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A Bound on Local Minima of Arrangements that Implies the Upper Bound Theorem

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Abstract. This paper shows that the i-level of an arrangement of hyperplanes in E^d has at most $\binom{i+d-1}{d-1}$ local minima. This bound follows from ideas previously used to prove bounds on $(\leq k)$-sets. Using linear programming duality, the Upper Bound Theorem is obtained as a corollary, giving yet another proof of this celebrated bound on the number of vertices of a simple polytope in E^d with n facets.

1. Introduction

We need some terminology for arrangements, similar to that in Edelsbrunner's text [3]. Let $\mathscr{A}(H)$ be a simple arrangement of a set H of n hyperplanes in E^d. For $h \in H$, let the upper half-space h^+ be the open half-space bounded by h that contains $(\infty, 0, \ldots, 0)$, and let the lower half-space h^- be the other open half-space bounded by h. Say that $x \in E^d$ is above $h \in H$ if $x \in h^+$, and below h if $x \in h^-$. The i-level of $\mathscr{A}(H)$ is the boundary of the set of points that are below no more than i hyperplanes of H. Thus, for example, the 0-level of $\mathscr{A}(H)$ is the boundary of the convex polytope $\mathscr{P}(H) = \bigcap_{h \in H}(h^+ \cup h)$. The maximum number of vertices of $\mathscr{A}(H)$ on its i-level is a combinatorial problem of long standing. While some results have long been known for $d = 2$ [4], and recently sharpened slightly [8], only relatively recently have nontrivial bounds been known for the general problem in higher dimensions. These results are stated in a dual form, concerning *k-sets* of sets of points. One related result is that the maximum total number of vertices on all i-levels, for $i \leq k$, is $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$, a $(\leq k)$-*set* bound [2].

Using similar techniques, Mulmuley then showed that the number of local minima on levels $i \le k$ is $O(k^d)$, where a local minimum is the point of the i-level such that all points on the i-level in a neighborhood of the point have a larger x_1 coordinate. Call a local minimum on the i-level an *i-minimum*, or a $(\le k)$-minimum if $i \le k$. An i-minimum is a vertex, and so Mulmuley's result is a bound on a class of vertices of the i-level. Note that the 0-minimum of H is the solution $x^*(H)$ of the linear programming problem $\min\{x_1 | x \in \mathscr{P}(H)\}$. In addition to bounding the number of $(\le k)$-minima, Mulmuley showed some bounds on related quantities, and conjectured that the number of i-minima is $O(i^{d-1})$ for every i [7]. This conjecture is confirmed here by the bound $\binom{i+d-1}{d-1}$, proven in the next section using the same technique as for bounds on $(\le k)$-sets and $(\le k)$-minima.

This i-minima bound is of course not new for $i = 0$ and $i = 1$, and it is not even new for $i = n/2$: using (projective or polar) duality, it is equivalent to the preliminary observation for $d = 2$ that forms the basis of a bound on the number of vertices on the $n/2$-level in E^2 [4]. Thus the contribution here is mostly one of observed connections and new proofs, and not new theorems.

Section 3 uses ideas of linear programming duality to show that the bound on i-minima readily implies the celebrated Upper Bound Theorem for convex polytopes [6], [1]. Here we mean only the upper bound of that theorem, and do not characterize the polytopes for which the bound is tight.

2. The Bound for *i*-Minima

Some preliminary notation: for a set S, let $\binom{S}{k}$ denote the collection of subsets of S of size k, so

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}.$$

We sometimes use the coordinatewise partial order on E^d where $x, y \in E^d$ have $x > y$ if $x_i > y_i$ for $i = 1, \ldots, d$.

The bound for i-minima follows from the following well-known properties of solutions of linear programming problems.

Lemma 2.1. *Any arrangement $\mathscr{A}(H)$ has at most one 0-minimum $x^*(H)$, and if it exists, there is $B \subset H$ of size d with $x^*(B) = x^*(H)$.*

Proof. Omitted; the second statement follows from Helly's theorem, as applied to the upper half-spaces of H and the half-spaces $\{x_1 \le q\}$, for all q smaller than the first coordinate of $x^*(H)$. □

Call the set B promised by the lemma a *basis* $b(H)$ of H. We can extend the notations $\mathscr{P}(H)$, $x^*(H)$, and $b(H)$ to subsets of H in the obvious way; however, for many $G \subseteq H$, the linear programming problem $\mathscr{LP}(G)$, of finding

$$\min\{x_1 | x \in \mathscr{P}(G)\},$$

may be unbounded, or have many solutions, and even if $x^*(G)$ is unique, there may not be a unique basis $b(G)$. To apply the lemma and bound i-minima, the definitions of $x^*(G)$ and $b(G)$ are extended below to all $G \subseteq H$, using lexicographic orders, such that every $G \subseteq H$ has a unique basis.

A point $x = (x_1, \ldots, x_d)$ is lexicographically (lex) smaller than point $y = (y_1, \ldots, y_d)$, written $x \prec y$, if $x_i < y_i$ for the smallest i at which their coordinates differ. For sufficiently small $\varepsilon > 0$ we have $x \prec y$ if and only if $x \cdot b_\varepsilon < y \cdot b_\varepsilon$, where $b_\varepsilon = (1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{d-1})$.

We broaden the definition of local minimum to include vertices that have lexicographically minimal (lexmin) coordinates in a neighborhood on the i-level. Thus for $G \subset H$, if the associated linear programming problem $\mathscr{LP}(G)$ has a bounded solution, then $x^*(G)$ exists and is unique. Note that a basis $b(G)$ yielding $x^*(G)$ also exists.

Extend the definition of $x^*(G)$ to the unbounded case as follows: choose a sufficiently small value K so that all vertices v of $\mathscr{A}(H)$ have all coordinates larger than K. Define $\underline{x}^*(G)$ as the lexmin point in $\mathscr{P}(G)$ with all coordinates no smaller than K.

With these definitions, all $G \subseteq H$ have a 0-minimum $\underline{x}^*(G)$, which is the same as the initial definition when $\mathscr{LP}(G)$ has a unique vertex with minimum x_1 coordinate. It remains to extend the notion of basis $b(G)$ appropriately. Here again lexicography is useful.

Given a set S of integers $\{i | 1 \le i \le n\}$, the lexicographic order on $\binom{S}{k}$ is as follows: for $A, B \in \binom{S}{k}$, order A and B so that $A = \{a_1, \ldots, a_k\}$ and $a_1 < a_2 < \cdots < a_k$ and similarly order $B = \{b_1, \ldots, b_k\}$. Now $A \prec B$ if and only if $a_i < b_i$ at the smallest index i at which they differ.

We impose a lexicographic order on $\binom{H}{d}$ by numbering the hyperplanes of H arbitrarily from 1 to n and then saying $A, B \in \binom{H}{d}$ have $A \prec B$ if and only if the associated sets of numbers A' and B' have $A' \prec B'$.

To define the basis $b(G)$ for $G \subset H$, let $b(G)$ denote the lexmin $B \in \binom{G}{d}$ so that $\underline{x}^*(B) = \underline{x}^*(G)$. Note that some of the hyperplanes determining $\underline{x}^*(G)$ may be of the form $x_i \ge K$, if $\mathscr{LP}(G)$ is unbounded and $x^*(G)$ does not exist; they are replaced in $b(G)$ by the smallest-numbered elements of G that are not above $\underline{x}^*(G)$.

An i-basis is defined as follows. For $B \in \binom{H}{d}$, note that $b(B) = B$, and define

$$I_B \equiv \{h \in H | b(B \cup \{h\}) \ne B\}.$$

That is, an element $h_j \in I_B$ is either above $\underline{x}^*(B)$, or there is some $h_k \in B$ with $j < k$ and

$$\underline{x}^*(B \backslash \{h_k\} \cup \{h_j\}) = \underline{x}^*(B),$$

so a lexicographically smaller subset with the same minimum can be obtained. If

I_B has i members, call B an i-basis. Note that every i-minimum has a corresponding i-basis. We count the i-minima by counting the i-bases.

Let $g_i(H)$ denote the number of i-minima of H, and let $g'_i(H)$ denote the number of i-bases. We have the following theorem.

Theorem 2.2. *If $\mathscr{A}(H)$ is an arrangement of n hyperplanes in E^d, then*

$$g_i(H) \leq g'_i(H) = \binom{i+d-1}{d-1}.$$

Proof. As discussed above, each i-minimum of $\mathscr{A}(H)$ has a corresponding i-basis, and each i-basis determines at most one i-minimum, so $g_i(H) \leq g'_i(H)$ and it suffices to count the i-bases. Consider a random $R \in \binom{H}{r}$, where $d \leq r \leq n$. Here each element of $\binom{H}{r}$ is equally likely. Any subset has exactly one basis. On the other hand, we can express the expected number of bases of R as

$$\sum_{B \in \binom{H}{d}} \operatorname{Prob}\{B \subset R,\ R \subseteq H \backslash I_B\},$$

since $B \in \binom{H}{d}$ is the basis of R if and only if $B \subset R$ and no elements of I_B appear in R. If B is an i-basis, the number of subsets $R \in \binom{H}{r}$ with $b(R) = B$ is $\binom{n-i-d}{r-d}$, since B must be in R, and the remaining $r-d$ choices of elements of R must be from $H \backslash B \backslash I_B$. Therefore the probability that i-basis B is the basis of R is $\binom{n-i-d}{r-d}/\binom{n}{r}$, and we have

$$1 = \sum_{0 \leq i \leq n-d} \frac{\binom{n-i-d}{r-d}}{\binom{n}{r}} g'_i(H) \tag{1}$$

for $d \leq r \leq n$. This equation is a special case of Lemma 2.1 of [2]. Since the matrix corresponding to this system of $n-d+1$ linear equations in $n-d+1$ unknowns can be rearranged to be triangular with positive diagonal elements, the system can be solved, and the reader can verify that the solution is $\binom{i+d-1}{d-1}$. □

This bound for $g_i(H)$ is not very good for large i; for example, there is at most one $(n-d)$-minimum, while there are $\binom{n-1}{d-1}$ $(n-d)$-bases. However, it is easy to show that a set B of d hyperplanes yields a minimum point x if and only if x is a maximum point in $\bigcap_{h \in B}(h^- \cup h)$. Hence $g_i(H) = g_{n-d-i}(H)$, and we have the following theorem.

Theorem 2.3. *For any simple arrangement $\mathscr{A}(H)$ of n hyperplanes in E^d, the number of i-minima $g_i(H)$ satisfies*

$$g_i \leq \min\left\{\binom{i+d-1}{d-1}, \binom{n-i-1}{d-1}\right\}.$$

3. The Upper Bound Theorem

The g-Vector of a Polytope. Suppose $\mathscr{P}$ is a simple d-polytope with at most n facets, and is the set of points $\{x \in E^d \mid Ax \leq b\}$, where A is an $n \times d$ matrix, x and b are column n-vectors, and $b \geq 0$. Since all entries of b are nonnegative, the origin is in $\mathscr{P}$. We also write the inequalities as $a_j x \leq b_j$, for $j = 1, \ldots, n$. Suppose w is an *admissible* row n-vector for $\mathscr{P}$, meaning that $wv \neq wv'$ for any two distinct vertices v and v' of $\mathscr{P}$. Orient the edges of the $\mathscr{P}$ in the direction of increasing w (*upward*) and let $g_i(\mathscr{P})$ denote the number of vertices with outdegree i, so that i of their incident edges point up. If $f_k(\mathscr{P})$ is the number of k-faces of $\mathscr{P}$, then

$$f_k(\mathscr{P}) = \sum_i \binom{i}{k} g_i(\mathscr{P}), \tag{2}$$

since each k-face F has a unique bottom vertex v, with all k edges in F incident to v pointing up. To bound the quantities $f_k(\mathscr{P})$ it is enough to bound $g_i(\mathscr{P})$. (The above condenses the discussion in Brøndsted's text of McMullen's proof of the Upper Bound Theorem [6], [1].)

The LP-Dual Arrangement. The linear programming problem

$$\max\{wx \mid x \in \mathscr{P}\}$$

has the dual problem

$$\min\{yb \mid y \in \mathscr{P}'\},$$

where

$$\mathscr{P}' = \{y \in E^n \mid y \in \mathscr{F},\ y \geq 0\},$$

and

$$\mathscr{F} = \{y \in E^n \mid yA = w\}$$

is an $(n-d)$-flat. Letting $d' = n - d$, the d'-polytope $\mathscr{P}'$ is one cell in the arrangement $\mathscr{A}(H)$ induced by the collection H of n hyperplanes $h_j \equiv \{y \mid y_j = 0\}$, $j = 1, \ldots, n$, restricted to $\mathscr{F}$. (Note that while the previous section discussed arrangements in E^d, here we consider one in a d'-flat.) We can define local minima for this arrangement where we seek minima of yb. We have the following lemma. It is standard [5, Section 8.2], but for completeness a proof appears below (neglecting some issues of degeneracy).

Lemma 3.1. *There is a one-to-one correspondence between i-minima of $\mathscr{A}(H)$ and vertices of $\mathscr{P}$ with outdegree i, and so $g_i(\mathscr{P}) = g_i(H)$.*

Proof. If v is a vertex of $\mathscr{P}$, then v is the solution of $\hat{A}v = \hat{b}$, a subsystem of d rows of $Ax \leq b$. Suppose $v' \in \mathscr{F}$ has zero coordinates for all but those corresponding to the rows giving $\hat{A}$. Thus v' is a vertex of $\mathscr{A}(H)$: it is the intersection of d' hyperplanes of H with $\mathscr{F}$. The nonzero coordinates of v' are determined by $v'A = w$.

First observe that v' is a local minimum $x^*(G)$ for $G = \{h_j | v'_j = 0\}$: note that if $y \in \mathscr{F}$, so $yA = w$, then $yb - wx = yb - yAx = y(b - Ax)$. Thus $v'b - wv = v'(b - Av) = 0$ since $v'_j = 0$ if and only if $a_j v \neq 0$. (So v' and v has the same objective function values in the dual linear programming problems.) On the other hand, if $yA = w$ and $y_j \geq 0$ when $v'_j = 0$, we have $yb - wv = y(b - Av) \geq 0$ since $b - Av \geq 0$ and $a_j v = b_j$ when $v'_j \neq 0$. Thus if $y \in \mathscr{P}'(G)$, then $yb \geq v'b$. Note that the inequality is strict if $y_j > 0$ for some j with $a_j v < b_j$.

Next to show that if v has outdegree i, then v' is an i-minimum. Since $v'_j < 0$ if and only if v' is below h_j, we need to show that a coordinate $v'_j \neq 0$ corresponds to an oriented edge (v, q) where $wv - wq = w(v - q)$ has the same sign as v'_j. Suppose (v, q) is an edge of $\mathscr{P}$. Then $\hat{A}v = \hat{b} \geq \hat{A}q$, with one strict inequality $a_j v = b_j > a_j q$, and with equality for the other rows of $\hat{A}$. This implies that $w(v - q) = v'A(v - q) = v'_j a_j(v - q)$, and since $a_j(v - q) > 0$, v'_j and $w(v - q)$ have the same sign. □

We have the Upper Bound Theorem, missing the proof that the given bound is tight for dual neighborly polytopes.

Theorem 3.2. *The number of k-faces of a simple polytope in E^d with n facets is at most*

$$\sum_i \binom{i}{k} \min\left\{\binom{i + n - d - 1}{n - d - 1}, \binom{n - i - 1}{n - d - 1}\right\}.$$

Proof. The bound follows by applying the previous lemma, (2), and Theorem 2.3. □

4. Concluding Remarks

It is curious that the $(\leq k)$-set bounds of [2] both rely on the Upper Bound Theorem and are proven using an argument like the proof of Lemma 2.2. Perhaps some more direct argument for them exists.

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