$$

If $\rho = \beta + i\gamma$ is a zero of $\zeta_L(s)$ with $|\gamma| \leqq 1$, and $\rho \neq \beta_0$, then by Theorem 5.1

$$\beta \leqq 1 - c_{19} \frac{\log (1 - \beta_0)^{-1}}{\log d_L}.$$

In that case

$$|k_2(\rho)| \leqq x^{\beta^2 + \beta} \leqq x^{1 + \beta}$$
$$\leqq x^2 \exp \left[-c_{19} \frac{(\log x) \log (1 - \beta_0)^{-1}}{\log d_L} \right] = x^2 (1 - \beta_0)^{c_{19} \frac{\log x}{\log d_L}}.$$

For zeros $\rho = \beta + i\gamma$ with $|\gamma| \geqq 1$ we have

$$|k_2(\rho)| \leqq x^{2 - \gamma^2},$$

and, by Lemma 2.1, these zeros contribute

$$\sum_{\substack{\rho \\ |\gamma| > 1}} |k_2(\rho)| \ll x \log d_L.$$

We have shown that

$$k_2(1) - \sum_{\rho} |k_2(\rho)| \geqq \frac{x^2}{10} \operatorname{Min} \{1, (1 - \beta_0) \log x\}$$
$$- c_{20} x \log d_L - c_{21} x^2 (1 - \beta_0)^{c_{19} \frac{\log x}{\log d_L}} \log d_L.$$

From (3.16) and (3.24) we have

$$\sum_{\substack{P \in P(C) \\ NP < x^{10}}} (\log NP) \hat{k}_2(NP) \geqq \frac{x^2}{10} \frac{|C|}{|G|} \text{Min}\{1, (1-\beta_0) \log x\}$$

$$- c_{20} x \log d_L - c_{21} x^2 (1-\beta_0)^{c_{19} \frac{\log x}{\log d_L}} \log d_L - c_{22} x^{7/4} \log d_L. \tag{6.9}$$

We now choose $x = d_L^{c_{23}}$ for a sufficiently large absolute constant c_{23}. We note that by Corollary 5.2 we have then

$$|G| x^{7/4} (\log x^2)^2 \log d_L < x^2 (1-\beta_0)$$

and hence the right side of (6.9) is

$$\geqslant c_{24} \frac{|C|}{|G|} d_L^{2c_{23}} \text{Min}\{1, (1-\beta_0) c_{23} \log d_L\} \geqslant c_{24} \frac{|C|}{|G|} d_L^{c_{23}}.$$

Hence there is $P \in P(C)$ with

$$NP \leqq x^{10} \leqq d_L^{10c_{23}}.$$

Choosing $A_1 = \text{Max}(4c_{16}, 10c_{23})$ completes the proof of Theorem 1.1.

7. Proof of Theorem 1.4

In this section we sketch a proof of the unconditional upper bound for the density of $P \in P(C)$. This time we use the kernel

$$k(s) = x^s e^{s^2},$$

where $x > 1$, for which

$$\hat{k}(u) = (4\pi)^{-1/2} \exp\{-\tfrac{1}{4}(\log x - \log u)^2\}.$$

Since $\hat{k}(u) \geqq 1/10$ for $u \in [x/2, 2x]$, we have

$$I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_C(s) k(s) ds = \sum_P \sum_{m=1}^{\infty} \theta(P^m)(\log NP) \cdot \hat{k}(NP^m)$$

$$\geqq \frac{\log x}{20} \cdot \{\pi_C(x) - \pi_C(x/2)\}. \tag{7.1}$$

On the other hand, by shifting the line of integration to $\text{Re}(s) = -1/4$, we find that

$$I = \frac{|C|}{|G|} x e - \frac{|C|}{|G|} \sum_\chi \bar{\chi}(g) \sum_{\rho_\chi} x^\rho e^{\rho^2} + O\left(\frac{|C|}{|G|} x^{-1/4} \log d_L\right) + O\left(\frac{|C|}{|G|} n_L\right), \tag{7.2}$$

where $\rho = \rho_\chi$ runs over the nontrivial zeros of $L(s, \chi, E)$, the last term comes from the singularity at $s = 0$, and the penultimate summand is due to the integral along

$\mathrm{Re}(s) = -1/4$. We wish to prove that $|I| \ll |C| x/|G|$. Now if $x \geq d_L$, then

$$n_L \ll \log d_L \ll x, \tag{7.3}$$
$$x^{-1/4} \log d_L \ll 1. \tag{7.4}$$

The exceptional zero (if it exists) gives

$$|x^\rho e^{\rho^2}| \leq x e \ll x.$$

By Lemma 2.3 any other zero $\rho = \beta + i\gamma$ satisfies

$$\beta \leq 1 - (c_2 \log d_L (|\gamma| + 2)^{n_L})^{-1},$$

and so if $|\gamma| < \log \log d_L$, then

$$\beta \leq 1 - c_{25} (\log d_L)^{-1} (\log \log \log d_L e^{20})^{-1}.$$

Therefore

$$\sum_\rho |x^\rho e^{\rho^2}| \ll x + x \sum_{|\gamma| \geq \log \log d_L} e^{-\gamma^2}$$
$$+ x \exp \left\{ -\frac{c_{25} \log x}{\log d_L \log \log \log d_L e^{20}} \right\} \sum_{\substack{\rho \\ |\gamma| < \log \log d_L}} e^{-\gamma^2}$$
$$\ll x + x (\log d_L)^{-1} + x (\log d_L) \exp \left\{ -\frac{c_{25} \log x}{\log d_L \log \log \log d_L e^{20}} \right\}. \tag{7.5}$$

If

$$\log x \geq c_{26} (\log d_L)(\log \log d_L)(\log \log \log d_L e^{20}), \tag{7.6}$$

then the right side of (7.5) is $\ll x$. Combining this with (7.1)–(7.4), we find that

$$\pi_C(2x) - \pi_C(x) \ll \frac{|C|}{|G|} \frac{x}{\log x} \tag{7.7}$$

if (7.6) is satisfied. Now for any $x \geq 2$,

$$\pi_C(x) \leq n_L \pi(x) \ll n_L \frac{x}{\log x}. \tag{7.8}$$

Therefore if $A_4 = 10 c_{26}$, and

$$x \log x \geq A_4 (\log d_L)(\log \log d_L)(\log \log \log d_L e^{20}),$$

then (7.7) and (7.8) yield the assertion of Theorem 1.4.

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