A BOUND ON EMBEDDING DIMENSIONS OF GEOMETRIC GENERIC FIBERS

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Abstract

The author finds a limit on the singularities that arise in geometric generic fibers of morphisms between smooth varieties of positive characteristic by studying changes in embedding dimension under inseparable field extensions. This result is then used in the context of the minimal model program to rule out the existence of smooth varieties fibered by certain nonnormal del Pezzo surfaces over bases of small dimension.

1. Introduction

This paper investigates the singularities that arise in generic fibers of morphisms between smooth varieties in positive characteristic. In characteristic 0, any morphism between smooth varieties admits a dense open locus of the base over which all fibers are smooth. However, over fields of positive characteristic this is no longer the case, as there exist morphisms between smooth varieties in which every fiber is singular (that is, nonsmooth over its base field). A simple example, occurring over an arbitrary field k of characteristic 2 (respectively 3), is the morphism $f : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ given by $(x, y) \mapsto x^2 + y^3$. The fiber of f over any point $t_0 \in \mathbb{A}_k^1$ is the planar curve defined by the equation $x^2 + y^3 - t_0 = 0$, which clearly has a cuspidal singularity at the geometric point $(\sqrt{t_0}, 0)$ (respectively $(0, \sqrt[3]{t_0})$).

This phenomenon is more than just pathology, rather it is a feature of positive characteristic geometry that arises naturally when attempting to study a class of

smooth varieties via morphisms to other varieties. One instance of this occurs in Mumford and Bombieri's classification of fibrations in characteristic $p > 0$, within the context of the Enriques classification of surfaces (cf. [1, 2]). As the above example illustrates, when $p = 2$ or 3 there exist smooth surfaces fibered in cuspidal curves of arithmetic genus 1 (also known as *quasielliptic fibrations*).

Main results. Nonsmooth points in the generic fiber of a morphism lie under those in the *geometric* generic fiber, an algebraically closed field extension of the generic fiber, at which the stalk of the structure sheaf fails to be a regular local ring (cf. Definition 2.3). To measure this failure, it is useful to recall the following definition.

DEFINITION 1.1. The difference by which the embedding dimension (cf. Section 2) at a (possibly nonclosed) point z of a scheme Z exceeds the codimension of that point is called the *embedding codimension* of z in Z,

$$\operatorname{ecodim}_Z(z) := \operatorname{edim}(\mathcal{O}_{Z,z}) - \dim(\mathcal{O}_{Z,z}).$$

Clearly the embedding codimension of $z \in Z$ is nonnegative, and equals zero if and only if $\mathcal{O}_{Z,z}$ is a regular local ring, so we see that it does provide some measure of the singularity at z. The main result of this paper is the following bound on the embedding codimension of points in the geometric generic fiber of a morphism between smooth varieties, which thus limits the possible singularities that arise.

THEOREM 1.2. *Let $f : X \to S$ be a morphism between smooth varieties over a perfect field k. Then the generic fiber X_ξ is a regular variety over the function field of S, $\kappa := \kappa(\xi)$, and any point $\bar{x}$ in the geometric generic fiber $\overline{X}_\xi := X_\xi \times_\kappa \bar{\kappa}$ satisfies*

$$\operatorname{ecodim}_{\overline{X}_\xi}(\bar{x}) \leqslant \dim(S). \tag{1.3}$$

REMARK 1.4. The bound on embedding codimension asserted in Theorem 1.2 is immediate for *special* fibers. This is because the geometric fiber $\overline{X}_s := X_s \times_{\kappa(s)} \bar{k}$ over any closed point $s \in S$ embeds via a closed immersion into the smooth variety $\overline{X} := X \times_k \bar{k}$, so it follows that $\operatorname{edim}_{\overline{X}_s}(\bar{x}) \leqslant \operatorname{edim}_{\overline{X}}(\bar{x})$ for all $\bar{x} \in \overline{X}_s$, and consequently that

$$\operatorname{ecodim}_{\overline{X}_s}(\bar{x}) = \operatorname{edim}_{\overline{X}_s}(\bar{x}) - \dim(\mathcal{O}_{\overline{X}_s,\bar{x}})$$
$$\leqslant \operatorname{edim}_{\overline{X}}(\bar{x}) - \dim(\mathcal{O}_{\overline{X}_s,\bar{x}})$$

$$= \dim(\mathcal{O}_{\overline{X}, \bar{x}}) - \dim(\mathcal{O}_{\overline{X}_s, \bar{x}})$$
$$= \dim(\overline{X}) - \dim(\overline{X}_s)$$
$$\leqslant \dim(S).$$

The content of the theorem is that this inequality, which easily holds for all special fibers, also holds for the generic fiber.

Main application. Our primary application of the above theorem is in the setting of the minimal model program, where one studies a higher-dimensional variety via its morphisms to simpler varieties. A primary goal in the program is to construct, from a given variety X, a minimal model by contracting each extremal curve $C \subseteq X$ that pairs negatively with the canonical divisor in X. If the curve C is sufficiently mobile in X, then this contraction morphism may not be birational, and instead may be a fibration by Fano schemes.

In positive characteristic, Kollár demonstrated the existence of these contraction morphisms on smooth 3-folds X, extending a result of Mori from characteristic 0 (cf. [5, 9]). Furthermore, he gives a detailed classification of the geometry of the possible contractions $f : X \to X'$ in the case where f is birational (that is, when X' is a 3-fold). If X' is a surface then f is simply a conic bundle, but if X' is a curve then f is a fibration by del Pezzo surface schemes, and Kollár remarks that the geometry here could potentially be rather complicated. He raises the question of whether the geometric generic fibers of such f can be nonnormal (cf. [5, Remark 1.2]) and if so, could the generic fiber Y of a del Pezzo surface fibration satisfy $H^1(Y, \mathcal{O}_Y) \neq 0$ (cf. [6, Remark 5.7.1]). Over a perfect field, all normal del Pezzo surfaces Y satisfy $H^1(Y, \mathcal{O}_Y) = 0$ by a result of Hidaka and Watanabe (cf. [4, Corollary 2.5]), although in all positive characteristics $p > 0$, Reid exhibits nonnormal del Pezzo surfaces Y with $H^1(Y, \mathcal{O}_Y) \neq 0$ (cf. [10, Section 4.4]).

The author recently constructed two projective morphisms $f : X \to S$ between smooth varieties of characteristic 2 whose generic fibers are regular del Pezzo surfaces Y with $h^1(Y, \mathcal{O}_Y) = 1$ (cf. [7]). In one example, X is a 5-fold (it is actually possible to create a similar example with X a 4-fold, the details of which shall be included in a forthcoming paper) and the geometric generic fiber is integral but nonnormal. In the other example, X is a 6-fold and the geometric generic fiber is nonreduced. It remains an open question whether del Pezzo surfaces Y with $H^1(Y, \mathcal{O}_Y) \neq 0$ can arise as the generic fiber of a morphism from a smooth 3-fold to a curve, but it follows from the main result of this paper that, at least in characteristics greater than 3, such geometry is not possible.

COROLLARY 1.5. *Let $f : X \to C$ be a surjective morphism between a smooth 3-fold X and a curve C over a perfect field of characteristic $p > 3$. If the generic fiber Y is a del Pezzo surface (that is, if ω_Y^{-1} is ample), then $H^1(Y, \mathcal{O}_Y) = 0$.*

Connections to the literature. Our main theorem is related to one result by Schröer (cf. [**12**, Corollary 2.4]) which asserts that, in the case of a proper fibration $f : X \to S$, inequality (1.3) is strict if $x \in \overline{X}_\xi$ is the generic point.

THEOREM 1.6 (Schröer). *Let $f : X \to S$ be a proper morphism between integral normal algebraic k-schemes of positive dimension satisfying $f_*(\mathcal{O}_X) = \mathcal{O}_S$, and let $\xi \in S$ denote the generic point. Then the geometric generic embedding dimension of X_ξ (that is, the embedding codimension of the generic point of $\overline{X}_\xi$) is strictly less than $\dim(S)$.*

In the same work, Schröer observes that a k-scheme X is geometrically reduced (that is, $X_{\bar{k}}$ is reduced) if and only if the base change $X_{k^{1/p}}$ of X by the height-one field extension $k \subseteq k^{1/p}$ is reduced. The analogous property for geometric regularity is a well-known result of EGA (cf. [**3**, Theorem IV.0.22.5.8]). We refine that result by proving the following proposition.

PROPOSITION 1.7. *Let k denote a field of characteristic $p > 0$ and let $x \in X$ denote a point in a k-variety X. If $x' \in X_{k^{1/p}}$ and $x^{(\infty)} \in X_{k^{1/p^\infty}}$ denote the preimages of x under the natural bijections $X_{k^{1/p^\infty}} \to X_{k^{1/p}} \to X$, then*

$$\mathrm{edim}_{X_{k^{1/p}}}(x') = \mathrm{edim}_{X_{k^{1/p^\infty}}}(x^{(\infty)}).$$

2. Regularity and smoothness

We briefly recall the definitions of the notions of regularity and smoothness.

DEFINITION 2.1. The *embedding dimension* of a locally Noetherian scheme X at a point $x \in X$ is the embedding dimension of the local ring $\mathcal{O}_{X,x}$ at the maximal ideal $\mathfrak{m}_x$, that is, the dimension of the Zariski cotangent space over the residue field $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$,

$$\mathrm{edim}_X(x) = \mathrm{edim}(\mathcal{O}_{X,x}) = \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2.$$

DEFINITION 2.2. A scheme X is *regular* if it is locally Noetherian and for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring (that is, the embedding dimension of $\mathcal{O}_{X,x}$ is equal to its Krull dimension).

DEFINITION 2.3. A scheme X is *smooth* over a field k if it is locally of finite type and geometrically regular over k (that is, $X \times_k \bar{k}$ is regular).

REMARK 2.4. Any smooth scheme is regular, and any regular scheme is locally integral. Therefore, any connected, separated scheme of finite type over k that is smooth over k (or regular) is automatically a *variety over k*, which in this paper refers to an integral, separated scheme of finite type over k.

Over a perfect field, the notions of regularity and smoothness are equivalent. However, over imperfect fields (of positive characteristic), a scheme may be regular but not smooth. We have already seen such an example: the generic fiber of the morphism $f : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ described in Section 1 is regular at all points but is not smooth since a cuspidal singularity appears after an algebraic extension of the function field $k(t)$. It turns out that pretty much all examples of regular varieties arise in this way—as generic fibers of morphisms between smooth varieties—and therefore the study of the singularities appearing in the geometric generic fibers of morphisms between smooth varieties reduces to the study of the singularities appearing in the geometric (that is, algebraically closed) base changes of regular varieties.

PROPOSITION 2.5. *Let Y be a variety over a finitely generated field extension K of a perfect field k. Y is regular if and only if there exists a morphism of smooth k-varieties $f : X \to B$ so that K is the function field of B and Y is the generic fiber of f.*

Proof. See [11, Proposition 1.6]. □

Notice that if X is a regular variety that is not smooth over k, then there exists a closed point $\bar{x} \in \overline{X} := X \times_k \bar{k}$ sitting over some point $x \in X$ such that

$$\mathrm{edim}_{\overline{X}}(\bar{x}) > \dim(\overline{X}) = \dim(X)$$
$$= \mathrm{edim}_X(x).$$

In this way, the existence of regular but nonsmooth schemes is directly linked to 'jumps' in embedding dimension after a geometric extension of scalars $\bar{k}/k$.

3. Jumps in embedding dimension

For any purely inseparable field extension k'/k, the morphism of affine schemes $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ is a universal homeomorphism. In particular, any k-algebra R

is local if and only if $R \otimes_k k'$ is so, which shows the following definition to be well formed.

DEFINITION 3.1. Let R be a local, finite-type k-algebra and k'/k a purely inseparable field extension. We define the *embedding jump* of R over the extension k'/k to be the difference between the embedding dimensions

$$\text{ejump}_{k'/k}(R) := \text{edim}(R \otimes_k k') - \text{edim}(R).$$

The *embedding jump* $\text{ejump}_{k'/k}(x)$ of a finite-type k-scheme X at a point $x \in X$ is defined by $\text{ejump}_{k'/k}(x) := \text{ejump}_{k'/k}(\mathcal{O}_{X,x})$.

REMARK 3.2. We make two easy observations about embedding jumps:

(1) embedding jumps are nonnegative (cf. [3, 0.IV.22.5.2.1]);

(2) because R is of finite type, any purely inseparable field extension k'/k admits some finite subextension $k \subseteq k''$ for which $\text{ejump}_{k'/k}(R) = \text{ejump}_{k''/k}(R)$.

In the special case that $X = \text{Spec } K = \{x\}$, for a finite field extension K/k, the embedding dimension $\text{edim}_X(x)$ is zero and so the embedding jump is simply the embedding dimension of the Artin local ring $K \otimes_k k'$,

$$\text{ejump}_{k'/k}(x) = \text{edim}(K \otimes_k k').$$

This quantity was studied by Schröer in [12, Proposition 2.1], where he proved the following theorem, which implies the 0-dimensional case of our main result (cf. Theorem 4.3).

PROPOSITION 3.3 (Schröer). *Let K/k be an extension of fields of characteristic $p > 0$, and let k'/k be a field extension that contains $k^{1/p}$. Then the embedding dimension of $K \otimes_k k'$ equals that of $K \otimes_k k^{1/p}$, which also equals the difference between the p-degree and the transcendence degree of the field extension K/k.*

For the reader's convenience, we now recall the definition of p-degree: The sheaf of Kähler differentials $\Omega_{X/k}$ on a variety X is a locally free $\mathcal{O}_X$-module of rank equal to $\dim X$ if and only if X is smooth over k. In characteristic 0, the transcendence degree of a finitely generated field extension K/k is equal to the dimension of the K-vector space of Kähler differentials $\Omega_{K/k}$. In characteristic p, this is no longer the case, suggesting that transcendence degree is perhaps not best suited for discussions of smoothness.

DEFINITION 3.4. Let K/k be an extension of fields of characteristic $p > 0$. The *p-degree of K/k* is defined to be the dimension of the K-vector space $\Omega_{K/k}$.

REMARK 3.5. If K/k is an arbitrary extension of fields of characteristic $p > 0$, then

$$p\text{-deg}(K/k) = p\text{-deg}(K/k(K^p)),$$

where $k(K^p)$ denotes the subfield of K generated by k and K^p. This is because $\Omega_{K/k} = \Omega_{K/k(K^p)}$, which holds since $d(f^p) = pf^{p-1}df = 0$ for all $f \in K$.

For a finitely generated extension K of a perfect field $\mathbf{F}$, the notion of p-degree and transcendence degree actually agree, due to the existence of a separating transcendence basis (cf. [8, Theorems 26.2–3]).

PROPOSITION 3.6. *If $\mathbf{F}$ is a perfect field and K is a finitely generated field extension, then*

$$p\text{-deg}(K/\mathbf{F}) = \text{tr. deg}(K/\mathbf{F}).$$

4. A bound on embedding jumps

In this section we prove our main result, which asserts a bound on the embedding jump at an arbitrary point of a regular variety in terms of the p-degree and transcendence degree of the residue field at that point. We begin by proving an essential lemma that bounds the jump in embedding dimension of a regular local Noetherian ring by that of its residue field.

LEMMA 4.1. *Let R be a local Noetherian ring, over a field k, with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. If k'/k is a purely inseparable field extension, then*

$$\text{ejump}_{k'/k}(R) \leqslant \text{edim}(\kappa \otimes_k k').$$

Proof. Denote by R' the local ring $R \otimes_k k'$, by $\mathfrak{m}'$ its maximal ideal, and by κ' its residue field $R'/\mathfrak{m}'$. Consider the short exact sequence of κ'-vector spaces,

$$0 \to \mathfrak{m}R'/(\mathfrak{m}R' \cap \mathfrak{m}'^2) \to \mathfrak{m}'/\mathfrak{m}'^2 \to \mathfrak{m}'/(\mathfrak{m}R' + \mathfrak{m}'^2) \to 0. \qquad (4.2)$$

As a κ'-vector space, the dimension of the middle term is $\text{edim}(R')$, by definition. We next consider the right-hand term, first noting the isomorphism

$$(\mathfrak{m}'/\mathfrak{m}R') \otimes_{R'} \kappa' \cong \mathfrak{m}'/(\mathfrak{m}R' + \mathfrak{m}'^2).$$

Clearly $\mathfrak{m}'/\mathfrak{m}R'$ is the maximal ideal of $R'/\mathfrak{m}R' \cong \kappa \otimes_k k'$. Since κ' is its residue field, the κ'-dimension of $(\mathfrak{m}'/\mathfrak{m}R') \otimes_{R'} \kappa'$ is equal to $\mathrm{edim}(\kappa \otimes_k k')$, which therefore equals the dimension of the right-hand term of (4.2).

To analyze the left-hand term of (4.2), observe that

$$\mathfrak{m}R'/\mathfrak{m}\mathfrak{m}' \cong (\mathfrak{m}/\mathfrak{m}^2) \otimes_\kappa \kappa',$$

and therefore

$$\dim_{\kappa'}(\mathfrak{m}R'/\mathfrak{m}\mathfrak{m}') = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{edim}(R).$$

Because of the natural inclusion, $\mathfrak{m}\mathfrak{m}' \subseteq \mathfrak{m}R' \cap \mathfrak{m}'^2$, we have the inequality

$$\dim_{\kappa'}(\mathfrak{m}R'/(\mathfrak{m}R' \cap \mathfrak{m}'^2)) \leqslant \dim_{\kappa'}(\mathfrak{m}R'/\mathfrak{m}\mathfrak{m}').$$

From the short exact sequence (4.2), it then follows that

$$\mathrm{edim}(R') = \mathrm{edim}(\kappa \otimes_k k') + \dim_{\kappa'}(\mathfrak{m}R'/(\mathfrak{m}R' \cap \mathfrak{m}'^2))$$
$$\leqslant \mathrm{edim}(\kappa \otimes_k k') + \mathrm{edim}(R). \qquad \square$$

We now combine the lemma with a result of Schröer to prove our main theorem.

THEOREM 4.3. *Let X be a regular k-variety. If k'/k is a purely inseparable extension, then for any $x \in X$ with residue field $\kappa(x)$, the embedding jump satisfies*

$$\mathrm{ejump}_{k'/k}(x) \leqslant p\text{-}\deg(\kappa(x)/k) - \mathrm{tr.deg}(\kappa(x)/k).$$

Proof. By Lemma 4.1, the jump in embedding dimensions is bounded by

$$\mathrm{ejump}_{k'/k}(x) \leqslant \mathrm{edim}(\kappa(x) \otimes_k k').$$

Choose embeddings of k' and $k^{1/p}$ into the algebraic closure $\bar{k}$, and let $k'' \subseteq k$ denote the subfield generated by k' and $k^{1/p}$. By Remark 3.2, $\mathrm{edim}(\kappa(x) \otimes_k k') \leqslant \mathrm{edim}(\kappa(x) \otimes_k k'')$. Schröer's result (Proposition 3.3) implies that $\mathrm{edim}(\kappa(x) \otimes_k k'')$ equals $\mathrm{edim}(\kappa(x) \otimes_k k^{1/p})$ and also equals the difference between the p-degree and the transcendence degree of the extension $\kappa(x)/k$. $\qquad \square$

Our primary applications of the above result will be through the following geometric consequence.

COROLLARY 4.4. *Let $f : \mathcal{X} \to B$ be a morphism of smooth varieties over a perfect field $\mathbf{F}$. The embedding dimension $\mathrm{edim}_{\overline{X}}(\bar{x})$ at any point $\bar{x} \in \overline{X} := X \times_k \bar{k}$ satisfies*

$$\mathrm{edim}_{\overline{X}}(\bar{x}) \leqslant \mathrm{edim}_X(x) + \dim(B),$$

where $x \in X$ denotes the point lying under $\bar{x} \in \overline{X}$.

Proof. By Theorem 4.3, $\mathrm{ejump}_{\bar{k}/k}(x) \leqslant p\text{-deg}(\kappa(x)/k) - \mathrm{tr.\,deg}(\kappa(x)/k)$, where $\kappa(x)$ denotes the residue field of $x \in X$. Clearly

$$p\text{-deg}(\kappa(x)/k) \leqslant p\text{-deg}(\kappa(x)/\mathbf{F}) = \mathrm{tr.\,deg}(\kappa(x)/\mathbf{F}),$$

with the latter equality following from the perfection of $\mathbf{F}$ and Proposition 3.6. Therefore,

$$\begin{aligned}
\mathrm{edim}_{\overline{X}}(\bar{x}) - \mathrm{edim}_X(x) &\leqslant \mathrm{tr.\,deg}(\kappa(x)/\mathbf{F}) - \mathrm{tr.deg}(\kappa(x)/k) \\
&= \mathrm{tr.\,deg}(k/\mathbf{F}) \\
&= \dim(B). \qquad \square
\end{aligned}$$

5.   Regular del Pezzo surfaces

The primary motivation for this investigation was to determine which singular del Pezzo surfaces can occur as the geometric generic fiber of the contraction of an extremal curve class on a smooth 3-fold. Although we do not answer this question definitively, the above results do rule out the nasty examples in characteristics $p > 3$ of nonnormal del Pezzo surfaces X with $H^1(X, \mathcal{O}_X) \neq 0$.

PROPOSITION 5.1. *Let X be a regular del Pezzo surface over a finitely generated field extension $k/\mathbf{F}$ of a perfect field $\mathbf{F}$ of characteristic p and transcendence degree $\mathrm{tr.\,deg}(k/\mathbf{F}) = d$. If $d \leqslant 1$ then X is geometrically reduced. If $p > d + 2$ and X is geometrically reduced, then $H^1(X, \mathcal{O}_X) = 0$.*

Proof. If k is of transcendence degree at most 1 over the perfect field $\mathbf{F}$, then $\overline{X} := X \times_k \bar{k}$ is reduced (cf. [12]). By the classification of normal del Pezzo surfaces over an algebraically closed field (cf. [4]), the result is true if $\overline{X}$ is normal. This just leaves the case where $\overline{X}$ is integral but nonnormal (and hence where $d > 0$). Such examples were classified by Reid (cf. [10]). In particular, in characteristics $p > 3$, the nonvanishing $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \neq 0$ is only possible when there exist points $\bar{x} \in \overline{X}$ with $\mathrm{edim}_{\overline{X}}(\bar{x}) = p$ (cf. [10, Section 4.4]). By Corollary 4.4, $\mathrm{edim}_{\overline{X}}(\bar{x}) \leqslant d + 2$ for all $\bar{x} \in \overline{X}$, and therefore $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$, which implies $H^1(X, \mathcal{O}_X) = 0$. $\square$

6.   Jumping is a height-one phenomenon

An extension of characteristic p fields L/K is said to be of *height* one if $L^p \subseteq K$. As a consequence of Theorem 4.3, we show that jumps in embedding dimension are a strictly height-one phenomenon. As a corollary, we recover the well-known result [3, Theorem IV.0.22.5.8] that asserts that geometric regularity

may be checked over height-one field extensions. We set the following notation for this section.

NOTATION 6.1. *For an imperfect field k of characteristic p, an element $t \in k \backslash k^p$, and a k-algebra R, set:*

(1) $k_n := k(\sqrt[p^n]{t})$; *and*

(2) $R_n := R \otimes_k k_n$.

LEMMA 6.2. *Let $R = K$ be a finitely generated field extension of an imperfect field k of characteristic p. If $t \in k \backslash k^p$ and $m := \max\{k \in \mathbb{N} : t \in K^{p^k}\}$, then for all $0 < n \leqslant m$, the ring*

$$R_n \cong \begin{cases} K[\varepsilon_n]/(\varepsilon_n^{p^n}) & : if\ 0 \leqslant n \leqslant m \\ K(\sqrt[p^n]{t})[\varepsilon_n]/(\varepsilon_n^{p^m}) & : if\ m < n, \end{cases}$$

and the natural ring inclusion $R_{n-1} \subseteq R_n$ is given by

$$\begin{cases} \varepsilon_{n-1} \mapsto \varepsilon_n^p & : if\ 0 \leqslant n \leqslant m \\ \varepsilon_{n-1} \mapsto \varepsilon_n & : if\ m < n. \end{cases}$$

In particular, the residue field of R_n equals K if and only if $\sqrt[p^n]{t} \in K$.

Proof. If $0 \leqslant n \leqslant m$, then $\sqrt[p^n]{t} \in K$ and therefore $R_n = K \otimes_k k(\sqrt[p^n]{t})$ is isomorphic to the Artin local ring $K[\varepsilon_n]/(\varepsilon_n^{p^n})$, where $\varepsilon_n := \sqrt[p^n]{t} \otimes 1 - 1 \otimes \sqrt[p^n]{t}$. Moreover, it follows then that $\varepsilon_{n-1} = \varepsilon_n^p$. On the other hand, if $m < n$, then $\sqrt[p^n]{t} \notin R_{n-1}$. Moreover, since $k_m \subseteq K$, the result follows by composing the following isomorphisms:

$$\begin{aligned} K \otimes_k k(\sqrt[p^n]{t}) &\cong (K \otimes_k k_m) \otimes_{k_m} k_n \\ &\cong K[\varepsilon_m]/(\varepsilon_m^{p^m}) \otimes_{k_m} k_m(\sqrt[p^n]{t}) \\ &\cong K(\sqrt[p^n]{t})[\varepsilon_m]/(\varepsilon_m^{p^m}). \qquad \square \end{aligned}$$

PROPOSITION 6.3. *Let k be an imperfect field of characteristic p. If $t \in k \backslash k^p$ and R is a Noetherian local k-algebra, then for any $n \geqslant 1$,*

$$\mathrm{ejump}_{k_1/k}(R) = \mathrm{ejump}_{k_n/k}(R).$$

Proof. First, assume we have proven the result in the base case $n = 2$. By applying this to the field k_{n-2} and the Noetherian local k_{n-2}-algebra R_{n-2}, it would follow

that for any $n \geqslant 2$,

$$\mathrm{ejump}_{k_n/k_{n-2}}(R_{n-2}) = \mathrm{ejump}_{k_{n-1}/k_{n-2}}(R_{n-2}).$$

Using this equality, we derive the general result by observing

$$\begin{aligned}
\mathrm{ejump}_{k_n/k}(R) &= \mathrm{ejump}_{k_n/k_{n-2}}(R_{n-2}) + \mathrm{ejump}_{k_{n-2}/k}(R) \\
&= \mathrm{ejump}_{k_{n-1}/k_{n-2}}(R_{n-2}) + \mathrm{ejump}_{k_{n-2}/k}(R) \\
&= \mathrm{ejump}_{k_{n-1}/k}(R),
\end{aligned}$$

and then arguing inductively. Thus, it suffices to prove the result in the case $n = 2$.

Let $n = 2$ and note by Remark 3.2(1) and Theorem 4.3 that

$$0 \leqslant \mathrm{ejump}_{k_1/k}(R) \leqslant \mathrm{ejump}_{k_2/k}(R) \leqslant 1.$$

Equality follows immediately in the case $\mathrm{ejump}_{k_1/k}(R) = 1$, so we henceforth assume $\mathrm{ejump}_{k_1/k}(R) = 0$. It easily follows that $\mathrm{ejump}_{k_2/k}(R) = \mathrm{ejump}_{k_2/k_1}(R_1)$, and we finish the proof by showing that this quantity also is zero.

Let K, K_1, and K_2 be the residue fields of R, R_1, and R_2, respectively, and denote by $\mathfrak{m}$, $\mathfrak{m}_1$, and $\mathfrak{m}_2$ the corresponding maximal ideals. Clearly there are inclusions $K \subseteq K_1 \subseteq K_2$. As K_1 is a quotient of $K \otimes_k k_1$ and K_2 is a quotient of $K_1 \otimes_{k_1} k_2$, it follows that $[K_1 : K], [K_2 : K_1] \in \{1, p\}$. Moreover, $[K_1 : K] = p$ if and only if $K_1 = K \otimes_k k_1$, that is, if and only if $\mathfrak{m}_1 = \mathfrak{m} R_1$, and similarly, $[K_2 : K_1] = p$ if and only if $\mathfrak{m}_2 = \mathfrak{m}_1 R_2$. We shall conclude the proof by analyzing separately the following cases.

Case: $[K_2 : K_1] = p$. As noted above, this holds only if $\mathfrak{m}_2 = \mathfrak{m}_1 R_2$. It follows that $\mathrm{ejump}_{k_2/k_1}(R_1) = 0$.

Case: $[K_1 : K] = p$. Since K_1 is the residue field of $K \otimes_k k_1$ and $K_1 \neq K$, Lemma 6.2 implies that $\sqrt[p]{t} \notin K$ and hence $\sqrt[p^2]{t} \notin K$. This means that K_2, which contains $\sqrt[p^2]{t}$ and is at most a p^2-dimensional vector space over K, must be precisely $K_2 = K(\sqrt[p^2]{t})$ with $[K_2 : K_1] = p$. It follows that $\mathfrak{m}_2 = \mathfrak{m}_1 R_2$, and hence that $\mathrm{ejump}_{k_2/k_1}(R_1) = 0$.

Case: $[K_1 : K] = [K_2 : K_1] = 1$. Since $K = K_2$ is the residue field of $K \otimes_k k_2$, Lemma 6.2 implies that $\sqrt[p^2]{t} \in K$. Another application of Lemma 6.2 yields the isomorphisms

$$K \otimes_k k_1 \cong R_1/\mathfrak{m} \cong K[\varepsilon_1]/(\varepsilon_1^p), \tag{6.4}$$

$$K \otimes_k k_2 \cong R_2/\mathfrak{m} \cong K[\varepsilon_2]/(\varepsilon_2^{p^2}), \tag{6.5}$$

where the natural inclusion $R_1/\mathfrak{m} \to R_2/\mathfrak{m}$ is given by $\varepsilon_1 \mapsto \varepsilon_2^p$. Choosing $f \in \mathfrak{m}_2$ to be any lift of ε_2, it follows that $f^p \in \mathfrak{m}_1$ is a lift of ε_1. Therefore $\mathfrak{m}_2 = \mathfrak{m} R_2 + (f)$

and $\mathfrak{m}_1 = \mathfrak{m}R_1 + (f^p)$. Notice that $f^p \notin \mathfrak{m}_1^2$ and so by Nakayama's lemma, it is included in some minimal set of generators $f^p, x_2, x_3, \ldots, x_m$ for the R_1-ideal $\mathfrak{m}_1$. Furthermore, we may choose these generators so that $x_2, \ldots, x_m \in \mathfrak{m}R_1$. Here $m = \text{edim}(R_1) = \text{edim}(R)$, since $\text{ejump}_{k_1/k}(R) = 0$. As ideals in R_2, we have

$$
\begin{aligned}
(f, x_2, \ldots, x_m) &= (f) + (f^p, x_2, \ldots, x_m) \\
&= (f) + \mathfrak{m}_1 R_2 \\
&= (f) + (f^p) + \mathfrak{m}R_2 \\
&= \mathfrak{m}_2.
\end{aligned}
$$

Therefore $\text{edim}(R) \leqslant \text{edim}(R_2) \leqslant m = \text{edim}(R)$, and hence $\text{ejump}_{k_2/k}(R) = 0$. $\qquad\square$

COROLLARY 6.6. *Let R be a local Noetherian k-algebra. If K/k is a purely inseparable field extension and $K' := K \cap k^{1/p}$, then*

$$
\text{ejump}_{K/k}(R) = \text{ejump}_{K'/k}(R).
$$

Proof. Since R is Noetherian, we may assume that K is finitely generated, so that $K = k(\sqrt[p^{n_1}]{t_1}, \ldots, \sqrt[p^{n_r}]{t_r})$ for certain $t_i \in k \backslash k^p$. It follows by inducting on r and applying Proposition 6.3 that $\text{ejump}_{K/k}(R) = \text{ejump}_{K'/k}(R)$, where $K' := k(\sqrt[p]{t_1}, \ldots, \sqrt[p]{t_r}) = K \cap k^{1/p}$. $\qquad\square$

We recover, as a further corollary, the following result (cf. [3, Theorem IV.0.22.5.8]).

THEOREM 6.7 (EGA). *Let X be a regular variety over a field k. X is smooth over k if and only if $X \times_k k^{1/p}$ is a regular variety over $k^{1/p}$.*

Proof. Let $R = \mathcal{O}_{X,x}$, for an arbitrary point $x \in X$. It follows from Corollary 6.6 that $\text{ejump}_{k^{1/p^\infty}/k}(R) = \text{ejump}_{k^{1/p}/k}(R)$. Since $\bar{k}/k^{1/p^\infty}$ is a separable field extension, $R \otimes_k \bar{k}$ is regular if and only if $R \otimes_k k^{1/p^\infty}$ is regular. The result then follows from the observation that for any field extension k'/k and regular local ring R, the base change $R \otimes_k k'$ is regular if and only if $\text{ejump}_{k'/k}(R) = 0$. $\qquad\square$

7.   Future directions

We leave an open question for future research.

QUESTION 7.1. Does there exist a regular del Pezzo surface X with $H^1(X, \mathcal{O}_X) \neq 0$ over a field of transcendence degree 1 over a perfect field?

By Proposition 5.1, if an example does exist, it occurs in characteristic 2 or 3. The author has constructed examples in characteristic 2 of regular del Pezzo surfaces X with $H^1(X, \mathcal{O}_X) \neq 0$ over fields of transcendence degree at least 3 (cf. [7]) and shall describe a similar example over a field of transcendence degree 2 in a forthcoming paper.

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